Name ____________________ Class Time _____

Math 2250 Extra Credit Problems
Chapter 3
January 2008

Due date: See the internet due date for 5.1, which is the due date for these problems. Records are locked on that date and only corrected, never appended.

Submitted work. Please submit one stapled package per problem. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number, e.g., Xc3.1-16. You may attach this printed sheet to simplify your work.

Problem XcL2.1. (maple lab 2)

You may submit this problem only for score increases on maple lab 2.

Consider the linear differential equation $u' + ku = ka(t)$, $u(0) = u_0$, where $a(t) = 1 + \sin(\pi(t-3)/12)$. Solve the equation for $u(t)$ and check your answer in maple. Use maple assist for integration.

Problem XcL2.2. (maple lab 2)

You may submit this problem only for score increases on maple lab 2.

Consider the linear differential equation $u' + ku = ka(t)$, $u(0) = u_0$, where $a(t) = 1 + \sin(\pi(t-3)/12)$. Find the steady-state periodic solution of this equation and check your answer in maple.

Problem Xc3.1-16. (Elimination)

Solve the system below using frame sequences and report one of the three possibilities: no solution, unique solution, infinitely many solutions.

$$\begin{array}{rcrcrcl} x & + & 5y & + & 6z & = & 3, \\ 5x & + & 2y & - & 10z & = & 1, \\ 8x & + & 17y & + & 8z & = & 5. \end{array}$$

Problem Xc3.1-26. (systems of equations)

Give an example of a 3×3 system of equations which illustrates three planes, two of which intersect in a line, and that line lies entirely in the third plane.

Problem Xc3.2-14. (Echelon systems)

Solve the system below using frame sequences and report one of the three possibilities: no solution, unique solution, infinitely many solutions. Use variable list order x, y, z, w.

$$\begin{array}{rcrcrcrcl} 3x & - & 6y & + & z & + & 13w & = & 15, \\ 3x & - & 6y & + & 3z & + & 21w & = & 21, \\ 2x & - & 4y & + & 5z & + & 26w & = & 23. \end{array}$$

Problem Xc3.2-24. (Three possibilities with symbols)

Solve the system below for all values of a, b using frame sequences and report one of the three possibilities: no solution, unique solution, infinitely many solutions. If the system has a solution, then report the general solution.

$$\begin{array}{rcrcl} x & + & ay & = & b, \\ ax & + & (a-b)y & = & a. \end{array}$$

Problem Xc3.3-10. (RREF)

Show the frame sequence steps to **rref**(A) and attach a maple answer check (or do the whole problem in maple).

$$A = \begin{pmatrix} 1 & -4 & -2 \\ 3 & -12 & 1 \\ 2 & -8 & 5 \end{pmatrix}$$

Problem Xc3.3-20. (RREF)

Show the frame sequence steps to **rref**(A) and attach a maple answer check (or do the whole problem in maple).

$$A = \begin{pmatrix} 1 & -4 & -2 & 4 & 0 \\ 3 & -12 & 1 & 5 & 0 \\ 2 & -8 & 5 & 5 & 1 \end{pmatrix}$$

Problem Xc3.4-20. (Vector general solution)

Find the general solution in vector form $\mathbf{x}$, expressed as a linear combination of column vectors using symbols t_1, t_2, t_3 $\ldots$ (as many symbols as needed for the free variables).

$$\begin{array}{rcrcrcrcrcl} x_1 & - & x_2 & & & + & 7x_4 & + & 3x_5 & = & 0, \\ & & & & x_3 & - & x_4 & - & 2x_5 & = & 0, \\ & & & & & & & & 0 & = & 0, \\ & & & & & & & & 0 & = & 0, \\ & & & & & & & & 0 & = & 0. \end{array}$$

Problem Xc3.4-40. (Superposition)

(a) Add the two systems below to prove that sums of solutions are again solutions. You will show that $\mathbf{x} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ is a solution, given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are solutions of the homogeneous equation.

$$\left\{ \begin{array}{rcrcl} ax_1 & + & by_1 & = & 0, \\ cx_1 & + & dy_1 & = & 0. \end{array} \right. \qquad \left\{ \begin{array}{rcrcl} ax_2 & + & by_2 & = & 0, \\ cx_2 & + & dy_2 & = & 0. \end{array} \right.$$

(b) Add the two systems below to prove the superposition principle. You will show that $\mathbf{x} = \begin{pmatrix} x_1 + x_3 \\ y_1 + y_3 \end{pmatrix}$ is a solution, given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ solves the homogeneous problem and $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$ solves the non-homogeneous problem.

$$\left\{ \begin{array}{rcrcl} ax_1 & + & by_1 & = & 0, \\ cx_1 & + & dy_1 & = & 0. \end{array} \right. \qquad \left\{ \begin{array}{rcrcl} ax_3 & + & by_3 & = & e, \\ cx_3 & + & dy_3 & = & f. \end{array} \right.$$

Problem Xc3.5-16. (Inverse by frame sequence)

Calculate the frame sequence from $C = ((: A), I)$ to **rref**(C) and report A^{-1}. Perform a hand answer check for the inverse matrix. No maple please, all with pencil and paper.

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

Problem Xc3.5-44a. (Inverses and frame sequences I)

(a) Suppose A is 8×8 and 60 entries are ones. Explain why A^{-1} does not exist.

(b) Suppose that A is invertible and 3×3. A frame sequence is started with A and gives final frame (not the **rref**)

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

The steps used to arrive at the final frame are (1) `combo(1,2,-3)`, (2) `swap(2,3)`, (3) `combo(1,2,-1)`, (4) `combo(2,3,1)`, (5) `mult(2,-1)`. Find the matrix A.

Problem Xc3.5-44b. **(Inverses and frame sequences II)**

Invent a particular 3×3 invertible matrix A_1 and display a frame sequence A_1 to A_6 (or slightly longer) involving documented steps of combo, swap and mult (one of each at least). Then write the frame sequence in the form

$$A_6 = E_5 E_4 E_3 E_2 E_1 A_1$$

where $E_1, \ldots, E_5$ are the elementary matrices representing the combo, swap and mult operations. Finally, check your answer by multiplying out the right side of the above identity, showing the multiplication gives A_6 (which should be $\mathbf{rref}(A_1) = I$).

Example. The same problem but for 2×2 matrix A_1.

$A_1 = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix}$

$A_2 = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, `mult(2,1/2)`, $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$

$A_3 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, `swap(1,2)`, $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A_4 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, `mult(2,1/3)`, $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}$

$A_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, `combo(2,1,-2)`, $E_4 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

Then

$$\begin{aligned} A_5 &= E_4 A_4 \\ &= E_4 E_3 A_3 \\ &= E_4 E_3 E_2 A_2 \\ &= E_4 E_3 E_2 E_1 A_1 \end{aligned}$$

Multiply out the four elementary matrices by hand to get

$$E_4 E_3 E_2 E_1 = \begin{pmatrix} -2/3 & 1/2 \\ 1/3 & 0 \end{pmatrix}$$

and then

$$E_4 E_3 E_2 E_1 A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

This last check can be done in maple by defining each 2×2 elementary matrix, e.g., `A1:=matrix([[0,3],[2,4]]);` and then

```
with(linalg):
evalm(E4&*E3&*E2&*E1&*A1);
```

The last line gives the identity, which is A_5, and that completes the answer check.

Problem Xc3.6-6. **(Determinants and the four rules)**

Calculate $\det(A)$ using only the four rules *triang, swap, combo, mult.* Check the answer in maple.

$$A = \begin{pmatrix} 1 & -4 & -2 & 4 & 0 \\ 3 & -12 & 1 & 5 & 0 \\ 2 & -8 & 5 & 5 & 1 \\ 0 & -8 & 5 & 5 & 1 \\ 0 & 0 & 5 & 5 & 1 \end{pmatrix}$$

Problem Xc3.6-20. **(Determinants, hybrid rules)**

Calculate $\det(A)$ using the four rules *triang, swap, combo, mult* plus the cofactor rule. Check the answer in maple.

$$A = \begin{pmatrix} 1 & -4 & -2 & 4 & 0 \\ 3 & -12 & 1 & 5 & 0 \\ 0 & -12 & 0 & 5 & 0 \\ 0 & -12 & 1 & 0 & 0 \\ 2 & -8 & 5 & 5 & 1 \end{pmatrix}$$

Problem Xc3.6-32. **(Cramer's Rule)**

Calculate x, y and z using Cramer's rule. Check the answer in maple.

$$\begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Problem Xc3.6-40. **(Adjugate formula)**

Find the inverse of the matrix A using the formula $A^{-1} = \frac{\text{adjugate}}{\text{determinant}}$.

$$A = \begin{pmatrix} 1 & -4 & -2 & 4 \\ 3 & -1 & 1 & 5 \\ 0 & -1 & 0 & 1 \\ 2 & 0 & -1 & 0 \end{pmatrix}$$

Problem Xc3.6-40. **(Adjugate formula)**

Find the entry in row 4 and column 2 of the adjugate matrix for A, using only determinants.

$$A = \begin{pmatrix} 1 & -4 & -2 & 4 \\ 3 & -1 & -1 & 3 \\ 0 & -1 & 0 & 1 \\ 2 & 0 & -1 & 0 \end{pmatrix}$$

Problem Xc3.6-60. (Induction)

Assume that $B_1 = 1$ and $B_2 = 2$. Assume $B_{k+2} = 2B_k + B_{k+1}$ for each integer $k = 1, 2, 3, \ldots$.

Let $\mathcal{Q}_n$ denote the statement that $B_k = 2^{k-1}$ for $1 \le k \le n$. Prove by mathematical induction that all statements $\mathcal{Q}_n$ are true.

Problem note: You must prove that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are true, individually. Mathematical induction then applies to the sequence of statements $\mathcal{Q}_3$, $\mathcal{Q}_4$, $\ldots$, in short, to statements $\mathcal{P}_j = \mathcal{Q}_{j+2}$, $j = 1, 2, 3, \ldots$.

End of extra credit problems chapter 3.