```
for n = 3:k-2
   b(n+2) = ((n-1)*b(n) + b(n-2))/(n*(n+1));
   end
format rat, b
```

give the same results, *except that* the coefficient b_{10} of x^9 is shown as 73/31801 rather than the correct value 119/51840 shown in Eq. (4). It happens that

$$\frac{73}{31801} \approx 0.0022955253 \quad \text{while} \quad \frac{119}{51840} \approx 0.0022955247,$$

so the two rational fractions agree when rounded to 9 decimal places. The explanation is that (unlike *Mathematica* and *Maple*) MATLAB works internally with decimal rather than exact arithmetic. But at the end its **format rat** algorithm converts a correct 14-place approximation for b_{10} into an incorrect rational fraction that's "close but no cigar."

You can substitute $b_1 = 1$, $b_2 = 0$ and $b_1 = 0$, $b_2 = 1$ separately (instead of $b_1 = b_2 = 1$) in the commands shown here to derive partial sums of the two linearly independent solutions displayed in Eqs. (18) and (19) of Example 7. This technique can be applied to any of the examples and problems in this section.

11.3 Frobenius Series Solutions

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We now investigate the solution of the homogeneous second-order linear equation

$$A(x)y'' + B(x)y' + C(x)y = 0 \tag{1}$$

near a singular point. Recall that if the functions A, B, and C are polynomials having no common factors, then the singular points of Eq. (1) are simply those points where $A(x) = 0$. For instance, $x = 0$ is the only singular point of the Bessel equation of order n,

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

whereas the Legendre equation of order n,

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

has the two singular points $x = -1$ and $x = 1$. It turns out that some of the features of the solutions of such equations of the most importance for applications are largely determined by their behavior near their singular points.

We will restrict our attention to the case in which $x = 0$ is a singular point of Eq. (1). A differential equation having $x = a$ as a singular point is easily transformed by the substitution $t = x - a$ into one having a corresponding singular point at 0. For example, let us substitute $t = x - 1$ into the Legendre equation of order n. Because

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt},$$

$$y'' = \frac{d^2y}{dx^2} = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]\frac{dt}{dx} = \frac{d^2y}{dt^2},$$

and $1 - x^2 = 1 - (t+1)^2 = -2t - t^2$, we get the equation

$$-t(t+2)\frac{d^2y}{dt^2} - 2(t+1)\frac{dy}{dt} + n(n+1)y = 0.$$

This new equation has the singular point $t = 0$ corresponding to $x = 1$ in the original equation; it has also the singular point $t = -2$ corresponding to $x = -1$.

Types of Singular Points

A differential equation having a singular point at 0 ordinarily will *not* have power series solutions of the form $y(x) = \sum c_n x^n$, so the straightforward method of Section 11.2 fails in this case. To investigate the form that a solution of such an equation might take, we assume that Eq. (1) has analytic coefficient functions and rewrite it in the standard form

$$y'' + P(x)y' + Q(x)y = 0, \tag{2}$$

where $P = B/A$ and $Q = C/A$. Recall that $x = 0$ is an ordinary point (as opposed to a singular point) of Eq. (2) if the functions $P(x)$ and $Q(x)$ are analytic at $x = 0$; that is, if $P(x)$ and $Q(x)$ have convergent power series expansions in powers of x on some open interval containing $x = 0$. Now it can be proved that each of the functions $P(x)$ and $Q(x)$ *either* is analytic *or* approaches $\pm\infty$ as $x \to 0$. Consequently, $x = 0$ is a singular point of Eq. (2) provided that either $P(x)$ or $Q(x)$ (or both) approaches $\pm\infty$ as $x \to 0$. For instance, if we rewrite the Bessel equation of order n in the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0,$$

we see that $P(x) = 1/x$ and $Q(x) = 1 - (n/x)^2$ both approach infinity as $x \to 0$.

We will see presently that the power series method can be generalized to apply near the singular point $x = 0$ of Eq. (2), provided that $P(x)$ approaches infinity no more rapidly than $1/x$, and $Q(x)$ no more rapidly than $1/x^2$, as $x \to 0$. This is a way of saying that $P(x)$ and $Q(x)$ have only "weak" singularities at $x = 0$. To state this more precisely, we rewrite Eq. (2) in the form

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0, \tag{3}$$

where

$$p(x) = xP(x) \quad \text{and} \quad q(x) = x^2Q(x). \tag{4}$$

DEFINITION Regular Singular Point

The singular point $x = 0$ of Eq. (3) is a **regular singular point** if the functions $p(x)$ and $q(x)$ are both analytic at $x = 0$. Otherwise it is an **irregular singular point**.

In particular, the singular point $x = 0$ is a *regular* singular point if $p(x)$ and $q(x)$ are both polynomials. For instance, we see that $x = 0$ is a regular singular point of Bessel's equation of order n by writing that equation in the form

$$y'' + \frac{1}{x}y' + \frac{x^2 - n^2}{x^2}y = 0,$$

noting that $p(x) \equiv 1$ and $q(x) = x^2 - n^2$ are both polynomials in x.

By contrast, consider the equation

$$2x^3y'' + (1 + x)y' + 3xy = 0,$$

which has the singular point $x = 0$. If we write this equation in the form of (3), we get

$$y'' + \frac{(1+x)/(2x^2)}{x}y' + \frac{\frac{3}{2}}{x^2}y = 0.$$

Because

$$p(x) = \frac{1+x}{2x^2} = \frac{1}{2x^2} + \frac{1}{2x} \to \infty$$

as $x \to 0$ (although $q(x) \equiv \frac{3}{2}$ is a polynomial), we see that $x = 0$ is an irregular singular point. We will not discuss the solution of differential equations near irregular singular points; this is a considerably more advanced topic than the solution of differential equations near regular singular points.

Example 1 Consider the differential equation

$$x^2(1+x)y'' + x(4-x^2)y' + (2+3x)y = 0.$$

In the standard form $y'' + Py' + Qy = 0$ it is

$$y'' + \frac{4-x^2}{x(1+x)}y' + \frac{2+3x}{x^2(1+x)}y = 0.$$

Because

$$P(x) = \frac{4-x^2}{x(1+x)} \quad \text{and} \quad Q(x) = \frac{2+3x}{x^2(1+x)}$$

both approach ∞ as $x \to 0$, we see that $x = 0$ is a singular point. To determine the nature of this singular point we write the differential equation in the form of Eq. (3):

$$y'' + \frac{(4-x^2)/(1+x)}{x}y' + \frac{(2+3x)/(1+x)}{x^2}y = 0.$$

Thus

$$p(x) = \frac{4-x^2}{1+x} \quad \text{and} \quad q(x) = \frac{2+3x}{1+x}.$$

Because a quotient of polynomials is analytic wherever the denominator is nonzero, we see that $p(x)$ and $q(x)$ are both analytic at $x = 0$. Hence $x = 0$ is a *regular* singular point of the given differential equation. ■

It may happen that when we begin with a differential equation in the general form in Eq. (1) and rewrite it in the form in (3), the functions $p(x)$ and $q(x)$ as given in (4) are indeterminate forms at $x = 0$. In this case the situation is determined by the limits

$$p_0 = p(0) = \lim_{x\to 0} p(x) = \lim_{x\to 0} xP(x) \tag{5}$$

and

$$q_0 = q(0) = \lim_{x\to 0} q(x) = \lim_{x\to 0} x^2 Q(x). \tag{6}$$

If $p_0 = 0 = q_0$, then $x = 0$ may be an ordinary point of the differential equation $x^2y'' + xp(x)y' + q(x)y = 0$ in (3). Otherwise:

- If both the limits in (5) and (6) exist and are *finite*, then $x = 0$ is a regular singular point.
- If either limit fails to exist or is infinite, then $x = 0$ is an irregular singular point.

Remark: The most common case in applications, for the differential equation written in the form

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0, \tag{3}$$

is that the functions $p(x)$ and $q(x)$ are *polynomials.* In this case $p_0 = p(0)$ and $q_0 = q(0)$ are simply the constant terms of these polynomials, so there is no need to evaluate the limits in Eqs. (5) and (6). ■

Example 2 To investigate the nature of the point $x = 0$ for the differential equation

$$x^4y'' + (x^2 \sin x)y' + (1 - \cos x)y = 0,$$

we first write it in the form in (3):

$$y'' + \frac{(\sin x)/x}{x}y' + \frac{(1 - \cos x)/x^2}{x^2}y = 0.$$

Then l'Hôpital's rule gives the values

$$p_0 = \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

and

$$q_0 = \lim_{x\to 0}\frac{1 - \cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}$$

for the limits in (5) and (6). Since they are not both zero, we see that $x = 0$ is not an ordinary point. But both limits are finite, so the singular point $x = 0$ is regular. Alternatively, we could write

$$p(x) = \frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

and

$$\begin{aligned} q(x) = \frac{1 - \cos x}{x^2} &= \frac{1}{x^2}\left[1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\right] \\ &= \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots. \end{aligned}$$

These (convergent) power series show explicitly that $p(x)$ and $q(x)$ are analytic and moreover that $p_0 = p(0) = 1$ and $q_0 = q(0) = \frac{1}{2}$, thereby verifying directly that $x = 0$ is a regular singular point. ■

The Method of Frobenius

We now approach the task of actually finding solutions of a second-order linear differential equation near the regular singular point $x = 0$. The simplest such equation is the constant-coefficient *equidimensional equation*

$$x^2y'' + p_0xy' + q_0y = 0 \tag{7}$$

to which Eq. (3) reduces when $p(x) \equiv p_0$ and $q(x) \equiv q_0$ are constants. In this case we can verify by direct substitution that the simple power function $y(x) = x^r$ is a solution of Eq. (7) if and only if r is a root of the quadratic equation

$$r(r-1) + p_0r + q_0 = 0. \tag{8}$$

In the general case, in which $p(x)$ and $q(x)$ are power series rather than constants, it is a reasonable conjecture that our differential equation might have a solution of the form

$$y(x) = x^r\sum_{n=0}^{\infty} c_nx^n = \sum_{n=0}^{\infty} c_nx^{n+r} = c_0x^r + c_1x^{r+1} + c_2x^{r+2} + \cdots \tag{9}$$

—the product of x^r and a power series. This turns out to be a very fruitful conjecture; according to Theorem 1 (soon to be stated formally), every equation of the form in (1) having $x = 0$ as a regular singular point does, indeed, have at least one such solution. This fact is the basis for the **method of Frobenius**, named for the German mathematician Georg Frobenius (1848–1917), who discovered the method in the 1870s.

An infinite series of the form in (9) is called a **Frobenius series**. Note that a Frobenius series is generally *not* a power series. For instance, with $r = -\frac{1}{2}$ the series in (9) takes the form

$$y = c_0x^{-1/2} + c_1x^{1/2} + c_2x^{3/2} + c_3x^{5/2} + \cdots;$$

it is not a series in *integral* powers of x.

To investigate the possible existence of Frobenius series solutions, we begin with the equation

$$x^2y'' + xp(x)y' + q(x)y = 0 \tag{10}$$

obtained by multiplying the equation in (3) by x^2. If $x = 0$ is a regular singular point, then $p(x)$ and $q(x)$ are analytic at $x = 0$, so

$$\begin{aligned} p(x) &= p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots, \\ q(x) &= q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots. \end{aligned} \tag{11}$$

Suppose that Eq. (10) has the Frobenius series solution

$$y = \sum_{n=0}^{\infty} c_nx^{n+r}. \tag{12}$$

We may (and always do) assume that $c_0 \neq 0$ because the series must have a first nonzero term. Termwise differentiation in Eq. (12) leads to

$$y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} \tag{13}$$

and

$$y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}. \tag{14}$$

Substitution of the series in Eqs. (11) through (14) in Eq. (10) now yields

$$\begin{aligned}\left[r(r-1)c_0x^r + (r+1)rc_1x^{r+1} + \cdots\right] \\ + \left[p_0x + p_1x^2 + \cdots\right]\cdot\left[rc_0x^{r-1} + (r+1)c_1x^r + \cdots\right] \\ + \left[q_0 + q_1x + \cdots\right]\cdot\left[c_0x^r + c_1x^{r+1} + \cdots\right] = 0.\end{aligned} \tag{15}$$

Upon multiplying initial terms of the two products on the left-hand side here and then collecting coefficients of x^r, we see that the lowest-degree term in Eq. (15) is $c_0[r(r-1)+p_0r+q_0]x^r$. If Eq. (15) is to be satisfied identically, then the coefficient of this term (as well as those of the higher-degree terms) must vanish. But we are assuming that $c_0 \neq 0$, so it follows that r must satisfy the quadratic equation

$$r(r-1) + p_0r + q_0 = 0 \tag{16}$$

of precisely the same form as that obtained with the equidimensional equation in (7). Equation (16) is called the **indicial equation** of the differential equation in (10), and its two roots (possibly equal) are the **exponents** of the differential equation (at the regular singular point $x = 0$).

Our derivation of Eq. (16) shows that *if* the Frobenius series $y = x^r \sum c_nx^n$ is to be a solution of the differential equation in (10), *then* the exponent r must be one of the roots r_1 and r_2 of the indicial equation in (16). If $r_1 \neq r_2$, it follows that there are two possible Frobenius series solutions, whereas if $r_1 = r_2$ there is only one possible Frobenius series solution; the second solution cannot be a Frobenius series. The exponents r_1 and r_2 in the possible Frobenius series solutions are determined (using the indicial equation) by the values $p_0 = p(0)$ and $q_0 = q(0)$ that we have discussed. In practice, particularly when the coefficients in the differential equation in the original form in (1) are polynomials, the simplest way of finding p_0 and q_0 is often to write the equation in the form

$$y'' + \frac{p_0 + p_1x + p_2x^2 + \cdots}{x}y' + \frac{q_0 + q_1x + q_2x^2 + \cdots}{x^2}y = 0. \tag{17}$$

Then inspection of the series that appear in the two numerators reveals the constants p_0 and q_0.

Example 3 Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y'' + 3x(1+x)^3y' - (1-x^2)y = 0.$$

Solution We divide each term by $2x^2(1+x)$ to recast the differential equation in the form

$$y'' + \frac{\frac{3}{2}(1+2x+x^2)}{x}y' + \frac{-\frac{1}{2}(1-x)}{x^2}y = 0,$$

and thus see that $p_0 = \frac{3}{2}$ and $q_0 = -\frac{1}{2}$. Hence the indicial equation is

$$r(r-1) + \tfrac{3}{2}r - \tfrac{1}{2} = r^2 + \tfrac{1}{2}r - \tfrac{1}{2} = (r+1)(r-\tfrac{1}{2}) = 0,$$

with roots $r_1 = \frac{1}{2}$ and $r_2 = -1$. The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty} b_n x^n. \qquad \blacksquare$$

Frobenius Series Solutions

Once the exponents r_1 and r_2 are known, the coefficients in a Frobenius series solution are determined by substitution of the series in Eqs. (12) through (14) in the differential equation, essentially the same method as was used to determine coefficients in power series solutions in Section 11.2. If the exponents r_1 and r_2 are complex conjugates, then there always exist two linearly independent Frobenius series solutions. We will restrict our attention here to the case in which r_1 and r_2 are both real. We also will seek solutions only for $x > 0$. Once such a solution has been found, we need only replace x^{r_1} with $|x|^{r_1}$ to obtain a solution for $x < 0$. The following theorem is proved in Chapter 4 of Coddington's *An Introduction to Ordinary Differential Equations*.

THEOREM 1 Frobenius Series Solutions

Suppose that $x = 0$ is a regular singular point of the equation

$$x^2y'' + xp(x)y' + q(x)y = 0. \tag{10}$$

Let $\rho > 0$ denote the minimum of the radii of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Let r_1 and r_2 be the (real) roots, with $r_1 \geqq r_2$, of the indicial equation $r(r-1) + p_0 r + q_0 = 0$. Then

(a) For $x > 0$, there exists a solution of Eq. (10) of the form

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \qquad (a_0 \neq 0) \tag{18}$$

corresponding to the larger root r_1.

(b) If $r_1 - r_2$ is neither zero nor a positive integer, then there exists a second linearly independent solution for $x > 0$ of the form

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n \qquad (b_0 \neq 0) \tag{19}$$

corresponding to the smaller root r_2.

The radii of convergence of the power series in Eqs. (18) and (19) are each at least ρ. The coefficients in these series can be determined by substituting the series in the differential equation

$$x^2 y'' + x p(x) y' + q(x) y = 0.$$

We have already seen that if $r_1 = r_2$, then there can exist only one Frobenius series solution. It turns out that, if $r_1 - r_2$ is a positive integer, then there may or may not exist a second Frobenius series solution of the form in Eq. (19) corresponding to the smaller root r_2. Examples 4 through 6 illustrate the process of determining the coefficients in those Frobenius series solutions that are guaranteed by Theorem 1.

Example 4 Find the Frobenius series solutions of

$$2x^2 y'' + 3xy' - (x^2 + 1)y = 0. \tag{20}$$

Solution First we divide each term by $2x^2$ to put the equation in the form in (17):

$$y'' + \frac{\frac{3}{2}}{x} y' + \frac{-\frac{1}{2} - \frac{1}{2}x^2}{x^2} y = 0. \tag{21}$$

We now see that $x = 0$ is a regular singular point, and that $p_0 = \frac{3}{2}$ and $q_0 = -\frac{1}{2}$. Because $p(x) \equiv \frac{3}{2}$ and $q(x) = -\frac{1}{2} - \frac{1}{2}x^2$ are polynomials, the Frobenius series we obtain will converge for all $x > 0$. The indicial equation is

$$r(r-1) + \tfrac{3}{2}r - \tfrac{1}{2} = \left(r - \tfrac{1}{2}\right)(r+1) = 0,$$

so the exponents are $r_1 = \frac{1}{2}$ and $r_2 = -1$. They do not differ by an integer, so Theorem 1 guarantees the existence of two linearly independent Frobenius series solutions. Rather than separately substituting

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = x^{-1} \sum_{n=0}^{\infty} b_n x^n$$

in Eq. (20), it is more efficient to begin by substituting $y = x^r \sum c_n x^n$. We will then get a recurrence relation that depends on r. With the value $r_1 = \frac{1}{2}$ it becomes a recurrence relation for the series for y_1, whereas with $r_2 = -1$ it becomes a recurrence relation for the series for y_2.

When we substitute

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1},$$

and

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

in Eq. (20)—the original differential equation, rather than Eq. (21)—we get

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + 3\sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty}c_n x^{n+r+2} - \sum_{n=0}^{\infty}c_n x^{n+r} = 0. \tag{22}$$

At this stage there are several ways to proceed. A good standard practice is to shift indices so that each exponent will be the same as the smallest one present. In this example, we shift the index of summation in the third sum by -2 to reduce its exponent from $n + r + 2$ to $n + r$. This gives

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + 3\sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=2}^{\infty}c_{n-2} x^{n+r} - \sum_{n=0}^{\infty}c_n x^{n+r} = 0. \tag{23}$$

The common range of summation is $n \geqq 2$, so we must treat $n = 0$ and $n = 1$ separately. Following our standard practice, the terms corresponding to $n = 0$ will always give the indicial equation

$$[2r(r-1) + 3r - 1]c_0 = 2\left(r^2 + \tfrac{1}{2}r - \tfrac{1}{2}\right)c_0 = 0.$$

The terms corresponding to $n = 1$ yield

$$[2(r+1)r + 3(r+1) - 1]c_1 = (2r^2 + 5r + 2)c_1 = 0.$$

Because the coefficient $2r^2 + 5r + 2$ of c_1 is nonzero whether $r = \frac{1}{2}$ or $r = -1$, it follows that

$$c_1 = 0 \tag{24}$$

in either case.

The coefficient of x^{n+r} in Eq. (23) is

$$2(n+r)(n+r-1)c_n + 3(n+r)c_n - c_{n-2} - c_n = 0.$$

We solve for c_n and simplify to obtain the recurrence relation

$$c_n = \frac{c_{n-2}}{2(n+r)^2 + (n+r) - 1} \qquad \text{for } n \geqq 2. \tag{25}$$

CASE 1: $r_1 = \frac{1}{2}$. We now write a_n in place of c_n and substitute $r = \frac{1}{2}$ in Eq. (25). This gives the recurrence relation

$$a_n = \frac{a_{n-2}}{2n^2 + 3n} \qquad \text{for } n \geqq 2. \tag{26}$$

With this formula we can determine the coefficients in the first Frobenius solution y_1. In view of Eq. (24) we see that $a_n = 0$ whenever n is odd. With $n = 2, 4$, and 6 in Eq. (26), we get

$$a_2 = \frac{a_0}{14}, \quad a_4 = \frac{a_2}{44} = \frac{a_0}{616}, \quad \text{and} \quad a_6 = \frac{a_4}{90} = \frac{a_0}{55{,}440}.$$

Hence the first Frobenius solution is

$$y_1(x) = a_0 x^{1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \frac{x^6}{55{,}440} + \cdots\right).$$

CASE 2: $r_2 = -1$. We now write b_n in place of c_n and substitute $r = -1$ in Eq. (25). This gives the recurrence relation

$$b_n = \frac{b_{n-2}}{2b^2 - 3n} \qquad \text{for } n \geqq 2. \tag{27}$$

Again, Eq. (24) implies that $b_n = 0$ for n odd. With $n = 2, 4$, and 6 in (27), we get

$$b_2 = \frac{b_0}{2}, \quad b_4 = \frac{b_2}{20} = \frac{b_0}{40}, \quad \text{and} \quad b_6 = \frac{b_4}{54} = \frac{b_0}{2160}.$$

Hence the second Frobenius solution is

$$y_2(x) = b_0 x^{-1}\left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \frac{x^6}{2160} + \cdots\right).$$ ■

Example 5 Find a Frobenius solution of Bessel's equation of order zero,

$$x^2 y'' + x y' + x^2 y = 0. \tag{28}$$

Solution In the form of (17), Eq. (28) becomes

$$y'' + \frac{1}{x} y' + \frac{x^2}{x^2} y = 0.$$

Hence $x = 0$ is a regular singular point with $p(x) \equiv 1$ and $q(x) = x^2$, so our series will converge for all $x > 0$. Because $p_0 = 1$ and $q_0 = 0$, the indicial equation is

$$r(r-1) + r = r^2 = 0.$$

Thus we obtain only the single exponent $r = 0$, and so there is only one Frobenius series solution

$$y(x) = x^0 \sum_{n=0}^{\infty} c_n x^n$$

of Eq. (28); it is in fact a power series.

Thus we substitute $y = \sum c_n x^n$ in (28); the result is

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} = 0.$$

We combine the first two sums and shift the index of summation in the third by -2 to obtain

$$\sum_{n=0}^{\infty} n^2 c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0.$$

The term corresponding to x^0 gives $0 = 0$: no information. The term corresponding to x^1 gives $c_1 = 0$, and the term for x^n yields the recurrence relation

$$c_n = -\frac{c_{n-2}}{n^2} \qquad \text{for } n \geqq 2. \tag{29}$$

Because $c_1 = 0$, we see that $c_n = 0$ whenever n is odd. Substituting $n = 2, 4,$ and 6 in Eq. (29), we get

$$c_2 = -\frac{c_0}{2^2}, \quad c_4 = -\frac{c_2}{4^2} = \frac{c_0}{2^2 \cdot 4^2}, \quad \text{and} \quad c_6 = -\frac{c_4}{6^2} = -\frac{c_0}{2^2 \cdot 4^2 \cdot 6^2}.$$

Evidently, the pattern is

$$c_{2n} = \frac{(-1)^n c_0}{2^2 \cdot 4^2 \cdots (2n)^2} = \frac{(-1)^n c_0}{2^{2n}(n!)^2}.$$

The choice $c_0 = 1$ gives us one of the most important special functions in mathematics, the **Bessel function of order zero of the first kind**, denoted by $J_0(x)$. Thus

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots. \tag{30}$$

In this example we have not been able to find a second linearly independent solution of Bessel's equation of order zero. We will derive that solution in Section 11.4; it will not be a Frobenius series. ■

When $r_1 - r_2$ Is an Integer

Recall that, if $r_1 - r_2$ is a positive integer, then Theorem 1 guarantees only the existence of the Frobenius series solution corresponding to the larger exponent r_1. Example 6 illustrates the fortunate case in which the series method nevertheless yields a second Frobenius series solution. An example in which the second solution is not a Frobenius series will be discussed in Section 11.4.

Example 6 Find the Frobenius series solutions of

$$xy'' + 2y' + xy = 0. \tag{31}$$

Solution In standard form the equation becomes

$$y'' + \frac{2}{x}y' + \frac{x^2}{x^2}y = 0,$$

so we see that $x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = 0$. The indicial equation

$$r(r-1) + 2r = r(r+1) = 0$$

has roots $r_1 = 0$ and $r_2 = -1$, which differ by an integer. In this case when $r_1 - r_2$ is an integer, it is better to depart from the standard procedure of Example 4 and begin our work with the *smaller* exponent. As you will see, the recurrence relation will then tell us whether or not a second Frobenius series solution exists. If it does exist, our computations will simultaneously yield *both* Frobenius series solutions. If the second solution does not exist, we begin anew with the larger exponent $r = r_1$ to obtain the one Frobenius series solution guaranteed by Theorem 1.

Hence we begin by substituting

$$y = x^{-1}\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n-1}$$

in Eq. (31). This gives

$$\sum_{n=0}^{\infty}(n-1)(n-2)c_n x^{n-2} + 2\sum_{n=0}^{\infty}(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We combine the first two sums and shift the index by -2 in the third to obtain

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} c_{n-2}x^{n-2} = 0. \tag{32}$$

The cases $n = 0$ and $n = 1$ reduce to

$$0 \cdot c_0 = 0 \quad \text{and} \quad 0 \cdot c_1 = 0.$$

Hence we have *two* arbitrary constants c_0 and c_1 and therefore can expect to find a general solution incorporating two linearly independent Frobenius series solutions. If, for $n = 1$, we had obtained an equation such as $0 \cdot c_1 = 3$, which can be satisfied for *no* choice of c_1, this would have told us that no second Frobenius series solution could exist.

Now knowing that all is well, from (32) we read the recurrence relation

$$c_n = -\frac{c_{n-2}}{n(n-1)} \qquad \text{for } n \geqq 2. \tag{33}$$

The first few values of n give

$$c_2 = -\frac{1}{2\cdot 1}c_0, \qquad c_3 = -\frac{1}{3\cdot 2}c_1,$$

$$c_4 = -\frac{1}{4\cdot 3}c_2 = \frac{c_0}{4!}, \qquad c_5 = -\frac{1}{5\cdot 4}c_3 = \frac{c_1}{5!},$$

$$c_6 = -\frac{1}{6\cdot 5}c_4 = -\frac{c_0}{6!}, \qquad c_7 = -\frac{1}{7\cdot 6}c_6 = -\frac{c_1}{7!};$$

evidently the pattern is

$$c_{2n} = \frac{(-1)^n c_0}{(2n)!}, \qquad c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}$$

for $n \geqq 1$. Therefore, a general solution of Eq. (31) is

$$\begin{aligned} y(x) &= x^{-1}\sum_{n=0}^{\infty} c_n x^n \\ &= \frac{c_0}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + \frac{c_1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \frac{c_0}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} + \frac{c_1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}. \end{aligned}$$

Thus

$$y(x) = \frac{1}{x}(c_0 \cos x + c_1 \sin x).$$

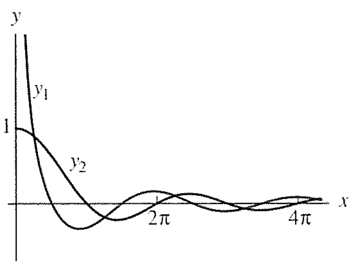

FIGURE 11.3.1. The solutions $y_1(x) = \dfrac{\cos x}{x}$ and $y_2(x) = \dfrac{\sin x}{x}$ in Example 6.

We have thus found a general solution expressed as a linear combination of the two Frobenius series solutions

$$y_1(x) = \frac{\cos x}{x} \quad \text{and} \quad y_2(x) = \frac{\sin x}{x}. \tag{34}$$

As indicated in Fig. 11.3.1, one of these Frobenius series solutions is bounded but the other is unbounded near the regular singular point $x = 0$—a common occurrence in the case of exponents differing by an integer. ■

Summary

When confronted with a linear second-order differential equation

$$A(x)y'' + B(x)y' + C(x)y = 0$$

with analytic coefficient functions, in order to investigate the possible existence of series solutions we first write the equation in the standard form

$$y'' + P(x)y' + Q(x)y = 0.$$

If $P(x)$ and $Q(x)$ are both analytic at $x = 0$, then $x = 0$ is an ordinary point, and the equation has two linearly independent power series solutions.

Otherwise, $x = 0$ is a singular point, and we next write the differential equation in the form

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0.$$

If $p(x)$ and $q(x)$ are both analytic at $x = 0$, then $x = 0$ is a regular singular point. In this case we find the two exponents r_1 and r_2 (assumed real, and with $r_1 \geqq r_2$) by solving the indicial equation

$$r(r-1) + p_0 r + q_0 = 0,$$

where $p_0 = p(0)$ and $q_0 = q(0)$. There always exists a Frobenius series solution $y = x^{r_1}\sum a_n x^n$ associated with the larger exponent r_1, and if $r_1 - r_2$ is not an integer, the existence of a second Frobenius series solution $y_2 = x^{r_2}\sum b_n x^n$ is also guaranteed.

11.3 Problems

In Problems 1 through 8, determine whether $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point. If it is a regular singular point, find the exponents of the differential equation at $x = 0$.

1. $xy'' + (x - x^3)y' + (\sin x)y = 0$
2. $xy'' + x^2y' + (e^x - 1)y = 0$
3. $x^2y'' + (\cos x)y' + xy = 0$
4. $3x^3y'' + 2x^2y' + (1 - x^2)y = 0$
5. $x(1 + x)y'' + 2y' + 3xy = 0$
6. $x^2(1 - x^2)y'' + 2xy' - 2y = 0$
7. $x^2y'' + (6 \sin x)y' + 6y = 0$
8. $(6x^2 + 2x^3)y'' + 21xy' + 9(x^2 - 1)y = 0$

If $x = a \neq 0$ is a singular point of a second-order linear differential equation, then the substitution $t = x - a$ transforms it into a differential equation having $t = 0$ as a singular point. We then attribute to the original equation at $x = a$ the behavior of the new equation at $t = 0$. Classify (as regular or irregular) the singular points of the differential equations in Problems 9 through 16.

9. $(1 - x)y'' + xy' + x^2y = 0$
10. $(1 - x)^2y'' + (2x - 2)y' + y = 0$
11. $(1 - x^2)y'' - 2xy' + 12y = 0$
12. $(x - 2)^3y'' + 3(x - 2)^2y' + x^3y = 0$
13. $(x^2 - 4)y'' + (x - 2)y' + (x + 2)y = 0$
14. $(x^2 - 9)^2y'' + (x^2 + 9)y' + (x^2 + 4)y = 0$
15. $(x - 2)^2y'' - (x^2 - 4)y' + (x + 2)y = 0$
16. $x^3(1 - x)y'' + (3x + 2)y' + xy = 0$

Find two linearly independent Frobenius series solutions (for $x > 0$) of each of the differential equations in Problems 17 through 26.

17. $4xy'' + 2y' + y = 0$
18. $2xy'' + 3y' - y = 0$
19. $2xy'' - y' - y = 0$
20. $3xy'' + 2y' + 2y = 0$
21. $2x^2y'' + xy' - (1 + 2x^2)y = 0$
22. $2x^2y'' + xy' - (3 - 2x^2)y = 0$
23. $6x^2y'' + 7xy' - (x^2 + 2)y = 0$
24. $3x^2y'' + 2xy' + x^2y = 0$
25. $2xy'' + (1 + x)y' + y = 0$
26. $2xy'' + (1 - 2x^2)y' - 4xy = 0$

Use the method of Example 6 to find two linearly independent Frobenius series solutions of the differential equations in Problems 27 through 31. Then construct a graph showing their graphs for $x > 0$.

27. $xy'' + 2y' + 9xy = 0$
28. $xy'' + 2y' - 4xy = 0$
29. $4xy'' + 8y' + xy = 0$
30. $xy'' - y' + 4x^3y = 0$
31. $4x^2y'' - 4xy' + (3 - 4x^2)y = 0$

In Problems 32 through 34, find the first three nonzero terms of each of two linearly independent Frobenius series solutions.

32. $2x^2y'' + x(x + 1)y' - (2x + 1)y = 0$
33. $(2x^2 + 5x^3)y'' + (3x - x^2)y' - (1 + x)y = 0$
34. $2x^2y'' + (\sin x)y' - (\cos x)y = 0$
35. Note that $x = 0$ is an irregular point of the equation

$$x^2y'' + (3x - 1)y' + y = 0.$$

(a) Show that $y = x^r \sum c_n x^n$ can satisfy this equation only if $r = 0$. **(b)** Substitute $y = \sum c_n x^n$ to derive the "formal" solution $y = \sum n!x^n$. What is the radius of convergence of this series?

36. **(a)** Suppose that A and B are nonzero constants. Show that the equation $x^2y'' + Ay' + By = 0$ has at most one solution of the form $y = x^r \sum c_n x^n$. **(b)** Repeat part (a) with the equation $x^3y'' + Axy' + By = 0$. **(c)** Show that the equation $x^3y'' + Ax^2y' + By = 0$ has no Frobenius series solution. (*Suggestion*: In each case substitute $y = x^r \sum c_n x^n$ in the given equation to determine the possible values of r.)

37. **(a)** Use the method of Frobenius to derive the solution $y_1 = x$ of the equation $x^3y'' - xy' + y = 0$. **(b)** Verify by substitution the second solution $y_2 = xe^{-1/x}$. Does y_2 have a Frobenius series representation?

38. Apply the method of Frobenius to Bessel's equation of order $\frac{1}{2}$,

$$x^2y'' + xy' + \left(x^2 - \tfrac{1}{4}\right)y = 0,$$

to derive its general solution for $x > 0$,

$$y(x) = c_0\frac{\cos x}{\sqrt{x}} + c_1\frac{\sin x}{\sqrt{x}}.$$

Figure 11.3.2 shows the graphs of the two indicated solutions.

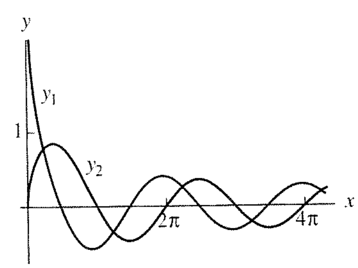

FIGURE 11.3.2. The solutions $y_1(x) = \dfrac{\cos x}{\sqrt{x}}$ and $y_2(x) = \dfrac{\sin x}{\sqrt{x}}$ in Problem 38.

39. **(a)** Show that Bessel's equation of order 1,

$$x^2y'' + xy' + (x^2 - 1)y = 0,$$

has exponents $r_1 = 1$ and $r_2 = -1$ at $x = 0$, and that the Frobenius series corresponding to $r_1 = 1$ is

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!\,(n+1)!\,2^{2n}}.$$

(b) Show that there is no Frobenius solution corresponding to the smaller exponent $r_2 = -1$; that is, show that it is impossible to determine the coefficients in

$$y_2(x) = x^{-1}\sum_{n=0}^{\infty} c_n x^n.$$

40. Consider the equation $x^2y'' + xy' + (1-x)y = 0$. **(a)** Show that its exponents are $\pm i$, so it has complex-valued Frobenius series solutions

$$y_+ = x^i\sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad y_- = x^{-i}\sum_{n=0}^{\infty} q_n x^n$$

with $p_0 = q_0 = 1$. **(b)** Show that the recursion formula is

$$c_n = \frac{c_{n-1}}{n^2 + 2rn}.$$

Apply this formula with $r = i$ to obtain $p_n = c_n$, then with $r = -i$ to obtain $q_n = c_n$. Conclude that p_n and q_n are complex conjugates: $p_n = a_n + ib_n$ and $q_n = a_n - ib_n$, where the numbers $\{a_n\}$ and $\{b_n\}$ are real. **(c)** Deduce from part (b) that the differential equation given in this problem has real-valued solutions of the form

$$y_1(x) = A(x)\cos(\ln x) - B(x)\sin(\ln x),$$
$$y_2(x) = A(x)\sin(\ln x) + B(x)\cos(\ln x),$$

where $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$.

41. Consider the differential equation

$$x(x-1)(x+1)^2y'' + 2x(x-3)(x+1)y' - 2(x-1)y = 0$$

that appeared in an advertisement for a symbolic algebra program in the March 1984 issue of the *American Mathematical Monthly*. **(a)** Show that $x = 0$ is a regular singular point with exponents $r_1 = 1$ and $r_2 = 0$. **(b)** It follows from Theorem 1 that this differential equation has a power series solution of the form

$$y_1(x) = x + c_2x^2 + c_3x^3 + \cdots.$$

Substitute this series (with $c_1 = 1$) in the differential equation to show that $c_2 = -2$, $c_3 = 3$, and

$$c_{n+2} = \frac{(n^2 - n)c_{n-1} + (n^2 - 5n - 2)c_n - (n^2 + 7n + 4)c_{n+1}}{(n+1)(n+2)}$$

for $n \geqq 2$. **(c)** Use the recurrence relation in part (b) to prove by induction that $c_n = (-1)^{n+1}n$ for $n \geqq 1$ (!). Hence deduce (using the geometric series) that

$$y_1(x) = \frac{x}{(1+x)^2}$$

for $0 < x < 1$.

42. This problem is a brief introduction to Gauss's **hypergeometric equation**

$$x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0, \quad (35)$$

where α, β, and γ are constants. This famous equation has wide-ranging applications in mathematics and physics. **(a)** Show that $x = 0$ is a regular singular point of Eq. (35), with exponents 0 and $1 - \gamma$. **(b)** If γ is not zero or a negative integer, it follows (why?) that Eq. (35) has a power series solution

$$y(x) = x^0\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^n$$

with $c_0 \neq 0$. Show that the recurrence relation for this series is

$$c_{n+1} = \frac{(\alpha+n)(\beta+n)}{(\gamma+n)(1+n)}c_n$$

for $n \geqq 0$. **(c)** Conclude that with $c_0 = 1$ the series in part (b) is

$$y(x) = 1 + \sum_{n=0}^{\infty}\frac{\alpha_n\beta_n}{n!\,\gamma_n}x^n, \quad (36)$$

where $\alpha_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ for $n \geqq 1$, and β_n and γ_n are defined similarly. **(d)** The series in (36) is known as the **hypergeometric series** and is commonly denoted by $F(\alpha, \beta, \gamma, x)$. Show that

(i) $F(1, 1, 1, x) = \dfrac{1}{1-x}$ (the geometric series);

(ii) $xF(1, 1, 2, -x) = \ln(1+x)$;

(iii) $xF\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right) = \tan^{-1} x$;

(iv) $F(-k, 1, 1, -x) = (1+x)^k$ (the binomial series).

11.3 Application Automating the Frobenius Series Method

Here we illustrate the use of a computer algebra system such as *Maple* to apply the method of Frobenius. More complete versions of this application—illustrating the use of *Maple*, *Mathematica*, and MATLAB—can be found in the applications