Name ________________________________

Partial Differential Equations 3150

Sample Midterm Exam 1

Exam Date: Wednesday, 27 February

Instructions: This exam is timed for 50 minutes. Up to 60 minutes is possible. No calculators, notes, tables or books. Problems use only chapters 1 and 2 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Vibration of a Finite String)
The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes.

Solve the finite string vibration problem on $0 \le x \le 1$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= 2\sin(\pi x) - 3\sin(5\pi x), \\ u_t(x,0) &= 0. \end{aligned}$$

Answer:

Because the wave initial velocity is zero, then the only normal modes are sine times cosine. The initial wave shape can be modified to a solution by inserting the missing cosine factors present in the corresponding normal mode. Then $u(x,t) = 2\sin(\pi x)\cos(c\pi t) - 3\sin(5\pi x)\cos(5c\pi t)$. We check it is a solution.

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2. (Periodic Functions)

(a) [30%] Find the period of $f(x) = \sin 2x \cos 2x$.

(b) [40%] Let $T = 2$. If $f(x)$ is the T-periodic extension of the function $f_0(x) = x(x-2)$ on $0 \le x \le 2$, then find $f(-3)$.

(c) [30%] Is $f(x) = \cos(\sin(x))$ an even periodic function?

Answer:

(a) $f(x) = (1/2)\sin(4x)$ by a trig identity. Then period $= 2\pi/4$.
(b) $f(-3) = f(-3+T+T) = f(1) = f_0(1) = -1$.
(c) Yes. Details: $f(-x) = \cos(\sin(-x)) = \cos(-\sin(x)) = \cos(\sin(x)) = f(x)$.

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3. (Fourier Series)
Let $f_0(x) = 1$ on the interval $0 < x < \pi$, $f_0(x) = -1$ on $-\pi < x < 0$, $f_0(x) = 0$ for $x = 0, \pi, -\pi$. Let $f(x)$ be the 2π-periodic extension of f_0 to the whole real line.

(a) [80%] Compute the Fourier coefficients for the terms $\sin(5x)$ and $\cos(4x)$.

(b) [20%] Which values of x in $|x| < 3\pi$ might exhibit Gibb's phenomenon?

Answer:

(a) Because $f_0(x)$ is odd, then $f(x)$ is odd. Then the coefficient of $\cos(4x)$ is zero, without computation, because all cosine terms in the Fourier series of f have zero coefficient. The coefficient of $\sin(5x)$ is given by the formula

$$b_5 = \frac{1}{\pi}\int_{-\pi}^{\pi} f_0(x)\sin(5x)dx = \frac{2}{\pi}\int_0^{\pi}\sin(5x)dx = \frac{4}{5\pi}.$$

(b) The jump discontinuities in $|x| < 3\pi$, which are at $0, \pm\pi, \pm 2\pi$.

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4. (Cosine and Sine Series)
Find the second nonzero term in the cosine series expansion of $f(x)$, formed as the even 2π-periodic extension of the base function $|\cos(2x)|$ on $0 < x < \pi$. Leave the Fourier coefficient in integral form, unevaluated, unless you need to compute the value.

Answer:

The first nonzero coefficient is a_0. The fifth coefficient a_4 is the next nonzero coefficient:

$$a_4 = \frac{2}{\pi}\int_0^\pi |\cos(2x)|\cos(4x)dx = \frac{4}{3\pi}.$$

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5. (Convergence of Fourier Series)

(a) [30%] Display Dirichlet's kernel formula.

(b) [40%] State the Fourier Convergence Theorem for piecewise smooth functions.

(c) [30%] Give an example of a function $f(x)$ which does not have a Gibb's over-shoot.

Answer:

(a) $\frac{1}{2} + \cos(x) + \cdots + \cos(nx) = \frac{\sin(nx + x/2)}{2\sin(x/2)}$

(b) Let f be a 2π-periodic piecewise smooth function on $(-\infty, \infty)$. Then for all values of x,

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where the Fourier coefficients a_0, a_n, b_n are given by the Euler formulas:

$$a_0 = \frac{1}{2\pi}\int_{-pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-pi}^{\pi} f(x)\cos(nx)dx,$$

$$b_n = \frac{1}{\pi}\int_{-pi}^{\pi} f(x)\sin(nx)dx.$$

(c) Any continuously differentiable function will work.

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