Name _____

Math 3150 Problems
Chapter 1

Due date: See the internet due date. Problems are collected once a week. Records are locked when the stack is returned. Records are only corrected, never appended.

Submitted work. Please submit one package per problem set. Label each problem with its corresponding problem number, e.g., Prob3.1-4 or Xc1.2-4 . Kindly label extra credit problems with label **Extra Credit** . You may attach this printed sheet to simplify your work.

Labeling. The label Probx.y-z means the problem is for chapter x , section y , problem z . When $y = 0$, then the problem does not have a textbook analog, it is a **background problem**. Otherwise, the problem number should match a corresponding problem in the textbook. The same labeling applies to extra credit problems, e.g., Xc1.0-4 , Xc1.1-2 .

Chapter 1: 1.1-1.2 − Introduction

Prob1.0-1. (**Partial Derivatives**)
Compute the partial derivatives.

$\partial_x(1 + t + x^2)$	$\partial_x(1 + xt + (tx)^2)$	$\partial_x(\sin xt + t^2 + tx^2)$
$\partial_t(\sinh t \sin 2x)$	$\partial_t(e^{2t+x}\sin(t+x))$	$\partial_t(e^{3t+2x}(e^t \sin x + e^x \cos t))$

Prob1.0-2. (**Jacobian**)

Find the Jacobian matrix J and the Jacobian determinant $|J|$ for the transformation $\mathbf{F}(x, y) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, where $f_1(x, y) = \cos(2y)\sin(3x)$, $f_2(x, y) = e^y \sin(y + x)$.

Background: J is the augmented 2×2 matrix of column vectors $\partial_x \mathbf{F}(x, y)$ and $\partial_y \mathbf{F}(x, y)$. For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is defined by $|A| = ad - bc$.

Prob1.0-3. (**Directional Derivative**)
Compute the gradient vector $\mathbf{grad}(f)$ and the directional derivative of f in the direction of $\mathbf{v} = 2\vec{\imath} - \vec{\jmath}$ at $x = y = 1$.

Background: $\mathbf{grad}(f) = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}$, $DD(f) = \mathbf{grad}(f) \cdot \mathbf{v}/|\mathbf{v}|$.

$f = x + y + 1$	$f = x^2 + y^2$	$f = ye^{x-1}$

Prob1.0-4. (**First Order ODE**)
Solve the following first order ODE problems. Reference: Appendix A of Asmar's text. Edwards-Penney DE+LA Chapter 1. The first four are linear, the remaining separable.

$y' - y = 0$	$y' + 2y = 0$	$y' + 2y = 4$
$xy' + y = x + 1$	$y' = y^2$	$y' = xy$

Prob1.0-5. (**Wronskian**)
Compute the Wronskian of the given functions at $x = 0$.

Reference: Appendix A of Asmar's text, *PDE and BVP*. Edwards-Penney, *DE and LA*, Chapter 5.

$1, x$	$1, e^x$	$\cos x, \sin x$
e^x, e^{2x}, e^{3x}	$1, \cosh x, \sinh x$	$1, \ln|1 + x|, x\ln|1 + x|$

Prob1.0-6. (Newton's Laws)

(a) State Newton's second law $F = ma$ in calculus terms for a motion with position $x(t)$.

(b) Use the statement in (a) to derive the spring-mass ODE $mx''(t) + kx(t) = 0$. Assume no external forces and a Hooke's restoring force on the spring of magnitude $kx(t)$.

References: Asmar section 1.2 and Edwards-Penney DE+LA Chapter 5. The ODE in (b) is called the **harmonic oscillator**.

Prob1.1-3. (PDE Solution)

Verify that $u(x,t) = e^{-(x-t)^2}$ is a solution of the initial value problem $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0$, $u(0,t) = e^{-t^2}$. Draw four (4) waveform snapshots of the solution u, for $x = 0, 1, 2, 3$. Each snapshot is a 2-dimensional plot in the tu-plane with t-range -2 to 5.

Prob1.1-6. (Change of Variables)

Show the details in solving the PDE $2\dfrac{\partial u}{\partial t} + 3\dfrac{\partial u}{\partial x} = 0$ by using a change of variables. Check your answer.

Problem notes: Let $r = at + bx$, $s = ct + dx$ be the change of variables $(t, x) \to (r, s)$ and let $w(r, s) = u(t, x)$. The problem is to determine a, b, c, d such that the PDE is changed into the simpler PDE $\dfrac{\partial w}{\partial r} = 0$.

Reference: Example 3, section 1.1 of Asmar's text.

Answer: $u(x, t) = f(2x - 3t)$.

Prob1.1-12. (Method of Characteristics)

Show the details in solving $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ by the method of characteristics. Check your answer.

Problem notes: The ODE for characteristics is the linear ODE $\dfrac{dy}{dx} = \dfrac{y}{x}$. The answer is $u = f(y/x)$.

Prob1.2-4a. (d'Alembert's Solution)

Let $w(r, s) = u(x, t)$, $r = x + ct$, $s = x - ct$, where $u(x, t)$ is a solution to the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$. Show the details in the derivation of the equation $\dfrac{\partial^2 w}{\partial r \partial s} = 0$.

Xc1.2-4b. (d'Alembert's Solution)

Let $w(r, s) = u(x, t)$, $r = x + ct$, $s = x - ct$, where $u(x, t)$ is a solution to the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$. Assume $\dfrac{\partial^2 w}{\partial r \partial s} = 0$. Solve this equation to verify $u(x, t) = F(x + ct) + G(x - ct)$ where F and G are sufficiently smooth functions.

Prob1.2-5. (Vibrating Infinite String)

Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$ with initial conditions $u(x, 0) = \dfrac{1}{1 + x^2}$, $u_t(x, 0) = 0$, $-\infty < x < \infty$, $t \geq 0$. Use d'Alembert's solution.

Prob1.2-17a. (Vibrating Finite String)

Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$ with boundary and initial conditions $u(0, t) = u(L, t) = 0$, $u(x, 0) = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L} + \frac{2}{5}\sin\frac{7\pi x}{L}$, $u_t(x, 0) = 0$, $0 < x < L$, $t \geq 0$. Use the series formula $u(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$.

References: Asmar's text, *PDE and BVP*, section 1.2 and Edwards-Penney, *DE and BVP*, section 9.6 (2280 textbook).

Xc1.2-17b. (Loudness)

The fraction of the loudness associated with the fundamental tone (b_1-term in the series) is the quotient

$$F_1 = \frac{(n^2 b_n^2)\big|_{n=1}}{\sum_{k=1}^{\infty} k^2 b_k^2}$$

Find an approximation to the percentage $100F_1$.

References: Prob1.2-17a. The discussion of music in E&P includes a derivation of the formula for the percentage loudness $100F_n$.