Linear Algebra 2270-2

Due in Week 9

The ninth week finishes chapter 4 and starts the work from chapter 5. Here's the list of problems, problem notes and answers.

Problem week9-1. Define a function T from $\mathcal{R}^n$ to $\mathcal{R}^m$ by the matrix multiply formula $T(\vec{x}) = A\vec{x}$. Prove that for all vectors $\vec{u}, \vec{v}$ and all constants c, (a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, (b) $T(c\vec{u}) = cT(\vec{u})$. *Definition: T is called a* **linear transformation** *if T maps $\mathcal{R}^n$ into $\mathcal{R}^m$ and satisfies (a) and (b).*

Problem week9-2. Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^n$ that satisfies $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x}$. Prove that the $n \times n$ matrix A of T is orthogonal, that is, $A^T A = I$, which means the columns of A are **orthonormal**:

$$\mathbf{col}(A,i) \cdot \mathbf{col}(A,j) = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \mathbf{col}(A,i) \cdot \mathbf{col}(A,i) = 1.$$

Problem week9-3. Let T be a linear transformation given by $n \times n$ orthogonal matrix A. Then $\|T(\vec{x})\| = \|\vec{x}\|$ holds. Construct an example of such a matrix A for dimension $n = 3$, which corresponds to holding the z-axis fixed and rotating the xy-plane 45 degrees counter-clockwise. Draw a 3D-figure which shows the action of T on the unit cube $S = \{(x,y,z) \ : \ 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$.

Section 4.4. Exercises 3, 6, 10, 13, 15, 18, 30, 32

Section 5.1. Exercises 1, 3, 8, 14, 21, 27, 33

Section 5.2. Exercises 2, 5, 7, 11, 12, 16, 20, 23, 32

Problem Notes

Problem week9-1. Write out both sides of identities (a) and (b), replacing $T(\vec{w})$ by matrix product $A\vec{w}$ for various choices of $\vec{w}$. The compare sides to finish the proof.

Problem week9-2. Equation $\|T(\vec{x})\| = \|\vec{x}\|$ means lengths are preserved by T. It also means $\|A\vec{x}\| = \|\vec{x}\|$, which applied to $\vec{x} = \mathbf{col}(I,k)$ means $\mathbf{col}(A,k)$ has length equal to $\mathbf{col}(I,k)$ (=1). Write $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w} = \vec{w}^T \vec{w}$ (the latter a matrix product). The write out the equation $\|A\vec{x}\|^2 = \|\vec{x}\|^2$, to see what you get, for various choices of unit vectors $\vec{x}$.

Problem week9-3. The equations for such a transformation can be written as plane rotation equations in x, y plus the identity in z. They might look like $x' = x\cos\theta - y\sin\theta$, $y' =$ similar, $z' = z$. Choose θ then test it by seeing what happens to $x = 1$, $y = 0$, $z = 0$, the answer for which is a rotation of vector $(1,0,0)$. The answer for A is obtained by writing the scala equations as a matrix equation $(x', y', z')^T = A(x,y,z)^T$.

Issues for Strang's problems will be communicated here. For help, please send email, call 581-6879, or visit JWB 113.

4.4-18. It helps to find explicitly Q and R, which can be quickly checked in `Maple`.

5.2-2. Use exercise 1 in part (a).

5.2-12. Find the cofactor matrix for A. Then compare the inverse of A with AC^T.

Some Answers

4.4. Exercises 3, 6, 13, 15, 18, 30, 32 have textbook answers.

4.4-10. (a) If q_1, q_2, q_3 are orthonormal then the dot product of q_1 with $c_1q_1 + c_2q_2 + c_3q_3 = 0$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$. Independent q's. (b) $Qx = 0$ implies $x = 0$ implies $x = 0$.

5.1. Exercises 1, 8, 14, 21, 27 have textbook answers.

5.1-3. (a) False: $\det(I + I)$ is not $1 + 1$ (b) True: The product rule extends to ABC (use it twice) (c) False: $\det(4A)$ is $4^n \det(A)$ (d) False: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $AB - BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is invertible.

5.1-33. I now know that maximizing the determinant for 1, -1 matrices is Hadamard's problem (1893): see Brenner in American Math. Monthly volume 79 (1972) 626-630. Neil Sloane's wonderful On-Line Encyclopedia of Integer Sequences (`research.att.com/~njas`) includes the solution for small n (and more references) when the problem is changed to 0, 1 matrices. That sequence `A003432` starts from $n = 0$ with $1, 1, 1, 2, 3, 5, 9$. Then the 1, -1 maximum for size n is 2^{n-1} times the 0, 1 maximum for size $n - 1$ (so $(32)(5) = 160$ for $n = 6$ in sequence `A003433`). To reduce the 1, -1 problem from 6 by 6 to the 0, 1 problem for 5 by 5, multiply the six rows by ± 1 to put $+1$ in column 1. Then subtract row 1 from rows 2 to 6 to get a 5 by 5 submatrix S of -2, 0 and divide S by -2. Here is an advanced MATLAB code and a 1, -1 matrix with largest $\det(A) = 48$ for $n = 5$:

```
 n=5; p=(n-1)^2; A0=ones(n); maxdet=0;
for k=0 : 2^p - 1
 Asub=rem(floor(k. * 2.^(-p + 1: 0)),2);
 A=A0;
 A(2:n,2:n)= 1-2*reshape(Asub,n-1,n-1);
 if abs(det(A)>maxdet, maxdet=abs(det(A)); maxA=A;
 end
end
```

$$\text{Output:} \qquad maxA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{bmatrix}, \quad maxdet = 48$$

5.2. Exercises 2, 11, 12, 16, 20, 32 have textbook answers.

5.2-5. Four zeros in the same row guarantee $\det = 0$. $A = I$ has 12 zeros (maximum with $\det \neq 0$).

5.2-7. $5!/2 = 60$ permutation matrices have $\det = +1$. Move row 5 of I to the top; starting from rows in the order $(5, 1, 2, 3, 4)$, elimination to reach I will take four row exchanges.

5.2-23.

(a) If we choose an entry from B we must choose an entry from the zero block; result zero. This leaves entries from A times entries from D leading to $\det(A)\det(D)$.

(b) and (c) Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. See the solution to problem 25.