Name ____________________________

Math 3150 Problems
Chapter 3

Due date: See the internet due date. Problems are collected once a week. Records are locked when the stack is returned. Records are only corrected, never appended.

Submitted work. Please submit one package per problem. Label each problem with its corresponding problem number, e.g., $\boxed{\text{Prob3.1-4}}$ or $\boxed{\text{Xc1.2-4}}$. Kindly label extra credit problems with label $\boxed{\textbf{Extra Credit}}$. You may attach this printed sheet to simplify your work.

Labeling. The label $\boxed{\text{Probx.y-z}}$ means the problem is for chapter $\boxed{\text{x}}$, section $\boxed{\text{y}}$, problem $\boxed{\text{z}}$. When $y = 0$, then the problem does not have a textbook analog, it is a **background problem**. Otherwise, the problem number should match a corresponding problem in the textbook. The same labeling applies to extra credit problems, e.g., $\boxed{\text{Xc1.0-4}}$, $\boxed{\text{Xc1.1-2}}$.

Chapter 3: 3.1 – Examples in Physics and Engineering

Prob3.1-3. (Classification)

Classify $u_{xx} - u_t = 2u$ as linear, nonlinear, homogeneous, non-homogeneous, and report the order of the equation.

Prob3.1-7. (Laplace Equation)

Verify that $u(x, y) = e^y \cos x + x + y$ is a solution of Laplace's partial differential equation.

Chapter 3: 3.2-3.3 – One Dimensional Wave Equation

Prob3.2-1. (Wave Equation)

Derive the equation $u_{tt} = 10^5 u_{xx}$ for the vibrations of a stretched homogeneous string with linear density $\rho = 0.001$ kg/m and tension $\tau = 100$ N, with no forces other than the tension. State all assumptions used to obtain the model. Make the presentation brief, by referencing a textbook for derivation details and results.

Prob3.3-9a. (Separation of Variables)

Solve $u_{tt} = u_{xx}$, $u(0, t) = u(1, t) = 0$, $u(x, 0) = x(1 - x)$, $u_t(x, 0) = \sin \pi x$, $t \geq 0$, $0 \leq x \leq 1$. The model is for a guitar string of unit length.

Prob3.3-9b. (Filmstrip Plots)

Plot partial sums of the answer to the previous problem,

$$u(x,t) = \frac{1}{\pi}\sin(\pi x)\sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^3(2m+1)^3}\sin(2m\pi x + \pi x)\cos(2m\pi t + \pi t),$$

at $t = 0, 1, 2, 3$. Choose the number of series terms for the four graphics by making the first graphic match $x(1 - x)$ on $0 \leq x \leq 1$. This filmstrip has 4 frames, each frame corresponding to a time t. A frame has graph window $0 \leq x \leq 1$, $a \leq u \leq b$ (you must choose a, b).

Prob3.3-9c. (Surface Plot)

Plot a specific partial sum of the answer

$$u(x,t) = \frac{1}{\pi}\sin(\pi x)\sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^3(2m+1)^3}\sin(2m\pi x + \pi x)\cos(2m\pi t + \pi t)$$

on the domain $0 \leq x \leq 1$, $0 \leq t \leq 4$. Use all features possible of the 3D graphics program in order to produce the best plot with fine accuracy, view and colors.

Prob3.3-13. **(Damped Vibrations of a String)**
Solve the problem

$$\begin{aligned} u_{tt}(x,t)+u_t(x,t) &= u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(\pi,t) &= 0, \\ u(x,0) &= \sin x, \\ u_t(x,0) &= 0. \end{aligned}$$

Chapter 3: 3.4 – d'Alembert's Method

Prob3.4-15. **(d'Alembert's Solution)**
Consider the problem

$$\begin{aligned} u_{tt} &= u_{xx},\ 0\le x\le 1,\ t\ge 0, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= 0. \end{aligned}$$

Assume $f(x)=4x$ on $0\le x\le 0.25$, $f(x)=2-4x$ on $0.25<x\le 0.5$, $f(x)=0$ on $0.5<x\le 1$.

(a) Find a solution formula for $u(x,t)$ using d'Alembert's method.

(b) Plot a 3-frame filmstrip of the string shape at times $t=0, 0.25, 0.5$.

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# EXAMPLE. Let f(x)=4x on [0,.25], f(x)=2-4x on [.25,.5], f(x)=0 otherwise
# Asmar 3.4-15, D'Alembert's solution of the wave equation, f=pulses,g=0
pulse:=(x,a,b)->piecewise(x<a,0,x<b,1,0);
f:=x->4*x*pulse(x,0,1/4)+(2-4*x)*pulse(x,1/4,1/2); #plot(f(x),x=0..1);
F0:=x->piecewise(x<0,-f(-x),f(x)); # Odd extension of f(x)
plot(F0(x),x=-1..1);
F:=x->F0(x-2*floor(x/2+1/2)) # plot(F(x),x=-2..3);
u:=(x,t)->(1/2)*(F(x+t)+F(x-t)); #plot(u(x,0.7),x=-2..2);
plots[animate]( plot, [u(x,t),x=-3..3], t=0..1.5, trace=0, frames=50 );
```

See the web site links for updates to this sample maple code.

Xc3.4-18. **(Energy Conservation and d'Alembert's Solution)**
Define

$$E(t)=\frac{1}{2}\int_0^L \left(u_t^2(x,t)+c^2u_x^2(x,t)\right)dx.$$

Prove the energy conservation law, which says that the energy during free vibrations of a string is constant for all time.

Hint: Show $dE/dt=0$.

Chapter 3: 3.5-3.6 – One Dimensional Heat Equation

Prob3.5-13. **(Nonhomogeneous Heat Equation)**
Consider the one-dimensional heat conduction problem

$$\begin{aligned} u_t &= u_{xx},\ 0\le x\le \pi,\ t>0, \\ u(0,t) &= 100, \\ u(\pi,t) &= 50, \\ u(x,0) &= f(x). \end{aligned}$$

Assume $f(x)=33x$ on $0<x\le\pi/2$, $f(x)=33\pi-33x$ on $\pi/2<x<\pi$. Find a solution formula for the temperature $u(x,t)$.

Prob3.6-3. **(Heat Conduction in an Insulated Bar)**
Consider the one-dimensional heat conduction problem

$$\begin{aligned} u_t &= u_{xx},\ 0 \le x \le 1,\ t > 0, \\ u_x(0,t) &= 0, \\ u_x(1,t) &= 0, \\ u(x,0) &= \cos \pi x \end{aligned}$$

Find a solution formula for the temperature $u(x,t)$ at location x along the bar at time t. Hint: Don't integrate!
Remark. The book's problem 3.6-3 has a piecewise example, using $u(x,0) = f(x)$. See the maple advice for problem 3.5-13, to handle that case.

Chapter 3: 3.7 – Two Dimensional Equations

Prob3.7-5a. **(Vibrations of a Membrane)**
Consider the rectangular drumhead problem, in which we assume $0 < x < 1$, $0 < y < 1$, $t > 0$:

$$\begin{aligned} u_{tt}(x,y,t) &= \frac{1}{\pi^2}\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right), \\ u(0,y,t) &= 0, \\ u(1,y,t) &= 0, \\ u(x,0,t) &= 0, \\ u(x,1,t) &= 0, \\ u(x,y,0) &= 0, \\ u_t(x,y,0) &= 1. \end{aligned}$$

Solve for the drumhead defection $u(x,y,t)$.

Prob3.7-5b. **(Membrane Snapshots)**
Consider the solution of the rectangular drumhead problem given by the series

$$u(x,y,t) = \frac{16}{\pi^2} \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1}{nm\sqrt{n^2+m^2}} \sin(m\pi x)\sin(n\pi y)\sin\left(t\sqrt{m^2+n^2}\right).$$

Illustrate the various shapes of the drumhead during vibration, by plotting suitable surface snapshots at times $t = 1, 2, 3$. The snapshot at $t = 0$ should be the initial flat membrane shape $u = 0$. Choose suitable partial sums to reveal adequate detail in the plots.

Prob3.7-12. **(Heat Conduction in a Plate)**
Consider the rectangular plate heat conduction problem in which we assume $0 \le x \le 1$, $0 < y < 1$, $t > 0$:

$$\begin{aligned} u_t(x,y,t) &= u_{xx}(x,y,t) + u_{yy}(x,y,t), \\ u(0,y,t) &= 0, \\ u(1,y,t) &= 0, \\ u(x,0,t) &= 0, \\ u(x,1,t) &= 0, \\ u(x,y,0) &= x(1-x)y(1-y). \end{aligned}$$

Solve for the plate temperature $u(x,y,t)$.

Xc3.7-12. **(Heat Conduction in a Plate)**
Find a series estimate for the solution $u(x,y,t)$ of the rectangular plate heat conduction problem which shows that $|u(x,y,t)| \le Me^{-\alpha t}$ for some number $M > 0$ and some constant $\alpha > 0$. Then conclude that

$$\lim_{t\to\infty} u(x,y,t) = 0,$$

which implies the plate temperature u stabilizes to the edge temperature $u = 0$ as t approaches infinity.

Problem notes.

The Cauchy-Schwartz inequality is used to find a upper estimate of $|u|^2$ as a product of two positive series. One series is numeric, and Bessel's inequality can be used to determine an upper bound M_1 for it. The other series in the product is a series of functions, each function an exponential function bounded above by $e^{-\beta t}$, where $\beta > 0$ is a fixed constant. A clever analysis of the exponential factors, using the geometric series formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, shows that the second series is bounded by some constant $M_2 > 0$ times $e^{-\rho t}$, where $\rho = \beta/2$. Taking square roots across $|u|^2 \le M_1 M_2 e^{-\rho t}$ implies that constants $M = \sqrt{M_1 M_2} > 0$ and $\alpha = \rho/2 > 0$ satisfy $|u| \le M e^{-\alpha t}$.

Chapter 3: 3.8-3.9 – Laplace's and Poisson's Equations

Prob3.8-2. **(Steady-State Temperature in a Plate)**

Consider the rectangular plate steady-state heat conduction problem in which we assume the plate is given by $0 \le x \le 1$, $0 < y < 1$:

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= 0, \\ u(x,0) &= 0, \\ u(x,1) &= 100, \\ u(0,y) &= 0, \\ u(1,y) &= 100. \end{aligned}$$

(a) Draw a figure for the Dirichlet problem, showing the edge temperatures on the plate. Break the problem into two subproblems, decomposing $u = u_1 + u_2$. Draw figures for each subproblem.

(b) Solve for the temperatures $u_1(x,y)$ and $u_2(x,y)$.

(c) Report the solution to the original problem, $u = u_1 + u_2$.

Notes. In the general problem of Nakhle's section 3.8, $f_1(x) = g_1(y) = 0$ and $f_2(x) = g_2(y) = 100$. In the summary shaded display before the 3.8 exercises, $A_n = C_n = 0$ and B_n, D_n are computed from equations (5), (6). Example 2 in Nakhle's section 3.8 solves for B_n, therefore you have an easy answer check for half the problem.

Prob3.9-3. **(Poisson Problem)**

Consider the rectangular plate steady-state Poisson heat conduction problem in which we assume the plate is given by $0 \le x \le 1$, $0 < y < 1$:

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= \sin \pi x, \\ u(x,0) &= 0, \\ u(x,1) &= x, \\ u(0,y) &= 0, \\ u(1,y) &= 0. \end{aligned}$$

(a) Draw a figure for the Poisson problem with zero boundary conditions [see (b)]. Draw a second figure for the corresponding Dirichlet problem with identical boundary conditions [see (c)].

(b) Solve for the temperature $u_1(x,y)$ satisfying the Poisson problem

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= \sin \pi x, \\ u(x,0) &= 0, \\ u(x,1) &= 0, \\ u(0,y) &= 0, \\ u(1,y) &= 0. \end{aligned}$$

(c) Solve for the temperature $u_2(x,y)$ satisfying the Dirichlet problem

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= 0, \\ u(x,0) &= 0, \\ u(x,1) &= x, \\ u(0,y) &= 0, \\ u(1,y) &= 0. \end{aligned}$$

(d) Report the solution to the original Poisson problem, which is $u = u_1 + u_2$.

Notes. Problem (c) is solved in section 3.8 of Nakhle's textbook, with summary in the shaded display just before the exercises 3.8. In this display, $A_n = C_n = D_n = 0$ and B_n must be computed from (5) using $f_2(x) = x$. Problem (b) is solved from Nakhle's section 3.9 equations (2) and (4). A similar problem is solved in Example 1. The challenge is the double integration in (4) with $f(x, y) = \sin \pi x$. Luckily, this is an iterated double integral, evaluated by two successive one-variable integrations:

$$E_{mn} = \frac{-4}{\lambda_{mn}} \int_0^1 \sin \pi x \sin m\pi x dx \int_0^1 \sin n\pi y dy.$$