Differential Equations 2280
Midterm Exam 3
Wednesday, 22 April 2009

Instructions: This in-class exam is 15 minutes. Do one problem only. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Each problem is scored 100.

Please discard this sheet after reading it.

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1. (ch7) Do enough to make 100%

(1a) [50%] Derive the formula $\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}(-tf(t))$.

(1b) [50%] Solve $x'' + 2x' = 0$, $x(0) = 0$, $x'(0) = 1$ by Laplace's Method.

(1c) [50%] Solve the system $x' = x + y$, $y' = -y$, $x(0) = 1$, $y(0) = 2$ by Laplace's Method.

Answer: (1a) $\frac{d}{ds}\mathcal{L}(f(t)) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty f(t)\frac{d}{ds}(e^{-st})dt = \int_0^\infty f(t)(-t)e^{-st}dt = \mathcal{L}(f(t)(-t))$.

(1b) $L(x) = 1/(s^2 + 2s) = \frac{a}{s} + \frac{b}{s+2} = \mathcal{L}(a + be^{-2t})$ implies $x(t) = a + be^{-2t}$. Partial fractions applied to $\frac{1}{s^2+2s} = \frac{a}{s} + \frac{b}{s+2}$ implies $a = 1/2$, $b = -1/2$.

(1c) Transform the equations with $\mathcal{L}$ and collect into a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$. A shortcut is Laplace's resolvent method. Then

$$\begin{pmatrix} s-1 & -1 \\ 0 & s+1 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solve by Cramer's rule to obtain $\mathcal{L}(x) = \frac{s+3}{(s-1)(s+1)}$, $\mathcal{L}(y) = \frac{2}{s+2}$. Then partial fractions and the backward Laplace table imply $x(t) = 2e^t - e^{-t}$, $y(t) = 2e^{-t}$.

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2. (ch5) Do both

The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t} + c_4\mathbf{v}_4 e^{\lambda_4 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3, 4$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 5 & 1 & 1 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix},$$

then

(2a) [75%] Display eigenanalysis details for A.

(2b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Answer: (2a) Use cofactor expansion on the last row of $\det(A - \lambda I)$ to obtain the expansion $(7-\lambda)^2(4-\lambda)(6-\lambda)$. Then $\lambda = 4, 6, 7, 7$. Three sequences of **rref** computations are required on augmented matrices constructed from $A - 4I$, $A - 6I$, $A - 7I$ to find the eigenpairs

$$\left(4, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(7, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right), \quad \left(7, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right).$$

(2b) $\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{6t}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{7t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 e^{7t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$

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3. (ch5) Do enough to make 100%

(3a) [50%] The eigenvalues are 3, 5 for the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Putzer's spectral formula. Don't expand matrix products, in order to save time. However, do compute the coefficient functions r_1, r_2.

(3b) [50%] Using the same matrix A from part (a), display the solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors.

(3c) [50%] Using the same matrix A from part (a), compute the exponential matrix e^{At}) by any known method, for example, the formula $e^{At} = \Phi(t)\Phi^{-1}(0)$.

Answer: (3a) $\mathbf{u}(t) = e^{At}\mathbf{x}(0)$, $e^{At} = e^{3t}I + \frac{e^{3t}-e^{5t}}{3-5}(A - 3I)$. Functions r_1, r_2 are computed from $r_1' = 3r_1$, $r_1(0) = 1$, $r_2' = 5r_2 + r_1$, $r_2(0) = 0$.

(3b) $\mathbf{u}(t) = e^{3t}\vec{\mathbf{c}}_1 + e^{5t}\vec{\mathbf{c}}_2$. Differentiate once and use $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, then set $t = 0$. The resulting system is

$$\begin{aligned} \vec{\mathbf{u}}_0 &= e^0\vec{\mathbf{c}}_1 + e^0\vec{\mathbf{c}}_2 \\ A\vec{\mathbf{u}}_0 &= 3e^0\vec{\mathbf{c}}_1 + 5e^0\vec{\mathbf{c}}_2 \end{aligned}$$

(3c) From Putzer's result of (3a),

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^{3t} + e^{5t} & e^{5t} - e^{3t} \\ e^{5t} - e^{3t} & e^{3t} + e^{5t} \end{pmatrix}.$$

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4. (ch5) Do both

(4a) [50%] Display the solution of $\mathbf{u}' = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ according to the Laplace Resolvent Method.

(4b) [50%] Display the variation of parameters formula for the system below, given $e^{At} = \begin{pmatrix} e^{2t} & e^{2t} - e^t \\ 0 & e^t \end{pmatrix}$. Then integrate to find $\mathbf{u}_p(t)$ for $\mathbf{u}' = A\mathbf{u}$.

$$\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

Answer: (4a) The resolvent equation $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}(0)$ is the system

$$\begin{pmatrix} s-3 & -1 \\ 0 & s-1 \end{pmatrix}\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

The system is solved by Cramer's rule for unknowns $\mathcal{L}(x)$, $\mathcal{L}(y)$ to obtain

$$\mathcal{L}(x) = \frac{2}{(s-3)(s-1)}, \quad \mathcal{L}(y) = \frac{2}{s-1}.$$

Partial fractions $\frac{2}{(s-3)(s-1)} = \frac{a}{s-3} + \frac{b}{s-1}$ and the backward Laplace table imply

$$x(t) = ae^{3t} + be^t, \quad y(t) = 2e^t.$$

The values of the constants are $a = 1$, $b = -1$.

(4b) $\vec{\mathbf{u}}_p(t) = e^{At}\int_0^t e^{-Au}\begin{pmatrix} e^u \\ 0 \end{pmatrix} du = e^{At}\int_0^t \begin{pmatrix} e^{-u} \\ 0 \end{pmatrix} du = \begin{pmatrix} e^t - 1 \\ 0 \end{pmatrix}$.

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5. (ch6) Do enough to make 100%

(5a) [30%] Define *asymptotically stable equilibrium* for $\mathbf{u}' = \mathbf{f}(\mathbf{u})$, a nonlinear 2-dimensional system in which $\mathbf{f}$ is continuously differentiable.

(5b) [30%] Give an example of a linear 2-dimensional system with a stable spiral at equilibrium point $x = y = 0$. Draw a representative phase diagram about $x = y = 0$.

(5c) [40%] Give an example of a nonlinear 2-dimensional system with exactly two equilibria.

(5d) [40%] Display a formula for the general solution of the equation $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{u}$.

Then explain why the system has a center at $(0, 0)$.

Answer: (5a) It is a constant solution $t \to \vec{\mathbf{u}}_0$. The equilibrium solution must be stable. Further, $\lim_{t\to\infty} \|\vec{\mathbf{u}}(t) - \vec{\mathbf{u}}_0\| = 0$ for all solutions $\vec{\mathbf{u}}(t)$ such that $\|\vec{\mathbf{u}}(0) - \vec{\mathbf{u}}_0\|$ is sufficiently small.

(5b) Required are characteristic roots like $-1 \pm i$. Let $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$. Then $\det(A - \lambda I) = (\lambda + 1)^2 + 1$, which gives the desired roots and classification of a stable spiral.

(5c) There are many examples, but none of them are linear $\mathbf{u}' = A\mathbf{u}$, because in this case $\det(A) \neq 0$ is required for classification, and then $(0, 0)$ is the only equilibrium point. **Example**: The nonlinear system $x' = y$, $y' = x(x - 1)$ has exactly two equilibrium points $(0, 0)$, $(1, 0)$.

(5d) The characteristic equation $\det(A - \lambda I) = 0$ is $\lambda^2 + 1 = 0$ with complex roots $\pm i$ and corresponding atoms $\cos t$, $\sin t$. Then the Cayley-Hamilton Method implies

$$\vec{\mathbf{u}}(t) = \cos t \vec{\mathbf{c}}_1 + \sin t \vec{\mathbf{c}}_2.$$

First explanation, why the classification is a center. Such solutions are 2π-periodic and wrap around the origin. Trajectories form either an ellipse or a circle, depending on initial data. **Second explanation**, why the classification is a center. The answer is a spiral or an ellipse, because of the complex roots, which indicate wrapping of the trajectories around the origin. It can't be a spiral, because the solution formula does not limit to the zero vector at either $t = \infty$ nor $t = -\infty$. So it must be a center.

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