# Construction of the general solution from a list of distinct atoms.

The general solution $y$ of a homogeneous constant-coefficient linear differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

is a formal linear combination of the atoms of this equation, using symbols $c_1$, . . . , $c_n$ for the coefficients:

$$y = c_1(\text{atom } 1) + \cdots + c_n(\text{atom } n).$$

Described here is how to construct the atoms.

Euler's theorem says that the $y = e^{r_1 x}$ is a solution of the differential equation if and only $r = r_1$ is a root (real or complex) of the **characteristic equation**

$$r^n + p_{n-1}r^{n-1} + \cdots + p_1 r + p_0 = 0.$$

More generally, $e^{r_1 x}$, . . . , $x^j e^{r_1 x}$ are solutions if and only if $(r - r_1)^{j+1}$ divides the characteristic polynomial. These solutions are atoms if $r_1$ is real. If $r_1$ is complex, then they are linear combinations of atoms; see below for distillation of atoms in this case.

The list of distinct atoms is generated from the roots of the characteristic equation. We do not digress here on how to solve such equations but rather report that the **Fundamental Theorem of Algebra** supplies, at least theoretically, exactly $n$ roots $r$, real or complex, for an $n$th order polynomial equation.

The atom list for the differential equation is the set of all distinct atom corresponding to roots of the characteristic equation. Picard's theorem says that this list is a **basis** for the solution space of the differential equation, provided it contains $n$ elements. Because the characteristic equation has exactly $n$ roots, then the list of atoms has $n$ elements, and it is a **basis**.

**Euler's Theorem**. *The function $y = x^j e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the $n$th order if and only if $(r - r_1)^{j+1}$ divides the characteristic polynomial.*

The atom list for either a real root or a complex root is generated as follows.

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**1**. If $r_1$ is a real root, then the atom list for $r_1$ begins with $e^{r_1 x}$. The revised atom list is

$$e^{r_1 x}, xe^{r_1 x}, \ldots, x^{k-1}e^{r_1 x}$$

provided $r_1$ is a root of multiplicity $k$, that is, $(r - r_1)^k$ divides the characteristic polynomial, but $(r - r_1)^{k+1}$ does not.

**2**. If $r_1 = \alpha + i\beta$, with $\beta > 0$, is a complex root along with its conjugate root $r_2 = \alpha - i\beta$, then the atom list for this pair of roots (both $r_1$ and $r_2$ counted) begins with

$$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x.$$

If the roots have multiplicity $k$, then the list of $2k$ atoms are

$$\begin{array}{llll} e^{\alpha x}\cos\beta x, & xe^{\alpha x}\cos\beta x, & \ldots, & x^{k-1}e^{\alpha x}\cos\beta x, \\ e^{\alpha x}\sin\beta x, & xe^{\alpha x}\sin\beta x, & \ldots, & x^{k-1}e^{\alpha x}\sin\beta x. \end{array}$$

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Explanation of steps **1** and **2**.

In step **1**, root $r_1$ always produces atom $e^{r_1 x}$, but if the multiplicity is $k > 1$, then $e^{r_1 x}$ is multiplied by 1, $x$, . . . , $x^{k-1}$.

In step **2**, the expected first terms $e^{r_1 x}$ and $e^{r_2 x}$ [$e^{\alpha x + i\beta x}$ and $e^{\alpha x - i\beta x}$] are **not atoms**, but they are **linear combinations of atoms**:

$$e^{\alpha x \pm i\beta x} = e^{\alpha x} \cos \beta x \pm i e^{\alpha x} \sin \beta x.$$

The atom list for a complex conjugate pair of roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ is obtained by multiplying the two *real* atoms

$$e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x$$

by the powers

$$1, x, \ldots, x^{k-1}$$

to obtain the $2k$ distinct *real* atoms in item **2** above.