Differential Equations Preliminary Examination

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Instructions: The examination consists of two parts. **Part A** consists of exercises concerning *Ordinary differential Equations* and **Part B** consists of exercises concerning *Partial Differential Equations*.

To obtain full credit, please complete three exercises from part A and three exercises from part B, a total of six (6) exercises. All exercises are equally weighted and partial credit applies to each. A passing score is 60% of the total possible score.

Sound and detailed solutions are expected, but bear in mind that too many details are time-consuming. Judgement of what is essential will be an important factor in determining the final score.

Part A
Ordinary Differential Equations
Do three (3) exercises from Part A for full credit.

Exercise A-1. Consider the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0$$

on the set X: $|t - t_0| \leq a$, $|x - x_0| \leq b$. Below, *outline the proof* means a sequence of statements and lemmas and very brief details. In the outlines, you may freely use the statement and details of proof from the Picard–Lindelöf existence and uniqueness theorem, which says that there exists a unique solution $x(t)$ defined on the interval $|t - t_0| \le \alpha = \min(a, b/m)$, $m = \max\{|f(t,x)| : (t,x) \in X\}$, provided f satisfies some special conditions. Select with a check–mark [√] and solve either A-1.I or A-1.II:

[ ] A-1.I. Let $f(t,x)$ be measurable in t for each fixed x, continuous in x for fixed t a.e. and for some $m \in L^1(t_0 - a, t_0 + a)$, $|f(t,x)| \le m(t)$ whenerever $(t,x) \in X$. Outline the proof of Carathéodory's existence theorem: There exists an absolutely continuous function $x(t)$ defined on an interval $|t - t_0| \le h$ with $h \le a$ such that $x'(t) = f(t, x(t))$ a.e., and $x(t_0) = x_0$.

[ ] A-1.II. Assume that f is continuous on X. Outline the proof of the Peano existence theorem: There exists for some $h > 0$ at least one solution $x(t)$ defined on the interval $|t - t_0| < h$.

Solution:

A-1.I: It will be shown that the initial value problem has a solution $x(t)$ continuously differentiable on $J = \{t : |t - t_0| \le \alpha\}$ for some $\alpha \le a$, to be determined later on, by an application of the Schauder fixed point theorem. Construct $Tx = x_0 + \int_{t_0}^t f(r, x(r))\, dr$ as in the Picard proof. Let $M = \{x(t) \in C(J) : |x(t) - x_0| \le b\}$. Let E be the Banach space $C(J)$ with $\|x\| = \max\{|x(t)| : t \in J\}$. The set M is closed, bounded and convex. The Schauder fixed point theorem will be applied to the operator T, giving a solution $x(t)$ continuous on $|t - t_0| \le \alpha$. The solution $x(t)$ has by virtue of the integral equation additional smoothness, hence it is a continuously differentiable solution of the initial value problem.

Three lemmas have to be established, to complete the proof.

Lemma. The mapping T is continuous on M.

Details: The composite $t \to f(t, x(t))$ is proved measurable by consideration of characteristic functions of intervals first, then simple functions, then measurable functions. Lebesgue's dominated convergence theorem applies to prove the continuity of T.

Lemma. The mapping T maps M into M.

Details: The domain measure 2α for interval $J = \{t : |t - t_0| \le \alpha\}$ is reduced until $\int_J m(r)dr < b$. Then $|Tx(t) - x_0| \le \int_J |f(r, x(r))|\, dr \le \int_J m(r)dr < b$ proves that T maps M into M.

Lemma. If $\{x_n\}$ is a bounded sequence in M, then $\{Tx_n\}$ is uniformly bounded in E and it satifies the equicontinuity inequality $\|Tx_n(t_1) - Tx_n(t_2)\| \le \left|\int_{t_1}^{t_2} m(r)dr\right|$, hence it has a norm-convergent subsequence.

Details: From above, $\|Tx_n - x_0\| \le b$, so the sequence is E-bounded, and by the equicontinuity inequality, it is equicontinuous. Apply the Arzela-Ascoli theorem to the sequence.

A-1.II: It will be shown that the initial value problem has a solution $x(t)$ continuously differentiable on $J = \{t : |t - t_0| \le \alpha\}$ by an application of the Schauder fixed point theorem. Construct $Tx = x_0 + \int_{t_0}^t f(r, x(r))\,dr$ as in the Picard proof. Let $M = \{x(t) \in C(J) : |x(t) - x_0| \le b\}$. Let E be the Banach space $C(J)$ with $\|x\| = \max\{|x(t)| : |t - t_0| \le \alpha\}$. The set M is closed, bounded and convex. The Schauder fixed point theorem will be applied to the operator T, giving a solution $x(t)$ continuous on $|t - t_0| \le \alpha$. This solution has by virtue of the integral equation additional smoothness, hence it is a solution of the initial value problem. The details of proof requires the following lemmas.

Lemma. T maps M into M

Details: $|Tx(t) - x_0| \le \left|\int_{t_0}^t |f(r, x(r))|\,dr\right| \le \alpha m \le b$. This work duplicates the Picard proof, and hence the details could be omitted.

Lemma. The operator T is continuous on M, that is, $\lim_{n\to\infty} \|x_n - x\| = 0$ with $x_n \in M$ and $x \in M$ implies $\lim_{n\to\infty} \|Tx_n - Tx\| = 0$.

Details: Apply Lebesgue's dominated convergence theorem.

Lemma. If $\{x_n\}$ is a bounded sequence in M, then $\{Tx_n\}$ is uniformly bounded and satifies the equicontinuity inequality $\|Tx_n(t_1) - Tx_n(t_2)\| \le m|t_1 - t_2|$, hence it has a norm-convergent subsequence.

Details: The inequality follows by repeating the inequality steps used to prove T maps M into M. Apply the Arzela-Ascoli theorem.

Exercise A-2. Assume that the eigenvalues of a real $n \times n$ matrix A have negative real part. Select with a check–mark $\boxed{\surd}$ and solve either A-2.I or A-2.II:

☐ A-2.I.

(a) Prove that positive constants M and α exist such that for all $x \in R^n$ and $t \ge 0$

$$\|e^{At}x\| \le M\|x\|e^{-\alpha t}.$$

(b) Prove that the zero solution of $u' = Au$ is asymptotically stable.

☐ A-2.II. Prove that for $h(t)$ continuous and T-periodic, the equation $u' = Au + h(t)$ has a unique T-periodic solution $u(t)$.

Solution:

A-2.I(a): Using $e^{At} = Pe^{Jt}P^{-1}$ for real Jordan form $J = P^{-1}AP$, it suffices to prove the inequality for A a real Jordan form. Each Jordan block B corresponds to a block e^{Bt} in the exponential. Write $e^{Bt} = e^{\mathcal{R}\mathrm{e}(\lambda)t}e^{Ct}$ where C is a Jordan block with purely complex eigenvalue $\mathcal{I}\mathrm{m}(\lambda)$. While e^{Ct} may contain polynomial terms, the negative exponential factor $e^{\mathcal{R}\mathrm{e}(\lambda)t}$ implies the desired inequality, by choosing α such that $\mathcal{R}\mathrm{e}(\lambda) < -\alpha < 0$ for all eigenvalues λ. In the development of this proof, the following lemmas are used.

Lemma. $|Ax| \le \|A\||x|$ where $\|A\|^2 = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2$.

Lemma. $\lim_{t\to\infty} P(t)e^{-\alpha t} = 0$ for any polynomial P.

A-2.I(b): Solutions of $u' = Au$ already exist for $t \ge 0$, therefore the issue is limit zero at infinity for any nonzero solution. Part (a) implies that $|u(t)| = \left|e^{At}u(0)\right| \le M|u(0)|e^{-\alpha t}$, hence the result is proved.

A-2.II: First, the uniqueness. Let $u_0(t)$ be a T-periodic solution. Then $u(t) = e^{At}c + u_0(t)$ is the general solution; here c is a constant vector. It follows that the difference $x(t) = u_0(t) - u_1(t)$ of T-periodic solutions u_0 and u_1 satisfies $x(t) = e^{At}d$ for some constant vector d. Because $\lim |x(t)| = 0$ at $t = \infty$, the T-periodicity of $x(t)$ implies $x(t) = 0$. Hence $u_0 = u_1$ and the periodic solution is unique.

Existence of the T-periodic solution will be proved by finding an initial value $u(0)$ such that $u(t)$ satisfies the periodicity requirement $u(0) = u(T)$. The T-periodic extension of $u(t)$ to $(-\infty, \infty)$ will then be of class C^1, it will satisfy the differential equation and hence it will be the desired T-periodic solution.

The requirement $u(0) = u(T)$ is translated via the variation of parameters formula into the relation

$$u(0) = e^{AT}u(0) + \int_0^T e^{A(T-s)}h(s)\,ds.$$

The matrix $I - e^{AT}$ has determinant equal to the product of its eigenvalues $\mu = 1 - e^{\lambda T}$, which are nonzero due to $\mathcal{R}\mathrm{e}(\lambda) < 0$. Invertibility of $I - e^{AT}$ therefore implies that $u(0)$ exists such that $u(0) = u(T)$, and the proof is complete.

Exercise A-3. Let the system $x' = f(x)$ define a C^1 flow ϕ_t on the open set E contained in R^n. Prove that the positive limit set $\Gamma^+(v)$ of a trajectory $x(t)$ with $x(0) = v$ is closed. Then, select with a check–mark $\boxed{\surd}$ and solve either A-3.I or A-3.II:

☐ A-3.I. Consider the autonomous planar dynamical system

$$x' = 6x - 2xy - 6x^2, \quad y' = -7y + 2xy - y^2.$$

(a) Compute the four rest points (=equilibrium or stationary points) of the system and the linearization about each rest point.

(b) Make a table in which each row contains a rest point, the classification stable or unstable, and the geometric classification node, spiral, center or hyperbolic point.

(c) Sketch the phase diagram showing the rest points and the local behavior of solution curves (rough and brief!).

☐ A-3.II. Let $r^2 = x^2 + y^2$, $w = (r^2 - 1)(r^2 - 4)$ and consider the planar system $x' = -y + xw$, $y' = x + yw$ ($r' = rw$, $\theta' = 1$ in polar coordinates). Apply the Poincaré-Bendixson theorem to prove that $r = 1$ and $r = 2$ are limit cycles (a periodic orbit γ with $\gamma = \Gamma^+(v)$ or $\gamma = \Gamma^-(v)$ for nearby v).

Solution:

$\Gamma^+(v)$ is closed: The set $\Gamma^+(v)$ is the set of all limit points $\lim_{n\to\infty} x(t_n)$ where $t_n \ge 0$, $\lim_{n\to\infty} t_n = \infty$ and $x(t)$ is the unique solution of $x' = f(x)$, $x(0) = v$. Let x_0 be a limit point of the set. Then $x_0 = \lim_{n\to\infty} x_n$ where x_n belongs to $\Gamma^+(v)$. Write $x_n = \lim_{k\to\infty} x(t_{nk})$ where $\lim_{k\to\infty} t_{nk} = \infty$. Choose $s_1 = t_{11}$. Inductively, for each $n > 1$, choose $s_n = t_{nk}$ with $k = k(n)$ such that $t_{nk} > s_{n-1}$ and $|x_n - x(t_{nk})| < 1/n$. Then $x_0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x(s_n) + \lim_{n\to\infty}(x_n - x(s_n)) = \lim_{n\to\infty} x(s_n)$, so x_0 belongs to $\Gamma^+(v)$.

A-3.I(a): The rest points satisfy the factored equations $x(6 - 2y - 6x) = 0$, $y(-7 + 2x - y) = 0$. Solving gives rest points $(0,0)$, $(0,-7)$, $(1,0)$, $(2,-3)$.

The linearization about a rest point is $u' = Au$ where A is the Jacobian matrix of the nonlinear system evaluated at the rest point.

The Jacobian matrix is

$$J = \begin{pmatrix} 6 - 2y - 12x & -2x \\ 2y & -7 - 2y + 2x \end{pmatrix}.$$

At the four rest points this matrix becomes respectively

$$J(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & -7 \end{pmatrix}, \quad J(1,0) = \begin{pmatrix} -6 & -2 \\ 0 & -5 \end{pmatrix},$$

$$J(0,-7) = \begin{pmatrix} 20 & 0 \\ -14 & 7 \end{pmatrix}, \quad J(2,-3) = \begin{pmatrix} -12 & -4 \\ -6 & 3 \end{pmatrix}.$$

(b) The rest points, their stability and their classifications are:

$(0,0)$	unstable	hyperbolic point
$(1,0)$	stable	node
$(0,-7)$	unstable	node
$(2,-3)$	unstable	hyperbolic point

(c) The drawing, omitted here, should have a graph window at least as large as $-1 \le x \le 3$, $-8 \le y \le 1$. Appearing on the graph should be the rest points, their stability and their classification. Optionally, the semiaxes of the hyperbolas at the hyperbolic points can be drawn using eigenvector information. Arrows showing the direction of flow are optional.

A-3.II: The rest points of $u' = f(u)$ are found from the equations $0 = -y + xw$, $0 = x + yw$. Therefore, the only rest point is $x = y = 0$, that is, $f(u) = 0$ implies $u = (0,0)$.

The polar form of the system implies that $r = 1$ and $r = 2$ are invariant sets. They correspond exactly to periodic solutions $(\cos t, \sin t)$ and $(2\cos t, 2\sin t)$. These periodic solutions will be proved to be limit cyles according to the statement of the Poincaré–Bendixson theorem, which is:

Assume $f \in C^1(R)$. Let $\gamma^+(v)$ be the positive semiorbit of $x' = f(x)$, $x(0) = v$, assumed to not intersect itself and to be contained in a compact subset $K \subset D$ and let all points u_0 in the positive limit set $\Gamma^+(v)$ satisfy $f(u_0) \neq 0$. Then $\Gamma^+(v)$ is the orbit of a periodic solution of $u' = f(u)$ with smallest positive period T.

We freely use the fact that solutions of a C^1 autonomous two dimensional system cannot cross. Let v satisfy $0 < |v| < 1$. The polar form $r' = rw$ implies that $r'(t) > 0$, hence $r(t)$ increases near $t = 0$. The non-crossing property implies $r(t)$ increases to 1 as $t \to \infty$. Therefore, the trajectory of $u' = f(u)$, $u(0) = v$ for $t \geq 0$ does not intersect itself and it stays inside the compact set $x^2 + y^2 \leq 1$. Because $f(u_0) = 0$ implies $u_0 = (0, 0)$, we can satisfy the hypothesis of the Poincaré–Bendixson theorem by showing that $\Gamma^+(v)$ does not contain the origin. This follows directly form $r'(t) > 0$. The Poincaré–Bendixson theorem applies to show that $\Gamma^+(v)$ is a periodic orbit γ, that is, $\gamma = \Gamma^+(v)$ for all $0 < |v| < 1$. The argument can be repeated for $1 < |v| < 2$ to prove that $r(t)$ decreases to 1 as $t \to \infty$ and again $\gamma = \Gamma^+(v)$. That $|v| = 1$ for all $v \in \gamma$ follows from the argument just made, so γ is the unit circle, the same invariant set for all choices of v, $0 < |v| < 1$ or $1 < |v| < 2$. The invariant set $r = 1$ is a limit cycle, by the argument just presented.

The limit cycle result for $r = 2$ is obtained by applying the same kind of arguments to a related system obtained from the change of independent variable $s = -t$. The change replaces the system by $du/ds = -f(u)$ and the polar system by $dr/ds = -rw$, $d\theta/ds = -1$. Translating back to the original system, the set $r = 2$ is a limit cycle.

Exercise A-4. Select with a check–mark [√] and solve either A-4.I or A-4.II:

[ ] A-4.I: Assume $f : D \to R^N$ is continuous and f is bounded by a constant m on a subdomain $D_0 \subset D$. Let $u(t)$ be a solution of $u' = f(t, u)$ with $(t, u(t)) \in D_0$ on $a < t < b$.

(a) Prove that $u(t)$ satisfies a Lipschitz condition $|u(t_1) - u(t_2)| \leq m|t_1 - t_2|$.

(b) Prove that $u(t)$ has one-sided limits at $t = a$ and $t = b$: $\lim_{t \to a+} u(t)$ and $\lim_{t \to b-} u(t)$ exist and are finite.

(c) Explain the connection between (b) and the extension of solutions of initial value problems to a maximal interval of existence.

[ ] A-4.II: Let $f : [a, b] \to R^1$ be continuous and assume $f(a)f(b) \neq 0$. Verify the following properties of topological degree:
(a) If $f(b) > 0 > f(a)$, then $d(f, (a, b), 0) = 1$
(b) If $f(b) < 0 < f(a)$, then $d(f, (a, b), 0) = -1$
(c) If $f(a)f(b) > 0$, then $d(f, (a, b), 0) = 0$.

Solution:

A-4.I(a): The integral equation $u(t) = u_0 + \int_{t_0}^{t} f(r, u(r))\,dr$ implies that $|u(t_1) - u(t_2)| \le \left|\int_{t_1}^{t_2} |f(r, u(r))|\,dr\right| \le m|t_1 - t_2|$.

A-4.I(b): A sequence $\{u(t_n)\}$ with $\{t_n\}$ decreasing to $t = a$ will by (a) be a Cauchy sequence hence convergent. This proves the left-hand limit exists. The right-hand limit is done similarly.

A-4.I(c): The case of right extension will be discussed. It is assumed that $u_0(t)$ solves $u' = f(t, u)$ and it exists on $a < t < b$. Using the limiting value $u_0^* = \lim_{t \to b-} u_0(t)$, provided by (b), the initial value problem $u' = f(t, u)$, $u(b) = u_0^*$ is solved to give a solution $u_1(t)$ defined for $|t - b| < \alpha$. We then have the task of proving that the patched function

$$u(t) = \begin{cases} u_0(t) & a < t < b, \\ u_0^* & t = b, \\ u_1(t) & t \ge b \end{cases}$$

solves the differential equation $u' = f(t, u)$ on $(a, b + \alpha)$, or equivalently the integral equation $u(t) = u(t_0) + \int_{t_0}^{t} f(r, u(r))\,dr$ for some $a < t_0 < b$. Thus, the issue is the continuity of $u(t)$, which is settled directly by (b).

A-4.II: Define a linear function g with $g(a) = f(a)$, $g(b) = f(b)$. Then $d(g, (a, b), 0)$ is defined because $g(x) \ne 0$ on the boundary of (a, b). Direct calculation uses the definition

$$d(g, (a, b), 0) = \sum_{g(x)=0} \frac{g'(x)}{|g'(x)|}.$$

If $g(a)g(b) < 0$, then $d(g, (a, b), 0) = \pm 1$, because there is exactly one root of $g(x) = 0$ in (a, b). More precisely, $d(g, (a, b), 0) = 1$ for $g(a) < 0 < g(b)$ and $d(g, (a, b), 0) = -1$ for $g(a) > 0 > g(b)$. If $g(a)g(b) > 0$, then the linear equation $g(x) = 0$ has no roots in (a, b), therefore $d(g, (a, b), 0) = 0$ for $g(a)g(b) > 0$.

The homotopy $H(t, \lambda) = (1 - \lambda)f(t) + \lambda g(t)$, $a \le t \le b$, $0 \le \lambda \le 1$, equals $f(a)$ or $f(b)$ at a boundary point t of $[a, b]$, hence it is nonzero there, and by homotopy invariance of the topological degree, $d(f, (a, b), 0) = d(g, (a, b), 0)$.

A-4.II(a): From above, $d(f, (a, b), 0) = d(g, (a, b), 0) = 1$.

A-4.II(b): From above, $d(f, (a, b), 0) = d(g, (a, b), 0) = -1$.

A-4.II(c): From above, $d(f, (a, b), 0) = d(g, (a, b), 0) = 0$.

Part B

Partial Differential Equations

Do three (3) problems from Part B for full credit.

Exercise B-1. Consider the Sturm–Liouville problem $x^2(x^2u')' + \lambda u = 0$ on $1/2 \leq x \leq 1$ with boundary conditions $u(1/2) = u(1) = 0$.

(a) State without proof the main theorem on eigenfunction expansions which applies to this example.

(b) Use the change of variables $w(t) = u(1/t)$ to transform the differential equation into $d^2w/dt^2 + \lambda w = 0$. Then calculate the eigenvalues λ_n and eigenfunctions u_n, by citing without proof a result for the Sturm-Liouville problem $y'' + \lambda y = 0$, $y(a) = 0 = y(b)$.

(c) Sturm oscillation theory and the Prüfer transformation are used in the general theory to produce the candidate eigenvalues and eigenfunctions. Sketch briefly how this is accomplished, without proofs.

Solution:

B-1.I(a): Let $(p(t)y')' + (q(t) + \lambda r(t))y = 0$ have continuous coefficents p, q, r on $[a, b]$ with $p(t) > 0$ on $[a, b]$ and $r(t) > 0$ on (a, b). If boundary conditions $y(a) = y(b) = 0$ are imposed, then there exists an infinite increasing sequence $\{\lambda_n\}$ of eigenvalues and corresponding eigenfunctions $\{y_n\}$ such that $\lambda_n \to \infty$ and $n \to \infty$ and $\int_a^b y_n y_m r\, dt = 0$ for $n \neq m$. The eigenfunctions are complete in the Hilbert space of functions f satisfying $\int_a^b |f(t)|^2 r(t)\, dt < \infty$ equipped with inner product $(f, g) = \int_a^b f(t)\overline{g(t)}r(t)\, dt$ and norm $\|f\| = \sqrt{(f, f)}$.

B-1.I(b): The change of variables $w(t) = u(1/t)$, $x = 1/t$ satisfies for $' = d/dx$ the relation $dw/dt = u'(x)(-x^2)$ or $u'(x) = -t^2\, dw/dt$, therefore

$$\begin{aligned} x^2(x^2u'(x))' &= t^{-2}d(-dw/dt)/dx \\ &= -t^{-2}(-t^2 d^2w/dt^2) \\ &= d^w/dt^2. \end{aligned}$$

It follows that $x^2(x^2u')' + \lambda u = 0$ if and only if $dw/dt^2 + \lambda w = 0$. The boundary conditions transform to $w(1) = w(2) = 0$. We know this:

Lemma. The eigenpairs of $d^2y/dt^2 + \lambda y = 0$, $y(a) = 0 = y(b)$ are given by $\lambda_n = (n\pi/(b - a))^2$, $y_n = \sin(n\pi(t - a)/(b - a))$.

The eigenpairs of the original problem are therefore given by $\lambda_n = (n\pi)^2$, $u_n(x) = \sin(n\pi(x^{-1} - 1))$, $n \geq 1$.

B-1.I(c): The Prüfer tranformation $y = r(t)\cos\theta(t)$, $p(t)y'(t) = r(t)\sin\theta(t)$ is used to convert the question of existence of an eigenpair into the crossing of $\theta(t)$ with $n\pi$–lines. It is shown that $\theta(t)$ crosses with positive slope, and $\theta(t) \to \infty$, hence there are an infinite number of eigenvalues that limit at infinity. The corresponding eigenfunction is taken to be $y = r(t)\sin\theta(t)$, which has the required properties $y(a) = y(b) = 0$, $y \neq 0$.

Orthogonality of the eigenfunctions is not a part of this discussion, because it follows by a general argument, in which the differential equations satisfied by the two eigenfunctions are multiplied by an eigenfunction, then subtracted followed by an integration over $[a,b]$ to obtain an integral identity. This integral identity reduces to the desired orthogonality condition.

Exercise B-2. Select with a check–mark $\boxed{\surd}$ and solve either B-2.I or B-2.II:

$\square$ B-2.I: Define the Sobolev space $H^m(\Omega)$ for open $\Omega \subset R^n$. Then

(a) Prove that $H^m(\Omega)$ is a Hilbert space.

(b) Give an example of a sequence which shows that the subspace $C([0,1])$ in $L^1(0,1)$ is not complete in the L^1-norm.

(c) Compute the distributional derivatives ∂f and $\partial^2 f$ for $f(x) = |x|$ in $H^2(-\infty,\infty)$. Assume results for the Heaviside unit step and Delta.

$\square$ B-2.II: Define what it means for H to be a Hilbert space. Then:

(a) Explain the meaning of the formula $H = M \oplus M^\perp$ and give conditions on M for when it is true (do not give proofs).

(b) State the Riesz representation theorem and use (a) to prove it.

Solution:

B-2:I: Let Ω be open in R^n. The space $C^m(\overline{\Omega})$ is the space of restrictions to $\overline{\Omega}$ of functions in $C_0^m(R^n)$. Equip this linear space with the inner product

$$(f,g) = \sum_{\alpha \le m} \int_\Omega D^\alpha f \cdot \overline{D^\alpha g}\, dx$$

and let $\|f\| = \sqrt{(f,f)}$. Define $H^m(\Omega)$ to be the completion of the linear space $C^m(\overline{\Omega})$ with the norm $\|\cdot\|$.

B-2.I(a): The space H^m is already complete, so the issue is whether it is an inner product space. The inner product defined on $C^m(\overline{\Omega})$ extends uniquely under completion to a function $\langle f,g\rangle$ which agrees with (f,g) on $C^m(\overline{\Omega})$. The inner product properties to verify are (1) $\langle f_1+f_2,g\rangle = \langle f_1,g\rangle + \langle f_2,g\rangle$; (2) $\langle cf,g\rangle = c\langle f,g\rangle$; (3) $\langle f,g\rangle = \overline{\langle g,f\rangle}$; (4) $\langle f,f\rangle \ge 0$ and $\langle f,f\rangle = 0$ implies $f=0$. Properties (1),(2) and (3) follow by limiting, because the inner product on $C^m(\overline{\Omega})$ has these properties. Limiting also shows $\langle f,f\rangle \ge 0$ in (4). To prove the last part of (4), select a sequence $\{f_n\}$ in $C^m(\overline{\Omega})$ convergent in the H^m–norm to f with $\langle f,f\rangle = 0$. Then $\|f_n\| = \sqrt{(f_n,f_n)}$ converges to $\sqrt{\langle f,f\rangle} = 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} f_n = 0$ in $H^m(\Omega)$, which by the completion process implies $f = 0$.

B-2.I(b): Let

$$f_n(t) = \begin{cases} 0 & 0 \le t \le 0.5 - 1/n, \\ n(t-0.5)+1 & 0.5 - 1/n < t < 0.5, \\ 1 & 0.5 \le t \le 1. \end{cases}$$

For $k \geq 0$, $\|f_n - f_{n+k}\| \leq 1/n$, where $\|f\| = \int_0^1 |f(t)|\,dt$ is the usual norm in $L^1(0,1)$. Therefore, $\{f_n\}$ is Cauchy in $L^1(0,1)$. If the sequence limits to an element f of $C([0,1])$, then $f(t) = 0$ for $0 < t < 0.5$ and $f(t) = 1$ otherwise, which violates the continuity of f at $t = 1/2$. This proves that $\{f_n\}$ is Cauchy in L^1 but has an L^1–limit that does not belong to $C([0,1])$. Therefore, $C([0,1])$ fails to be complete in the L^1 norm.

B-2.I(c): A distributional derivative $F = \partial f$ is defined by the relation

$$\int_R -f(x)g'(x)\,dx = \int_R F(x)g(x)\,dx$$

for all g of class C_0^∞. We assume the result $\partial H = \delta$ for Heaviside's function H. It is expected that $\partial f = -1$ for $x < 0$ and $\partial f = 1$ for $x > 0$. The claim is that $(\partial f)(g) = \int_R F(x)g(x)\,dx$ where $F(x) = -1$ for $x < 0$, $F(x) = 1$ for $x \geq 0$, that is, $F(x) = -1+2H(x)$ or briefly, $\partial f = 2H-1$. The proof is by expansion of the two sides of the defining relation followed by integration by parts, which shows the two sides equal. Finally, $\partial^2 f = \partial(-1+2H) = 2\partial H = 2\delta$.

B-2.II: A linear space H is equipped with a norm $\|f\| = \sqrt{(f,f)}$ and inner product (f,g) satisfying the requirements (1) $(f_1+f_2,g) = (f_1,g)+(f_2,g)$, (2) $(cf,g) = c(f,g)$, (3) $(f,g) = \overline{(g,f)}$, (4) $(f,f) \geq 0$ and $(f,f) = 0$ implies $f = 0$. This inner product space is called a pre-Hilbert space, and if it is complete, then it is called a Hilbert space. Complete means that Cauchy sequences in the space converge to elements of the space.

B-2.II(a): The formula $H = M \oplus M^\perp$ applies when M is a closed subspace of H. It means that each element $f \in H$ can be uniquely expressed as $f = m + n$ where $m \in M$ and $n \in M^\perp = \{y \in H : (y,f) = 0, \quad f \in M\}$.

B-2.II(b): The Riesz theorem says that each continuous linear functional f on a Hilbert space H can be represented as $f(x) = (y,x)$ in terms of the inner product $(\cdot,\cdot)$ on H, for some unique $y \in H$.

Riesz proof: To prove uniqueness of y, write $(y_1,x) = (y_2,x)$ as $(y_1 - y_2, x) = 0$ and substitute $x = y_1 - y_2$. To prove existence of y, take $y = 0$ for the zero functional. Otherwise, $f \neq 0$ and the kernel $M = \{x : f(x) = 0\}$ is a closed subspace of H which does not equal H. It follows from the product theorem $H = M \oplus M^\perp$ that $M^\perp$ is nontrivial, so we choose $n \in M^\perp$ with $\|n\| = 1$ and $f(n) \neq 0$. Define $y = f(n)n$. We prove that the formula $f(x) = (y,x)$ is valid for all $x \in H$.

Given $x \in H$, write $x = m + z$ where $m \in M$ and $z \in M^\perp$. Define $w = f(n)z - \overline{f(z)}n$. Then $w \in M^\perp$, but also $f(w) = 0$ forces $w \in M$, hence by uniqueness, $w = 0$, that is, $f(n)z = \overline{f(z)}n$. The proof is completed by these steps:

$$\begin{aligned}
(y, x) &= \overline{f(n)}(n, x) \\
&= \overline{f(n)}(n, m + z) \\
&= 0 + \overline{f(n)}(n, z) \\
&= (n, f(n)z) \\
&= (n, f(z)n) \\
&= f(z)(n, n) \\
&= 0 + f(z) \\
&= f(m) + f(z) \\
&= f(x)
\end{aligned}$$

Exercise B-3. Select with a check–mark $\boxed{\checkmark}$ and solve either B-3.I or B-3.II:

$\square$ B-3.I: Let u, h denote elements of some Sobolev space and consider the distributional differential equation $-u'' + u = h$ with Dirichlet boundary conditions $u(0) = u(1) = 0$.

(a) Formulate an abstract boundary value problem $a(u, v) = (h, v)$, by defining the sequilinear form $a(u, v)$, the Hilbert space H and the inner product $(\cdot, \cdot)$.

(b) Discuss in detail how the Riesz theorem applies to solve the abstract boundary value problem.

(c) Is enough known about the Hilbert space solution u for it to be a solution of the distributional differential equation? Explain.

$\square$ B-3.II: Let $\phi \in H_0^1(G)$ and denote by $\|\cdot\|$ the usual norm in $L^2(G)$, G open in R^n. Assume $|x_1| \le K$ for all $x \in G$.

(a) Prove the Poincaré inequality $\|\phi\| \le 2K\|\partial_{x_1}\phi\|$.

(b) Explain, without proof, how to use generalizations of the Poincaré inequality to solve the abstract boundary value problem for distributional differential equations of the form $\Delta u = f$.

Solution:

B-3.I(a): Let H be the Hilbert space $H_0^1(0, 1)$ equipped with norm $\|f\| = \sqrt{(f, f)}$ and inner product

$$(f, g) = \int_0^1 \left(f g + \partial f \overline{\partial g}\right) dx.$$

Define the sesquilinear form by

$$a(u, v) = \int_0^1 \left(\partial u(x)\overline{\partial v(x)} + u(x)\overline{v(x)}\right) dx.$$

The abstract problem is to solve for $u \in H$ in the problem

$$a(u, v) = \int_0^1 h(x)\overline{v(x)}\, dx, \quad v \in H.$$

B-3.I(b): The abstract boundary value problem is solved by applying the Riesz theorem, because $a(u,v)$ is exactly (u,v). The Cauchy-Schwartz inequality implies that $\int_0^1 h(x)\overline{v(x)}\,dx$ defines a continuous conjugate linear functional on H. The space H contains the boundary conditions $u(0) = u(1) = 0$, in the abstract sense of the trace operator, as found in Showalter's textbook.

B-3.I(c): The abstract boundary value problem is solved for $u \in H_0^1(0,1)$, which is identified with the space of absolutely continuous functions on $[0,1]$ satisfying $u(0) = u(1) = 0$. If h belongs to $L^2(0,1)$, then u exists by the Riesz theorem and the integral equation

$$\int_0^1 \left(u'(x)v'(x) + u(x)v(x) - h(x)v(x)\right)\,dx = 0$$

is satisfied for all $v \in H$. This is a *weak form* of the original boundary value problem.

If u solves the distributional problem $-u''+u = h$, then $u'' = u-h \in L^2(0,1)$. Therefore, $u \in H^2$ will be necessary, if it is possible to recover the distributional differential equation from the weak formulation. Showalter (Ch 3) discusses how to do this, in a variety of ways.

Remark. A more classical approach on the problem would start with the variation of parameters relation

$$u(x) = c\sinh(x) - \int_0^x \sinh(x-r)h(r)\,dr.$$

The constant c is chosen to make $u(1) = 0$, possible because $\sinh(1) \neq 0$. Assuming h continuous makes u'' continuous and $-u''+u = h$.

B-3.II(a): Density implies that it is only required to prove the Poincaré inequality for $\phi \in C_0^\infty(G)$. Consider the identity

$$\partial_{x_1}\left(x_1\,\phi(x)\cdot\overline{\phi(x)}\right) = |\phi(x)|^2 + x_1\left(\partial_{x_1}\phi(x)\cdot\overline{\phi(x)} + \phi(x)\cdot\overline{\partial_{x_1}\phi(x)}\right).$$

This identity will be integrated over G. The divergence theorem applies to the term $\partial_{x_1}\left(x_1\,\phi(x)\cdot\overline{\phi(x)}\right)$ to obtain zero, because of $\phi(x) = 0$ for $x \in \partial G$. Integrating over G then gives the formula

$$-\int_G |\phi(x)|^2\,dx = \int_G x_1\left(\partial_{x_1}\phi(x)\cdot\overline{\phi(x)} + \phi(x)\cdot\overline{\partial_{x_1}\phi(x)}\right)\,dx.$$

Square both sides and estimate the integrand on the right by $2K|\phi|\,|\partial_{x_1}\phi|$. Apply the Cauchy-Schwartz inequality to obtain

$$\left(\int_G |\phi(x)|^2\,dx\right)^2 \le 4K^2\int_G |\phi(x)|^2\,dx \int_G |\partial_{x_1}\phi(x)|^2\,dx.$$

Divide and take square roots to finish the proof.

B-3.II(b): Generalizations of the Poincaré inequality take the form of an imbedding inequality of the form

$$\int_G |\phi(x)|^2\,dx \le L\sum_{|\alpha|=1}\int_G |\partial^\alpha\phi(x)|^2\,dx.$$

Such inequalities say that an equivalent norm to the H^1-norm can be obtained by dropping the terms for summation index $|\alpha| = 0$. The inner product can be replaced by a similar truncation of terms. The result is that a differential equation like $\Delta u = f$ has an abstract formulation $a(u,v) = \int_G fv\,dx$ where $a(u,v)$ is this re-defined inner product on H^1. The Riesz theorem applies directly, giving existence of a solution u to the abstract problem.

Exercise B-4. Sobolev proved an imbedding inequality of the form $\|f\|_B \le M\|f\|_A$, where $A = H^m(G)$ and B is the set of functions u such that $D^\alpha u$ is uniformly continuous on G for $|\alpha| \le k$. Give, without proof, the conditions on the open set $G \subset R^n$ and the integers m and k.

Select with a check–mark [√] and solve either B-4.I or B-4.II:

[ ] B-4.I: Under Sobolev's conditions on G, m and k, each $f \in H^m(G)$ satisfies $\partial^\alpha f = D^\alpha g$ a.e. for some k-times continuously differentiable function g, $|\alpha| \le k$.

(a) Prove this, assuming the imbedding inequality above.

(b) Determine for $n = 2$ the least m such that an element in $H^m(G)$ has 4 continuous derivatives (G as above).

[ ] B-4.II: Regularity theory implies that certain abstract boundary value problems $a(u,v) = (F,v)$ can be solved for $u \in H^{2+s}(G)$ provided $F \in H^s(G)$. Consider the Dirichlet problem for $\Delta u = F$, $x \in G$.

(a) Assume G is a disk in R^2. Give without proof a smoothness condition on F for the existence of C^2 solutions u.

(b) Assume G is bounded and open in R^2. Give without proof conditions on G and ∂G, and a smoothness condition on F, for the existence of C^2 solutions u.

Solution:

B-4 Conditions: The set G is open, bounded in R^n and G satisfies a cone condition, meaning that the radius $\rho > 0$ and volume $v > 0$ of a cone can be specified such that each point $y \in \overline{G}$ is the vertex of a cone of radius ρ and volume v that lies entirely in $\overline{G}$. The integers m and k must satisfy $m > k + n/2$.

B-4.I(a): Given $m > k + n/2$, G bounded open in R^n with cone condition, Sobolev imbedding says that

$$\sup_{x\in G} |D^\alpha u(x)| \le M\|u\|, \quad |\alpha| \le k,$$

where $\|\cdot\|$ is the norm in $H^m(G)$, for each $u \in A = C_u^k(\overline{G})$; the latter is the set of u such that $D^\alpha u$ is uniformly continuous on $\overline{G}$ for $|\alpha| \le k$. If $w \in H^m(G)$, then density implies w can be approximated by functions in $w_j \in C^m(\overline{G})$, that is, $w = \lim_{j\to\infty} w_j$. Because G is bounded and $m > k$, each w_j belongs to A. Set $u = w_{j+r} - w_j$ in the Sobolev imbedding inequality to show that $\{w_j\}$ is Cauchy in A. Let $g = \lim_{j\to\infty} w_j$ in the Banach space A. Then each distributional derivative $\partial^\alpha w$, being in L^2, has to agree a.e. with $D^\alpha g$, for $|\alpha| \leq k$.

B-4.I(b): Let $n = 2$. We determine the least m such that an element in $H^m(G)$ has 4 continuous derivatives. Because $m > k + n/2$ reduces to $m > k + 1$ and we want $k = 4$, then $m > 5$ is required. The least is $m = 6$.

B-4.II(a): We have $u \in H^{2+s}(G)$ provided $F \in H^s(G)$, according to regularity theory. To apply Sobolev imbedding with $n = 2$ and $k = 2$ (to get C^2 solutions) we need $s > k + n/2$ or $s > 3$. Already, the disk satisfies the cone condition. Likewise, the coefficient smoothness conditions and manifold conditions on G are satisfied. Therefore, $F \in H^4(G)$ implies $u \in C^2(\overline{G})$ for a disk G.

B-4.II(b): It is known that G bounded open satisfies the cone condition if ∂G is a C^1 manifold. Regularity theory requires $F \in H^4(G)$ to obtain $u \in C^2(\overline{G})$, as in (a), but we must add that ∂G is a C^{2+s} manifold where $s = 4$. No additional requirements surface for the differential equation, because its coefficients are of class C^∞.