2.3 Linear Equations I

An equation $y' = f(x, y)$ is called **first-order linear** or a **linear equation** provided it can be rewritten in the special form

$$y' + p(x)y = r(x) \tag{1}$$

for some functions $p(x)$ and $r(x)$. In most applications, p and r are assumed to be continuous. The function $p(x)$ is called the **coefficient of** y. The function $r(x)$ (r abbreviates *right side*) is called the **non-homogeneous term** or the **forcing term**. Engineering texts call $r(x)$ the **input** and the solution $y(x)$ the **output**.

A practical test:

> An equation $y' = f(x, y)$ with f continuously differentiable is **linear** provided $f_y(x, y)$ is independent of y.

Form (1) is obtained by defining $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$. Two examples:

$Ly' + Ry = E$	The LR-circuit equation with $p(x) = R/L$ and $r(x) = E/L$. Symbols L, R and E are respectively inductance, resistance and electromotive force.
$y' = -h(y - y_1)$	Newton's cooling equation with $p(x) = h$ and $r(x) = hy_1$. Oven temperature y_1 and meat thermometer reading $y(t)$ appear in the roast model.

Classifying Linear Equations

Algebraic complexity may make an equation $y' = f(x, y)$ appear to be **non-linear**, e.g., $y' = (\sin^2(xy) + \cos^2(xy))y$ simplifies to $y' = y$.

Computer algebra systems classify an equation $y' = f(x, y)$ as linear provided the identity $f(x, y) = f(x, 0) + f_y(x, 0)y$ is valid. Equivalently, $f(x, y) = r(x) - p(x)y$, where $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$. Automatic simplifications in computer algebra systems make this test practical. Hand verification can use the same method.

Elimination of an equation $y' = f(x, y)$ from the class of linear equations can be done from *necessary conditions*. The equality $f_y(x, y) = f_y(x, 0)$ implies two such conditions:

1. If $f_y(x, y)$ depends on y, then $y' = f(x, y)$ is not linear.

2. If $f_{yy}(x, y) \neq 0$, then $y' = f(x, y)$ is not linear.

For instance, either condition implies $y' = 1 + y^2$ is *not linear*.

Variation of Parameters and Integrating Factors

The initial value problem

$$y' + p(x)y = r(x), \quad y(x_0) = y_0, \tag{2}$$

where p and r are continuous in an interval containing $x = x_0$, has an explicit solution (justified on page 89)

$$y(x) = e^{-\int_{x_0}^{x} p(s)ds} \left(y_0 + \int_{x_0}^{x} r(t) e^{-\int_{x_0}^{t} p(s)ds}\, dt \right). \tag{3}$$

Formula (3) is called **variation of parameters**, for historical reasons. While (3) has some appeal, applications use the **integrating factor method** below, which is developed with indefinite integrals for computational efficiency. No one memorizes (3); they remember and study the *method*. See Example 11, page 87, for technical details.

Integrating Factor Identity

The technique called the **integrating factor method** uses the replacement rule (justified on page 89)

$$\text{The fraction} \quad \frac{\left(e^{\int p(x)dx} Y\right)'}{e^{\int p(x)dx}} \quad \text{replaces} \quad Y' + p(x)Y. \tag{4}$$

The factor $e^{\int p(x)dx}$ in (4) is called an **integrating factor**.

The Integrating Factor Method

Standard Form	Rewrite $y' = f(x, y)$ in the form $y' + p(x)y = r(x)$ where p, r are continuous. The method applies only in case this is possible.
Find W	Find a simplified formula for $W = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently.
Prepare for Quadrature	Obtain the new equation $\dfrac{(Wy)'}{W} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (4).
Method of Quadrature	Clear fractions to obtain $(Wy)' = rW$. Apply the method of quadrature to get $Wy = \int r(x)W(x)dx + C$. Divide by W to isolate the explicit solution $y(x)$.

In identity (4), functions p, Y and Y' are assumed continuous with p and Y *arbitrary* functions. The integral $\int p(x)dx$ equals $P(x)+C$, where $P(x)$ is some anti-derivative of $p(x)$. Because $e^{\int p(x)dx} = e^{P(x)}e^{C}$, then factor

e^C divides out of the fraction in (4). Applications therefore simplify the **integrating factor** $e^{\int p(x)dx}$ to $e^{P(x)}$, where $P(x)$ is *any suitable antiderivative* of $p(x)$ (effectively, we take $C = 0$).

Equation (4) is central to the method, because it collapses the two terms $y' + py$ into a single term $(Wy)'/W$; the method of quadrature applies to $(Wy)' = rW$. The literature calls the exponential factor W an **integrating factor** and equivalence (4) a **factorization** of $Y' + p(x)Y$.

Simplifying an integrating factor. Factor W is simplified by dropping constants of integration. To illustrate, if $p(x) = 1/x$, then $\int p(x)dx = \ln|x| + C$. The algebra rule $e^{A+B} = e^A e^B$ implies that $W = e^C e^{\ln|x|} = |x|e^C = (\pm e^C)x$. Let $c_1 = \pm e^C$. Then $W = c_1 W_1$ where $W_1 = x$. The fraction $(Wy)'/W$ reduces to $(W_1 y)'/W_1$, because c_1 cancels. In an application, we choose the simpler expression W_1. The illustration also shows that the exponential in W can sometimes be eliminated.

Superposition

Formula (3) can be decomposed into two expressions, called y_h and y_p, so that the **general solution** is expressed as $y = y_h + y_p$. The function y_h solves the homogeneous equation $y' + p(x)y = 0$ and y_p solves the non-homogeneous equation $y' + p(x)y = r(x)$. This observation is called the **superposition principle**.

Equation (3) implies the **homogeneous solution** y_h and a **particular solution** y_p^* can be defined by

$$(5) \qquad y_h = y_0 e^{-\int_{x_0}^{x} p(s)ds}, \quad y_p^* = e^{-\int_{x_0}^{x} p(s)ds} \int_{x_0}^{x} r(t) e^{\int_{x_0}^{t} p(s)ds} dt.$$

Verification amounts to setting $r = 0$ in (3) to determine y_h. The solution y_p^* depends on the forcing term $r(x)$, but y_h does not. Initial conditions of a problem are buried in y_h. Experimentalists view the computation of y_p^* as a *single experiment* in which the state y_p^* is determined by the forcing term $r(x)$ and zero initial data $y = 0$ at $x = x_0$.

Structure of Solutions

Formula (3), proved on page 89, directly establishes existence for the solution to the linear initial value problem (2). The proof also determines what other particular solutions might be used in the formula for a general solution:

Theorem 3 (Solution Structure)
Assume $p(x)$ and $r(x)$ are continuous on $a < x < b$ and $a < x_0 < b$. Let y_h and y_p^* be defined by equation (5). Let y be a solution of $y' + p(x)y = r(x)$ on $a < x < b$. Then y can be decomposed as $y = y_h + y_p^*$, where $y_0 = y(x_0)$.

In short, a linear equation has the solution structure *homogeneous plus particular*. In particular, two solutions of the non-homogeneous equation differ by some solution y_h of the homogeneous equation.

Examples

11 **Example (Integrating Factor Method)** Solve the equation $2y' + 6y = e^{-x}$ by the integrating factor method.

Solution: The solution is $y = \frac{1}{4}e^{-x} + ce^{-3x}$. An answer check appears in Example 13. The details:

$y' + 3y = 0.5e^{-x}$	Divide by 2 to get the standard form.
$W = e^{3x}$	Find the integrating factor $W = e^{\int 3dx}$.
$\dfrac{(e^{3x}y)'}{e^{3x}} = 0.5e^{-x}$	Replace the LHS by the integrating factor quotient; see page 85.
$(e^{3x}y)' = 0.5e^{2x}$	Clear fractions. Prepared for quadrature
$e^{3x}y = 0.5\int e^{2x}dx$	Method of quadrature applied.
$y = 0.5e^{-3x}\left(e^{2x}/2 + c_1\right)$	Evaluate the integral. Divide by $W = e^{3x}$.
$= \frac{1}{4}e^{-x} + ce^{-3x}$	Final answer, $c = 0.5c_1$.

12 **Example (Superposition)** Find a particular solution of $y' + 2y = 3e^x$ with fewest terms.

Solution: The answer is $y = e^x$. The first step solves the equation using the integrating factor method, giving $y = e^x + ce^{-2x}$; details below. A particular solution with fewest terms, $y = e^x$, is found by setting $c = 0$. The solution y_p^* of equation (5) has two terms: $y_p^* = e^x - e^{3x_0}e^{-2x}$. The reason for the extra term is the condition $y = 0$ at $x = 0$. The two particular solutions differ by the homogeneous solution y_0e^{-2x} where $y_0 = e^{3x_0}$.

Integrating factor method details:

$y' + 2y = 3e^x$	The standard form.
$W = e^{2x}$	Find the integrating factor $W = e^{\int 2dx}$.
$\dfrac{(e^{2x}y)'}{e^{3x}} = 3e^x$	Integrating factor identity applied on the left.
$e^{2x}y = 3\int e^{3x}dx$	Clear fractions and apply quadrature.
$y = e^{-2x}\left(e^{3x} + c\right)$	Evaluate the integral. Isolate y.
$= e^x + ce^{-2x}$	Solution found.

13 **Example (Answer Check)** Show the answer check details for $2y' + 6y = e^{-x}$ and candidate solution $y = \frac{1}{4}e^{-x} + ce^{-3x}$.

Solution: **Details**:

LHS $= 2y' + 6y$	Left side of the equation $2y' + 6y = e^{-x}$.
$= 2(-\frac{1}{4}e^{-x} - 3ce^{-3x}) + 6(\frac{1}{4}e^{-x} + ce^{-3x})$	Substitute for y.
$= e^{-x} + 0$	Simplify terms.
$=$ RHS	DE verified.

14 Example (Finding y_h and y_p) Find the homogeneous solution y_h and a particular solution y_p for the equation $2xy' + y = 4x^2$ on $x > 0$.

Solution: The solution by the integrating factor method is $y = 0.8x^2 + cx^{-1/2}$; details below. Then $y_h = cx^{-1/2}$ and $y_p = 0.8x^2$ give $y = y_h + y_p$.

The symbol y_p stands for *any* particular solution. Variation of parameters gives a *different* particular solution $y_p^* = 0.8x^2 - 0.8x_0^{5/2}x^{-1/2}$. It differs from the other particular solution $0.8x^2$ by a homogeneous solution $Kx^{-1/2}$.

Integrating factor method details:

$y' + 0.5y/x = 2x$	Standard form. Divided by $2x$.
$W = e^{0.5\int dx/x}$	The integrating factor is $W = e^{\int p}$.
$= e^{0.5\ln x}$	Simplify the integration constant.
$= x^{1/2}$	Used $\ln u^n = n\ln u$. Simplified W found.
$\dfrac{\left(x^{1/2}y\right)'}{x^{1/2}} = 2x$	Integrating factor identity applied on the left.
$x^{1/2}y = 2\int x^{3/2}dx$	Clear fractions. Apply quadrature.
$y = x^{-1/2}\left(4x^{5/2}/5 + c\right)$	Evaluate the integral. Divide to isolate y.
$= 4x^2 + cx^{-1/2}$	Solution found.

15 Example (Classification) Classify the equation $y' = x + \ln(xe^y)$ as linear or non-linear.

Solution: It's linear, with standard linear form $y' + (-1)y = x + \ln x$. To explain why, the term $\ln(xe^y)$ on the right expands into $\ln x + \ln e^y$, which in turn is $\ln x + y$, using logarithm rules. Because $e^y > 0$, then $\ln(xe^y)$ makes sense for only $x > 0$. Henceforth, assume $x > 0$.

Computer algebra test $f(x, y) = f(x, 0) + f_y(x, 0)y$. Expected is LHS − RHS $= 0$ after simplification. This example produced $\ln e^y - y$ instead of 0, evidence that limitations may exist.

```
assume(x>0):
f:=(x,y)->x+ln(x*exp(y)):
LHS:=f(x,y):
RHS:=f(x,0)+subs(y=0,diff(f(x,y),y))*y:
simplify(LHS-RHS);
```

If the test *passes*, then $y' = f(x, y)$ becomes $y' = f(x, 0) + f_y(x, 0)y$. This example gives $y' = x + \ln x + y$, which converts to the standard linear form $y' + (-1)y = x + \ln x$.

Details and Proofs

Justification of Formula (3): Define

$$\mathcal{Q}(x) = e^{-\int_{x_0}^{x} p(s)ds}, \quad \mathcal{R}(x) = \int_{x_0}^{x} \frac{r(t)}{\mathcal{Q}(t)} dt.$$

The calculus rule $(e^u)' = u'e^u$ and the fundamental theorem of calculus result $(\int_{x_0}^{x} G(t)dt)' = G(x)$ can be used to obtain the formulas

$$\mathcal{Q}' = (-p)\mathcal{Q}, \quad \mathcal{R}' = \frac{r}{\mathcal{Q}}.$$

Existence. Equation (3) is $y = \mathcal{Q}(y_0 + \mathcal{R})$. Existence will be established by showing that y satisfies $y' + py = r$, $y(x_0) = y_0$. The initial condition $y(x_0) = y_0$ follows from $\mathcal{Q}(x_0) = 1$ and $\mathcal{R}(x_0) = 0$. The steps below verify $y' + py = r$, completing existence.

$y' = [\mathcal{Q}(y_0 + \mathcal{R})]'$	Equation (3), using notation $\mathcal{Q}$ and $\mathcal{R}$.
$\quad = \mathcal{Q}'(y_0 + \mathcal{R}) + \mathcal{Q}\mathcal{R}'$	Sum and product rules applied.
$\quad = -p\mathcal{Q}(y_0 + \mathcal{R}) + \mathcal{Q}\mathcal{R}'$	Used $\mathcal{Q}' = (-p)\mathcal{Q}$.
$\quad = -p\mathcal{Q}(y_0 + \mathcal{R}) + r$	Used $\mathcal{R}' = r/\mathcal{Q}$.
$\quad = -py + r$	Apply $y = \mathcal{Q}(y_0 + \mathcal{R})$.

Uniqueness. It remains to show that the solution given by (3) is the only solution. Start by assuming Y is another, subtract them to obtain $u = y - Y$. Then $u' + pu = 0$, $u(x_0) = 0$. To show $y \equiv Y$, it suffices to show $u \equiv 0$.

According to the integrating factor method, the equation $u' + pu = 0$ is equivalent to $(Wu)' = 0$ where $W = e^{\mathbf{P}}$ and $\mathbf{P}(x) = \int_{x_0}^{x} p(t)dt$. Integrate $(Wu)' = 0$ from x_0 to x, giving $W(x)u(x) = W(x_0)u(x_0)$. Since $u(x_0) = 0$ and $W(x) \neq 0$, it follows that $u(x) = 0$ for all x. This completes the proof.

Remarks on Picard's Theorem. The Picard-Lindelöf theorem, page 61, implies existence-uniqueness, but only on a smaller interval, and furthermore it supplies no practical formula for the solution. Formula (3) is therefore an improvement over the results obtainable from the general theory.

Justification of Factorization (4): It is assumed that $Y(x)$ is a given but otherwise arbitrary differentiable function. Equation (4) will be justified in its fraction-free form

$$\left(e^{\mathbf{P}} Y\right)' = e^{\mathbf{P}} \left(Y' + pY\right), \quad \mathbf{P}(x) = \int p(x)dx. \tag{6}$$

$\text{LHS} = \left(e^{\mathbf{P}} Y\right)'$	The left side of equation (6).
$\quad = \left(e^{\mathbf{P}}\right)' Y + e^{\mathbf{P}} Y'$	Apply the product rule $(uv)' = u'v + uv'$.
$\quad = e^{\mathbf{P}} pY + e^{\mathbf{P}} Y'$	Use the chain rule $(e^u)' = e^u u'$ and $\mathbf{P}' = p$.
$\quad = e^{\mathbf{P}} \left(Y' + pY\right)$	The common factor is $e^{\mathbf{P}}$.
$\quad = \text{RHS}$	The right hand side of equation (6).

Exercises 2.3

Integrating Factor Method. Apply the integrating factor method, page 85, to solve the given linear equation. See the examples starting on page 87 for details.

1. $y' + y = e^{-x}$

2. $y' + y = e^{-2x}$

3. $2y' + y = e^{-x}$

4. $2y' + y = e^{-2x}$

5. $2y' + y = 1$

6. $3y' + 2y = 2$

7. $2xy' + y = x$

8. $3xy' + y = 3x$

9. $y' + 2y = e^{2x}$

10. $2y' + y = 2e^{x/2}$

11. $y' + 2y = e^{-2x}$

12. $y' + 4y = e^{-4x}$

13. $2y' + y = e^{-x}$

14. $2y' + y = e^{-2x}$

15. $4y' + y = 1$

16. $4y' + 2y = 3$

17. $2xy' + y = 2x$

18. $3xy' + y = 4x$

19. $y' + 2y = e^{-x}$

20. $2y' + y = 2e^{-x}$

Superposition. Find a particular solution with fewest terms. See Example 12, page 87.

21. $3y' = x$

22. $3y' = 2x$

23. $y' + y = 1$

24. $y' + 2y = 2$

25. $2y' + y = 1$

26. $3y' + 2y = 1$

27. $y' - y = e^x$

28. $y' - y = xe^x$

29. $xy' + y = \sin x$ $(x > 0)$

30. $xy' + y = \cos x$ $(x > 0)$

31. $y' + y = x - x^2$

32. $y' + y = x + x^2$

General Solution. Find y_h and a particular solution y_p. Report the general solution $y = y_h + y_p$. See Example 14, page 88.

33. $y' + y = 1$

34. $xy' + y = 2$

35. $y' + y = x$

36. $xy' + y = 2x$

37. $y' - y = x + 1$

38. $xy' - y = 2x - 1$

39. $2xy' + y = 2x^2$ $(x > 0)$

40. $xy' + y = 2x^2$ $(x > 0)$

Classification. Classify as linear or non-linear. Use the test $f(x, y) = f(x, 0) + f_y(x, 0)y$ and a computer algebra system, when available, to check the answer. See Example 15, page 88.

41. $y' = 1 + 2y^2$

42. $y' = 1 + 2y^3$

43. $yy' = (1 + x)\ln e^y$

44. $yy' = (1 + x)\,(\ln e^y)^2$

45. $y' \sec^2 y = 1 + \tan^2 y$

46. $y' = \cos^2(xy) + \sin^2(xy)$

47. $y'(1+y) = xy$

48. $y' = y(1+y)$

49. $xy' = (x+1)y - xe^{\ln y}$

50. $2xy' = (2x+1)y - xye^{-\ln y}$

Proofs and Details.

51. Prove directly without appeal to Theorem 3 that the difference of two solutions of $y' + p(x)y = r(x)$ is a solution of the homogeneous equation $y' + p(x)y = 0$.

52. Prove that y_p^* given by equation (5) and $y_p = Q^{-1}\int r(x)Q(x)dx$ given in the integrating factor method are related by $y_p = y_p^* + y_h$ for some solution y_h of the homogeneous equation.

53. Let $Q' = 1 + x$ and define $y = Q^{-1}\int e^x Q(x)dx$. Show directly, without appeal to theorems of this section, that $y' + (1+x)y = e^x$.

54. Let $Q' = x\ln(1+x^2)$ and define $y = Q^{-1}\int x^2 Q(x)dx$. Show directly, without appeal to theorems of this section, that $y' + x\ln(1+x^2)y = x^2$.

2.4 Linear Equations II

Studied here are the subjects of variation of parameters and undetermined coefficients for linear first order differential equations.

Variation of Parameters

A particular solution $y_p^*(x)$ of the non-homogeneous equation

$$y' + p(x)y = r(x) \tag{1}$$

is given by equation (5) in 2.3, reproduced below in (2). Literature calls it the **variation of parameters** formula or the **variation of constants** formula.

Theorem 4 (Variation of Parameters)
A particular solution of the differential equation $y' + p(x)y = r(x)$ is given by either of the formulae

$$y_p^*(x) = e^{-\int_{x_0}^{x} p(s)ds} \int_{x_0}^{x} r(t)e^{\int_{x_0}^{t} p(s)ds}\, dt, \tag{2}$$

$$y_p(x) = e^{-\int p(x)dx} \int r(x)e^{\int p(x)dx}\, dx. \tag{3}$$

Indefinite Integrals. The indefinite integral form (3) is used in science and engineering applications. The answers (2) and (3) differ by a solution of the homogeneous equation: $y_p^*(x) = y_p(x) + y_h(x)$ for some choice of the constant c in y_h. Both answers (2) and (3) are solutions of the nonhomogeneous differential equation, even though (2) generally contains an extra term. While (2) satisfies $y(x_0) = 0$, (3) may not.

Integrating Factor Formula. An integrating factor for (1) is

$$Q(x) = e^{\int p(x)dx}.$$

Formula (3) can be written in terms of $Q(x)$ as

$$y_p(x) = \frac{1}{Q(x)} \int r(x)Q(x)dx.$$

Compact Formula. Because $\int_x^t f = \int_x^{x_0} f + \int_{x_0}^t f$ and $\int_x^{x_0} f = -\int_{x_0}^x f$, the exponential factors in (2) can be re-written as

$$y_p(x) = \int_{x_0}^{x} r(t)e^{\int_x^t p(s)ds}\, dt. \tag{4}$$

The reader is warned that using indefinite integrals in (4) results in the wrong answer.

Terminology. The name *variation of parameters* comes from the idea of *varying the parameter* c in the homogeneous solution formula $y_h = c\mathbf{R}(x)$, where $\mathbf{R}(x) = e^{-\int p(x)dx}$. Historically, c is replaced by an unknown function $y_0(x)$, to define a *trial solution* $y(x) = y_0(x)\mathbf{R}(x)$ of (1). A derivation appears on page 97.

The Method of Undetermined Coefficients

The method applies to $y' + p(x)y = r(x)$. It finds a particular solution y_p *without* the integration steps present in variation of parameters. The requirements and limitations:

1. Coefficient $p(x)$ of $y' + p(x)y = r(x)$ is constant.
2. The function $r(x)$ is a sum of constants times atoms.

An **atom** is a term having one of the forms

$$x^m,\ x^m e^{ax},\ x^m \cos bx,\ x^m \sin bx,\ x^m e^{ax} \cos bx \quad \text{or} \quad x^m e^{ax} \sin bx.$$

The symbols a and b are real constants, with $b > 0$. Symbol $m \ge 0$ is an integer. The terms x^3, $x\cos 2x$, $\sin x$, e^{-x}, $x^6 e^{-\pi x}$ are atoms. Conversely, if $r(x) = 4\sin x + 5xe^x$, then split the sum into terms and drop the coefficients 4 and 5 to identify atoms $\sin x$ and xe^x; then $r(x)$ is a sum of constants times atoms.

The Method.

1. Repeatedly differentiate the atoms of $r(x)$ until no new atoms appear. Multiply the distinct atoms so found by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** y.
2. **Fixup rule**: if solution e^{-px} of $y' + py = 0$ appears in trial solution y, then replace in y matching atoms e^{-px}, xe^{-px}, ... by xe^{-px}, x^2e^{-px}, ... (other atoms appearing in y are unchanged). The modified expression y is called the **corrected trial solution**.
3. Substitute y into the differential equation $y' + py = r(x)$. Match coefficients of atoms left and right to write out linear algebraic equations for the undetermined coefficients $d_1, \ldots, d_k$.
4. Solve the equations. The trial solution y with evaluated coefficients $d_1, \ldots, d_k$ becomes the particular solution y_p.

Undetermined Coefficients Illustrated. We will solve

$$y' + 2y = xe^x + 2x + 1 + 3\sin x.$$

Solution:

Test Applicability. The right side $r(x) = xe^x + 2x + 1 + 3\sin x$ is a sum of terms constructed from the atoms xe^x, x, 1, $\sin x$. The left side is $y' + p(x)y$ with $p(x) = 2$, a constant. Therefore, the method of undetermined coefficients applies to find y_p.

Trial Solution. The atoms of $r(x)$ are subjected to differentiation. The distinct atoms so found are 1, x, e^x, xe^x, $\cos x$, $\sin x$ (drop coefficients to identify new atoms). The solution e^{-2x} of $y' + 2y = 0$ does not appear in the list of atoms, so the fixup rule does not apply. Then the trial solution is the expression

$$y = d_1(1) + d_2(x) + d_3(e^x) + d_4(xe^x) + d_5(\cos x) + d_6(\sin x).$$

Equations. To substitute the trial solution y into $y' + 2y$ requires a formula for y':

$$y' = d_2 + d_3e^x + d_4xe^x + d_4e^x - d_5\sin x + d_6\cos x.$$

Then

$$\begin{aligned}
r(x) &= y' + 2y \\
&= d_2 + d_3e^x + d_4xe^x + d_4e^x - d_5\sin x + d_6\cos x \\
&\quad + 2d_1 + 2d_2x + 2d_3e^x + 2d_4xe^x + 2d_5\cos x + 2d_6\sin x \\
&= (d_2 + 2d_1)(1) + 2d_2(x) + (3d_3 + d_4)(e^x) + (3d_4)(xe^x) \\
&\quad + (2d_5 + d_6)(\cos x) + (2d_6 - d_5)(\sin x)
\end{aligned}$$

Also, $r(x) \equiv 1 + 2x + xe^x + 3\sin x$. Coefficients of atoms on the left and right must match. For instance, constant term $d_2 + 2d_1$ in the expansion of $y' + 2y$ matches constant term 1 in $r(x)$. Writing out the matches gives the equations

$$\begin{array}{rl}
2d_1 + d_2 &= 1, \\
2d_2 &= 2, \\
3d_3 + d_4 &= 0, \\
3d_4 &= 1, \\
2d_5 + d_6 &= 0, \\
-d_5 + 2d_6 &= 3.
\end{array}$$

Solve. The first four equations can be solved by back-substitution to give $d_2 = 1$, $d_1 = 0$, $d_4 = 1/3$, $d_3 = -1/9$. The last two equations are solved by elimination or Cramer's rule (reviewed in Chapter 3) to give $d_6 = 6/5$, $d_5 = -3/5$.

Report y_p. The trial solution y with evaluated coefficients $d_1, \ldots, d_6$ becomes

$$y_p(x) = x - \frac{1}{9}e^x + \frac{1}{3}xe^x - \frac{3}{5}\cos x + \frac{6}{5}\sin x.$$

A Fixup Rule Illustration. Solve the equation

$$y' + 3y = 8e^x + 3x^2e^{-3x}$$

by the method of undetermined coefficients. Verify that the general solution $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = 2e^x + x^3e^{-3x}.$$

Solution: The right side $r(x) = 8e^x + 3x^2e^{-3x}$ is constructed from atoms e^x, x^2e^{-3x}. Repeated differentiation of these atoms identifies the new list of atoms e^x, e^{-3x}, xe^{-3x}, x^2e^{-3x}. The fixup rule applies because the solution e^{-3x} of $y' + 3y = 0$ appears in the list. The atoms of the form x^me^{-3x} are multiplied by x to give the new list of atoms e^x, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}. Readers should take note that atom e^x is unaffected by the fixup rule modification. Then the corrected trial solution is

$$y = d_1e^x + d_2xe^{-3x} + d_3x^2e^{-3x} + d_4x^3e^{-3x}.$$

The trial solution expression y is substituted into $y' + 3y = 2e^x + x^2e^{-3x}$ to give the equation

$$4d_1e^x + d_2e^{-3x} + 2d_3xe^{-3x} + 3d_4x^2e^{-3x} = 8e^x + 3x^2e^{-3x}.$$

Coefficients of atoms on each side of the preceding equation are matched to give the equations

$$\begin{aligned}
4d_1 \qquad\qquad\qquad &= 8,\\
d_2 \qquad\qquad &= 0,\\
2d_3 \qquad &= 0,\\
3d_4 &= 3.
\end{aligned}$$

Then $d_1 = 2$, $d_2 = d_3 = 0$, $d_4 = 1$ and the particular solution is reported to be $y_p = 2e^x + x^3e^{-3x}$.

Examples

16 Example (Variation of Parameters Method) Solve the equation $2y' + 6y = 4xe^{-3x}$ by the method of variation of parameters, verifying $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = x^2e^{-3x}.$$

Solution: Divide the equation by 2 to obtain the standard linear form

$$y' + 3y = 2xe^{-3x}.$$

Solution y_h. The homogeneous equation $y' + 3y = 0$ is solved by the *recipe* to give $y_h = ce^{-3x}$.

Solution y_p. Identify $p(x) = 3$, $r(x) = 2xe^{-3x}$ from the standard form. The mechanics: let $y' = f(x, y) \equiv 2xe^{-3x} - 3y$ and define $r(x) = f(x, 0)$, $p(x) = -f_y(x, y) = 3$. The variation of parameters formula is applied as follows. First, compute the integrating factor $Q(x) = e^{\int p(x)dx} = e^{3x}$. Then

$$\begin{aligned}
y_p(x) &= (1/Q(x)) \int r(x)Q(x)dx\\
&= e^{-3x} \int 2xe^{-3x}e^{3x}dx\\
&= x^2e^{-3x}.
\end{aligned}$$

It must be explained that all integration constants were set to zero, in order to obtain the shortest possible expression for y_p. Indeed, if $Q = e^{3x+c_1}$ instead of e^{3x}, then the factors $1/Q$ and Q contribute constant factors $1/e^{c_1}$ and e^{c_1}, which multiply to one; the effect is to set $c_1 = 0$. On the other hand, an integration constant c_2 added to $\int r(x)Q(x)dx$ adds the homogeneous solution c_2e^{-3x} to the expression for y_p. Because we seek the shortest expression which is a solution to the non-homogeneous differential equation, the constant c_2 is set to zero.

17 Example (Undetermined Coefficient Method) Solve the equation $2y' + 6y = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ by the method of undetermined coefficients, verifying $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Solution: The method applies, because the differential equation $2y' + 6y = 0$ has constant coefficients and the right side $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ is constructed from the list of atoms xe^{-x}, xe^{-3x}, $\sin x$.

List of Atoms. Differentiate the atoms xe^{-x}, xe^{-3x}, $\sin x$ to find the new list of atoms e^{-x}, xe^{-x}, e^{-3x}, xe^{-3x}, $\cos x$, $\sin x$. The solution e^{-3x} of $2y'+6y = 0$ appears in the list: the fixup rule applies. Then e^{-3x}, xe^{-3x} are replaced by xe^{-3x}, x^2e^{-3x} to give the corrected list of atoms e^{-x}, xe^{-x}, xe^{-3x}, x^2e^{-3x}, $\cos x$, $\sin x$. Please note that only two of the six atoms were corrected.

Trial solution. The corrected trial solution is

$$y = d_1e^{-x} + d_2xe^{-x} + d_3xe^{-3x} + d_4x^2e^{-3x} + d_5\cos x + d_6\sin x.$$

Substitute y into $2y' + 6y = r(x)$ to give

$$\begin{aligned} r(x) &= 2y' + 6y \\ &= (4d_1 + 2d_2)e^{-x} + 4d_2xe^{-x} + 2d_3e^{-3x} + 4d_4xe^{-3x} \\ &\quad +(2d_6 + 6d_5)\cos x + (6d_6 - 2d_5)\sin x. \end{aligned}$$

Equations. Matching atoms on the left and right of $2y' + 6y = r(x)$, given $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$, justifies the following equations for the undetermined coefficients; the solution is $d_2 = 1$, $d_1 = -1/2$, $d_3 = 0$, $d_4 = 1$, $d_6 = 3/4$, $d_5 = -1/4$.

$$\begin{aligned} 4d_1 + 2d_2 &= 0, \\ 4d_2 &= 4, \\ 2d_3 &= 0, \\ 4d_4 &= 4, \\ 6d_5 + 2d_6 &= 0, \\ -2d_5 + 6d_6 &= 5. \end{aligned}$$

Report. The trial solution upon substitution of the values for the undetermined coefficients becomes

$$y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Details

Historical Account of Variation of Parameters.

$r = y' + py$	Let $\mathbf{R}(x) = e^{-\int p(x)dx}$. Assume $y = y_0(x)\mathbf{R}(x)$ solves $y' + py = r$.
$= (y_0\mathbf{R})' + py_0\mathbf{R}$	Substitute $y = y_0(x)\mathbf{R}(x)$ but suppress x.
$= y_0'\mathbf{R} + y_0\mathbf{R}' + py_0\mathbf{R}$	Apply the product rule $(uv)' = u'v + uv'$.
$= y_0'\mathbf{R} - y_0 p\mathbf{R} + py_0\mathbf{R}$	Let $\mathcal{Q} = e^{\int p(x)dx}$. Apply $\mathcal{Q}' = -p\mathcal{Q}$.
$= y_0'/\mathcal{Q}$	Because $1/\mathbf{R} = \mathcal{Q}$.

The calculation gives $y_0'(x) = r(x)\mathcal{Q}(x)$. The method of quadrature applies to determine $y_0(x) = \int_{x_0}^{x}(r(t)\mathcal{Q}(t))dt$, because $y_0 = 0$ at $x = x_0$. Then $y = y_0\mathbf{R}$ duplicates the formula for y_p^* given in (5), which is equivalent to (2).

Exercises 2.4

Variation of Parameters I. Report the shortest particular solution given by the formula

$$y_p(x) = \frac{\int rQ}{Q}, \quad Q = e^{\int p}.$$

1. $y' = x + 1$

2. $y' = 2x - 1$

3. $y' + y = e^{-x}$

4. $y' + y = e^{-2x}$

5. $y' - 2y = 1$

6. $y' - y = 1$

7. $2y' + y = e^{x}$

8. $2y' + y = e^{-x}$

9. $xy' = x + 1$

10. $xy' = 1 - x^2$

Variation of Parameters II. Compute the particular solution given by

$$y_p^*(x) = \frac{\int_{x_0}^{x} rQ}{Q(t)}, \quad Q(t) = e^{\int_{x_0}^{t} p}.$$

11. $y' = x + 1$, $x_0 = 0$

12. $y' = 2x - 1$, $x_0 = 0$

13. $y' + y = e^{-x}$, $x_0 = 0$

14. $y' + y = e^{-2x}$, $x_0 = 0$

15. $y' - 2y = 1$, $x_0 = 0$

16. $y' - y = 1$, $x_0 = 0$

17. $2y' + y = e^{x}$, $x_0 = 1$

18. $2y' + y = e^{-x}$, $x_0 = 1$

19. $xy' = x + 1$, $x_0 = 1$

20. $xy' = 1 - x^2$, $x_0 = 1$

Atoms. Report the list of distinct atoms of the given function $f(x)$.

21. $x + e^{x}$

22. $1 + 2x + 5e^{x}$

23. $x(1 + x + 2e^{x})$

24. $x^2(2 + x^2) + x^2e^{-x}$

25. $\sin x \cos x + e^{x} \sin 2x$

26. $\cos^2 x - \sin^2 x + x^2 e^{x} \cos 2x$

27. $(1 + 2x + 4x^5)e^{x}e^{-3x}e^{x/2}$

28. $(1+2x+4x^5+e^{x} \sin 2x)e^{-3x/4}e^{x/2}$

29. $\dfrac{x + e^{x}}{e^{-2x}} \sin 3x + e^{3x} \cos 3x$

30. $\dfrac{x + e^{x} \sin 2x + x^3}{e^{-2x}} \sin 5x$

Initial Trial Solution. Differentiate repeatedly $f(x)$ and report the list of distinct atoms which appear in f and all its derivatives.

31. $12 + 5x^2 + 6x^7$

32. $x^6/x^{-4} + 10x^4/x^{-6}$

33. $x^2 + e^x$

34. $x^3 + 5e^{2x}$

35. $(1 + x + x^3)e^x + \cos 2x$

36. $(x + e^x)\sin x + (x - e^{-x})\cos 2x$

37. $(x + e^x + \sin 3x + \cos 2x)e^{-2x}$

38. $(x^2 e^{-x} + 4\cos 3x + 5\sin 2x)e^{-3x}$

39. $(1 + x^2)(\sin x \cos x - \sin 2x)e^{-x}$

40. $(8 - x^3)(\cos^2 x - \sin^2 x)e^{3x}$

Fixup Rule. Given the homogeneous solution y_h and an initial trial solution y, determine the final trial solution according to the fixup rule.

41. $y_h(x) = ce^{2x}$, $y = d_1 + d_2 x + d_3 e^{2x}$

42. $y_h(x) = ce^{2x}$, $y = d_1 + d_2 e^{2x} + d_3 x e^{2x}$

43. $y_h(x) = ce^{0x}$, $y = d_1 + d_2 x + d_3 x^2$

44. $y_h(x) = ce^{x}$, $y = d_1 + d_2 x + d_3 x^2$

45. $y_h(x) = ce^{x}$, $y = d_1 \cos x + d_2 \sin x + d_3 e^x$

46. $y_h(x) = ce^{2x}$, $y = d_1 e^{2x}\cos x + d_2 e^{2x}\sin x$

47. $y_h(x) = ce^{2x}$, $y = d_1 e^{2x} + d_2 x e^{2x} + d_3 x^2 e^{2x}$

48. $y_h(x) = ce^{-2x}$, $y = d_1 e^{-2x} + d_2 x e^{-2x} + d_3 e^{2x} + d_4 x e^{2x}$

49. $y_h(x) = cx^2$, $y = d_1 + d_2 x + d_3 x^2$

50. $y_h(x) = cx^3$, $y = d_1 + d_2 x + d_3 x^2$

Undetermined Coefficients: Trial Solution. Find the form of the **corrected** trial solution y but do not evaluate the undetermined coefficients.

51. $y' = x^3 + 5 + x^2 e^x(3 + 2x + \sin 2x)$

52. $y' = x^2 + 5x + 2 + x^3 e^x(2 + 3x + 5\cos 4x)$

53. $y' - y = x^3 + 2x + 5 + x^4 e^x(2 + 4x + 7\cos 2x)$

54. $y' - y = x^4 + 5x + 2 + x^3 e^x(2 + 3x + 5\cos 4x)$

55. $y' - 2y = x^3 + x^2 + x^3 e^x(2e^x + 3x + 5\sin 4x)$

56. $y' - 2y = x^3 e^{2x} + x^2 e^x(3 + 4e^x + 2\cos 2x)$

57. $y' + y = x^2 + 5x + 2 + x^3 e^{-x}(6x + 3\sin x + 2\cos x)$

58. $y' - 2y = x^5 + 5x^3 + 14 + x^3 e^x(5 + 7xe^{-3x})$

59. $2y' + 4y = x^4 + 5x^5 + 2x^8 + x^3 e^x(7 + 5xe^x + 5\sin 11x)$

60. $5y' + y = x^2 + 5x + 2e^{x/5} + x^3 e^{x/5}(7 + 9x + 2\sin(9x/2))$

Undetermined Coefficients. Compute a particular solution y_p according to the method of undetermined coefficients. Report (1) the initial trial solution, (2) the corrected trial solution, (3) the system of equations for the undetermined coefficients and finally (4) the formula for y_p.

61. $y' + y = x + 1$

62. $y' + y = 2x - 1$

63. $y' - y = e^x + e^{-x}$

64. $y' - y = xe^x + e^{-x}$

65. $y' - 2y = 1 + x + e^{2x} + \sin x$

66. $y' - 2y = 1 + x + xe^{2x} + \cos x$

67. $y' + 2y = xe^{-2x} + x^3$

68. $y' + 2y = (2 + x)e^{-2x} + xe^x$

69. $y' = x^2 + 4 + xe^x(3 + \cos x)$

70. $y' = x^2 + 5 + xe^x(2 + \sin x)$