Atoms

An **atom** is a term with coefficient 1 obtained by taking the real and imaginary parts of

$$x^j e^{ax+icx}, \quad j = 0, 1, 2, \ldots,$$

where a and c represent real numbers and $c \geq 0$.

Theorem 1 ((Independence of Atoms))
Any finite list of distinct atoms is linearly independent.

Details and Remarks

- The definition plus Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies that an atom is a term of one of the following types:

  $$x^n, \; x^n e^{ax}, \; x^n e^{ax}\cos bx, \; x^n e^{ax}\sin bx.$$

  The symbol n is an integer $0, 1, 2, \ldots$ and a, b are real numbers with $b > 0$.
- In particular, $1, x, x^2, \ldots, x^k$ are atoms and this list is independent.
- The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom e^x. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms x and x^2 are indeed atoms. Terms like e^{x^2}, $\ln|x|$ and $x/(1 + x^2)$ are not atoms, nor are they constructed from atoms.

Construction of the general solution from a list of distinct atoms

- The general solution y of a homogeneous constant-coefficient linear differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

  is known to be a formal linear combination of the atoms of this equation, using symbols c_1, …, c_n for the coefficients:

$$y = c_1(\text{atom } 1) + \cdots + c_n(\text{atom } n).$$

  In particular, each atom listed is itself a solution of the differential equation.

- **Euler's theorem** *infra* explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation

$$r^n + p_{n-1}r^{n-1} + \cdots + p_1 r + p_0 = 0.$$

- The **Fundamental Theorem of Algebra** states that there are exactly n roots r, real or complex, for an nth order polynomial equation. The result explains how we know that the characteristic equation has exactly n roots.

- **Picard's theorem** says that the constructed atom list is a **basis** for the solution space of the differential equation, provided it contains n independent elements.

  Because the list of atoms constructed by Euler's theorem has n distinct elements, which are independent, then these atoms form a **basis** for the general solution of the differential equation.

Euler's Theorem

Theorem 2 (L. Euler)
The function $y = x^j e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the nth order if and only if $(r - r_1)^{j+1}$ divides the characteristic polynomial.

The Atom List

1. If r_1 is a real root, then the atom list for r_1 begins with $e^{r_1 x}$. The revised atom list is

$$e^{r_1 x}, xe^{r_1 x}, \ldots, x^{k-1} e^{r_1 x}$$

   provided r_1 is a root of multiplicity k, that is, $(r - r_1)^k$ divides the characteristic polynomial, but $(r - r_1)^{k+1}$ does not.

2. If $r_1 = \alpha + i\beta$, with $\beta > 0$, is a complex root along with its conjugate root $r_2 = \alpha - i\beta$, then the atom list for this pair of roots (both r_1 and r_2 counted) begins with

$$e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x.$$

   If the roots have multiplicity k, then the list of $2k$ atoms are

$$\begin{array}{llll} e^{\alpha x} \cos \beta x, & xe^{\alpha x} \cos \beta x, & \ldots, & x^{k-1} e^{\alpha x} \cos \beta x, \\ e^{\alpha x} \sin \beta x, & xe^{\alpha x} \sin \beta x, & \ldots, & x^{k-1} e^{\alpha x} \sin \beta x. \end{array}$$

Explanation of steps 1 and 2

1. Root r_1 always produces atom $e^{r_1 x}$, but if the multiplicity is $k > 1$, then $e^{r_1 x}$ is multiplied by $1, x, \ldots, x^{k-1}$.
2. The expected first terms $e^{r_1 x}$ and $e^{r_2 x}$ $[e^{\alpha x + i\beta x}$ and $e^{\alpha x - i\beta x}]$ are **not atoms**, but they are **linear combinations of atoms**:

$$e^{\alpha x \pm i\beta x} = e^{\alpha x} \cos \beta x \pm i e^{\alpha x} \sin \beta x.$$

The atom list for a complex conjugate pair of roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ is obtained by multiplying the two *real* atoms

$$e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x$$

by the powers

$$1, x, \ldots, x^{k-1}$$

to obtain the $2k$ distinct *real* atoms in item **2** above.