Fourier Transform for Partial Differential Equations

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Introduction: Fourier Transform

The Fourier transform creates another representation of a signal, specifically a representation as a weighted sum of complex exponentials. It is designed for non-periodic signals that decay at infinity, the condition that $\int_{-\infty}^{\infty} |f(x)| dx$ is finite.

Because of Euler's formula

$$e^{iq} = \cos(q) + i \sin(q)$$

where $i^2 = -1$, the Fourier transform produces a representation of a signal (or an image) as a weighted sum of sines and cosines.

Given a signal (or image) a and its Fourier transform A, then the **forward Fourier transform** goes from the spatial domain, either continuous or discrete, to the frequency domain, which is always continuous. The **inverse Fourier transform** goes from the frequency domain back to the spatial domain.

$$\text{Forward}: \quad A = F(a), \quad \text{Inverse}: \quad a = F^{-1}(A)$$

Definition: Fourier Transform

$$F(w) = FT[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$$f(x) = FT[f]^{-1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w) e^{iwx} dw$$

The **Reciprocity Relation** connects the two similar formulas:

$$\begin{aligned} f(-u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w) e^{-iwu} dw \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-iux} dx \\ &= FT[F](u) \end{aligned}$$

In words, setting $x = -u$ in the inverse Fourier transform equation produces the forward Fourier transform equation for the function F.

Fourier Transform Properties

1. Linearity $FT[f+g]=FT[f]+FT[g]$ and $FT[cf]=cFT[f]$
2. x-differentiation $FT[f']=(iw)FT[f]$
3. w-differentiation $FT[xf(x)]=i\dfrac{d}{dw}FT[f]$
4. Convolution $FT[f*g]=FT[f]FT[g]$, where
   $f*g(x)=g*f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-t)g(t)dt$
5. x-shifting $FT[f(x-a)]=e^{-iaw}FT[f]$
6. w-shifting $FT[e^{iax}f(x)](w)=FT[f](w-a)$

Parseval's Energy Identity

The square $|f(t)|^2$ of the time signal represents how the energy contained in the signal distributes over time t, while the spectrum squared $|F(w)|^2$ represents how the energy distributes over frequency (the power density spectrum). The same amount of energy is contained in either time or frequency domain, because of Parseval's formula:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(w)|^2 dw$$

To evaluate what this means graphically, compute both integrands and graph them on a large interval. Use the Gaussian example

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

Fourier Sine and Cosine Integral Representations

The theory applies to functions $f(x)$ defined only on $0 < x < \infty$.

Definition. The Fourier Cosine Integral Representation

$$f(x) = \int_0^\infty A(w)\cos(wx)dw, \quad A(w) = \frac{2}{\pi}\int_0^\infty f(t)\cos(wt)dt$$

Definition. The Fourier Sine Integral Representation

$$f(x) = \int_0^\infty B(w)\sin(wx)dw, \quad B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin(wt)dt$$

Fourier Sine and Cosine Transforms

Definition. The Fourier Cosine and Sine Forward Transforms

$$FCT[f](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos(wt) dt,$$

$$FST[f](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(wt) dt$$

Definition. The Fourier Cosine and Sine Inverse Transforms

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty FCT[f](w) \cos(wx) dw,$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty FST[f](w) \sin(wx) dw$$

Fourier Sine and Cosine Transform Properties

Theorem 1 (Properties)

- $FCT[f](w) = FT[f](w)$ for $w \geq 0$, provided f is even on $(-\infty, \infty)$.
- $FST[f](w) = FT[f](w)$ for $w \geq 0$, provided f is odd on $(-\infty, \infty)$.
- Both FCT and FST satisfy the Fourier transform's linearity property.
- $FCT[f'] = w\, FST[f] - \sqrt{2/\pi} f(0)$, provided $\lim_{x\to\infty} f(x) = 0$.
- $FST[f'] = -w\, FCT[f]$, provided $\lim_{x\to\infty} f(x) = 0$.
- $FCT[xf(x)] = \frac{d}{dw} FST[f]$
- $FST[xf(x)] = -\frac{d}{dw} FCT[f]$