The eighth week starts the work from chapter 4. Here’s the list of problems, followed by problem notes and a few answers.

**Section 4.1.** Exercises 5, 9, 11, 12, 16, 17, 19, 20, 21, 26

**Section 4.2.** Exercises 1, 2, 3, 11, 12, 17, 21, 27, 31

**Section 4.3.** Exercises 1, 6, 12, 17, 18, 21

**Problem Notes**

Issues for Strang’s problems will be communicated here. For help, please send email, call 581-6879, or visit JWB 113.

**Some Answers**

**4.1**

Exercises 9, 12, 16, 21 have textbook answers.

**4.1-5.** The problem is about the equation \( \text{nullspace} = \text{rowspace} \perp \), valid for both matrix \( A \) and \( A^T \). The Fundamental Theorem of Linear Algebra, Part II, can be summarized as the text

\[ \text{The nullspace is perpendicular to the rowspace.} \]

The text is justified from equality of vectors: \( Ax = 0 \) is equivalent to scalar equations \( \text{row}(A,1) \cdot x = 0, \ldots, \text{row}(A,m) \cdot x = 0 \), which says that \( x \) is perpendicular to all rows of \( A \), hence perpendicular to the rowspace of \( A \).

(a) Apply the equation to \( A^T \). Then \( \text{nullspace}(A^T) \perp \text{rowspace}(A^T) \), equivalent to \( \text{nullspace}(A^T) \perp \text{colspace}(A) \), implies any solution \( y \) to \( A^T y = 0 \) is perpendicular to any \( Ax \). Since \( b = Ax \), then \( y \perp b \) or \( y^T b = 0 \).

(b) If \( A^T y = (1,1,1) \) has a solution, then \( y \) is in \( \text{rowspace}(A) \). Then \( \text{nullspace}(A) \perp \text{rowspace}(A) \) implies \( y \cdot x = 0 \) for all \( x \) in the nullspace of \( A \).

**4.1-9.** Answers: colspace, perpendicular. See problem 4.1-5 above for nullspace \( \perp \) rowspace, applied here for matrix \( A^T \).

**Prove** \( A^T Ax = 0 \) implies \( Ax = 0 \).

Because \( y = Ax \) is a linear combination of the columns of \( A \), then \( y \) is in \( \text{colspace}(A) = \text{rowspace}(A^T) \). If \( A^T Ax = 0 \), then \( A^T y = 0 \), which implies \( y \) is in \( \text{nullspace}(A^T) \). Use \( \text{nullspace}(A^T) \perp \text{rowspace}(A^T) \). Then \( \text{nullspace}(A) \) and \( \text{rowspace}(A^T) \) meet only in the vector \( y = 0 \), which says \( y = Ax = 0 \).

**Prove** \( Ax = 0 \) implies \( A^T Ax = 0 \).

First, assume \( Ax = 0 \). Multiply by \( A^T \) to get \( A^T Ax = A^T 0 \). The right side is the zero vector, which gives \( A^T Ax = 0 \).

See also problem 4.2-27, which repeats this same argument.

**4.1-11.**

For \( A \): The nullspace is spanned by \((-2,1)\), the row space is spanned by \((1,2)\). The column space is the line through \((1,3)\) and \( N(A^T) \) is the line through \((3,-1)\). In each case,

For \( B \): The nullspace of \( B \) is is a line spanned by \((0,1)\), the row space is a line spanned by \((1,0)\). The column space and left nullspace are the same as for \( A \). As in (a), the line pairs are perpendicular.

**4.1-17.** If \( S \) is the subspace of \( \mathbb{R}^3 \) containing only the zero vector, then \( S^\perp \) is \( \mathbb{R}^3 \). If \( S \) is spanned by \((1,1,1)\), then \( S^\perp \) is the plane spanned by any two independent vectors perpendicular to \((1,1,1)\). For example, the
vectors $(1, -1, 0)$ and $(1, 0, -1)$. If $S$ is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, then $S^\perp$ is the line spanned by $(0, 1, 0)$, computed as the cross product of the two vectors, then scaled to be a unit vector.

4.1-26. $A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$

This example shows a matrix with perpendicular columns. Then $A^T A = 9I$ is diagonal: $(A^T A)_{ij} = (\text{column } i \text{ of } A) \cdot (\text{column } j \text{ of } A)$. When the columns are unit vectors, then $A^T A = I$.

4.2. Exercises 1, 3, 11, 21, 31 have textbook answers.

4.2-2.
(a) The projection of $b = (\cos \theta, \sin \theta)$ onto $a = (1, 0)$ is $p = (\cos \theta, 0)$.
(b) The projection of $b = (1, 1)$ onto $a = (1, -1)$ is $p = (0, 0)$ since $a^T b = 0$.

4.2-12. $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$ projection matrix onto the column space of $A$ (the $xy$ plane)

$P_2 = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \\ 0.5 & 0.5 & 1 \end{pmatrix} =$ Projection matrix onto the second column space. Certainly $(P_2)^2 = P_2$.

4.2-17. If $P_2 = P$ then $(I - P)^2 = (I - P)(I - P) = I^2 - PI - IP + P^2 = I - P$. When $P$ projects onto the column space, then $I - P$ projects onto the left nullspace.

4.2-27. If $A^T A x = 0$ then $A x$ is a vector in the nullspace of $A^T$. But $A x$ is a vector in the column space of $A$. To be in both of those perpendicular spaces, $A x$ must be zero. So $A$ and $A^T A$ have the same nullspace.

4.3. Exercises 1, 18, 21 have textbook answers.

4.3-6. $a = (1, 1, 1, 1)$ and $b = (0, 8, 8, 20)$ give $\hat{x} = \frac{a^T b}{a^T a} = 9$ and the projection is $\hat{x} a = p = (9, 9, 9, 9)$. Then $e^T a = (-9, -1, -1, 11)^T (1, 1, 1, 1) = 0$ and $\|e\| = \sqrt{204}$.

4.3-12.
(a) $a = (1, \ldots, 1)$ has $a^T a = m$, $a^T b = b_1 + \cdots + b_m$. Therefore $\hat{x} = a^T b / m$ is the mean of the b’s
(b) $e = b - \hat{x} a$, $b = (1, 2, b)$, $\|e\| = \sum_{i=1}^{m} (b_1 - \hat{x})^2 = \text{variance}$
(c) $p = (3, 3, 3)$, $e = (-2, -1, 3)$, $P^T e = 0$. $P = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

4.3-17. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$. The solution $\bar{e} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$ comes from $\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$. 