Math 2270 Extra Credit Problems
Chapter 3
December 2011

These problems were created for Bretscher’s textbook, but apply for Strang’s book, except for the division by chapter. To find the background for a problem, consult Bretscher’s textbook, which can be checked out from the math library or the LCB Math Center.

Due date: See the internet due dates. Records are locked on that date and only corrected, never appended.

Submitted work: Please submit one stapled package. Kindly label problems Extra Credit. Label each problem with its corresponding problem number. You may attach this printed sheet to simplify your work.

Problem XC3.1-12. (Image and kernel)
For the matrix $C$ below, display a frame sequence from $C$ to $\text{rref}(C)$. Write the image of $C$ as the span of the pivot columns of $C$. Write the kernel of $C$ as the list of partial derivatives $\partial x/\partial c_i$, etc, where $x$ is the vector general solution to $Cx = 0$.

$$C = \begin{pmatrix}
1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 0 & 3 & 1 \\
1 & -1 & -2 & 0 & 4 \\
1 & -1 & -3 & 4 & 6
\end{pmatrix}$$

Problem XC3.1-22. (Geometry of a linear transformation)
Give an example of a $3 \times 3$ matrix $A$ such that $T(x) = Ax$ has image equal to the plane through the three points $(0,0,0)$, $(0,1,1)$, $(1,1,0)$.

Problem XC3.1-38. (Image and kernel)
Express the image of the matrix $A$ as the kernel of a matrix $B$. The matrix $B$ can be a different size than $A$.

$$A = \begin{pmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 0 & 3 \\
1 & -1 & -2 & 0 \\
1 & -1 & -3 & 4
\end{pmatrix}$$

Problem XC3.1-50. (Kernel and image)
Let $A$ be an $n \times n$ matrix and $B = \text{rref}(A)$. Do $A$ and $B$ have the same kernels and images? Prove each assertion or give a counterexample.

Problem XC3.2-18. (Redundant vectors)
Identify the redundant vectors in the list, by application of the pivot theorem.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 12 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$ 

Problem XC3.2-22. (Pivot columns)
Express the non-pivot columns of $A$ as linear combinations of the pivot columns of $A$.

$$A = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
1 & 1 & 2 & 3 \\
1 & 2 & 3 & 4
\end{pmatrix}$$
Problem XC3.2-46. (Basis for \( \ker(A) \))
Find a basis for the kernel of \( A \), using frame sequence methods.

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 2 & 3 \\
0 & -1 & 2 & 3
\end{pmatrix}
\]

Problem XC3.2-48. (Independence with symbols)
Determine all values of the symbols \( a \) and \( b \) such that the following vectors are linearly independent.

\[
\begin{pmatrix}
a \\
0 \\
0 \\
2b
\end{pmatrix}, \quad
\begin{pmatrix}
a \\
2b \\
0 \\
\end{pmatrix}, \quad
\begin{pmatrix}
1 \\
3b \\
0 \\
a-b
\end{pmatrix}
\]

Problem XC3.3-10. (Basis for \( \ker(A) \) and \( \dim(A) \))
Find redundant columns by inspection and then find a basis for \( \dim(A) \) and \( \ker(A) \).

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 0 & 0 \\
2 & 2 & 2
\end{pmatrix}
\]

Problem XC3.3-24. (Basis for \( \ker(A) \) and \( \dim(A) \))
Find \( \text{rref}(A) \) and then a basis for \( \ker(A) \) and \( \dim(A) \).

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 3 \\
0 & 0 & 1 & 3
\end{pmatrix}
\]

Problem XC3.3-52. (Row space basis)
Find a basis for the row space of \( A \) consisting of columns of \( A^T \).

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
3 & 3 & 1 & 3 \\
5 & 3 & 1 & 3
\end{pmatrix}
\]

Problem XC3.3-64. (Kernels of matrices)
Prove or disprove that \( \ker(A) = \ker(B) \).

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
3 & 3 & 1 & 3 & 0 \\
5 & 3 & 1 & 3 & 0
\end{pmatrix}, \quad
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
3 & 3 & 1 & 3 & 0 \\
5 & 3 & 1 & 3 & 0
\end{pmatrix}
\]

Problem XC3.4-18. (Coordinates and spanning sets)
Let \( V = \text{span}\{v_1, v_2, v_3\} \), where the vectors are displayed below. Test for \( x \) in \( V \), and if true, then report the coordinates of \( x \) relative to the the vectors \( v_1, v_2, v_3 \).

\[
v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \quad x = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}
\]
Problem XC3.4-30. (Matrix of a linear transformation)
Find the matrix $B$ of the linear transformation $T$ relative to the basis $v_1, v_2, v_3$.

$$T(x) = \begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{pmatrix},$$

$$v_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

Problem XC3.4-46. (Basis of a plane)
Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$. Choose a basis $v_1, v_2$ for the plane, arbitrarily, your choice. Then determine the vector $x$ which has coordinates $2, -1$ relative to this basis.

End of extra credit problems chapter 3.