Basis, Dimension, Kernel, Image

- Definitions: Pivot, Basis, Rank and Nullity
- Main Results: Dimension, Pivot Theorem
- Main Results: Rank-Nullity, Row Rank, Pivot Method
- Definitions: Kernel, Image, `rowspace`, `colspace`
- How to Compute: Nullspace, Rowspace, Colspace
- Dimension, Kernel and Image
- Testing Bases for Equivalence
- Equivalent Bases: Computer Illustration
- A False Test for Equivalent Bases

Definitions: Pivot and Basis

Pivot of A	A column in $\mathbf{rref}(A)$ which contains a leading one has a corresponding column in A, called a pivot column of A.
Basis of V	It is an independent set $\mathbf{v}_1, \ldots, \mathbf{v}_k$ from data set V whose linear combinations generate all data items in V. .

Definitions: Rank and Nullity

$\mathbf{rank}(A)$	The number of leading ones in $\mathbf{rref}(A)$
$\mathbf{nullity}(A)$	The number of columns of A minus $\mathbf{rank}(A)$

Main Results: Dimension, Pivot Theorem

Theorem 1 (Dimension)
If a vector space V has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and also a basis $\mathbf{u}_1, \ldots, \mathbf{u}_q$, then $p = q$. The **dimension** of V is this unique number p.

Theorem 2 (The Pivot Theorem)

- The pivot columns of a matrix A are linearly independent.
- A non-pivot column of A is a linear combination of the pivot columns of A.

The proofs can be found in web documents and also in the textbook by E & P. Self-contained proofs of the statements of the pivot theorem appear in these slides.

Lemma 1 Let B be invertible and $\mathbf{v}_1, \ldots, \mathbf{v}_p$ independent. Then $B\mathbf{v}_1, \ldots, B\mathbf{v}_p$ are independent.

Proof of Independence of the Pivot Columns

Consider the fundamental frame sequence identity $\mathbf{rref}(\boldsymbol{A}) = \boldsymbol{E}\boldsymbol{A}$ where $\boldsymbol{E} = \boldsymbol{E}_k \cdots \boldsymbol{E}_2\boldsymbol{E}_1$ is a product of elementary matrices. Let $\boldsymbol{B} = \boldsymbol{E}^{-1}$. Then

$$\mathbf{col}(\mathbf{rref}(\boldsymbol{A}), \boldsymbol{j}) = \boldsymbol{E}\,\mathbf{col}(\boldsymbol{A}, \boldsymbol{j})$$

implies that a pivot column $\boldsymbol{j}$ of $\boldsymbol{A}$ satisfies

$$\mathbf{col}(\boldsymbol{A}, \boldsymbol{j}) = \boldsymbol{B}\,\mathbf{col}(\boldsymbol{I}, \boldsymbol{j}).$$

Because the columns of $\boldsymbol{I}$ are independent, then also the pivot columns of $\boldsymbol{A}$ are independent, by the Lemma.

Proof of Non-Pivot Column Dependence

Using matrix B from the previous proof, $\vec{\mathbf{u}} = B\vec{\mathbf{v}}$ holds for a non-pivot column $\vec{\mathbf{u}}$ of A and its corresponding non-pivot column $\vec{\mathbf{v}}$ in $C = \mathbf{rref}(A)$. Because each nonzero row of C has a leading one, if a component $v_i \neq 0$, then row i of C has a leading one in column $j_i < i$. Then $\mathbf{col}(C, j_i)$ is a column of the identity I and

$$\vec{\mathbf{v}} = \sum_{v_i \neq 0} v_i \,\mathbf{col}(C, j_i).$$

Multiply the preceding display by B to give

$$\begin{aligned}
\vec{\mathbf{u}} &= B\vec{\mathbf{v}} \\
&= \sum_{v_i \neq 0} v_i B \,\mathbf{col}(C, j_i) \\
&= \sum_{v_i \neq 0} v_i \,\mathbf{col}(A, j_i).
\end{aligned}$$

Then $\vec{\mathbf{u}}$ is a linear combination of pivot columns of A.

Main Results: Rank-Nullity, Row Rank, Pivot Method

Theorem 3 (Rank-Nullity Equation)
$\mathbf{rank}(A) + \mathbf{nullity}(A) =$ column dimension of A

Theorem 4 (Row Rank Equals Column Rank)
The number of independent rows of a matrix A equals the number of independent columns of A. Equivalently, $\mathbf{rank}(A) = \mathbf{rank}(A^T)$.

Theorem 5 (Pivot Method)
Let A be the augmented matrix of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Let the leading ones in $\mathbf{rref}(A)$ occur in columns $i_1, \ldots, i_p$. Then a largest independent subset of the k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the set

$$\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_p}.$$

Proof that $\mathbf{rank}(\boldsymbol{A}) = \mathbf{rank}(\boldsymbol{A}^{\boldsymbol{T}})$

Let $\boldsymbol{S}$ denote the set of all linear combinations of the rows of $\boldsymbol{A}$. Then $\boldsymbol{S}$ is a subspace, known as the row space of $\boldsymbol{A}$. A frame sequence from $\boldsymbol{A}$ to $\mathbf{rref}(\boldsymbol{A})$ consists of combination, swap and multiply operations on the rows of $\boldsymbol{A}$. Therefore, each nonzero row of $\mathbf{rref}(\boldsymbol{A})$ is a linear combination of the rows of $\boldsymbol{A}$. Because these rows are independent and span $\boldsymbol{S}$, then they are a basis for $\boldsymbol{S}$. The size of the basis is $\mathbf{rank}(\boldsymbol{A})$.

The pivot theorem applied to $\boldsymbol{A}^{\boldsymbol{T}}$ implies that each vector in $\boldsymbol{S}$ is a linear combination of the pivot columns of $\boldsymbol{A}^{\boldsymbol{T}}$. Because the pivot columns of $\boldsymbol{A}^{\boldsymbol{T}}$ are independent and span $\boldsymbol{S}$, then they are a basis for $\boldsymbol{S}$. The size of the basis is $\mathbf{rank}(\boldsymbol{A}^{\boldsymbol{T}})$.

The two competing bases for $\boldsymbol{S}$ have sizes $\mathbf{rank}(\boldsymbol{A})$ and $\mathbf{rank}(\boldsymbol{A}^{\boldsymbol{T}})$, respectively. But the size of a basis is unique, called the dimension of the subspace $\boldsymbol{S}$, hence the equality

$$\mathbf{rank}(\boldsymbol{A}) = \mathbf{rank}(\boldsymbol{A}^{\boldsymbol{T}}).$$

Definitions: Kernel, Image, rowspace, colspace

$\text{kernel}(A) = \text{nullspace}(A) = \{\mathrm{x} : A\mathrm{x} = 0\}$.
$\text{Image}(A) = \text{colspace}(A) = \{\mathrm{y} : \mathrm{y} = A\mathrm{x} \text{ for some } \mathrm{x}\}$.
$\text{rowspace}(A) = \text{colspace}(A^T) = \{\mathrm{w} : \mathrm{w} = A^T\mathrm{y} \text{ for some } \mathrm{y}\}$.

How to Compute Nullspace, Rowspace and Colspace

Null Space. Compute $\mathbf{rref}(\boldsymbol{A})$. Write out the general solution $\mathbf{x}$ to $\boldsymbol{A}\mathbf{x} = \mathbf{0}$, where the free variables are assigned parameter names $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k$. Report the basis for $\mathbf{nullspace}(\boldsymbol{A})$ as the list $\boldsymbol{\partial}_{t_1}\mathbf{x}, \ldots, \boldsymbol{\partial}_{t_k}\mathbf{x}$.

Column Space. Compute $\mathbf{rref}(\boldsymbol{A})$. Identify the pivot columns $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_k$. Report the basis for $\mathbf{colspace}(\boldsymbol{A})$ as the list of columns $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_k$ of $\boldsymbol{A}$.

Row Space. Compute $\mathbf{rref}(\boldsymbol{A}^T)$. Identify the pivot columns $\boldsymbol{j}_1, \ldots, \boldsymbol{j}_\ell$ of $\boldsymbol{A}^T$. Report the basis for $\mathbf{rowspace}(\boldsymbol{A})$ as the list of rows $\boldsymbol{j}_1, \ldots, \boldsymbol{j}_\ell$ of $\boldsymbol{A}$.

Alternatively, compute $\mathbf{rref}(\boldsymbol{A})$, then $\mathbf{rowspace}(\boldsymbol{A})$ has a *different* basis consisting of the list of nonzero rows of $\mathbf{rref}(\boldsymbol{A})$.

Dimension, Kernel and Image

Symbol $\dim(V)$ equals the number of elements in a basis for V.

Theorem 6 (Dimension Identities)

(a) $\dim(\mathrm{nullspace}(A)) = \dim(\mathrm{kernel}(A)) = \mathrm{nullity}(A)$

(b) $\dim(\mathrm{colspace}(A)) = \dim(\mathrm{Image}(A)) = \mathrm{rank}(A)$

(c) $\dim(\mathrm{rowspace}(A)) = \mathrm{rank}(A)$

(d) $\dim(\mathrm{kernel}(A)) + \dim(\mathrm{Image}(A)) =$ column dimension of A

(e) $\dim(\mathrm{kernel}(A)) + \dim(\mathrm{kernel}(A^T)) =$ column dimension of A

Testing Bases for Equivalence

Theorem 7 (Equivalence Test for Bases)
Define augmented matrices

$$\boldsymbol{B} = \mathbf{aug}(\mathbf{v}_1, \ldots, \mathbf{v}_k), \quad \boldsymbol{C} = \mathbf{aug}(\mathbf{u}_1, \ldots, \mathbf{u}_\ell), \quad \boldsymbol{W} = \mathbf{aug}(\boldsymbol{B}, \boldsymbol{C}).$$

Then relation $\boldsymbol{k} = \boldsymbol{\ell} = \mathbf{rank}(\boldsymbol{B}) = \mathbf{rank}(\boldsymbol{C}) = \mathbf{rank}(\boldsymbol{W})$ implies

1. $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an independent set.

2. $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$ is an independent set.

3. $\mathbf{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathbf{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$

In particular, $\mathbf{colspace}(\boldsymbol{B}) = \mathbf{colspace}(\boldsymbol{C})$ and each set of vectors is an equivalent basis for this vector space.

Proof: Because $\mathbf{rank}(\boldsymbol{B}) = \boldsymbol{k}$, then the first $\boldsymbol{k}$ columns of $\boldsymbol{W}$ are independent. If some column of $\boldsymbol{C}$ is independent of the columns of $\boldsymbol{B}$, then $\boldsymbol{W}$ would have $\boldsymbol{k}+\mathbf{1}$ independent columns, which violates $\boldsymbol{k} = \mathbf{rank}(\boldsymbol{W})$. Therefore, the columns of $\boldsymbol{C}$ are linear combinations of the columns of $\boldsymbol{B}$. Then vector space $\mathbf{colspace}(\boldsymbol{C})$ is a subspace of vector space $\mathbf{colspace}(\boldsymbol{B})$. Because both vector spaces have dimension $\boldsymbol{k}$, then $\mathbf{colspace}(\boldsymbol{B}) = \mathbf{colspace}(\boldsymbol{C})$. The proof is complete.

Equivalent Bases: Computer Illustration

The following maple code applies the theorem to verify that two bases are equivalent:

1. The basis is determined from the colspace command in maple.

2. The basis is determined from the pivot columns of $\boldsymbol{A}$.

In maple, the report of the column space basis is identical to the nonzero rows of $\mathbf{rref}(\boldsymbol{A}^T)$.

```
with(linalg):
A:=matrix([[1,0,3],[3,0,1],[4,0,0]]);
colspace(A);            # Solve Ax=0, basis v1,v2 below
v1:=vector([2,0,-1]);v2:=vector([0,2,3]);
rref(A);                # Find the pivot cols=1,3
u1:=col(A,1); u2:=col(A,3); # pivot col basis
B:=augment(v1,v2); C:=augment(u1,u2);
W:=augment(B,C);
rank(B),rank(C),rank(W); # Test requires all equal 2
```

A False Test for Equivalent Bases

The relation

$$\mathbf{rref}(\boldsymbol{B}) = \mathbf{rref}(\boldsymbol{C})$$

holds for a substantial number of matrices $\boldsymbol{B}$ and $\boldsymbol{C}$. However, it does not imply that each column of $\boldsymbol{C}$ is a linear combination of the columns of $\boldsymbol{B}$.

For example, define

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{rref}(\boldsymbol{B}) = \mathbf{rref}(\boldsymbol{C}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

but $\mathbf{col}(\boldsymbol{C}, 2)$ is not a linear combination of the columns of $\boldsymbol{B}$. This means $\mathbf{colspace}(\boldsymbol{B}) \neq \mathbf{colspace}(\boldsymbol{C})$.

Geometrically, the column spaces are planes in $\boldsymbol{R}^3$ which intersect only along the line $\boldsymbol{L}$ through the two points $(0, 0, 0)$ and $(1, 0, 1)$.