MATH 2270-003 Exam 2 Fall 2010

Solutions

1. (10 points) V is the span of the given vectors in $\mathbb{R}^4$. Find orthonormal vectors whose span is V.

$$\bar{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 3\sqrt{2} \end{pmatrix}$$

Solution:

Rescale $\bar{v}_1$ to make it a unit vector:

$$\bar{b}_1 = \frac{\bar{v}_1}{|\bar{v}_1|} = \frac{1}{\sqrt{5}}\bar{v}_1$$

Check if $\bar{b}_1$ is orthogonal to $\bar{v}_2$:

$$\bar{b}_1 \cdot \bar{v}_2 = \frac{1}{\sqrt{5}}(\bar{v}_1 \cdot \bar{v}_2) = \frac{1}{\sqrt{5}} \neq 0$$

They are not orthogonal, so subtract off the $\bar{b}_1$–part of $\bar{v}_2$:

$$\bar{v}_2 - \frac{1}{\sqrt{5}}\bar{b}_1 = \bar{v}_2 - \frac{1}{5}\bar{v}_1 = \begin{pmatrix} 3 \\ \frac{6}{5} \\ \frac{3}{5} \\ 3\sqrt{2} \end{pmatrix}$$

This vector is orthogonal to $\bar{b}_1$, but needs to be rescaled to be a unit vector:

$$\left| \begin{pmatrix} 3 \\ \frac{6}{5} \\ \frac{3}{5} \\ 3\sqrt{2} \end{pmatrix} \right| = \frac{12}{\sqrt{5}}$$

An orthonormal basis is $\{\bar{b}_1, \bar{b}_2\}$ where $\bar{b}_1$ is as above and $\bar{b}_2$ is

$$\bar{b}_2 = \frac{\sqrt{5}}{12} \begin{pmatrix} 3 \\ \frac{6}{5} \\ \frac{3}{5} \\ 3\sqrt{2} \end{pmatrix}$$

2. (15 points) For the subspace V in the previous problem, give the matrix that projects $\mathbb{R}^4$ to V and the matrix that projects $\mathbb{R}^4$ to $V^\perp$. (Note: It is acceptable to give the answer as a product of matrices, you do not need to perform the matrix multiplication.)

Solution:

The formula for the projection matrix is $P_V = A(A^TA)^{-1}A^T$, where A is a matrix whose columns are a basis for V. If we use the orthonormal basis for V found in the previous problem, then $A^TA = ID$, so we have $P_V = AA^T$ where:

$$A = \begin{pmatrix} 3 & \frac{\sqrt{5}}{12}3 \\ \frac{6}{5} & \frac{\sqrt{5}}{12}\frac{6}{5} \\ \frac{2}{5} & \frac{\sqrt{5}}{12}\frac{3}{5} \\ 3\sqrt{2} & \frac{\sqrt{5}}{12}3\sqrt{2} \end{pmatrix}$$

Projection to $V^\perp$ is given by: $P_{V^\perp} = ID - P_V$

3. (15 points) Find the least squares best fit line for the points $(0,1)$, $(1,2)$, $(2,3)$, $(4,4)$.

Solution:

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}$ and $\bar{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

$A\bar{x} = \bar{b}$ has no solutions. The least squares best fit line is given by:

$$\begin{aligned}
\bar{x} &= (A^TA)^{-1}A^T\bar{b} \\
&= \left(\begin{pmatrix} 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 21 & 7 \\ 7 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 24 \\ 10 \end{pmatrix} \\
&= \frac{1}{35} \begin{pmatrix} 4 & -7 \\ -7 & 21 \end{pmatrix} \begin{pmatrix} 24 \\ 10 \end{pmatrix} = \begin{pmatrix} \frac{26}{35} \\ \frac{42}{35} \end{pmatrix}
\end{aligned}$$

So, the least squares best fit line is: $y = \frac{26}{35}x + \frac{42}{35}$

4. (15 points) For the following matrix, find the eigenvalues and the maximum number of linearly independent eigenvectors. Find this many linearly independent eigenvectors.

$$A = \begin{pmatrix} 4 & 1 \\ -5 & -2 \end{pmatrix}$$

Solution:

The characteristic polynomial is $(\lambda - 3)(\lambda + 1)$, so there are two distinct real eigenvalues, 3 and -1. There is an eigenvector for each eigenvalue, and eigenvectors for distinct eigenvectors form a linearly independent set, so there are at least two linearly independent eigenvectors. We are in $\mathbb{R}^2$, so there are at most two vectors in a linearly independent set. Thus, there are at most two linearly independent eigenvectors.

An eigenvector for A corresponding to eigenvalue 3 is a null vector for the matrix $A - 3ID = \begin{pmatrix} 1 & 1 \\ -5 & -5 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a null vector for this matrix.

An eigenvector for A corresponding to eigenvalue -1 is a null vector for the matrix $A + 1ID = \begin{pmatrix} 5 & 1 \\ -5 & -1 \end{pmatrix}$, and $\begin{pmatrix} -\frac{1}{5} \\ 1 \end{pmatrix}$ is a null vector for this matrix.

5. (5 points) For the following matrix, find the eigenvalues and the maximum number of linearly independent eigenvectors. Find this many linearly independent eigenvectors.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Solution:

This matrix is in Jordan Form. The eigenvalues are the diagonal entries: 1, 2, 3. There are four Jordan Blocks, so there are at most four linearly independent eigenvectors. They are:

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$$

6. (5 points) Find the determinant of the following matrix:

$$\begin{pmatrix}3 & 2 & 3\\1 & 0 & 1\\0 & 1 & -1\end{pmatrix}$$

Solution:

There are several methods to compute this. The answer is 2.

7. (15 points) Describe the orbits of the discrete linear dynamical system $\bar{v}_{i+1} = A\bar{v}_i$ for the matrix

$$A = \begin{pmatrix}0 & -1\\2 & 3\end{pmatrix}$$

Solution:

A has eigenvalues 2 and 1 with corresponding eigenvectors $\bar{x} = \begin{pmatrix}-1\\2\end{pmatrix}$ and $\bar{y} = \begin{pmatrix}-1\\1\end{pmatrix}$, respectively.

The two linearly independent eigenvectors form a basis for $\mathbb{R}^2$, so any vector $\bar{v}_0$ can be written as a linear combination $\bar{v}_0 = a\bar{x} + b\bar{y}$ of these. The orbit of $\bar{v}_0$ consists of the points $A^k\bar{v}_0 = 2^k a\bar{x} + 1^k b\bar{y}$. Therefore, if $a = 0$ then the orbit is a single fixed point of A. Otherwise, the orbit consists of points escaping to infinity exponentially quickly and moving parallel to the vector $\bar{x}$.

8. (5 points) Suppose V is a 4 dimensional subspace of $\mathbb{R}^9$. Let P_V be the matrix that projects $\mathbb{R}^9$ onto the subspace V. What are the dimensions and rank of the matrix P_V?

Solution:

P_V takes a vector in $\mathbb{R}^9$ and gives back a vector in $\mathbb{R}^9$, so it is a 9×9 matrix.

The column space of P_v is V, so the column space is 4 dimensional. The rank of P_V is equal to the dimension of the column space, so P_V is rank 4.

9. (10 points) Suppose A is a 3×3 matrix whose entries all have absolute value less than or equal to 2. Find such a matrix that has $Det(A) \geq 30$. Is it possible to find such a matrix with $Det(A) \geq 50$? Find one or explain why it is impossible.

Solution:

There are many ways to find such a matrix with determinant 32. One way would be to make two entries along the diagonal equal to -2, and make all the other entries equal to 2.

It is not possible to have $Det(A) \geq 50$. The Big Formula for the determinant has $3! = 6$ entries, each of which is a product of three entries of the matrix. Thus, we have 6 terms each of which is at most 8, so the determinant could not possibly be greater than 48.

10. (15 points)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

Find a matrix C such that $C^{-1}AC$ is diagonal.

Solution:

Find three linearly independent eigenvectors of A. Let C be the matrix whose columns are these linearly independent eigenvectors. Then C is invertible and $C^{-1}AC$ is diagonal.

The characteristic polynomial of A is $(2 - \lambda)(\lambda^2 - 5\lambda)$ which has roots 0, 2, and 5.

(Note that A has a 0 eigenvalue; it is not invertible, but it is still diagonalizable.)

The eigenvector corresponding to 2 is just the first coordinate vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Column 3 is equal to 2 times column 3, so a null vector, an eigenvector with eigenvalue 0, is $\begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$. Finally, and eigenvector for eigenvalue 5 is a null vector for the matrix $A - 5ID = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{pmatrix}$

A null vector for this matrix is $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

So let $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 2 \\ 0 & 1 & 1 \end{pmatrix}$

No new questions beyond this point.