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Introduction to Linear Algebra 2270-1
Midterm Exam 2 Fall 2007
Exam Date: 31 October

Instructions. This exam is designed for 50 minutes. Calculators, books, notes and computers are not allowed.

1. **(Matrices and independence)** Do **two** parts only.

(a) [50%] Let E be an $n \times n$ invertible matrix. Assume vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathcal{R}^n$ are given such that $E\mathbf{v}_1, \ldots, E\mathbf{v}_k$ are columns of some invertible $n \times n$ matrix F. Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

(b) [50%] Suppose A and B are both $n \times n$ invertible matrices. Prove or give a counterexample: the row spaces of A and B are identical.

(c) [50%] **If you did (a) and (b), then you have 100% – stop!**
Suppose matrices A and B are both $n \times m$ and the leading ones in $\mathbf{rref}(A)$, $\mathbf{rref}(B)$ are in exactly the same locations. Explain why $\mathbf{rank}(A) = \mathbf{rank}(B)$, but their column spaces could be different.

(a) Solve $\sum_1^k c_i v_i = 0$ for $c_1, \ldots, c_k$ by multiplication by E, obtaining $\sum_1^k c_i E v_i = E0 = 0$. Independence of $\{Ev_i\}_1^k \Rightarrow c_1 = \cdots = c_k = 0$.

(b) Because $rref(A) = rref(B) = I$, then both row spaces are spanned by the rows of I.

(c) The rank is the count of leading ones in rref, which implies the ranks are equal. If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then both have rref equal to A but their column spaces are different.

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2. (Kernel and similarity) Do **two** parts.

(a) [50%] Illustrate the relation $\mathbf{rref}(A) = E_k \cdots E_2 E_1 A$ by a frame sequence and explicit elementary matrices for the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

(b) [50%] Prove or disprove: If $B = S^{-1}AS$ for an invertible square matrix S, then $\mathbf{im}(A)$ is isomorphic to $\mathbf{im}(B)$.

(c) [50%] **If you did (a) and (b), then you have 100% – stop!**
Prove or disprove: $\mathbf{im}(B) = \mathbf{im}(A)$, for all frames A and B in any frame sequence.

(a)

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix} \text{swap}(1,3)$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \text{combo}(1,2,-1)$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \text{combo}(2,3,-1)$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \text{combo}(2,1,-1)$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = E_4 E_3 E_2 E_1 \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) First prove $x \in \ker(B) \Leftrightarrow Sx \in \ker(A)$. Similarly, $y \in \ker(A) \Leftrightarrow S^{-1}y \in \ker(B)$. Then $\dim(\ker(A)) = \dim(\ker(B))$, which implies $\dim(\mathrm{Im}(A)) = \dim(\mathrm{Im}(B))$. Let $v_1, \ldots, v_k$ be a basis of $\mathrm{Im}(A)$ and $u_1, \ldots, u_k$ be a basis of $\mathrm{Im}(B)$. Define T to map $\sum c_i v_i$ into $\sum c_i u_i$. Then T is an isomorphism $\mathrm{Im}(A)$ to $\mathrm{Im}(B)$.

(c) Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ obtained by a swap. Then $\mathrm{Im}(A) = \left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$, $\mathrm{Im}(B) = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ are not equal.

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3. **(Independence and bases)** Do **two** parts.

(a) [50%] Let A be a 6×6 matrix such that A^3 is the zero matrix. Prove or give a counterexample: $\dim(\mathbf{ker}(A)) \le 3$.

(b) [50%] Let V be the vector space of all polynomials $p(x) = c_0 + c_1x + c_2x^2$ under function addition and scalar multiplication. Let S be the subspace of V satisfying the relations $p(0) = p(1)$, $\int_{-1}^{1} p(x)dx = p(1)$. Find $\dim(S)$ and display a basis for S.

(b) [50%] **If you did (a) and (b), then you have 100% – stop!**
Let V be the vector space of all functions $f(x) = c_0 + c_1x + c_2e^x$ under function addition and scalar multiplication. Prove that $1 - x$, $2x$, $x - e^x$ form a basis of V.

(a) $A =$ zero matrix has $\ker(A) = \mathbb{R}^6$ of dim $= 6$.

(b) The relations imply $\begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then $S \cong \mathrm{span}\left\{\begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix}\right\}$ under the isomorphism $c_0 + c_1x + c_2x^2 \to \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$. So $\dim(S) = 1$ and a basis is $(-2/3) + (-1)x + (1)x^2$.

(c) Already we know $\dim(V) = 3$ with basis equal to the three atoms $1, x, e^x$. It suffices to work with the isomorphism $T: c_0 + c_1x + c_2x^2 \to \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$ and show the 3 given vectors in V map to independent fixed vectors

$$A = \mathrm{aug}\left(T(1-x), T(2x), T(x-e^x)\right)$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$\det(A) \ne 0 \Rightarrow$ cols of A independent

$\Rightarrow 1-x, 2x, x-e^x$ indep

$\Rightarrow$ They are a basis of V

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4. **(Linear transformations)** Do **two** parts.

(a) [50%] Let L be a line through the origin in $\mathcal{R}^5$ with unit direction $\mathbf{u}$. Let $T(\mathbf{x})$ be the orthogonal projection of $\mathbf{x}$ onto L. Define T precisely. Display its representation matrix A, i.e., $T(\mathbf{x}) = A\mathbf{x}$.

(b) [50%] Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^m$. Let C be the $n \times n$ identity matrix, with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Let A be the matrix whose columns are $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$. Prove that $T(\mathbf{x}) = A\mathbf{x}$.

(a) $T(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$

$A = \text{aug}(u_1\vec{u}, u_2\vec{u}, u_3\vec{u}, u_4\vec{u}, u_5\vec{u})$

(b) any $\vec{x} \in \mathbb{R}^n$ is written as $\vec{x} = \sum_1^n x_i \vec{v}_i$. Then

$$\begin{aligned} T(\vec{x}) &= T\left(\sum_1^n x_i\vec{v}_i\right) \\ &= \sum_1^n x_i\, T(\vec{v}_i) \qquad \text{by linearity of } T \\ &= \text{l.c. of the cols of } A \\ &= A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= A\vec{x}. \end{aligned}$$

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5. (Linear spaces) Do all three parts.

(a) [50%] Let V be the linear space of all 2×3 matrices A. Display a basis of V in which each basis element has at least three nonzero entries. Check your answer by using the standard isomorphism T from V to $\mathcal{R}^6$ and frame sequences.

(b) [20%] Let $S = \left\{ \begin{pmatrix} a & b+c \\ -2a & 2b+2c+a \end{pmatrix} : a, b, c \text{ real} \right\}$. Find a basis for S.

(c) [30%] Prove by means of the Subspace Criterion that the kernel of an $m \times n$ matrix A,

$$S = \{\mathbf{x} \ : \ A\mathbf{x} = \mathbf{0}\},$$

is a subspace of the linear space $\mathcal{R}^n$.

(a) Define $T\left(\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\right) = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$, an isomorphism $V \to \mathbb{R}^6$.

The matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ has $\det(A) = 4$, so A^{-1} exists, meaning A has 6 independent columns. This can also be checked by the pivot theorem. Then T^{-1} maps the columns of A to a basis of V:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)
- Zero is in S, because $A\vec{0} = \vec{0}$.
- If $\vec{x}, \vec{y}$ are in S, then $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$ implies $\vec{x}+\vec{y}$ is in S
- If $\vec{x}$ is in S and c = scalar, then $A(c\vec{x}) = cA\vec{x} = c \cdot \vec{0} = \vec{0}$ implies $c\vec{x}$ is in S.

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