Name ____________________________ Class Time _____

Math 2250 Maple Project 5: Linear Algebra
August 2007

Due date: See the internet due dates. Maple lab 5 has problems L5.1, L5.2, L5.3.

References: Code in `maple` appears in `2250mapleL5-F2007.txt` at URL `http://www.math.utah.edu/~gustafso/`. This document: `2250mapleL5-F2007.pdf`.

Problem L5.1. (Matrix Algebra)

Define $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$. Create a worksheet in `maple` which states this problem in text, then defines the four objects. The worksheet should contain text, `maple` code and displays. Continue with this worksheet to answer **(1)**–**(7)** below. Submit problem L5.1 as a worksheet printed on 8.5 by 11 inch paper. See Example 1 for maple commands.

(1) Compute AB and BA. Are they the same?

(2) Compute $A+B$ and $B+A$. Are they the same?

(3) Let $C = A + B$. Compare C^2 to $A^2 + 2AB + B^2$. Explain why they are different.

(4) Compute transposes $C_1 = (AB)^T$, $C_2 = A^T$ and $C_3 = B^T$. Find an equation for C_1 in terms of C_2 and C_3. Verify the equation.

(5) Solve for $\mathbf{X}$ in $B\mathbf{X} = \mathbf{v}$ by three different methods.

(6) Solve $A\mathbf{Y} = \mathbf{v}$ for $\mathbf{Y}$. Do an answer check.

(7) Solve $A\mathbf{Z} = \mathbf{w}$. Explain your answer using the three possibilities for a linear system.

Problem L5.2. (Row space)

Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 0 & 1 & -4 & -3 & -15 \\ 1 & 2 & -3 & -1 & -9 \end{pmatrix}$. Find two different bases for the row space of A, using the following three methods.

1. The method of Example 2, below.
2. The `maple` command `rowspace(A)`.
3. The **rref**-method: select rows from **rref**(A).

Two of the methods produce exactly the same basis. Verify that the two bases $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ are **equivalent**. This means that each vector in $\mathcal{B}_1$ is a linear combination of the vectors in $\mathcal{B}_2$, and conversely, that each vector in $\mathcal{B}_2$ is a linear combination of the vectors in $\mathcal{B}_1$.

Problem L5.3. (Matrix Equations)

Let $A = \begin{pmatrix} 8 & 10 & 3 \\ -3 & -5 & -3 \\ -4 & -4 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Let P denote a 3×3 matrix. Assume the following result:

> **Lemma 1**. The equality $AP = PT$ holds if and only if the columns $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of P satisfy $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = -2\mathbf{v}_2$, $A\mathbf{v}_3 = 5\mathbf{v}_3$. [proved after Example 4]

(a) Determine three specific columns for P such that $\det(P) \neq 0$ and $AP = PT$. Infinitely many answers are possible. See Example 4 for the maple method that determines a column of P.

(b) After reporting the three columns, check the answer by computing $AP - PT$ (it should be zero) and $\det(P)$ (it should be nonzero).

Staple this page on top of the maple work sheets. Examples and theory on the next page ...

Example 1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$. Create a `maple` work sheet. Define and display matrix A and vector $\mathbf{b}$. Then compute

(1) The inverse of A.

(2) The augmented matrix $C = \mathbf{aug}(A, \mathbf{b})$.

(3) The reduced row echelon form $R = \mathbf{rref}(C)$.

(4) The column $\mathbf{X}$ of R which solves $A\mathbf{X} = \mathbf{b}$.

(5) The matrix A^3.

(6) The transpose of A.

(7) The matrix $A - 3A^2$.

(8) The solution $\mathbf{X}$ of $A\mathbf{X} = \mathbf{b}$ by two methods different than (4).

Solution: A lab instructor can help you to create a blank work sheet in `maple`, enter code and print the work sheet. The code to be entered appears below. To get help, enter `?linalg` into a worksheet, then select commands that match ones below.

```
with(linalg):
A:=matrix([[1,2,3],[2,-1,1],[3,0,-1]]);
b:=vector([9,8,3]);
print("(1)"); inverse(A);
print("(2)"); C:=augment(A,b);
print("(3)"); R:=rref(C);
print("(4)"); X:=col(R,4);
print("(5)"); evalm(A^3);
print("(6)"); transpose(A);
print("(7)"); evalm(A-3*(A^2));
print("(8)"); X:=linsolve(A,b); X:=evalm(inverse(A) &* b);
```

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$.

(1) Find a basis for the column space of A.

(2) Find a basis for the row space of A.

(3) Find a basis for the nullspace of A.

(4) Find $\mathbf{rank}(A)$ and $\mathbf{nullity}(A)$.

(5) Find the dimensions of the nullspace, row space and column space of A.

Solution: The theory applied: *The columns of B corresponding to the leading ones in* $\mathbf{rref}(B)$ *are independent and form a basis for the column space of B.* These columns are called the **pivot columns** of B. Results for the row space can be obtained by applying the above theory to the transpose of the matrix.
The `maple` code which applies is

```
with(linalg):
A:=matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
print("(1)"); C:=rref(A); # leading ones in columns 1,2,4
               BASIScolumnspace=col(A,1),col(A,2),col(A,4);
```

```
print("(2)"); F:=rref(transpose(A)); # leading ones in columns 1,2,3
              BASISrowspace=row(A,1),row(A,2),row(A,3);
print("(3)"); nullspace(A); linsolve(A,vector([0,0,0,0]));
print("(4)"); RANK=rank(A); NULLITY=coldim(A)-rank(A);
print("(5)"); DIMnullspace=coldim(A)-rank(A); DIMrowspace=rank(A);
              DIMcolumnspace=rank(A);
```

Example 3. Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$. Verify that the following column space bases of A are equivalent.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix},$$

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 17 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -9 \end{pmatrix}.$$

Solution: We will use this result:

Lemma 2. Bases $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are equivalent bases if and only if the augmented matrices $F = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, $G = \mathbf{aug}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and $H = \mathbf{aug}(F, G)$ satisfy the rank condition $\mathbf{rank}(F) = \mathbf{rank}(G) = \mathbf{rank}(H) = 3$.

The proof appears below.

The `maple` code which applies is

```
with(linalg):
A:=matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
v1:=vector([1,2,3,4]); v2:=vector([1,3,5,3]); v3:=vector([2,1,1,2]);
w1:=vector([1, 0, 0, -3]); w2:=vector([0, 1, 0, 17]); w3:=vector([0, 0, 1, -9]);
F:=augment(v1,v2,v3);
G:=augment(w1,w2,w3);
H:=augment(F,G);
rank(F); rank(G); rank(H);
```

We remark that the two bases in the example were discovered from the `maple` code

```
rref(A); # pivot cols 1,2,4
v1:=col(A,1); v2:=col(A,2); v3:=col(A,4);
B:=rref(transpose(A)); # pivot cols 1,2,3
w1:=row(B,1); w2:=row(B,2); w3:=row(B,3);
```

Proof of Lemma 2.

Proof: The test appears in the online pdf documents at the course web site. Let's justify the test here, independently, showing only half the proof: $\mathbf{rank}(F) = \mathbf{rank}(G) = \mathbf{rank}(H) = n$ implies the bases are equivalent.

The equation $\mathbf{rref}(F) = EF$ holds for E a product of elementary matrices. Then EH has to have n lead variables, because of F in the first n columns, and the remaining rows are zero, because $\mathbf{rank}(H) = n$. Therefore, the first n columns of H are the pivot columns of H. The non-pivots of H are just the columns of G, and by the pivot theory, they are linear combinations of the pivot columns, namely, the columns of F. We can multiply H by a permutation matrix P which effectively swaps F and G. Already, HP has the n independent columns of F, so its rank is at least n. But its other columns are linear combinations of these columns, so the rank is exactly n. Now we argue by symmetry that the columns of F are linear combinations of the columns of G, using HP instead of H.

The proof is complete.

Example 4. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}$. Solve the equation $A\mathbf{x} = -3\mathbf{x}$ for $\mathbf{x}$.

Solution. Let $\lambda = -3$. The idea is to write the equation $A\mathbf{x} = \lambda\mathbf{x}$ as a homogeneous problem $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Define $B = A - \lambda I$. The homogeneous equation $B\mathbf{x} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$. It has a nonzero solution $\mathbf{x}$ if and only if there are infinitely many solutions, in which case the solutions are found by a frame sequence to $\mathbf{rref}(B)$. The `maple` details appear below. The basis vectors for $B\mathbf{x} = \mathbf{0}$ are obtained in the usual way, by taking partial derivatives on the general solution with respect to the symbols t_1, t_2, ... In this case, there is just one basis vector

$$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

```
with(linalg):
A:=matrix([[1,2,3],[2,-1,1],[3,0,0]]);
B:=evalm(A-(-3)*diag(1,1,1));
linsolve(B,vector([0,0,0]));
# ans: t_1*vector([-2,1,2])
# Basis == partial on t_1 == vector([-2,1,2])
```

Proof of Lemma 1. Define $r_1 = 1$, $r_2 = -2$, $r_3 = 5$. Assume $AP = PT$, $P = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $T = \mathbf{diag}(r_1, r_2, r_3)$. The definition of matrix multiplication implies that $AP = \mathbf{aug}(A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3)$ and $PT = \mathbf{aug}(r_1\mathbf{v}_1, r_2\mathbf{v}_2, r_3\mathbf{v}_3)$. Then $AP = PT$ holds if and only if the columns of the two matrices match, which is equivalent to the three equations $A\mathbf{v}_1 = r_1\mathbf{v}_1$, $A\mathbf{v}_2 = r_2\mathbf{v}_2$, $A\mathbf{v}_3 = r_3\mathbf{v}_3$. The proof is complete.

End of Maple Lab 5 Linear Algebra.