Differential Equations

and

Linear Algebra

A Course for
Science and Engineering

Solutions Manual July 2022

Part I: Chapters 1-7
Part II: Chapters 8-12

by
Grant B. Gustafson

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Chapter 1

Fundamentals

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1.1 Exponential Modeling

Growth-Decay Model

Solve the given initial value problem using the growth-decay formula; see page 3 and Example 1.1 page 7.

1. $y' = -3y$, $y(0) = 20$
Solution: $y(x) = 20\,e^{-3x}$ by the growth-decay formula page 3.

2. $y' = 3y$, $y(0) = 1$
Solution: $y(x) = e^{3x}$

3. $3A' = A$, $A(0) = 1$
Solution: $A(t) = e^{t/3}$

4. $4A' + A = 0,\ A(0) = 3$

5. $3P' - P = 0,\ P(0) = 10$
Solution: $P(t) = 10\, e^{t/3}$

6. $4P' + 3P = 0,\ P(0) = 11$

7. $I' = 0.005I,\ I(t_0) = I_0$
Solution: $I(t) = I_0\, e^{(t-t_0)/200}$

8. $I' = -0.015I,\ I(t_0) = I_0$

9. $y' = \alpha y,\ y(t_0) = 1$
Solution: $y(t) = e^{\alpha(t-t_0)}$

10. $y' = -\alpha y,\ y(t_0) = y_0$

Growth-decay Theory

11. Graph without a computer $y = 10(2^x)$ on $-3 \le x \le 3$.
Solution: The graph is made by graphics methods in Appendix A.2. The curve increases and passes through the points $(-3, 10/8)$, $(0, 10)$, $(3, 80)$.

12. Graph without a computer $y = 10(2^{-x})$ on $-3 \le x \le 3$.

13. Find the doubling time for the growth model $P = 100e^{0.015t}$.
Solution: Solve $P(t) = 2P(0)$ for t: this is the time t required to double the population size. The equation is $100e^{0.015t} = 200$. Solve it by applying ln across the equation: $\ln\left(e^{0.015t}\right) = \ln 2$. Simplify using $\ln(e^u) = u$. Then $0.015t = \ln 2$ gives $t = 46.20981204$.

14. Find the doubling time for the growth model $P = 1000e^{0.0195t}$.

15. Find the elapsed time for the decay model $A = 1000e^{-0.11237t}$ until $|A(t)| < 0.00001$.
Solution: Solve $A(t) = 0.00001$ for $t = 163.9288132$. A sane answer is 164, but no unique answer exists.

16. Find the elapsed time for the decay model $A = 5000e^{-0.01247t}$ until $|A(t)| < 0.00005$.

Newton Cooling Recipe

Solve the given cooling model. Follow Example 1.2 on page 8.

17. $u' = -10(u-4)$, $u(0) = 5$
Solution: $u = 4 + ce^{-10t}$, $c = 1$

18. $y' = -5(y-2)$, $y(0) = 10$

19. $u' = 1 + u$, $u(0) = 100$
Solution: $u = -1 + ce^{t}$, $c = 101$

20. $y' = -1 - 2y$, $y(0) = 4$

21. $u' = -10 + 4u$, $u(0) = 10$
Solution: Let $v = u - 10/4$. Then $v' = u' = -10 + 4u = 4v$ and $v = ce^{4t}$. Back-substitute: $u = 10/4 + v = 5/2 + ce^{4t}$ and $c = 15/2$.

22. $y' = 10 + 3y$, $y(0) = 1$

23. $2u' + 3 = 6u$, $u(0) = 8$
Solution: $u = \dfrac{1}{2} + \dfrac{15}{2}e^{3t}$

24. $4y' + y = 10$, $y(0) = 5$

25. $u' + 3(u+1) = 0$, $u(0) = -2$
Solution: $u = -1 - e^{-3t}$

26. $u' + 5(u+2) = 0$, $u(0) = -1$

27. $\alpha' = -2(\alpha - 3)$, $\alpha(0) = 10$
Solution: $\alpha(t) = 3 + 7e^{-2t}$

28. $\alpha' = -3(\alpha - 4)$, $\alpha(0) = 12$

Newton Cooling Model

The cooling model $u(t) = u_0 + A_0 e^{-ht}$ is applied; see page 4. Methods parallel those in the flask cooling example, page 9, and the baking example, page 10.

29. **(Ingot Cooling)** A metal ingot cools in the air at temperature 20C from 130C to 75C in one hour. Predict the cooling time to 23C.
Solution: Given: $u(t) = u_1 + (u_0 - u_1)e^{-kt}$, $u(0) = 130$, $u(1) = 75$, $u_1 = 20$, time t in hours. Then $u_1 = 20$, $u_0 = u(0) = 130$ and $u(t) = 20 + (130 - 20)e^{-kt}$ with k as yet unknown. Let $t = 1$ in the equation for $u(t)$ and use $u(1) = 75$ to obtain an equation for k: $75 = 20 + 110e^{-k}$. Solve for $k = \ln 2$. The time t when $u(t) = 23$ is called the cooling time. Find The value of t by solving the equation $23 = 20 + 110e^{-\ln(2)t}$. The cooling time is 5 hours and 12 minutes, approximately.

30. **(Rod Cooling)** A plastic rod cools in a large vat of 12-degree Celsius water from 75C to 20C in 4 minutes. Predict the cooling time to 15C.

Solution: Given: $u(t) = u_1 + (u_0 - u_1)e^{-kt}$, $u(0) = 75$, $u(4) = 20$, $u_1 = 12$, time t in minutes. Proceed as in the ingot problem above to find $k = -(1/4)\ln(8/63)$ and the cooling time 5 minutes and 54 seconds, approximately.

31. **(Murder Mystery)** A body discovered at 1:00 in the afternoon, March 1, 1929, had temperature 80F. Assume outdoor temperature 50F from 9am. Over the next hour the body's temperature dropped to 76F. Estimate the date and time of the murder.

Solution: Given: $u(t) = u_1 + (u_0 - u_1)e^{-kt}$, $u(0) = 80$, $u(1) = 76$, $u_1 = 50$, time t in hours. Then $u_1 = 50$, $u_0 = u(0) = 80$ and $u(t) = 50+(80-50)e^{-kt}$ with k as yet unknown. Let $t = 1$ in the equation for $u(t)$ and use $u(1) = 76$ to obtain an equation for k: $76 = 50 + 30e^{-k}$. Proceed as in the ingot problem above to find $k = -(1/4)\ln(8/63)$ and the cooling time 5 minutes and 54 seconds, approximately.

32. **(Time of Death)** A dead body found in a 40F river had body temperature 70F. The coroner requested that the body be left in the river for 45 minutes, whereupon the body's temperature was 63F. Estimate the time of death, relative to the discovery of the body.

Verhulst Model

Solve the given Verhulst logistic equation using formula (8). Follow Example 1.3 on page 8.

33. $P' = P(2 - P)$, $P(0) = 1$

Solution: The formula is $P(t) = \frac{aP(0)}{bP(0)+(a-bP(0))e^{-at}}$ with $a = 1$, $b = 2$. Then $P(t) = \frac{1}{2-e^{-t}}$.

34. $P' = P(4 - P)$, $P(0) = 5$

35. $y' = y(y - 1)$, $y(0) = 2$

Solution: The formula is $y(t) = \frac{ay(0)}{by(0)+(a-by(0))e^{-at}}$ with $a = -1$, $b = 1$. Then $y(t) = \frac{2}{2-3e^{t}}$.

36. $y' = y(y - 2)$, $y(0) = 1$

Solution: $y(t) = 1/(2 - e^t)$

37. $A' = A - 2A^2$, $A(0) = 3$

Solution: $A(t) = 3/(6 - e^t)$

38. $A' = 2A - 5A^2$, $A(0) = 1$

39. $F' = 2F(3 - F)$, $F(0) = 2$

Solution: $F(t) = 2/(6 - 4e^{-2t})$

40. $F' = 3F(2 - F)$, $F(0) = 1$

Inverse Modeling

Given the model, find the differential equation and initial condition.

41. $A = A_0 e^{4t}$

Solution: $A' = 4A$, $A(0) = A_0$

42. $A = A_0 e^{-3t}$

43. $P = 1000e^{-0.115t}$

Solution: $P'(t) = -0.115P(t)$, $P(0) = 1000$

44. $P = 2000e^{-7t/5}$

45. $u = 1 + e^{-3t}$

Solution: First, $u(0) = 2$. Second, $u'(t) = -3e^{-3t} = -3(u(t) - 1)$. Answer: $u' = 3(1 - u)$, $u(0) = 2$

46. $u = 10 - 2e^{-2t}$

47. $P = \dfrac{10}{10 - 8e^{-2t}}$

Solution: First, $P(0) = 10$.Define a, b by the equations $bP(0) = 10$, $a - bP(0) = -8$. Solve: $a = 2$, $b = 1$. Answer: $P' = 2P(1 - P)$, $P(0) = 10$.

48. $P = \dfrac{5}{15 - 14e^{-t}}$

49. $P = \dfrac{1}{5 - 4e^{-t}}$

Solution: First, $P(0) = 1$.Define a, b by the equations $bP(0) = 5$, $a - bP(0) = -4$. Solve: $a = 1$, $b = 5$. Answer: $P' = P(5 - P$, $P(0) = 1$.

50. $P = \dfrac{2}{4 - 3e^{-t}}$

Populations

Use Malthusian population theory page 6 and Malthusian model $P(t) = P_0 e^{kt}$. Methods appear in Examples 1.4 and 1.5 page 8.

51. (World Population) The world population of $5,500,000,000$ people was increasing at a rate of $250,000$ people per day in June of 1993. Predict the date when the population reaches 10 billion.

Solution: Population 10 billion is reached in June or July in 2029, approximately 36 years later. Model: $P(t) = kP(t)$, $P(0) = 5.5$ billion with t in days. Then $P(1) = P(0)e^{kt} = 5500250000/1000000000$ determines $k = 0.00004545396695$ and $P(t) = 10$ after $t = 13152.71293$ days.

52. (World Population) Suppose the world population at time $t = 0$ is 5.5 billion and increases at rate $250,000$ people per day. How many years before that was the population one billion?

Solution: About 103 years earlier, according to the Malthusian model in the previous exercise.

53. (Population Doubling) A population of rabbits increases by 10% per year. In how many years does the population double?

Solution: About 7.3 years, because of the model: $P(t) = P(0)e^{kt}$. Use $P(1) = 1.1P(0)$ to find $k = 0.09531017980$, then solve $P(t) = 2P(0)$ for $t = 7.272540898$.

54. (Population Tripling) A population of bacteria increases by 15% per day. In how many days does the population triple?

55. (Population Growth) Trout in a river are increasing by 15% in 5 years. To what population size does 500 trout grow in 15 years?

Solution: About 1006 after 15 years, because of the model: $P(t) = P(0)e^{kt}$, $P(0) = 500$ with t in years. Use $P(5) = 1.15P(0)$ to find $k = 0.02795238848$, then evaluate $P(15) = P(0)e^{15k} = 1005.678594$.

56. (Population Growth) A region of 400 acres contains 1000 forest mushrooms per acre. The population is decreasing by 150 mushrooms per acre every 2 years. Find the population size for the 400-acre region in 15 years.

Verhulst Equation

Write out the solution to the given differential equation and report the carrying capacity $M = \lim_{t\to\infty} P(t)$.

57. $P' = (1-P)P$

Solution: $P(t) = \dfrac{P(0)}{P(0) + (1-P(0))e^{-t}}$, $M = a/b = 1$.

58. $P' = (2-P)P$

59. $P' = 0.1(3-2P)P$

Solution: $P' = 0.2(1.5-P)P$,
$P(t) = \dfrac{0.2P(0)}{1.5P(0) + (0.2 - 1.5P(0))e^{-0.2t}}$, $M = a/b = 2/15$.
Check answers in `maple`:

```
y:=unapply(a*y0/(b*y0+(a-b*y0)*exp(-a*t)),(t,a,b,y0));
evalf(simplify(y(t,0.2,1.5,y0))); evalf(0.2/1.5);
limit(y(t,0.2,1.5,y0),t=infinity);
```

60. $P' = 0.1(4-3P)P$

61. $P' = 0.1(3+2P)P$

Solution: $P' = -0.2(-1.5-P)P$,
$P(t) = \dfrac{-0.2P(0)}{-1.5P(0) + (-0.2 + 1.5P(0))e^{0.2t}}$, $M = 0$.

62. $P' = 0.1(4+3P)P$

63. $P' = 0.2(5-4P)P$

64. $P' = 0.2(6-5P)P$

65. $P' = 11P - 17P^2$

66. $P' = 51P - 13P^2$

Logistic Equation

The following exercises use the Verhulst logistic equation $P' = (a - bP)P$, page 6. Some methods appear on page 11.

67. **(Protozoa)** Experiments on the protozoa *Paramecium* determined growth rate $a = 2.309$ and carrying capacity $a/b = 375$ using initial population $P(0) = 5$. Establish the formula $P(t) = \dfrac{375}{1 + 74e^{-2.309t}}$.

68. **(World Population)** Demographers projected the world population in the year 2000 as 6.5 billion, which was corrected by census to 6.1 billion. Use $P(1965) = 3.358 \times 10^9$, $a = 0.029$ and carrying capacity $a/b = 1.0760668 \times 10^{10}$ to compute the logistic equation projection for year 2000.

69. **(Harvesting)** A fish population satisfying $P' = (a - bP)P$ is subjected to harvesting, the new model being $P' = (a - bP)P - H$. Assume $a = 0.04$, $a/b = 5000$ and $H = 10$. Using algebra, rewrite it as $P' = a(\alpha - P)(P - \beta)$ in terms of the roots α, β of $ay - by^2 - H = 0$. Apply the change of variables $u = P - \beta$ to solve it.

Solution: The equation for u is $u' = bu(\alpha - \beta - u)$ where $\alpha = 263.9320225$, $\beta = 4736.067977$. Then $P(t) = u(t) + \beta = \dfrac{\alpha u(0)}{\beta u(0) + (\alpha - \beta u(0))e^{-\alpha t}} + \beta$.

70. **(Extinction)** Let an endangered species satisfy $P' = bP^2 - aP$ for $a > 0$, $b > 0$. The term bP^2 represents births due to chance encounters of males and females, while the term aP represents deaths. Use the change of variable $u = P/(bP - a)$ to solve it. Show from the answer that initial population sizes $P(0)$ below a/b become extinct.

Solution: The model equation is $P' = P(P - 1)$ with change of variables $u = P/(P - 1)$. Solve the simpler equation first, then generalize to $P' = P(bP - a)$.

71. **(Logistic Answer Check)** Let $P = au/(1 + bu)$, $u = u_0e^{at}$, $u_0 = P_0/(a - bP_0)$. Verify that $P(t)$ is a solution the differential equation $P' = (a - bP)P$ and $P(0) = P_0$.

Solution: Extract the details from the Logistic Solution Verification, located immediately above the exercises for this section.

72. **(Logistic Equation)** Let k, α, β be positive constants, $\alpha < \beta$. Solve $w' = k(\alpha - w)(\beta - w)$, $w(0) = w_0$ by the substitution $u = (\alpha - w)/(\beta - w)$, showing that $w = (\alpha - \beta u)/(1 - u)$, $u = u_0e^{(\alpha-\beta)kt}$, $u_0 = (\alpha - w_0)/(\beta - w_0)$. This equation is a special case of the harvesting equation $P' = (a - bP)P + H$.

Growth-Decay Uniqueness Proof

73. State precisely and give a calculus text reference for *Rolle's Theorem*, which says that a function vanishing at $x = a$ and $x = b$ must have slope zero at some point in $a < x < b$.

Solution: Rolle's Theorem can be found in most college level calculus textbooks. The hypothesis is $f(x)$ is differentiable on $a < x < b$ and $f(a) = f(b) = 0$. The conclusion: $f'(x) = 0$ for some point x between a, b.

74. Apply Rolle's Theorem to prove that a differentiable function $v(x)$ with $v'(x) = 0$ on $a < x < b$ must be constant.

1.2 Exponential Application Library

Light Intensity

The following exercises apply the theory of light intensity on page 16, using the model $I(t) = I_0 e^{-kx}$ with x in meters. Methods parallel Example 1.8 on page 21.

1. The light intensity is $I(x) = I_0 e^{-1.4x}$ in a certain swimming pool. At what depth x does the light intensity fall off by 50%?

Solution: Solve $I(x) = 0.5I_0$ for $x = 0.4951051290$ using logarithms.

2. The light intensity in a swimming pool falls off by 50% at a depth of 2.5 meters. Find the depletion constant k in the exponential model.

3. Plastic film is used to cover window glass, which reduces the interior light intensity by 10%. By what percentage is the intensity reduced, if two layers are used?

Solution: Let the plastic film have thickness X. Model $I(x) = I_0 e^{-kx}$ will be used where I_0 is the light intensity on the surface of the glass ($x = 0$). Given is $I(X) = 0.9I_0$. The task is to find $I(2X)$ as a percentage of I_0. Using logarithms on $e^{-kX} = 0.9$ gives $kX = 0.1053605157$, then $e^{-2kX} = 0.8099999999$. The answer: 81% for two layers. Misgivings: Should $x = 0$ be the surface of the glass or the surface of the plastic film or the surface of the glass where the materials are sandwiched together?

4. Double-thickness colored window glass is supposed to reduce the interior light intensity by 20%. What is the reduction for single-thickness colored glass?

RC-Electric Circuits

In the exercises below, solve for $Q(t)$ when $Q_0 = 10$ and graph $Q(t)$ on $0 \le t \le 5$.

5. $R = 1$, $C = 0.01$.

Solution: Model: $RQ' + Q/C = 0$ with solution $Q(t) = Q_0 e^{-kt}$, $k = 1/(RC)$. Then $k = 100$, $Q_0 = 10$, $Q(t) = 10e^{-10t}$. The graph is a strictly decreasing curve joining points $(0, 10)$, $(1, 0.06737947)$ and $(5, 0)$. Maple computed $Q(5) \approx 1.9/10^{21}$. See Appendix A for hand graphing of exponentials.

6. $R = 0.05$, $C = 0.001$.

7. $R = 0.05$, $C = 0.01$.

Solution: $Q(t) = 10e^{-2000t}$

8. $R = 5$, $C = 0.1$.

9. $R = 2$, $C = 0.01$.
Solution: $Q(t) = 10e^{-50t}$

10. $R = 4$, $C = 0.15$.

11. $R = 4$, $C = 0.02$.
Solution: $Q(t) = 10e^{-12.5t}$

12. $R = 50$, $C = 0.001$.

LR-Electric Circuits

In the exercises below, solve for $I(t)$ when $I_0 = 5$ and graph $I(t)$ on $0 \le t \le 5$.

13. $L = 1$, $R = 0.5$.
Solution: Model: $LI' + RI = 0$, $I_0 = 5$, $I(t) = I_0 e^{-Rt/L}$. Then $R/L = 1/2$ and $I(t) = 5e^{-t/5}$.

14. $L = 0.1$, $R = 0.5$.

15. $L = 0.1$, $R = 0.05$.
Solution: $I(t) = 5e^{-t/5}$

16. $L = 0.01$, $R = 0.05$.

17. $L = 0.2$, $R = 0.01$.
Solution: $I(t) = 5e^{-t/20}$

18. $L = 0.03$, $R = 0.01$.

19. $L = 0.05$, $R = 0.005$.
Solution: $I(t) = 5e^{-t/10}$

20. $L = 0.04$, $R = 0.005$.

Interest and Continuous Interest

Financial formulas which appear on page 18 are applied below, following the ideas in Examples 1.11, 1.12 and 1.13, pages 22 and 24.

21. (Total Interest) Compute the total daily interest and also the total continuous interest for a 10-year loan of $5,000$ dollars at 5% per annum.

Solution: Answer: 1366.889426 for daily interest and 1366.990456 for continuous interest. The difference is 10 cents.
Part (a): Daily Interest
The Auto Loan example in this section contains the formulas and ideas. Assume a month is 30 days and a year is 360 days. The problem can be viewed as follows: the \$5000 loan is a checking account with \$5000 deposit that accrues interest at 5% per annum compounded daily. The twist: a check of amount P is subtracted from the account every 30 days. The problem can then be phrased as follows:
(1) Find the amount P to be written as a monthly check so that the account balance is zero after 3600 days;
(2) Report the total interest added to the checking account over the 3600 days (10 years).
Let the daily simple interest rate be $R = 0.05/360$. Let $B(n)$ be the checking account balance after n days. Define $B(0) = 5000$, $Z = (1+R)^{30}$. The monthly check of amount P is posted at the end of day 30. Then $B(30) = B(0)Z - P$.
Similarly, $B(60) = B(30)Z - P$, $B(90) = B(60)Z - P$. Induction is used to obtain the formula $B(30k) = B(0)Z^k - P\left(1 + \cdots + Z^{k-1}\right)$. The geometric sum formula $1 + u + \cdots + u^n = \dfrac{u^{n+1}-1}{u-1}$ implies $B(30k) = B(0)Z^k - P\,\dfrac{Z^k - 1}{Z - 1}$.
The checking account has zero balance after 3600 days (10 years) provided the payment P satisfies the equation $B(30k) = 0$ for $k = 120$. Then $0 = B(0)Z^{120} - P\,\dfrac{Z^{120}-1}{Z-1}$. Solve for

$$P = B(0)Z^{120}\frac{Z-1}{Z^{120}-1}.$$

Substitute $B(0) = 5000$ and $Z = (1+R)^{30}$. Then $P = 53.05741190$, $120P = 6366.889426$ which implies the total interest paid over ten years would be that amount less \$5000: interest paid $= 1366.889426$.
Part (b): Continuous Interest
Following the Auto Loan example, the part (a) formulas are correct provided $Z = e^{30R} = 1.004175359$. The remaining details are unchanged from the computation above, which implies

$$P = B(0)Z^{120}\frac{Z-1}{Z^{120}-1}, \quad Z = e^{30R} = 1.004175359.$$

Then $P = 53.05825380$ and the interest paid $= 1366.990456$.

22. (Total Interest) Compute the total daily interest and also the total continuous interest for a 15-year loan of $7,000$ dollars at $5\frac{1}{4}\%$ per annum.

23. (Monthly Payment) Find the monthly payment for a 3-year loan of $8,000$ dollars at 7% per annum compounded continuously.

Solution: Payment = 216.2051540.

24. (Monthly Payment) Find the monthly payment for a 4-year loan of $7,000$ dollars at $6\frac{1}{3}\%$ per annum compounded continuously.

25. (Effective Yield) Determine the effective annual yield for a certificate of deposit at $7\frac{1}{4}\%$ interest per annum, compounded continuously.

Solution: Follow the **Effective Annual Yield** example. The answer for one year is $100(e^{0.0725} - 1) = 7.5192806\%$ based on 360 days and $100(e^{365(0.0725)/360} - 1) = 7.6276011\%$ for 365 days.

26. (Effective Yield) Determine the effective annual yield for a certificate of deposit at $5\frac{3}{4}\%$ interest per annum, compounded continuously.

27. (Retirement Funds) Assume a starting salary of $35,000$ dollars per year, which is expected to increase 3% per year. Retirement contributions are $10\frac{1}{2}\%$ of salary, deposited monthly, growing at $5\frac{1}{2}\%$ continuous interest per annum. Find the retirement amount after 30 years.

Solution: Answer: $396,588.1407$. Follow the **Retirement Funds** example. Maple code:

```
s:=0.055/12;P:=n->(35000/12)*(1.03)^(n-1);
R:=n->0.105*P(n);
X:=0;for j from 1 to 30 do
X:=X*exp(12*s)+R(j)*(exp(12*s)-1)/(1-exp(-s));end do:
X;
```

28. (Retirement Funds) Assume a starting salary of $45,000$ dollars per year, which is expected to increase 3% per year. Retirement contributions are $9\frac{1}{2}\%$ of salary, deposited monthly, growing at $6\frac{1}{4}\%$ continuous interest per annum. Find the retirement amount after 30 years.

29. (Actual Cost) A van is purchased for $18,000$ dollars with no money down. Monthly payments are spread over 8 years at $12\frac{1}{2}\%$ interest per annum, compounded continuously. What is the actual cost of the van?

Solution: Answer: Cost = $\$28,624.40733$, payment = $\$298.1709097$. Maple code:

```
R:=0.125/360;Z:=exp(30*R);T:=12*8;
P:=18000*(Z-1)*Z^T/(Z^T - 1);T*P;T*P-18000;
```

30. **(Actual Cost)** Furniture is purchased for $15,000$ dollars with no money down. Monthly payments are spread over 5 years at $11\frac{1}{8}\%$ interest per annum, compounded continuously. What is the actual cost of the furniture?

Radioactive Decay

Assume the decay model $A' = -kA$ from page 19. Below, $A(T) = 0.5A(0)$ defines the *half-life* T. Methods parallel Examples 1.14– 1.17 on pages 25 –26.

31. **(Half-Life)** Determine the half-life of a radium sample which decays by 5.5% in 13 years.

Solution: Answer: About 159 years. Follow the **Half–life of Radium** example. Solve for $k = -0.004351565499$ in $e^{13k} = 0.945$. Then solve for $t = 159.2868545$ in $e^{-0.004351565499\,t} = 0.5$.

32. **(Half-Life)** Determine the half-life of a radium sample which decays by 4.5% in 10 years.

33. **(Half-Life)** Assume a radioactive isotope has half-life 1800 years. Determine the percentage decayed after 150 years.

Solution: Answer: 5.6%. Follow the **Radium Disintegration** example. Solve $e^{1800\,k} = 0.5$ for $k = -0.0003850817670$. Evaluate $e^{-0.0003850817670(150)} = 0.9438743127$.

34. **(Half-Life)** Assume a radioactive isotope has half-life 1650 years. Determine the percentage decayed after 99 years.

35. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 5.5% in 13 years.

Solution: Answer: $k = -0.3498922950$. Follow the **Radium Disintegration** example. Solve $e^{-13k} = 94.5$ for $k = -0.3498922950$.

36. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 4.5% in 10 years.

37. **(Radiocarbon Dating)** A fossil found near the town of Dinosaur, Utah contains carbon-14 at a ratio of 6.21% to the atmospheric value. Determine its approximate age according to Libby's method.

Solution: Answer: $22,323.576$ years. Follow the **Radiocarbon Dating** example, assuming model $A(t) = A(0)e^{-kt}$, the half-life of carbon-14 is 5568 years and $k = \ln(2)/5568$. Known is $A(0) = 0.0621A(t)$ for some time t in the past. Solve $A(0) = 0.0621A(0)e^{-kt}$ for $t = \ln(0.0621)/k = -22323.576$.

38. **(Radiocarbon Dating)** A fossil found in Colorado contains carbon-14 at a ratio of 5.73% to the atmospheric value. Determine its approximate age according to Libby's method.

39. **(Radiocarbon Dating)** In 1950, the Lascaux Cave in France contained charcoal with 14.52% of the carbon-14 present in living wood samples nearby. Estimate by Libby's method the age of the charcoal sample.

Solution: Answer: 15500.68 years.

40. **(Radiocarbon Dating)** At an excavation in 1960, charcoal from building material had 61% of the carbon-14 present in living wood nearby. Estimate the age of the building.

41. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 5% in 12 years. By what percentage is it reduced in 99 years?

Solution: Answer: 34.5%. Follow the **Percentage of an Isotope** example. Model $A(t) = A(0)e^{-kt}$. Solve $0.95 = e^{-12k}$ for $k = 0.004274441199$. Evaluate $e^{-99k} = 0.6549674897$.

42. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 6.5% in $1,000$ years. By what percentage is it reduced in $5,000$ years?

Chemical Reactions

Assume below the model $A' = kA$ for a first-order reaction. See page 21 and Example 1.18, page 27.

43. **(First-Order $A + B \longrightarrow C$)** A chemical reaction produces $X(t)$ grams of product C from 50 grams of chemical A and 32 grams of catalyst B. The reaction uses 1 gram of A to 4 grams of B. Variable t is in minutes. Justify for some constant K the model $\dfrac{dX}{dt} = K\left(50 - \frac{1}{5}X\right)\left(32 - \frac{4}{5}X\right)$ and calculate $\lim_{t\to\infty} X(t) = 40$.

Solution: The rate of change of $X(t)$ is proportional to the product of the amounts present of A and B. These amounts are $50 - \frac{1}{5}X$ and $32 - \frac{4}{5}X$. Fractions $\frac{1}{5}$, $\frac{4}{5}$ mean that from 5 grams of C there is 1 gram of A used (supply=50) and 4 grams of B used (supply=32). Proportionality constant K times the product of the two amounts of A and B then equals $\frac{dX}{dt}$. Factor out the two fractions from the two amounts to obtain the new form $\dfrac{dX}{dt} = \frac{4}{25}K(40 - X)(250 - X)$ and define $\alpha = 40$, $\beta = 250$, $k = \frac{4K}{25}$ (re-arranged to insure $\alpha < \beta$). Follow the subsection on **Chemical Reactions**. The amount $X(t)$ of product C satisfies

$$X(t) = \frac{\alpha - \beta u(t)}{1 - u(t)}, \quad u(t) = u_0 e^{(\alpha-\beta)kt}, \quad u_0 = \frac{\alpha - X_0}{\beta - X_0}. \tag{1}$$

Then
$$X(t) = \frac{40 - 250\, u_0\, \mathrm{e}^{-210\, t}}{1 - u_0\, \mathrm{e}^{-210\, t}}$$
and $\lim_{t\to\infty} X(t) = 40$, because $\lim_{t\to\infty} e^{ct} = 0$ for c negative.

44. (First-Order $A + B \longrightarrow C$) A first order reaction produces product C from chemical A and catalyst B. Model the production of C using a grams of A and b grams of B, assuming initial amounts M of A and N of B, $M < N$.

45. (Law of Mass-Action) Consider a second-order chemical reaction $X(t)$ with $k = 0.14$, $\alpha = 1$, $\beta = 1.75$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 2$.

Solution: Follow the **Chemical Reaction** example. The amount $X(t)$ of product C satisfies

$$X(t) = \frac{\alpha - \beta u(t)}{1 - u(t)}, \quad u(t) = u_0 e^{(\alpha-\beta)kt}, \quad u_0 = \frac{\alpha - X_0}{\beta - X_0}. \tag{2}$$

Substitute $k = 0.14 = \frac{14}{100}$, $\alpha = 1$, $\beta = 1.75 = \frac{7}{4}$, $X(0) = 0$. Then $u_0 = \alpha/\beta = \frac{4}{7}$, $u(t) = \frac{4}{7}e^{-3kt/4}$ and

$$X(t) = \frac{1 - \frac{7}{4}u(t)}{1 - u(t)} = \frac{1 - e^{-21t/200}}{1 - \frac{4}{7}e^{-21t/200}}$$

The plot on $0 \le t \le 2$ is a strictly increasing curve from $(0, 0)$ to $(2, 0.353)$.

46. (Law of Mass-Action) Consider a second-order chemical reaction $X(t)$ with $k = 0.015$, $\alpha = 1$, $\beta = 1.35$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 10$.

47. (Mass-Action Derivation) Let k, α, β be positive constants, $\alpha < \beta$. Solve $X' = k(\alpha - X)(\beta - X)$, $X(0) = X_0$ by the substitution $u = (\alpha - X)/(\beta - X)$, showing that $X = (\alpha - \beta u)/(1 - u)$, $u = u_0 e^{(\alpha-\beta)kt}$, $u_0 = (\alpha - X_0)/(\beta - X_0)$.

Solution: Algebra on $u = (\alpha - X)/(\beta - X)$ gives $X = (\alpha - \beta u)/(1 - u)$. Compute $u' = \dfrac{-X'(\beta - X) + (\alpha - X)X'}{(\beta - X)^2}$ by the quotient rule in calculus. Used is $\alpha' = \beta' = 0$ by the constant rule in calculus. Simplify the fraction:

$$\begin{aligned} u' &= \frac{(\alpha - \beta)X'}{(\beta - X)^2} \\ &= (\alpha - \beta)X' \frac{X}{(\beta - X)^2} \end{aligned}$$

$$= (\alpha-\beta)k(\alpha-X)(\beta-X)\frac{X}{(\beta-X)^2}$$
$$= (\alpha-\beta)k\frac{\alpha-X}{\beta-X}$$
$$= (\alpha-\beta)ku$$

Exponential modeling for $u' = cu$ gives $u = u_0e^{ct} = u_0e^{(\alpha-\beta)kt}$. ■

48. **(Mass-Action Derivation)** Let k, α, β be positive constants, $\alpha < \beta$. Define $X = (\alpha-\beta u)/(1-u)$, where $u = u_0e^{(\alpha-\beta)kt}$ and $u_0 = (\alpha - X_0)/(\beta - X_0)$. Verify by calculus computation that (1) $X' = k(\alpha-X)(\beta-X)$ and (2) $X(0) = X_0$.

Drug Dosage

Employ the drug dosage model $D(t) = D_0e^{-ht}$ given on page 21 . Apply the techniques of Example 1.19, page 27 .

49. **(Injection Dosage)** Bloodstream injection of a drug into an animal requires a minimum of 20 milligrams per pound of body weight. Predict the dosage for a 12-pound animal which will maintain a drug level 3% higher than the minimum for two hours. Assume half-life 3 hours.

Solution: Answer: 393 milligrams. Follow the **Drug Dosage** example. The drug model is $D(t) = D_0e^{-ht}$, where D_0 is the initial dosage and h is the elimination constant. A half-life of three hours means $D_0e^{-3h} = \frac{1}{2}D_0$, which determines $h = \frac{1}{3}\ln(2) = 0.2310490602$. Constant D_0 is unknown. The requirement on D_0 is inequality $D(t) > 1.03(12)(20)$, valid for $t = 0$ to $t = 2$ hours. Depletion of the drug in the bloodstream means the drug levels are always decreasing, so it is enough to require that the level at 2 hours exceeds 1.03(12)(20). The critical value of dosage D_0 then occurs when $D(2) = 1.03(12)(20) = 247.20$ or $D_0e^{-2h} = 247.20$. Then $D_0 = 247.20e^{2h} = 392.4055401$ milligrams.

50. **(Injection Dosage)** Bloodstream injection of an antihistamine into an animal requires a minimum of 4 milligrams per pound of body weight. Predict the dosage for a 40-pound animal which will maintain an antihistamine level 5% higher than the minimum for twelve hours. Assume half-life 3 hours.

51. **(Oral Dosage)** An oral drug with half-life 2 hours is fully absorbed into the bloodstream in 45 minutes, blood level 63% of the dose. Assume 500 milligrams in the first dose is fully absorbed at $t = 0$. A second dose is taken 1 hour later to maintain a blood level of at least 180 milligrams for 2.5 hours. Explain why 1 hour might be reasonable.

Solution: Follow the **Drug Dosage** example. A typical drug brand is Tylenol, 500 milligrams per tablet. A 45-minute absorption means the blood level is $(0.63)500 = 315$ milligrams at time $t = 0$ hours. Then the body starts to eliminate the drug according to drug model $D(t) = 315e^{-ht}$, where h is the elimination constant. The half-life information implies $e^{-2h} = 0.5$ and then $h = \ln(2)/2 = 0.3465735903$. The problem: predict the time T in hours at which the second dose of 500 milligrams should be ingested. A guess for the answer T is provided by the blood level 315 depleting to 180, which happens when $D(t) = 180$. Equation $315e^{-hT} = 180$ has solution $T = 1.614709844$ hours. When the second dose is taken, about 45 minutes is required for the blood level to return to 315. In 45 minutes after dose two (taken at the one hour mark), the blood level from dose one falls to $D(1.075) = 315e^{-(1+0.75)h} = 171.7549679$. This contribution from dose one is slightly below 180, while contributions from dose two have maximized the blood level to 315. If 1.6 hours is used instead of one hour for dose two, then $D(1.6+0.75) = 315e^{-(1.6+0.75)h} = 139.5083842$, which means the blood level can drop below 180 for some time interval after dose two was ingested. The absorption rate of the drug affects blood levels significantly, but all that is known is 45 minutes to full absorption. Once the blood level is 315, then the previous analysis applies: $D(1.5) = 315e^{-(1.5)h} = 187.3001206$ insures blood level 180 for 2.5 hours.

52. (Oral Dosage) An oral drug with half-life 2 hours is fully absorbed into the bloodstream in 45 minutes, blood level 63% of the dose. Determine three (small) dosage amounts, and their administration time, which keep the blood level above 180 milligrams but below 280 milligrams over three hours.

1.3 Differential Equations of First Order

Solution Verification

Given the differential equation, initial condition and proposed solution y, verify that y is a solution. Don't try to *solve* the equation!

1. $\dfrac{dy}{dx} = y$, $y(0) = 2$, $y = 2e^x$

Solution: The details are an **answer check** with two panels.
Panel 1: Test DE.

$\text{LHS} = \dfrac{dy}{dx}$	Left side of DE $\frac{dy}{dx} = y$
$= \dfrac{d}{dx}(2e^x)$	Substitute expected answer $y = 2e^x$.
$= 2e^x$	Calculus constant rule and exponential rule.
$= y$	Definition $y = 2e^x$.
$= \text{RHS}$	Equal left and right side expressions for all symbols. DE verified.

Panel 2: Test IC.

$\text{LHS} = y(0)$	Left side of IC $y(0) = 2$
$= 2e^x\big\|_{x=0}$	Substitute expected answer $y = 2e^x$.
$= 2e^0$	Substitute $x = 0$.
$= 2 \cdot 1$	Use $e^0 = 1$.
$= \text{RHS}$	Left and right side of $y(0) = 2$ match for all symbols. IC verified.

2. $y' = 2y$, $y(0) = 1$, $y = e^{2x}$

3. $y' = y^2$, $y(0) = 1$, $y = (1-x)^{-1}$

Solution: Follow Exercise 1. In panel 1, dy/dx is found by the calculus power rule $(u^n)' = nu^{n-1}u'$ as $y' = ((1-x)^{-1})' = (-1)(1-x)^{-2}(-1)$. The RHS $= y^2 = (1-x)^{-2}$.

4. $\dfrac{dy}{dx} = y^3$, $y(0) = 1$,
$y = (1-2x)^{-1/2}$

5. $D^2y(x) = y(x)$, $y(0) = 2$,
$Dy(0) = 2$, $y = 2e^x$

Solution: Follow Exercise 1.

6. $D^2y(x) = -y(x)$, $y(0) = 0$,
$Dy(0) = 1$, $y = \sin x$

7. $y' = \sec^2 x$, $y(0) = 0$, $y = \tan x$

Solution: Follow Exercise 1. Needed in panel 1 is calculus identity $(\tan(x))' = \sec^2(x)$ and trig identities $\tan x = \sin x / \cos x$, $\sin 0 = 0$, $\cos 0 = 1$.

8. $y' = -\csc^2 x$, $y(\pi/2) = 0$,
$y = \cot x$

9. $y' = e^{-x}$, $y(0) = -1$, $y = -e^{-x}$

Solution: Follow Exercise 1. Needed in panel 1 is calculus identity $(e^u)' = e^u\, u'$. In panels 1,2 use pre-calculus identity $e^0 = 1$.

10. $y' = 1/x$, $y(1) = 1$, $y = \ln x$

Explicit and Implicit Solutions

Identify the given solution as *implicit* or *explicit*. If *implicit*, then solve for y in terms of x by college algebra methods.

11. $y = x + \sin x$

Solution: Explicit. The test: y isolated left, right side independent of symbol y.

12. $y = x + \sin x$

Solution: Explicit.

13. $2y + x^2 + x + 1 = 0$

Solution: Implicit. Left side is not y alone.

14. $x - 2y + \sin x + \cos x = 0$

Solution: Implicit.

15. $y = e^{\pi}$

Solution: Explicit. The test: y isolated left, right side independent of symbol y.

16. $e^y = \pi$

Solution: Implicit.

17. $e^{2y} = \ln(1+x)$

Solution: Implicit. Left side is not y alone but a composition involving y.

18. $\ln|1+y^2| = e^x$

Solution: Implicit.

19. $\tan y = 1 + x$

Solution: Implicit. Left side is not y alone but a composition involving y.

20. $\sin y = (x-1)^2$

Solution: Explicit.

Tables and Explicit Equations

For the given explicit equation, make a table of values $x = 0$ to $x = 1$ in steps of 0.2.

21. $y = x^2 - 2x$

Solution:

x	y
0.	0.
0.2	-0.36
0.4	-0.64
0.6	-0.84
0.8	-0.96
1.0	-1.00

```
# Maple code
Y:=x->x^2-2*x;
seq([0+n*0.2,Y(0+n*0.2)],n=0..5);
```

22. $y = x^2 - 3x + 1$

23. $y = \sin \pi x$

Solution: Follow exercise 21.

24. $y = \cos \pi x$

25. $y = e^{2x}$

Solution: Follow exercise 21.

26. $y = e^{-x}$

27. $y = \ln(1 + x)$

Solution: Follow exercise 21.

28. $y = x \ln(1 + x)$

Tables and Approximate Equations

Make a table of values $x = 0$ to $x = 1$ in steps of 0.2 for the given approximate equation. Identify precisely the *recursion* formulas applied to obtain the next table pair from the previous table pair.

29. $y(x + 0.2) \approx y(x) + 0.2(1 - y(x))$, $y(0) = 1$

Solution: The idea is to replace $\approx$ by $=$, then replace x by $0.2n$, for $n = 0, \ldots, 5$ in order for x to exhaust $x = 0$ to 1 in steps of 0.2. Define $y_n = y(0.2n)$. Then the recursion is $y_{n+1} = y_n + 0.2(1 - y_n)$, $y_0 = 1$.

30. $y(x + 0.2) \approx y(x) + 0.2(1 + y(x))$, $y(0) = 1$

31. $y(x + 0.2) \approx y(x) + 0.2(x - y(x))$, $y(0) = 0$

Solution: $y_{n+1} = y_n + 0.2(0.2n - y_n)$, $y_0 = 0$.

32. $y(x + 0.2) \approx y(x) + 0.2(2x + y(x))$, $y(0) = 0$

33. $y(x + 0.2) \approx y(x) + 0.2(\sin x + xy(x))$, $y(0) = 2$

Solution: $y_{n+1} = y_n + 0.2(\sin(0.2n) + 0.2ny_n)$, $y_0 = 2$.

34. $y(x + 0.2) \approx y(x) + 0.2(\sin x - x^2 y(x))$, $y(0) = 2$

35. $y(x + 0.2) \approx y(x) + 0.2(e^x - 7y(x))$, $y(0) = -1$

Solution: $y_{n+1} = y_n + 0.2(e^{0.2n} - 7y_n)$, $y_0 = -1$.

36. $y(x + 0.2) \approx y(x) + 0.2(e^{-x} - 5y(x))$, $y(0) = -1$

37. $y(x + 0.2) \approx y(x) + 0.1(e^{-2x} - 3y(x))$, $y(0) = 2$

Solution: $y_{n+1} = y_n + 0.1(e^{-0.4n} - 3y_n)$, $y_0 = 2$.

38. $y(x + 0.2) \approx y(x) + 0.2(\sin 2x - 2y(x))$, $y(0) = 2$

Hand Graphing

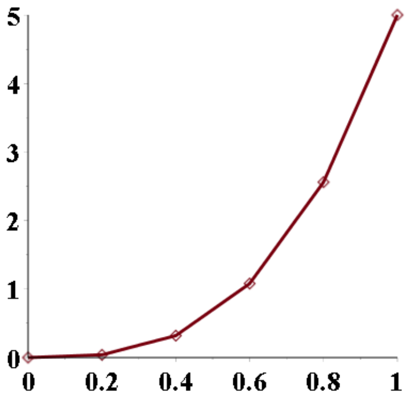
Make a graphic by hand on engineering paper, using 6 data points. Cite the divisions assigned horizontally and vertically. Label the axes and the center of coordinates. Supply one sample hand computation per graph. Employ a computer program or calculator to obtain the data points.

39. $y = 5x^3$, $x = 0$ to $x = 1$.

Solution: Maple:

```
Y:=x->5*x^3;L:=seq([0+n*0.2,Y(0+n*0.2)],n=0..5);
# [0., 0.], [0.2, 0.040], [0.4, 0.320], [0.6, 1.080],
# [0.8, 2.560], [1.0, 5.000]
plot([L],font=[times,bold,20],symbolsize=20,
     style=pointline,thickness=3);
```

x	y
0.0	0.0
0.2	0.04
0.4	0.32
0.6	1.08
0.8	2.56
1.0	5.0



A hand-drawn graphic is expected, using the methods in Appendix A.

40. $y = 3x$, $x = 0$ to $x = 1$.

41. $y = 2x^5$, $x = 0$ to $x = 1$.

Solution: Follow exercise 39.

42. $y = 3x^7$, $x = 0$ to $x = 1/2$.

43. $y = 2x^4$, $x = 0$ to $x = 1$.

Solution: Follow exercise 39.

44. $y = 3x^6$, $x = 0$ to $x = 1$.

45. $y = \sin x$, $x = 0$ to $x = \pi/4$.
Solution: Follow exercise 39.

46. $y = \cos x$, $x = 0$ to $x = \pi/4$.

47. $y = \dfrac{x+1}{x+2}$, $x = 0$ to $x = 1$.
Solution: Follow exercise 39.

48. $y = \dfrac{x-1}{x+1}$, $x = 0$ to $x = 1$.

49. $y = \ln(1+x)$, $x = 0$ to $x = 1$.
Solution: Follow exercise 39.

50. $y = \ln(1+2x)$, $x = 0$ to $x = 1$.

1.4 Direction Fields

Window and Grid

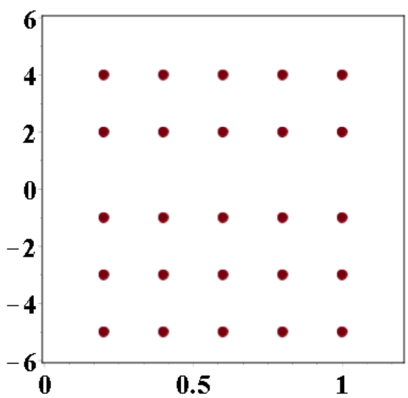
Find the equilibrium solutions, then determine a graph window which includes them and construct a 5×5 uniform grid. Follow Example 1.25.

1. $y' = 2y$

Solution: **Equilibrium solution**: $y = 0$.
Equilibrium solutions are found by substitution into the DE ($y' = 2y$ in the present case) using substitution $y = c$. The same substitution is used for every DE, where c is a constant. Then $(c)' = 2c$ determines c. There is just one value $c = 0$ that satisfies the equation $(c)' = 2c$, because $(c)' = 0$ for any constant c. It is possible for some DE that no value of c exists or that multiple values of c exist. In the present case: $y = 0$ results from $y = c$ after substitution of the answer(s) for c. Equilibrium solution $y = 0$ is reported.

The graph window could be $-0.2 \le x \le 1.2$, $-5 \le y \le 5$. It contains edge-to-edge curve $y = 0$, the equilibrium solution. There is no unique graph window to report: there are infinitely many choices, all correct.
The grid points are then selected for a 5×5 uniform grid (25 grid points). For instance, $x_k = 0.2k$, $k = 1, \ldots, 5$ and $y_k = k(-1)^k$ for $k = 1, \ldots, 5$. The grid point pairs (x, y) are:

```
(0.2, -1), (0.4, -1), (0.6, -1), (0.8, -1), (1.0, -1),
(0.2, 2), (0.4, 2), (0.6, 2), (0.8, 2), (1.0, 2),
(0.2, -3), (0.4, -3), (0.6, -3), (0.8, -3), (1.0, -3),
(0.2, 4), (0.4, 4), (0.6, 4), (0.8, 4), (1.0, 4),
(0.2, -5), (0.4, -5), (0.6, -5), (0.8, -5), (1.0, -5)
# Maple code
L:=seq(seq( [0.2*k,j*(-1)^j ],k=1..5),j=1..5);
plot([L],style=point,font=[courier,bold,20],
     view=[0 ..1.2,-6..6]);
```



2. $y' = 3y$

3. $y' = 2y + 2$

Solution: Follow Exercise 1. Equilibrium solution $y = -1$.

4. $y' = 3y - 2$

5. $y' = y(1 - y)$

Solution: Follow Exercise 1. Equilibrium solutions $y = 0$ and $y = 1$.

6. $y' = 2y(3 - y)$

7. $y' = y(1 - y)(2 - y)$

Solution: Follow Exercise 1. Equilibrium solutions $y = 0$, $y = 1$ and $y = 2$. Suitable graph window: $-1 \le x \le 1$, $-0.2 \le y \le 2.2$.

8. $y' = 2y(1 - y)(1 + y)$

9. $y' = 2(y - 1)(y + 1)^2$

Solution: Follow Exercise 1. Equilibrium solutions $y = 1$ and $y = -1$.

10. $y' = 2y^2(y - 1)^2$

11. $y' = (x + 1)(y + 1)(y - 1)y$

Solution: Follow Exercise 1. Equilibrium solutions $y = -1$ and $y = 1$. The factor $(x + 1)$ is canceled from the solution process for c, because the equation $(c)' = (x+1)(c+1)(c-1)$ is valid for all x. For instance, at $x = 0$ it says $0 = (c+1)(c-1)$, which results in the two answers $c = -1$ and $c = 1$.

12. $y' = 2(x + 1)y^2(y + 1)(y - 1)^2$

13. $y' = (x + 2)y(y - 3)(y + 2)$

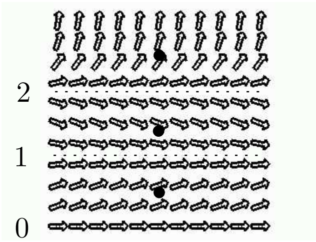
Solution: Follow Exercise 1. Equilibrium solutions $y = 0$, $y = 3$ and $y = -2$. Factor $(x + 2)$ cancels from the solution process for c; see Exercise 11.

14. $y' = (x + 1)y(y - 2)(y + 3)$

Threading Solutions

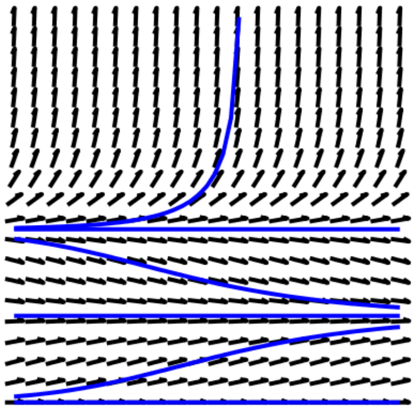
Each direction field below has window $0 \le x \le 3$, $0 \le y \le 3$. Start each threaded solution at a black dot and continue it left and right across the field. Dotted horizontal lines are equilibrium solutions. See Example 1.26.

15.



Solution: A computer-generated plot is not expected, just a hand-sketched drawing made over a paper print of the figure in Exercise 15. Drawing details expected: the curve has to go through the solid black dot; the curve's slope must match the slope of each arrow it passes.

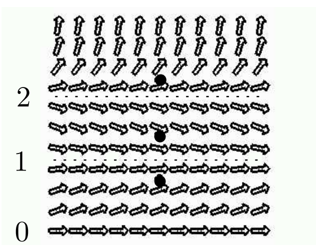
The computer plot:



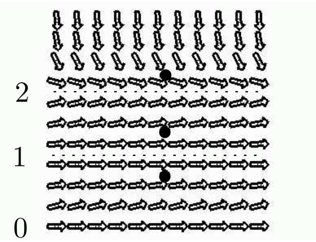
```
with(DEtools): # maple
phaseportrait((D(y))(x) = y(x)*(2-y(x))*(1-y(x)), y(x),
x = 0 .. 3, [[y(0) = 0], [y(0) = 1], [y(0) = 2],
[y(1.5) = 0.5], [y(1.5) = 1.4], [y(1.5) = 2.5]]);
```

Phase portrait packages make it possible to efficiently generate threaded curves by mouse-click.

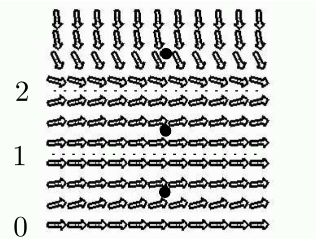
16.



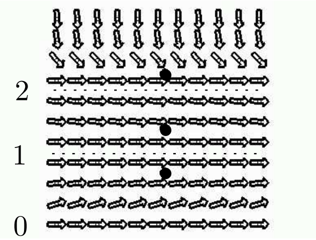
17.



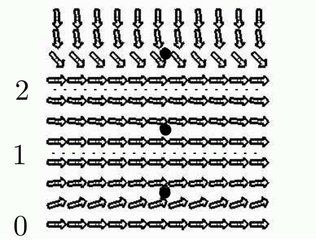
18.



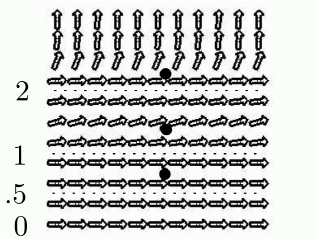
19.



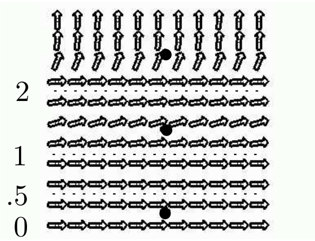
20.



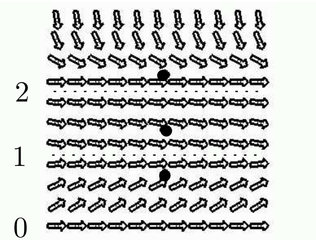
21.



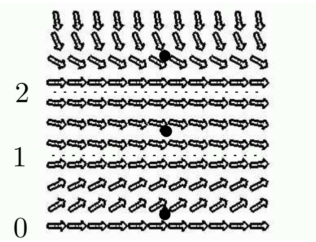
22.



23.



24.

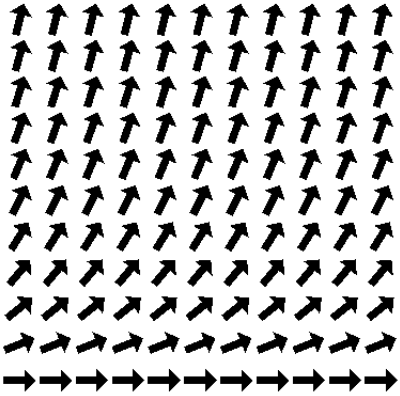


Uniform Grid Method

Apply the uniform grid method as in Example 1.27, page 45 to make a direction field of 11×11 grid points for the given differential equation on $-1 \leq x \leq 1$, $-2 \leq y \leq 2$. If using a computer program, then use about 20×20 grid points.

25. $y' = 2y$

Solution: The computer plot:



```
with(DEtools): # maple
dfieldplot(diff(y(x),x) = 2*y(x), y(x), x = 0 .. 2,
 y = 0..2,color=black,dirfield=[11,11],
 arrows=THICK,axes=none);
```

26. $y' = 3y$

27. $y' = 1 + y$

28. $y' = 2 + 3y$

29. $y' = x + y(2 - y)$

30. $y' = x + y(1 - 2y)$

31. $y' = 1 + y(2 - y)$

32. $y' = 1 + 2y(2 - y)$

33. $y' = x - y$

34. $y' = x + y$

35. $y' = y - \sin(x)$

36. $y' = y + \sin(x)$

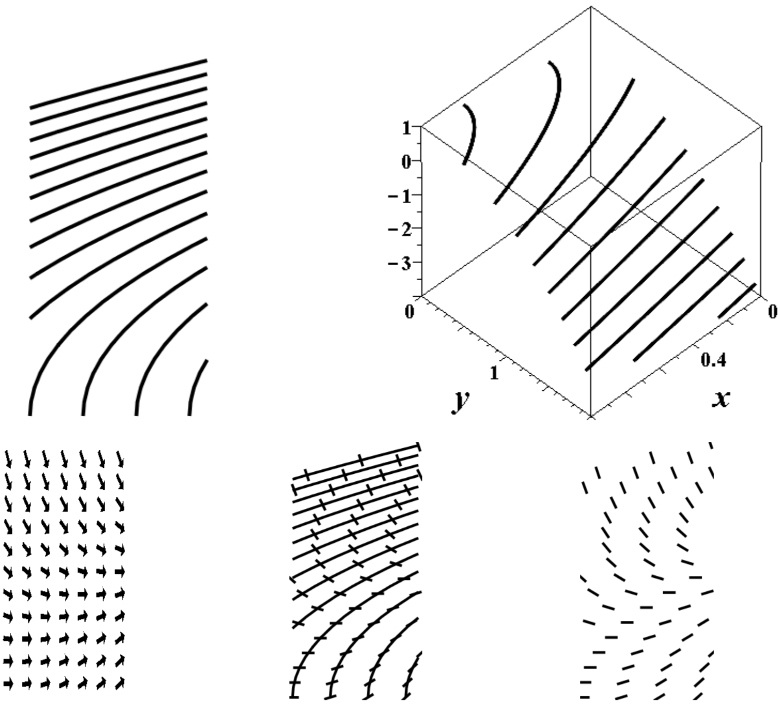
Isocline Method

Apply the isocline method as in Example 1.28, page 47 to make a direction field of about 11×11 points for the given differential equation on $0 \le x \le 1$,

$0 \le y \le 2$. Computer programs are used on these kinds of problems to find grid points as intersections of isoclines and horizontal lines. Graphics are expected to be done by hand. Extra isoclines can fill large white spaces.

37. $y' = x - y^2$

Solution: Needed are 11 or more isoclines $x - y^2 = M$ that remain mostly inside the graph window. Values of M are chosen by successive trial and error. Isoclines curves are standard curve library parabolas $y - y_0 = x^2$ which can be drawn by tracing and vertex translation. The five figures below show:
(1) a set of isoclines
(2) a 3D interpretation of the contours (z equals M),
(3) computer-generated direction field,
(4) isoclines plus lineal elements,
(5) lineal elements only.



A lineal element drawn in the direction field has constant value M along an isocline $x - y^2 = M$. There are only 14 different slopes to draw. The clumsy part of the effort is matching the lineal element slope M to the correct isocline in the figure, the plan being to duplicate the lineal element along the isocline curve by rigid translation (easel and T-square work on paper).

```
# Maple isocline plot
MM:=[seq(0.9-k/10,k=0..18)]:evalf(MM,1);
plots[implicitplot]([seq(x-y*y=M,M in MM)],x=0..1,y=0..2,
  thickness=4,color=black,axes=none,scaling=constrained);
# Maple contour plot of isoclines
plot3d(x-y*y,x=0..1,y=0..2,style=contour,
  thickness=4,color=black,font=[courier,bold,16],
  labelfont=[courier,bold,24]);
# Maple direction field plot
DEtools[dfieldplot](diff(y(x),x) = x-y(x)*y(x), y(x),
  x = 0 .. 1, y = 0..2,color=black,dirfield=[7,11],
  arrows=THICK,axes=none,scaling=constrained);
```

38. $y' = 2x - y^2$

39. $y' = 2y/(x+1)$

40. $y' = -y^2/(x+1)^2$

41. $y' = \sin(x-y)$

42. $y' = \cos(x-y)$

43. $y' = xy$

44. $y' = x^2y$

45. $y' = xy + 2x$

46. $y' = x^2y + 2x^2$

1.5 Phase Line Diagrams

Stability-Instability Test

Find all equilibria for the given differential equation and then apply Theorem 1.3, page 55, to obtain a classification of each equilibrium as a **source**, **sink** or **node**. Do not draw a phase line diagram.

1. $P' = (2 - P)P$

Solution: Equilibria $P = 0$, $P = 2$. Let $f(y) = (2 - y)y$. Then samples $f(-1) = -3$, $f(1) = 1$, $f(3) = -3$ show that f changes from minus to plus at $y = 0$ and from plus to minus at $y = 2$. Theorem 1.3 applies: $y = 0$ is a source and $y = 2$ is a sink.

2. $P' = (1 - P)(P - 1)$

3. $y' = y(2 - 3y)$

Solution: $y = 0$ is a source and $y = 2/3$ is a sink

4. $y' = y(1 - 5y)$

5. $A' = A(A - 1)(A - 2)$

Solution: $A = 0$ is a source, $A = 1$ is a sink, $A = 2$ is a source.

6. $A' = (A - 1)(A - 2)^2$

7. $w' = \dfrac{w(1 - w)}{1 + w^2}$

Solution: The sign of $f(y) = y(1-y)/(1+y^2)$ alternates from minus to plus to minus crossing the equilibria $y = 0, 1$. Then $w = 0$ is a source and $w = 1$ is a sink.

8. $w' = \dfrac{w(2 - w)}{1 + w^4}$

9. $v' = \dfrac{v(1 + v)}{4 + v^2}$

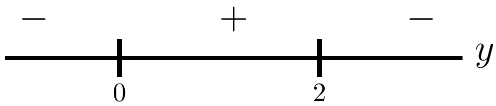
Solution: Sink $v = 0$, source $v = -1$.

10. $v' = \dfrac{(1 - v)(1 + v)}{2 + v^2}$

Phase Line Diagram

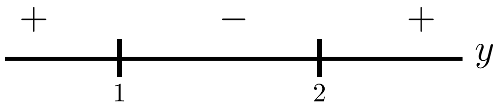
Draw a phase line diagram, with detail similar to Figure 20.

11. $y' = y(2 - y)$

Solution: 

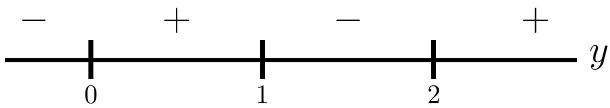
12. $y' = (y + 1)(1 - y)$

13. $y' = (y - 1)(y - 2)$

Solution: 

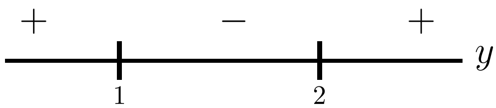
14. $y' = (y - 2)(y + 3)$

15. $y' = y(y - 2)(y - 1)$

Solution: 

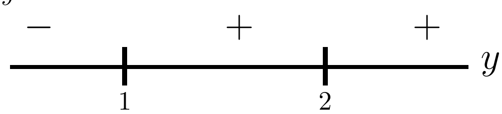
16. $y' = y(2 - y)(y - 1)$

17. $y' = \dfrac{(y - 2)(y - 1)}{1 + y^2}$

Solution: 

18. $y' = \dfrac{(2 - y)(y - 1)}{1 + y^2}$

19. $y' = \dfrac{(y - 2)^2(y - 1)}{1 + y^2}$

Solution: 

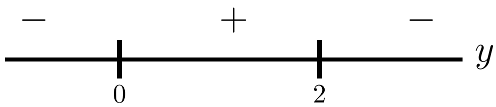
20. $y' = \dfrac{(y - 2)(y - 1)^2}{1 + y^2}$

Phase Portrait

Draw a phase portrait of threaded curves, using the phase line diagram constructed in the previous ten exercises.

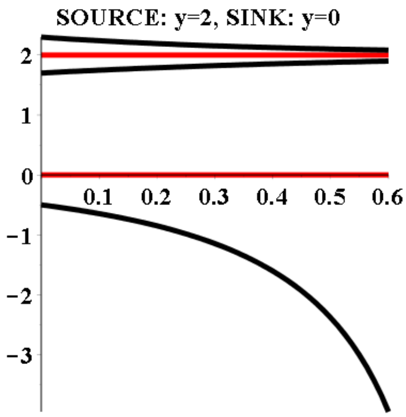
21. $y' = y(2 - y)$

Solution:



The curves drawn by hand should be either increasing or decreasing. The phase portrait contains horizontal lines $y = 0$ and $y = 2$. A threaded curve started at $x = 0$, $y < 0$ is decreasing and exits the bottom edge of the graphic. A threaded curve started at $x = 0$ with $0 < y < 1$ will increase and be trapped between the lines $y = 0$ and $y = 1$, limiting at infinity to the line $y = 1$. A threaded curve started at $x = 0$ with $y > 1$ will decrease and limit at infinity to the line $y = 1$. SPOUT: $y = 0$, FUNNEL: $y = 1$. Duplicate labels are SOURCE and SINK. The expected figure is drawn from the phase line diagram above using the three rules for constructing a phase portrait:

1. Equilibrium solutions are horizontal lines.
   Plotted equilibria are $y = 2$, $y = 0$, RED in the graphic.
2. Threaded solutions of $y' = f(y)$ don't cross.
   These are the BLACK curves in the graphic.
3. A threaded non-equilibrium solution that starts at $x = 0$ at a point y_0 must be increasing if $f(y_0) > 0$, and decreasing if $f(y_0) < 0$.
   Initial values used: $y(0) = -1/2$, $y(0) = 1.7$, $y(0) = 2.3$.




```
with(DEtools): # maple
phaseportrait((D(y))(x) = y(x)*(2-y(x)), y(x),
x = 0 .. 0.6, [[y(0) = -0.5], [y(0) = 0],
[y(0)=1.7],[y(0)=2],[y(0)=2.3]],arrows=none,
linecolor=[black,red,black,red,black],
thickness=5,font=[courier,bold,18],labels=["",""]);
```

22. $y' = (y + 1)(1 - y)$

23. $y' = (y-1)(y-2)$

24. $y' = (y-2)(y+3)$

25. $y' = y(y-2)(y-1)$

26. $y' = y(2-y)(y-1)$

27. $y' = \dfrac{(y-2)(y-1)}{1+y^2}$

28. $y' = \dfrac{(2-y)(y-1)}{1+y^2}$

29. $y' = \dfrac{(y-2)^2(y-1)}{1+y^2}$

30. $y' = \dfrac{(y-2)(y-1)^2}{1+y^2}$

Bifurcation Diagram

Draw a stack of phase line diagrams and construct from it a succinct bifurcation diagram with abscissa k and ordinate $y(0)$. Don't justify details at a bifurcation point.

31. $y' = (2-y)y - k$

Solution: Follow the **Bifurcation Diagram** example. Exercise 23 below will be solved as a second distinct example.

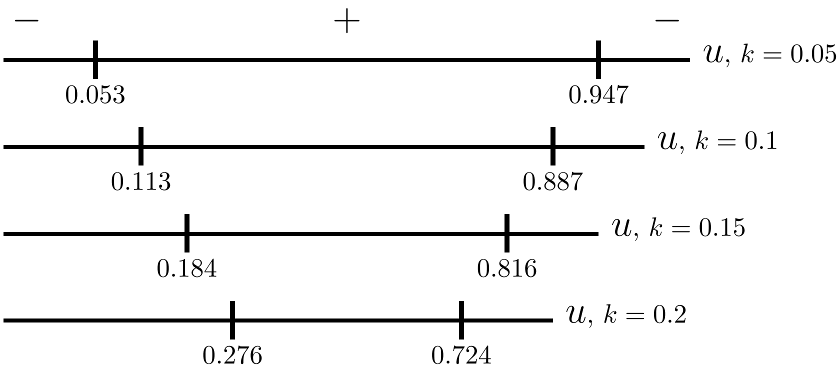
32. $y' = (3-y)y - k$

33. $y' = (2-y)(y-1) - k$

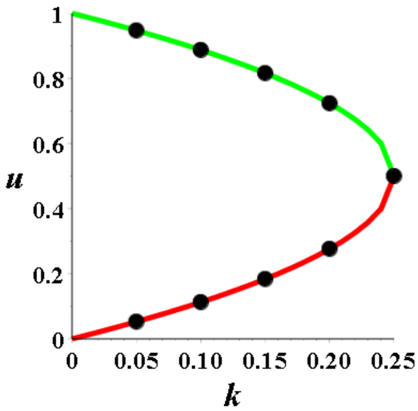
Solution: Follow the **Bifurcation Diagram** example. The change of variables $u = y - 1$ changes the y-equation into $u' = (1-u)u - k$, which is an autonomous differential equation similar to Exercise 21. Let $f(u) = (1-u)u - k$, where k is a parameter that controls the harvesting rate per annum. A phase line diagram will be made for each relevant value of k. First, the *equilibria* are computed as the roots $u = a(k)$, $u = b(k)$ of $f(u) = 0$ by the quadratic formula in college algebra:

$$a(k) = \frac{1}{2} + \frac{1}{2}\sqrt{1-4\,k}, \qquad b(k) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4\,k}$$

The roots are real only in case $1-4k \geq 0$ or $k \leq 0.25$. A double root occurs at $k = 0.25$.



The phase line diagrams are rotated counter-clockwise 90 degrees and assembled into a bifurcation diagram with connect-the-dots applied to the equilibria. The bifurcation point is at $k = 0.25$.




```
f:=u->u*(1-u)-k;
w:=unapply([solve(f(y)=0,y)],k);w(0.05),w(0.1),w(0.15),w(0.2);
with(plots):
eqs := [y=1/2 + sqrt(1 - 4*k)/2, y=1/2 - sqrt(1 - 4*k)/2]
Plot1:=implicitplot(eqs,k=0..0.25,y=0..2,font=[courier,bold,18],
 thickness=5,color=[red,green],labelfont=[courier,bold,24]);
Plot2:=pointplot([[0.25,1/2],[0.05, 0.95], [.1, 0.89],
 [0.15, 0.82], [0.2, 0.72], [0.05,0.053], [0.1,.11],
 [0.15,0.18], [0.2,0.28]],symbol=solidcircle,symbolsize=30)
display(Plot1,Plot2,labels=[k,u])
```

The bifurcation diagram for the original equation $y' = (2 - y)(y - 1) - k$ is obtained by translation $y = u + 1$.

34. $y' = (3 - y)(y - 2) - k$

35. $y' = y(0.5 - 0.001y) - k$

Solution: Factor out 0.001. Then follow the **Bifurcation Diagram** example.

36. $y' = y(0.4 - 0.045y) - k$

Details and Proofs

Supply details for the following statements.

37. (Stability Test)
Verify **(b)** of Theorem 1.3, page 55, by altering the proof given in the text for **(a)**.

Solution: Replace f by $-f$ in the proof of part (a) to discover the proof for part (b).

38. (Stability Test)
Verify **(b)** of Theorem 1.3, page 55, by means of the change of variable $x \to -x$.

39. (Autonomous Equations)
Let $y' = f(y)$ have solution $y(x)$ on $a < x < b$. Then for any c, $a < c < b$, the function $z(x) = y(x + c)$ is a solution of $z' = f(z)$.

Solution: The chain rule gives $z'(x) = y'(x + c)(x + c)' = y'(x + c)\,1 = f(y(x + c)) = f(z(x))$. Therefore, z is a solution if y is a solution.

40. (Autonomous Equations)
The method of isoclines can be applied to an autonomous equation $y' = f(y)$ by choosing equally spaced horizontal lines $y = c_i$, $i = 1, \ldots, k$. Along each horizontal line $y = c_i$ the slope is a constant $M_i = f(c_i)$, and this determines the set of invented slopes $\{M_i\}_{i=1}^{k}$ for the method of isoclines.

1.6 Computing and Existence

Multiple Solution Example

Define $f(x,y) = 3(y-1)^{2/3}$. Consider $y' = f(x,y)$, $y(0) = 1$.

1. Do an answer check for $y(x) = 1$. Do a second answer check for $y(x) = 1+x^3$.

Solution: A formal 2-panel answer check will be given for both solutions. It is expected that everyone will abbreviate the answer check, but essential details are expected.

Define $f(x,y) = 3(y-1)^{2/3}$ and $y(x) = 1$.
Panel 1: Verify DE.

$$\begin{aligned} \mathsf{LHS} &= y' \\ &= (1)' \\ &= 0 \\ \mathsf{RHS} &= f(x,y) \\ &= f(x,1) \\ &= 3(0)^{2/3} \\ &= 0 \qquad \text{DE verified.} \end{aligned}$$

Panel 2: Verify IC.

$$\begin{aligned} \mathsf{LHS} &= y(0) \\ &= (1)\,|_{x=0} \\ &= 1 \qquad \text{IC verified.} \end{aligned}$$

Define $f(x,y) = 3(y-1)^{2/3}$ and $y(x) = 1+x^3$.
Panel 1: Verify DE.

$$\begin{aligned} \mathsf{LHS} &= y' \\ &= (1+x^3)' \\ &= 3x^2 \\ \mathsf{RHS} &= f(x,y) \\ &= f(x,1+x^3) \\ &= 3(x^3)^{2/3} \\ &= 3x^2 \qquad \text{DE verified.} \end{aligned}$$

Panel 2: Verify IC.

$$\begin{aligned} \mathsf{LHS} &= y(0) \\ &= (1+x^3)\big|_{x=0} \\ &= 1 \qquad \text{IC verified.} \end{aligned}$$

2. Let $y(x) = 1$ on $0 \le x \le 1$ and $y(x) = 1 + (x-1)^3$ for $x \ge 1$. Do an answer check for $y(x)$.

3. Does $f_y(x,y)$ exist for all (x,y)?

Solution: Let $f(x,y) = 3(y-1)^{2/3}$. Then $f_y(x,y) = 2(y-1)^{-1/3}$. There is a divide by zero error at $y = 1$. Answer: No, it does not exist for all $(x.y)$.

4. Verify that Picard's theorem does not apply to $y' = f(x,y)$, $y(0) = 1$, due to discontinuity of f_y.

5. Verify that Picard's theorem applies to $y' = f(x,y)$, $y(0) = 2$.

Solution: Let $f(x,y) = 3(y-1)^{2/3}$, $y(0) = 2$. Then f is everywhere continuous and $f_y(x,y) = 2(y-1)^{-1/3}$ is continuous near $y = 2$ (the initial condition is $y(0) = 2$). The hypotheses of Picard's theorem are satisfied. The theorem applies.

Be advised that *a theorem applies* just means that the theorem's hypotheses should be checked for validity. This particular question has been often misinterpreted, the question left unanswered while providing details for a closed-form solution found with calculus and differential equations methods. Such details do not check hypotheses, instead they find a formula for the solution, a question not asked.

6. Let $y(x) = 1 + (x+1)^3$. Do an answer check for $y' = f(x,y)$, $y(0) = 2$. Does another solution exist?

Discontinuous Equation Example

Consider $y' = \dfrac{2y}{x-1}$, $y(0) = 1$. Define $y_1(x) = (x-1)^2$ and $y_2(x) = c(x-1)^2$. Define $y(x) = y_1(x)$ on $-\infty < x < 1$ and $y(x) = y_2(x)$ on $1 < x < \infty$. Define $y(1) = 0$.

7. Do an answer check for $y_1(x)$ on $-\infty < x < 1$. Do an answer check for $y_2(x)$ on $1 < x < \infty$. Skip condition $y(0) = 1$.

Solution: Define $f(x,y) = \dfrac{2y}{x-1}$, $y(x) = (x-1)^2$.

Panel 1: Verify DE.

$$\begin{aligned}
\text{LHS} &= y' \\
&= ((x-1)^2)' \\
&= 2(x-1) && \text{Chain rule and power rule.} \\
\text{RHS} &= f(x,y) \\
&= f(x,(x-1)^2) \\
&= \frac{2(x-1)^2}{x-1} \\
&= 2(x-1) && \text{DE verified.}
\end{aligned}$$

Define $f(x,y) = \dfrac{2y}{x-1}$, $y(x) = c(x-1)^2$.

$$\begin{aligned}
\text{LHS} &= y' \\
&= (c(x-1)^2)' \\
&= 2c(x-1) && \text{Chain rule and constant rule.} \\
\text{RHS} &= f(x,y) \\
&= f(x,c(x-1)^2) \\
&= \frac{2c(x-1)^2}{x-1} \\
&= 2c(x-1) && \text{DE verified.}
\end{aligned}$$

8. Justify one-sided limits $y(1+) = y(1-) = 0$. The functions y_1 and y_2 join continuously at $x = 1$ with common value zero and the formula for $y(x)$ gives one continuous formal solution for each value of c (∞-many solutions).

9. (a) For which values of c does $y_2'(1)$ exist? (b) For which values of c is $y_2(x)$ continuously differentiable?

Solution: Define $y_2(x) = c(x-1)^2$. Then $y_2'(x) = 2c(x-1)$, which is continuous. Answer: For all values of constant c.

10. Find all values of c such that $y_2(x)$ is a continuously differentiable function that satisfies the differential equation and the initial condition.

Finite Blowup Example

Consider $y' = 1 + y^2$, $y(0) = 0$. Let $y(x) = \tan x$.

11. Do an answer check for $y(x)$.

Solution: Let $f(x, y = 1 + y^2$, $y(x) = \tan x$.
Panel 1: Verify DE.

$$\begin{aligned}
\text{LHS} &= y' \\
&= (\tan x)' \\
&= \sec^2 x && \text{Derivative table, calculus.} \\
\text{RHS} &= f(x,y) \\
&= f(x, \tan x) \\
&= 1 + \tan^2 x \\
&= \sec^2 x && \text{Trig identity. DE verified.}
\end{aligned}$$

Panel 2: Verify IC.

$$\begin{aligned}
\text{LHS} &= y(0) \\
&= (\tan x)\,|_{x=0} \\
&= \tan 0 \\
&= 0 && \text{Trig identity. IC verified.}
\end{aligned}$$

12. Find the partial derivative f_y for $f(x, y) = 1 + y^2$. Justify that f and f_y are everywhere continuous.

13. Justify that Picard's theorem applies, hence $y(x)$ is the only possible solution to the initial value problem.

Solution: Both $f(x, y) = 1 + y^2$ and its derivative $f_y(x, y) = 2y$ are everywhere continuous. The hypotheses of Picard's theorem are satisfied.

14. Justify for $a = -\pi/2$ and $b = \pi/2$ that $y(a+) = -\infty$, $y(b-) = \infty$. Hence $y(x)$ blows up for finite values of x.

Numerical Instability Example

Let $f(x, y) = y - 2e^{-x}$.

15. Do an answer check for $y(x) = e^{-x}$ as a solution of the initial value problem $y' = f(x, y)$, $y(0) = 1$.

Solution: Let $f(x, y) = y - 2e^{-x}$, $y(x) = e^{-x}$.
Panel 1: Verify DE.

$$\begin{aligned} \text{LHS} &= y' \\ &= (e^{-x})' \\ &= -e^{-x} && \text{Chain rule } (e^u)' = u'e^u. \\ \text{RHS} &= f(x, y) \\ &= f(x, e^{-x}) \\ &= e^{-x} - 2e^{-x} \\ &= -e^{-x} && \text{DE verified.} \end{aligned}$$

Panel 2: Verify IC.

$$\begin{aligned} \text{LHS} &= y(0) \\ &= (e^{-x})\,|_{x=0} \\ &= e^0 \\ &= 1 && \text{Exponential identity. IC verified.} \end{aligned}$$

16. Do an answer check for $y(x) = ce^x + e^{-x}$ as a solution of $y' = f(x, y)$.

Multiple Solutions

Consider the initial value problem $y' = 5(y - 2)^{4/5}$, $y(0) = 2$.

17. Do an answer check for $y(x) = 2$. Do a second answer check for $y(x) = 2 + x^5$.

Solution: The answer check for $y(x) = 2$ will be skipped, because it parallels the one supplied below.
Let $f(x, y) = 5(y - 2)^{4/5}$, $y(x) = 2 + x^5$.
Panel 1: Verify DE $y' = 5(y - 2)^{4/5}$.

$$\begin{aligned} \text{LHS} &= y' \\ &= (2 + x^5)' \\ &= 5x^4 && \text{Power rule.} \\ \text{RHS} &= f(x, y) \\ &= f(x, 2 + x^5) \end{aligned}$$

$$\begin{aligned} &= 5(2 + x^5 - 2)^{4/5} \\ &= 5x^4 && \text{DE verified.} \end{aligned}$$

Panel 2: Verify IC $y(0) = 2$.

$$\begin{aligned} \text{LHS} &= y(0) \\ &= (2 + x^5)\,|_{x=0} \\ &= 2 && \text{IC verified.} \end{aligned}$$

18. Verify that the hypotheses of Picard's theorem fail to apply.

19. Find a formula which displays infinitely many solutions to $y' = f(x, y)$, $y(0) = 2$.

Solution: The initial value problem is $y' = 5(y - 2)^{4/5}$, $y(0) = 2$. Define $y = 2$ on $-1 \le x \le c$ and $y = 2 + (x - c)^5$ for $x > c > 0$. By definition, $y(0) = 2$. Let's focus on verifying the DE. For $-1 \le x \le c$ with $c > 0$ the answer check is identical to exercise 17. For $x > c$, the details are:
Panel 1: Verify DE $y' = 5(y - 2)^{4/5}$.

$$\begin{aligned} \text{LHS} &= y' \\ &= (2 + (x - c)^5)' \\ &= 5(x - c)^4 && \text{Power rule.} \\ \text{RHS} &= f(x, y) \\ &= f(x, 2 + (x - c)^5) \\ &= 5(2 + (x - c)^5 - 2)^{4/5} \\ &= 5(x - c)^4 && \text{DE verified.} \end{aligned}$$

20. Verify that the hypotheses of Peano's theorem apply.

Solution: Suggestion: Use continuity of compositions of continuous functions.

Discontinuous Equation

Consider $y' = \dfrac{y}{x - 1}$, $y(0) = 1$. Define $y(x)$ piecewise by $y(x) = -(x - 1)$ on $-\infty < x < 1$ and $y(x) = c(x - 1)$ on $1 < x < \infty$. Leave $y(1)$ undefined.

21. Do an answer check for $y(x)$. The initial condition $y(0) = 1$ applies only to the domain $-\infty < x < 1$.

Solution: To be checked: $y' = \dfrac{y}{x - 1}$, $y(0) = 1$.

Part I.
Let $y(x) = -(x - 1)$ on $-\infty < x < 1$. Because $y(0) = -(0 - 1) = 1$, just the DE will be verified.

Panel 1: Verify DE $y' = \dfrac{y}{x - 1}$ for $y(x) = -(x - 1)$.

$$\begin{aligned}
\text{LHS} &= y' \\
&= (-(x-1))' \\
&= -1 && \text{Constant rule.} \\
\text{RHS} &= f(x,y) \\
&= f(x,-(x-1)) \\
&= \frac{-(x-1)}{x-1} \\
&= -1 && \text{DE verified.}
\end{aligned}$$

Part II.
Let $y(x) = c(x-1)$ on $1 < x < \infty$. Because $x = 0$ is not in the domain, just the DE will be verified.

Panel 1: Verify DE $y' = \dfrac{y}{x-1}$ for $y(x) = c(x-1)$.

$$\begin{aligned}
\text{LHS} &= y' \\
&= (c(x-1))' \\
&= c && \text{Constant rule.} \\
\text{RHS} &= f(x,y) \\
&= f(x,c(x-1)) \\
&= \frac{c(x-1)}{x-1} \\
&= c && \text{DE verified.}
\end{aligned}$$

22. Justify one-sided limits $y(1+) = y(1-) = 0$. The piecewise definitions of $y(x)$ join continuously at $x = 1$ with common value zero and the formula for $y(x)$ gives one continuous formal solution for each value of c (∞-many solutions).

23. (a) For which values of c does $y'(1)$ exist? (b) For which values of c is $y(x)$ continuously differentiable?

Solution: (a) All $x \neq 1$. (b) All $x \neq 1$.

24. Find all values of c such that $y(x)$ is a continuously differentiable function that satisfies the differential equation and the initial condition.

Picard Iteration

Find the Picard iterates y_0, y_1, y_2, y_3.

25. $y' = y + 1$, $y(0) = 2$

Solution: Answer:
$y_0 = 2$,

$y_1 = 2 + 3\,x,$
$y_2 = 2 + 3\,x + 3/2\,x^2,$
$y_3 = 2 + 3\,x + 3/2\,x^2 + 1/2\,x^3$

```
y0:=2:f:=(x,y)->y+1:# Maple
y1:=x->y0+int(f(t,y0),t=0..x):
y2:=x->y0+int(f(t,y1(t)),t=0..x):
y3:=x->y0+int(f(t,y2(t)),t=0..x):
u[0]:=y0;u[1]:=y1(x);u[2]:=y2(x);u[3]:=y3(x);
ANS:=seq(y[i]=u[i],i=0..3);# List of iterates
SOL:=u[0]+sum(u[i]-u[i-1],i=1..3);# approximates y(x)
#
# Test series solution against iterate SOL
de:=diff(y(x),x)=f(x,y(x)): ic:=y(0)=y0:
dsolve({de,ic},y(x)); dsolve({de,ic},y(x),series);
```

26. $y' = 2y + 1$, $y(0) = 0$

27. $y' = y^2$, $y(0) = 1$

Solution: Answer:
$y_0 = 1,$
$y_1 = x + 1,$
$y_2 = 2/3 + 1/3\,(x + 1)^3,$
$y_3 = 1 + x + \frac{x^7}{63} + 1/9\,x^6 + 1/3\,x^5 + 2/3\,x^4 + x^3 + x^2$
The exact solution is $1/(1 - x) = 1 + x + x^2 + x^3 + \cdots$. Picard iteration performs poorly on this example, requiring many iterations to obtain $1 + x + x^2 + x^3 + x^4$ in the expansion SOL.

```
y0:=1:f:=(x,y)->y^2:# Maple
y1:=x->y0+int(f(t,y0),t=0..x):
y2:=x->y0+int(f(t,y1(t)),t=0..x):
y3:=x->y0+int(f(t,y2(t)),t=0..x):
u[0]:=y0;u[1]:=y1(x);u[2]:=y2(x);u[3]:=y3(x);
ANS:=seq(y[i]=u[i],i=0..3);# List of iterates
SOL:=u[0]+sum(u[i]-u[i-1],i=1..3);# approximates y(x)
```

28. $y' = y^2$, $y(0) = 2$

29. $y' = y^2 + 1$, $y(0) = 0$

Solution: Answer: The exact solution is $y(x) = \tan x$. Iterates:
$y_0 = 0,$
$y_1 = x,$
$y_2 = 1/3\,x^3 + x,$
$y_3 = x + \frac{x^7}{63} + 2/15\,x^5 + 1/3\,x^3$

```
y0:=0:f:=(x,y)->y^2+1:# Maple
y1:=x->y0+int(f(t,y0),t=0..x):
y2:=x->y0+int(f(t,y1(t)),t=0..x):
y3:=x->y0+int(f(t,y2(t)),t=0..x):
u[0]:=y0;u[1]:=y1(x);u[2]:=y2(x);u[3]:=y3(x);
ANS:=seq(y[i]=u[i],i=0..3);# List of iterates
SOL:=u[0]+sum(u[i]-u[i-1],i=1..3);# approximates y(x)
```

30. $y' = 4y^2 + 4$, $y(0) = 0$

31. $y' = y + x$, $y(0) = 0$

Solution: Answer:
$y_0 = 0$,
$y_1 = 1/2\,x^2$,
$y_2 = 1/2\,x^2 + 1/6\,x^3$,
$y_3 = 1/2\,x^2 + 1/24\,x^4 + 1/6\,x^3$

```
y0:=0:f:=(x,y)->y+x:# Maple
y1:=x->y0+int(f(t,y0),t=0..x):
y2:=x->y0+int(f(t,y1(t)),t=0..x):
y3:=x->y0+int(f(t,y2(t)),t=0..x):
u[0]:=y0;u[1]:=y1(x);u[2]:=y2(x);u[3]:=y3(x);
ANS:=seq(y[i]=u[i],i=0..3);# List of iterates
SOL:=u[0]+sum(u[i]-u[i-1],i=1..3);# approximates y(x)
```

32. $y' = y + 2x$, $y(0) = 0$

Picard Iteration and Taylor Series

Find the Taylor polynomial $P_n(x) = y(0) + y'(0)x + \cdots + y^{(n)}(0)x^n/n!$ and compare with the Picard iterates. Use a computer algebra system, if possible.

33. $y' = y$, $y(0) = 1$, $n = 4$,
$y(x) = e^x$

Solution: Answer: Taylor polynomial $(1/2)x^2 + (1/6)x^3 + (1/24)x^4 + (1/120)x^5 + O(x^6)$

Solution from the iterates $1/6\,x^3 + \frac{x^5}{120} + 1/24\,x^4 + 1/2\,x^2$, which matches the Taylor polynomial except for ordering of terms.

```
y0:=1:f:=(x,y)->y:# Maple
y1:=x->y0+int(f(t,y0),t=0..x):
y2:=x->y0+int(f(t,y1(t)),t=0..x):
y3:=x->y0+int(f(t,y2(t)),t=0..x):
u[0]:=y0;u[1]:=y1(x);u[2]:=y2(x);u[3]:=y3(x);
ANS:=seq(y[i]=u[i],i=0..3);# List of iterates
SOL:=u[0]+sum(u[i]-u[i-1],i=1..3);# approximates y(x)
taylor(exp(x),x=0,4);
```

34. $y' = 2y$, $y(0) = 1$, $n = 4$,
$y(x) = e^{2x}$

35. $y' = x - y$, $y(0) = 1$, $n = 4$,
$y(x) = -1 + x + 2e^{-x}$

Solution: Answer: Taylor polynomial $1 - x + x^2 - (1/3)x^3 + O(x^4)$
Solution from the iterates $1 + x^2 - x - (1/3)x^3 + (1/24) * x^4$, which matches the Taylor polynomial except for ordering of terms.

```
y0:=1:f:=(x,y)->x-y:# Maple
y1:=x->y0+int(f(t,y0),t=0..x):
y2:=x->y0+int(f(t,y1(t)),t=0..x):
y3:=x->y0+int(f(t,y2(t)),t=0..x):
u[0]:=y0;u[1]:=y1(x);u[2]:=y2(x);u[3]:=y3(x);
ANS:=seq(y[i]=u[i],i=0..3);# List of iterates
SOL:=u[0]+sum(u[i]-u[i-1],i=1..3);# approximates y(x)
taylor(-1 + x + 2*exp(-x),x=0,4);
```

36. $y' = 2x - y$, $y(0) = 1$, $n = 4$,
$y(x) = -2 + 2x + 3e^{-x}$

Numerical Instability

Use a computer algebra system or numerical laboratory. Let $f(x, y) = y - 2e^{-x}$.

37. Solve $y' = f(x, y)$, $y(0) = 1$ numerically for $y(30)$.

Solution: Answer: At $x = 30$, $y(x) = -1533016.91678766$, which is about 1.5 million.

```
y0:=1:f:=(x,y)->y-2*exp(-x):# Maple
de:=diff(y(x),x)=f(x,y(x)): ic:=y(0)=y0:
Y:=dsolve({de,ic},y(x),numeric): Y(30);
```

38. Solve $y' = f(x, y)$, $y(0) = 1 + 0.0000001$ numerically for $y(30)$.

Solution: At $x = 30$, $y(x) = -464432.443214007$, which is about 0.5 million.

Closed–Form Existence

Solve these initial value problems using a computer algebra system.

39. $y' = y$, $y(0) = 1$

Solution: Answer: $y = e^x$.

```
de:=diff(y(x),x)=y(x);ic:=y(0)=1;# Maple
dsolve([de,ic],y(x));
```

40. $y' = 2y$, $y(0) = 2$

41. $y' = 2y + 1$, $y(0) = 1$

Solution: Answer: $y = -\frac{1}{2} + \frac{3}{2}e^{2x}$.

```
de:=diff(y(x),x)=2*y(x)+1;ic:=y(0)=1;# Maple
dsolve([de,ic],y(x));
```

42. $y' = 3y + 2$, $y(0) = 1$

43. $y' = y(y-1)$, $y(0) = 2$

Solution: Answer: $y = \dfrac{2}{2 - e^x}$.

```
de:=diff(y(x),x)=y(x)*(y(x)-1);ic:=y(0)=2;# Maple
dsolve([de,ic],y(x));
```

44. $y' = y(1-y)$, $y(0) = 2$

45. $y' = (y-1)(y-2)$, $y(0) = 3$

Solution: Answer: $y = \dfrac{e^x - 4}{e^x - 2}$.

```
de:=diff(y(x),x)=(y(x)-1)*(y(x)-2);ic:=y(0)=3;# Maple
dsolve([de,ic],y(x));
```

46. $y' = (y-2)(y-3)$, $y(0) = 1$

47. $y' = -10(1-y)$, $y(0) = 0$

Solution: Answer: $y = 1 - e^{10x}$.

```
de:=diff(y(x),x)=(-10)*(1-y(x));ic:=y(0)=0;# Maple
dsolve([de,ic],y(x));
```

48. $y' = -10(2-3y)$, $y(0) = 0$

Lipschitz Condition

Justify the following results.

49. The function $f(x,y) = x - 10(2-3y)$ satisfies a Lipschitz condition on the whole plane.

Solution: $f(x,y_1) - f(x,y_2) = 30(y_1 - y_2)$ implies $|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2|$ for $M = 30$.

50. The function $f(x,y) = ax + by + c$ satisfies a Lipschitz condition on the whole plane.

51. The function $f(x,y) = xy(1-y)$ satisfies a Lipschitz condition on $D = \{(x,y) : |x| \le 1, \quad |y| \le 1\}$.

Solution: Details using the triangle inequality:

$$\begin{aligned}|f(x,y_1) - f(x,y_2)| &= |x||(y_2 - y_1)(y_2 + y_1) - (y_2 - y_1)| \\ &\le |x|(|y_2| + |y_1| + 1)|y_2 - y_1)| \\ &\le 1 \cdot (1 + 1 + 1)|y_2 - y_1)| \\ &= M|y_2 - y_1)| \text{ for } M = 3.\end{aligned}$$

52. The function $f(x,y) = x^2y(a - by)$ satisfies a Lipschitz condition on $D = \{(x,y) : x^2 + y^2 \le R^2\}$.

53. If f_y is continuous on D and the line segment from (x, y_1) to (x, y_2) is in D, then $f(x, y_1) - f(x, y_2) = \int_{y_1}^{y_2} f_y(x, u)du$.

Solution: Let $G(y) = f_y(x, y)$ for this fixed value of x, $y_1 \le y \le y_2$. Then G is a continuous function of y and the integral $\int_{y_1}^{y_2} G(y)dy$ is defined. Further, $G(y) = F'(y)$ where $F(y) = f(x, y)$. Apply the fundamental theorem of calculus: $\int_{y_1}^{y_2} F'(y)dy = F(y_1) - F(y_1) = f(x, y_1) - f(x, y_1)$. ■

54. If f and f_y are continuous on a disk D, then f is Lipschitz with $M = \max_D\{|f_y(x, u)|\}$.

Chapter 2

First Order Differential Equations

Contents

2.1 Quadrature Method

Quadrature

Find a candidate solution for each initial value problem and verify the solution. See Example 2.1 and Example 2.2, page 76.

1. $y' = 4e^{2x}$, $y(0) = 0$.

Solution: Answer: $y(x) = -2 + 2\,\mathrm{e}^{2\,x}$.
Solution steps:

$y' = 4e^{2x}$	Given DE.
$\int y'dx = \int 4e^{2x}dx$	Method of quadrature: Multiply by dx across the DE and integrate.
$y(x) + c_1 = \int 4e^{2x}dx$	FTC left, c_1=constant.
$y(x) + c_1 = 4\int e^{2x}dx$	Constant rule.
$y(x) + c_1 = \dfrac{4e^{2x}}{2} + c_2$	Integral table, c_2=constant.
$y(x) = 2e^{2x} + c$	Isolate y left, c=$c_2 - c_1$=constant.
$0 = y(0) = 2e^0 + c$	Substitute $x = 0$. Solve for $c = -2$.
$y(x) = 2e^{2x} - 2$	Isolate y left, c=$c_2 - c_1$=constant. Candidate solution.

It remains to do an answer check. For illustration, `maple` will be used to verify the solution instead of a handwritten 2-panel answer check.

```
y0:=0;F:=x->4*exp(2*x); y=y0+int(F(t),t=0..x);
# ANS := -2+2*exp(2*x)
```

2. $y' = 2e^{4x}$, $y(0) = 0$.

3. $(1 + x)y' = x$, $y(0) = 0$.

Solution: Answer: $y(x) = x - \ln(x + 1)$

```
y0:=0;F:=x->x/(1+x);
ANS:=y0+int(F(t),t=0..x) assuming x > -1;
# ANS := x - ln(x + 1)
```

4. $(1 - x)y' = x$, $y(0) = 0$.

5. $y' = \sin 2x$, $y(0) = 1$.

Solution: Answer: $y(x) = 3/2 - 1/2\,\cos(2\,x)$

```
y0:=1;F:=x->sin(2*x); ANS:=y0+int(F(t),t=0..x);
# ANS := 3/2-(1/2)*cos(2*x)
```

6. $y' = \cos 2x$, $y(0) = 1$.

7. $y' = xe^x$, $y(0) = 0$.

Solution: Answer: $y(x) = \mathrm{e}^x x - \mathrm{e}^x + 1$

```
y0:=0;F:=x->x*exp(x); ANS:=y0+int(F(t),t=0..x);
# ANS := exp(x)*x-exp(x)+1
```

8. $y' = xe^{-x^2}$, $y(0) = 0$.

9. $y' = \tan x$, $y(0) = 0$.

Solution: Answer: $y(x) = -\ln(\cos(x))$

```
y0:=0;F:=x->tan(x);
ANS:=y0+int(F(t),t=0..x) assuming x>0 and x < Pi/2;
# ANS := -ln(cos(x))
```

10. $y' = 1 + \tan^2 x$, $y(0) = 0$.

11. $(1 + x^2)y' = 1$, $y(0) = 0$.

Solution: Answer: $y(x) = \arctan(x)$

```
y0:=0;F:=x->1/(1+x^2); ANS:=y0+int(F(t),t=0..x);
# ANS := arctan(x)
```

12. $(1 + 4x^2)y' = 1$, $y(0) = 0$.

13. $y' = \sin^3 x$, $y(0) = 0$.

Solution: Answer: $y(x) = 2/3 - 1/3\ (\sin(x))^2 \cos(x) - 2/3\ \cos(x)$
Integration uses the trig identity $\sin^3(x) = \sin(x)(1 - \cos^2(x))$, obtained from $\cos^2(x) + \sin^2(x) = 1$.

```
y0:=0;F:=x->sin(x)^3; ANS:=y0+int(F(t),t=0..x);
# ANS := 2/3-(1/3)*sin(x)^2*cos(x)-(2/3)*cos(x)
```

14. $y' = \cos^3 x$, $y(0) = 0$.

15. $(1 + x)y' = 1$, $y(0) = 0$.

Solution: Answer: $y(x) = \ln(x + 1)$

```
y0:=0;F:=x->1/(1+x); ANS:=y0+int(F(t),t=0..x) assuming 1+x>0;
# ANS := ln(x+1)
```

16. $(2 + x)y' = 2$, $y(0) = 0$.

17. $(2 + x)(1 + x)y' = 2$, $y(0) = 0$.

Solution: Answer: $y(x) = 2\ \ln(2) - 2\ \ln(2 + x) + 2\ \ln(x + 1)$

```
y0:=0;F:=x->2/(1+x)/(2+x);
ANS:=y0+int(F(t),t=0..x) assuming 1+x>0 and 2+x > 0;
# ANS := 2*ln(2)-2*ln(2+x)+2*ln(x+1)
```

18. $(2 + x)(3 + x)y' = 3$, $y(0) = 0$.

19. $y' = \sin x \cos 2x$, $y(0) = 0$.

Solution: Answer: $y(x) = -1/3 - 1/6\cos(3x) + 1/2\cos(x)$

```
y0:=0;F:=x->sin(x)*cos(2*x); ANS:=y0+int(F(t),t=0..x);
# ANS := -1/3-(1/6)*cos(3*x)+(1/2)*cos(x)
```

20. $y' = (1 + \cos 2x)\sin 2x$, $y(0) = 0$.

River Crossing

A boat crosses a river of width w miles at v_b miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is v_c miles per hour. Find the transit time and the downstream drift to the opposite shore. See Example 2.3, page 78, and the details for (6).

21. $w = 1$, $v_b = 4$, $v_c = 12$

Solution: The simplest solution uses the equation in Example 2.3:

$$y(x) = \frac{4v_c}{v_b w^2}\left(-\frac{1}{3}x^3 + \frac{1}{2}wx^2\right). \tag{1}$$

Then

$$y(x) = \frac{4(12)}{4(1^2)}\left(-\frac{1}{3}x^3 + \frac{1}{2}4x^2\right).$$

The transit time is $1/v_b$ hours or 15 minutes.

The downstream drift is $y(1/v_b) = y(0.25) = \dfrac{4(12)}{4(1^2)}\left(-\dfrac{1}{3}\dfrac{1}{4^3} + \dfrac{1}{2}\dfrac{4}{4^2}\right) =$ 0.3125 miles.

```
Y:=(x,w,vb,vc) -> (4*vc)/(vb* w^2)*(-1/3* x^3+ 1/2* w*x^2 );
# DRIFT := Y(0.25,1,4,12) = 0.3125000000 or 1650 feet
```

22. $w = 1$, $v_b = 5$, $v_c = 15$

23. $w = 1.2$, $v_b = 3$, $v_c = 13$

Solution: The transit time is $1/v_b$ hours or 20 minutes.
The downstream drift is $y(1/v_b) = y(1/3) = 0.65386374$ miles or 3452.40 feet.

```
Y:=(x,w,vb,vc) -> (4*vc)/(vb*w^2)*(-1/3*x^3+1/2*w*x^2 );
# TRANSIT := 1/3.0; DRIFT := Y(1/3.0,1.2,3,13);
```

24. $w = 1.2$, $v_b = 5$, $v_c = 9$

25. $w = 1.5$, $v_b = 7$, $v_c = 16$

Solution: The transit time is $1/v_b$ hours or 8.57 minutes.
The downstream drift is $y(1/v_b) = y(1/7) = 0.05824733528$ miles or 307.55

feet.

```
Y:=(x,w,vb,vc) -> (4*vc)/(vb* w^2)*(-1/3* x^3+ 1/2* w*x^2 );
# TRANSIT := 1/7.0; DRIFT := Y(1/7.0,1.5,7,16);
```

26. $w = 2$, $v_b = 7$, $v_c = 10$

27. $w = 1.6$, $v_b = 4.5$, $v_c = 14.7$

Solution: The transit time is $1/v_b$ hours or 13.33 minutes.
The downstream drift is $y(1/v_b) = y(1/4.5) = 0.1176268861$ miles or 621.07 feet.

```
Y:=(x,w,vb,vc) -> (4*vc)/(vb* w^2)*(-1/3* x^3+ 1/2* w*x^2 );
# TRANSIT := 1/4.5; DRIFT := Y(1/4.5,1.6,7,14.7);
```

28. $w = 1.6$, $v_b = 5.5$, $v_c = 17$

Fundamental Theorem I

Verify the identity. Use the fundamental theorem of calculus part (b), page 75.

29. $\int_0^x (1+t)^3 dt = \frac{1}{4}\left((1+x)^4 - 1\right)$.

Solution: Let $F(x) = \frac{1}{4}\left((1+x)^4 - 1\right)$. It is enough to verify $F'(x) = (1+x)^3$, because the FTC gives $\int_0^x F'(t) = F(x) - F(0)$, which equals $F(x)$ because $F(0) = 0$. Details:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{1}{4}\left((1+x)^4 - 1\right)\right) \\ &= \frac{1}{4}\left(4(1+x)^3 - 0\right) \\ &= (1+x)^3. \end{aligned}$$

30. $\int_0^x (1+t)^4 dt = \frac{1}{5}\left((1+x)^5 - 1\right)$.

31. $\int_0^x te^{-t} dt = -xe^{-x} - e^{-x} + 1$.

Solution: As in exercise 29, it suffices to show that (RHS)$'$ = integrand.

32. $\int_0^x te^t dt = xe^x - e^x + 1$.

Fundamental Theorem II

Differentiate. Use the fundamental theorem of calculus part (b), page 75.

33. $\int_0^{2x} t^2 \tan(t^3) dt$.

Solution: The chain rule is required. Define $G(u) = \int_0^u t^2 \tan(t^3) dt$ and $u = 2x$. Then the integral $I = G(2x)$ and

$$\begin{aligned} \frac{dI}{dx} &= \frac{dG}{du}\frac{du}{dx} \\ &= u^2 \tan(u)(2x)' \\ &= 4x^2 \tan(2x)(2) = 8x^2 \tan(2x). \end{aligned}$$

34. $\int_0^{3x} t^3 \tan(t^2)dt$.

35. $\int_0^{\sin x} te^{t+t^2}dt$.

Solution: $\cos x \left(te^{t+t^2}\right)|_{t=\sin x} = (\cos x \sin x)e^{\sin x + \sin^2 x}$

36. $\int_0^{\sin x} \ln(1+t^3)dt$.

Fundamental Theorem III

Integrate $\int_0^1 f(x)dx$. Use the fundamental theorem of calculus part (a), page 75. Check answers with computer or calculator assist. Some require a clever u-substitution or an integral table.

37. $f(x) = x(x-1)$

Solution: Expand $f(x) = x^2 - x$, then $f'(x) = 2x - 1$.

38. $f(x) = x^2(x+1)$

39. $f(x) = \cos(3\pi x/4)$

Solution: The chain rule applies: $f'(x) = -\sin(3\pi x/4)\dfrac{3\pi}{4}$

40. $f(x) = \sin(5\pi x/6)$

41. $f(x) = \dfrac{1}{1+x^2}$

Solution: Power and chain rules apply:

$$\begin{aligned} f'(x) &= \left((1+x^2)^{-1}\right)' \\ &= (-1)(1+x^2)^{-2}(1+x^2)' \\ &= (-2x)(1+x^2)^{-2} \end{aligned}$$

42. $f(x) = \dfrac{2x}{1+x^4}$

43. $f(x) = x^2e^{x^3}$

Solution: Power rule, product rule, exponential rule, chain rule.
$f'(x) = 2\,x\mathrm{e}^{x^3} + 3\,x^4\mathrm{e}^{x^3}$

```
F:=x->x^2*exp(x^3);# Maple
ANS:=diff(F(x),x);
# ANS := 2*x*exp(x^3)+3*x^4*exp(x^3)
```

44. $f(x) = x(\sin(x^2) + e^{x^2})$

45. $f(x) = \dfrac{1}{\sqrt{-1+x^2}}$

Solution: Power rule, quotient rule, chain rule.

$f'(x) = -\dfrac{x}{\left(x^2-1\right)^{3/2}}$

```
F:=x->1/sqrt(x^2 - 1);# Maple
ANS:=diff(F(x),x);
# ANS := -x/(x^2-1)^(3/2)
```

46. $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

47. $f(x) = \dfrac{1}{\sqrt{1+x^2}}$

Solution: Power rule, chain rule.

$f'(x) = -\dfrac{x}{\left(x^2+1\right)^{3/2}}$

```
F:=x->1/sqrt(x^2 + 1);# Maple
ANS:=diff(F(x),x);
# ANS := -x/(x^2+1)^(3/2)
```

48. $f(x) = \dfrac{1}{\sqrt{1+4x^2}}$

49. $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

Solution: Power rule, quotient rule, chain rule.

$f'(x) = \dfrac{1}{\sqrt{x^2+1}} - \dfrac{x^2}{\left(x^2+1\right)^{3/2}}$

```
F:=x->x/sqrt(x^2 + 1);# Maple
ANS:=diff(F(x),x);
# ANS := 1/sqrt(x^2+1)-x^2/(x^2+1)^(3/2)
```

50. $f(x) = \dfrac{4x}{\sqrt{1-4x^2}}$

51. $f(x) = \dfrac{\cos x}{\sin x}$

Solution: Because $f(x) = \cot x$, then $f'(x) = -\csc^2 x$ from integral tables. Computer algebra systems give $-1 - \dfrac{\cos^2 x}{\sin^2 x}$ which equals $-1 - \cot^2 x$. Trig identity $1 + \cot^2 x = \csc^2 x$ explains the form of the answer from integral tables.

52. $f(x) = \dfrac{\cos x}{\sin^3 x}$

53. $f(x) = \dfrac{e^x}{1+e^x}$

Solution: Exponential rule, quotient rule.
$f'(x) = \dfrac{\mathrm{e}^x}{(1+\mathrm{e}^x)^2}$

```
F:=x->F:=x->exp(x)/(1+exp(x));# Maple
ANS:=diff(F(x),x);
# ANS := exp(x)/(1+exp(x))-(exp(x))^2/(1+exp(x))^2
```

54. $f(x) = \dfrac{\ln|x|}{x}$

55. $f(x) = \sec^2 x$

Solution: Power rule, chain rule.
$f'(x) = 2\sec(x)\sec(x)\tan(x)$

56. $f(x) = \sec^2 x - \tan^2 x$

57. $f(x) = \csc^2 x$

Solution: Power rule, chain rule.
$f'(x) = -2\csc(x)\csc(x)\cot(x)$

58. $f(x) = \csc^2 x - \cot^2 x$

59. $f(x) = \csc x \cot x$

Solution: Product rule.
$f'(x) = -\csc x \cot x \cot x - \csc x \csc^2 x$

60. $f(x) = \sec x \tan x$

Integration by Parts

Integrate $\int_0^1 f(x)dx$ by parts, $\int udv = uv - \int vdu$. Check answers with computer or calculator assist.

61. $f(x) = xe^x$

Solution: Let $u = x$, $dv = e^x dx$. Then $du = dx$, $v = -e^{-x}$. Parts gives

$$\begin{aligned}\int xe^x dx &= \int udv \\ &= uv - \int vdu \\ &= -xe^{-x} - \int -e^{-x}dx \\ &= -xe^{-x} - e^{-x} + c\end{aligned}$$

The answer is checked by differentiation:
$(-xe^{-x} - e^{-x} + c)' = -e^{-x} + xe^{-x} + e^{-x} = xe^{-x}$

62. $f(x) = xe^{-x}$

63. $f(x) = \ln|x|$

Solution: Let $u = \ln x$, $dv = dx$.

$\int f(x)dx = x\ln(x) - x$

64. $f(x) = x\ln|x|$

65. $f(x) = x^2e^{2x}$

Solution: Let $u = x^2$, $dv = e^{2x}dx$.

$\int f(x)dx = \frac{1}{4}(2x^2 - 2x + 1)e^{2x}$

66. $f(x) = (1 + 2x)e^{2x}$

67. $f(x) = x\cosh x$

Solution: Let $u = x$, $dv = \cosh(x)dx$. Then $v = \sinh x$.

$\int f(x)dx = x\sinh(x) - \cosh(x)$

68. $f(x) = x\sinh x$

69. $f(x) = x\arctan(x)$

Solution: Let $u = x$, $dv = \arctan(x)dx$. Then $v = \dfrac{1}{1+x^2}$.

$\int f(x)dx = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x)$

70. $f(x) = x\arcsin(x)$

Partial Fractions

Integrate f by partial fractions. Check answers with computer or calculator assist.

71. $f(x) = \dfrac{x+4}{x+5}$

Solution: Long division applies: $f(x) = 1 + \dfrac{-4}{x+5}$. Then integration is from tables:

$\int f(x)dx = x - 4\ln|x+5| + c$.

72. $f(x) = \dfrac{x-2}{x-4}$

73. $f(x) = \dfrac{x^2+4}{(x+1)(x+2)}$

Solution: Long division and partial fractions applies:
$f(x) = 1 - 8/(x+2) + 5/(x+1)$, then from integral tables
$\int f(x)dx = x + 5\,\ln(x+1) - 8\,\ln(x+2) + c$

The partial fraction steps:
1. Expand the denominator $(x+1)(x+2)$ into x^2+3x+2 and then perform long division:

$$f(x) = QUO + \frac{REM}{DENOM} = 1 + \frac{2-3x}{(x+1)(x+2))}$$

2. Expand REM/DENOM in partial fractions:

$$\frac{2-3x}{(x+1)(x+2))} = \frac{a}{x+1} + \frac{b}{x+2}$$

3. Clear fractions: multiply by DENOM.
4. Match coefficients to get equations for a, b, then solve for $a = 5$, $b = -8$.

```
F:=x->(x^2 + 4)/((x+1)*(x+2));# Maple
ANS:=int(F(x),x);
FRACTIONS:=convert(F(x),parfrac);
bot:=denom(F(x));top:=numer(F(x));
QUO:=quo(top,bot,x);REM:=rem(top,bot,x);
# QUO := 1,  REM := 2-3*x
# ANS := x+5*ln(x+1)-8*ln(x+2)
# FRACTIONS := 1-8/(x+2)+5/(x+1)
```

74. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)}$

75. $f(x) = \dfrac{x+4}{(x+1)(x+2)}$

Solution: Partial fractions applies:
$f(x) = -2/(x+2) + 3/(x+1)$, then from integral tables
$\int f(x)dx = 3\,\ln(x+1) - 2\,\ln(x+2) + c$

```
F:=x->(x+4)/((x+1)*(x+2));# Maple
ANS:=int(F(x),x);
FRACTIONS:=convert(F(x),parfrac);
# ANS := 3*ln(x+1)-2*ln(x+2)
# FRACTIONS := -2/(x+2)+3/(x+1)
```

76. $f(x) = \dfrac{x-1}{(x+1)(x+2))}$

77. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x+5)}$

Solution: Partial fractions applies:
$f(x) = \dfrac{-2/3}{x+2} + \dfrac{-1/12}{x+5} + \dfrac{3/4}{x+1}$, then from integral tables
$\int f(x)dx = 3/4 \ln(x+1) - 2/3 \ln(x+2) - 1/12 \ln(x+5) + c$

```
F:=x->(x+4)/((x+1)*(x+2)*(x+5));# Maple
ANS:=int(F(x),x);
FRACTIONS:=convert(F(x),parfrac);
# ANS := (3/4)*ln(x+1)-(2/3)*ln(x+2)-(1/12)*ln(x+5)
# FRACTIONS := -2/(3*(x+2))-1/(12*(x+5))+3/(4*(x+1))
```

78. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x+3)}$

79. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x-1)}$

Solution: Partial fractions applies:
$f(x) = \dfrac{2/3}{x+2} + \dfrac{5/6}{x-1} + \dfrac{-3/2}{x+1}$, then from integral tables
$\int f(x)dx = 2/3 \ln(x+2) + 5/6 \ln(x-1) - 3/2 \ln(x+1) + c$

```
F:=x->(x+4)/((x+1)*(x+2)*(x-1));# Maple
ANS:=int(F(x),x);
FRACTIONS:=convert(F(x),parfrac);
# ANS := (5/6)*ln(x-1)-(3/2)*ln(x+1)+(2/3)*ln(x+2)
# FRACTIONS := 2/(3*(x+2))-3/(2*(x+1))+5/(6*(x-1))
```

80. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x-1)}$

Special Methods

Integrate f by using the suggested u-substitution or method. Check answers with computer or calculator assist.

81. $f(x) = \dfrac{x^2+2}{(x+1)^2}$, $u = x+1$.

Solution: Answer: $\int f(x)dx = x - 2\ln(x+1) - 3\,(x+1)^{-1}$

Let $u = x+1$. Then $x = u-1$ and

$$f(x) = \frac{(u-1)^2+2}{(u)^2} = \frac{u^2-2u+3}{u^2}$$

$$= 1 - 2u^{-1} + 3u^{-2}$$

$$\int f(x)dx = \int (1 - 2u^{-1} + 3u^{-2})du$$

$$= u - 2\ln|u| - 3u^{-1} + c$$

$$= x + 1 - 2\ln|x+1| - 3/(x+1) + c$$

```
F:=x->(x^2+2)/(x+1)^2;# Maple
ANS:=int(F(x),x);
# ANS := (5/6)*ln(x-1)-(3/2)*ln(x+1)+(2/3)*ln(x+2)
```

82. $f(x) = \dfrac{x^2+2}{(x-1)^2}$, $u = x - 1$.

Solution: $\int f(x)dx = x + 2\ \ln(x-1) - 3\ (x-1)^{-1} + c$

83. $f(x) = \dfrac{2x}{(x^2+1)^3}$, $u = x^2 + 1$.

Solution: Let $u = x^2 + 1$. Then $du = 2xdx$:

$$f(x) = \frac{du}{u^3}$$

$$= u^{-3}du$$

$$\int f(x)dx = \int u^{-3}du$$

$$= u^{-2}/(-2) + c$$

$$= -\frac{1}{2}\frac{1}{(x^2+1)^2} + c$$

```
F:=x->(2*x)/(x^2+1)^3;# Maple
ANS:=int(F(x),x);
# ANS := -1/(2*(x^2+1)^2)
```

84. $f(x) = \dfrac{3x^2}{(x^3+1)^2}$, $u = x^3 + 1$.

85. $f(x) = \dfrac{x^3+1}{x^2+1}$, use long division.

Solution: Long division:

$$f(x) = \frac{x^3+1}{x^2+1}$$

$$= x + \frac{1-x}{x^2+1}$$

$$= x + \frac{1}{x^2+1} + \frac{-x}{x^2+1}$$

$$= x + \frac{1}{x^2+1} + \frac{-du/2}{u}, \quad \text{where } u = x^2+1$$

$$\textstyle\int f(x) = \int x dx + \int \frac{dx}{x^2+1} + \frac{-1}{2}\int \frac{du}{u}$$

$$= x^2/2 + \arctan(x) + \frac{-1}{2}\ln|u| + c$$

$$= x^2/2 + \arctan(x) + \frac{-1}{2}\ln|x^2+1| + c$$

```
F:=x->(x^3+ 1)/(x^2 + 1);# Maple
ANS:=int(F(x),x);
FRACTIONS:=convert(F(x),parfrac);
bot:=denom(F(x));top:=numer(F(x));
QUO:=quo(top,bot,x);REM:=rem(top,bot,x);
# ANS := (1/2)*x^2-(1/2)*ln(x^2+1)+arctan(x)
# FRACTIONS := x+(1-x)/(x^2+1)
```

86. $f(x) = \dfrac{x^4+2}{x^2+1}$, use long division.

2.2 Separable Equations

Separated Form Test

Test the given equation by the separated form test on page 83.

Report whether or not the equation *passes* or *fails*, as written. In this test, algebraic operations on the equation are disallowed. See Examples 2.4 and 2.5, page 86.

1. $y' = 2$

Solution: Passes. The left side has x absent and y' is a factor. The right side has y and y' absent.

2. $y' = x$

3. $y' + y = 2$

Solution: Fails. Left side fails to have factor y'.

4. $y' + 2y = x$

5. $yy' = 2 - x$

Solution: Passes.

6. $2yy' = x + x^2$

7. $yy' + \sin(y') = 2 - x$

Solution: Fails. Left side fails to have factor y'.

8. $2yy' + \cos(y) = x$

9. $2yy' = y' \cos(y) + x$

Solution: Fails. The right side contains y (and also y').

10. $(2y + \tan(y))y' = x$

Separated Equation

Determine the separated form $y'/G(y) = F(x)$ for the given separable equation. See Example 2.6, page 86.

11. $(1 + x)y' = 2 + y$

Solution: $\dfrac{y'}{2+y} = \dfrac{1}{1+x}$

12. $(1+y)y' = xy$

13. $y' = \dfrac{x+xy}{(x+1)^2-1}$

Solution: $\dfrac{y'}{1+y} = \dfrac{x}{(x+1)^2-1}$

14. $y' = \sin(x)\dfrac{1+y}{(x+2)^2-4}$

15. $xy' = y\sin(y)\cos(x)$

Solution: $\dfrac{y'}{y\sin(y)} = \dfrac{\cos(x)}{x}$

16. $x^2y' = y\cos(y)\tan(x)$

17. $y^2(x-y)y' = \dfrac{x^2-y^2}{x+y}$

Solution: Factor: $\dfrac{x^2-y^2}{x+y} = \dfrac{(x-y)(x+y)2}{x+y}$. Cancel like factors on the right. Then divide to get separated form $y^2y' = 1$.

18. $xy^2(x+y)y' = \dfrac{y^2-x^2}{x-y}$

19. $xy^2y' = \dfrac{y-x}{x-y}$

Solution: Cancel like factors on the right, then divide by x to get separated form $y^2y' = \dfrac{-1}{x}$

20. $xy^2y' = \dfrac{x^2-xy}{x-y}$

Equilibrium solutions

Determine the equilibria for the given equation. See Examples 2.7 and 2.9.

21. $y' = xy(1+y)$

Solution: Let $f(x,y) = xy(1+y)$. Substitute $y = c$ in equation $f(x,y) = 0$ to get $xc(1+c) = 0$. Cancel x, legal because x can be any number, e.g., $x = 1$. Solve for $c = 0$, $c = -1$. Substitute the answers for c back into the substitution $y = c$. Report the equilibria as $y = 0$ and $y = -1$

22. $xy' = y(1-y)$

23. $y' = \dfrac{1+y}{1-y}$

Solution: Let $f(x,y) = \frac{1+y}{1-y}$. Equilibria: $y = 1$. The often-reported *answer* $y = -1$ is a singular value, not an equilibrium: $y = -1$ makes $f(x,y) =$ infinity, not zero.

24. $xy' = \dfrac{y(1-y)}{1+y}$

25. $y' = (1+x)\tan(y)$

Solution: Equilibria: $y = n\pi$ for $n =$ any integer. The often-reported expression $x = -1$ is not an equilibrium. All equilibria have form $y =$ constant. Equation $y = c$ is required to be a solution, that is, $y = c$ passes a formal answer check. In the answer check, x is allowed to be any value.

26. $y' = y(1 + \ln y)$

27. $y' = xe^y(1+y)$

Solution: Equilibria: $y = -1$. Because $e^0 = 1$, then $y = 0$ is not an equilibrium.

28. $xy' = e^y(1-y)$

29. $xy' = e^y(1-y^2)(1+y)^3$

Solution: Equilibria: $y = -1$, $y = 1$. Let $f(x,y) = e^y(1-y^2)(1+y)^3$. In equation $f(x,c) = 0$, factor e^c cancels leaving $(1-y)(1+y)(1+y)^3 = 0$.

30. $xy' = e^y(1-y^3)(1+y^3)$

Non-Equilibrium Solutions

Find the non-equilibrium solutions for the given separable equation. See Examples 2.8 and 2.10 for details.

31. $y' = (xy)^{1/3}$, $y(0) = y_0$.

Solution: The separated form is $y^{-1/3}y' = x^{1/3}$. Apply quadrature:

$\int y^{-1/3}(x)y'(x)dx = \int x^{1/3}dx$

Non-equilibrium solution: $\dfrac{y^{2/3}(x)}{2/3} = \dfrac{x^{4/3}}{4/3} + c$

Equilibria: $y = 0$

Value c is determined by substitution of $x = 0$, $y = y_0$: $\dfrac{y_0^{2/3}}{2/3} = \dfrac{0^{4/3}}{4/3} + c$.

Then $c = \dfrac{y_0^{2/3}}{2/3}$.

32. $y' = (xy)^{1/5}$, $y(0) = y_0$.

33. $y' = 1 + x - y - xy$, $y(0) = y_0$.

Solution: Factor $1 + x - y - xy = (1 + x)(1 - y)$.

Separated form: $\dfrac{y'}{1 - y} = 1 + x$

Non-equilibrium solution: $-\ln|1 - y(x)| = x + x^2/2 + c$

Equilibria: $y = 1$

Value $c = -\ln|1 - y(0)|$ because $-\ln|1 - y(0)| = 0 + 0^2/2 + c$.

34. $y' = 1 + x + 2y + 2xy$, $y(0) = y_0$.

35. $y' = \dfrac{(x+1)y^3}{x^2(y^3 - y)}$, $y(1) = y_0 \neq 0$.

Solution: Factor $y^3 - y = y(y^2 - 1)$. Cancel factor y. Divide.

Separated form: $\dfrac{(y^2 - 1)y'}{y^2} = \dfrac{1 + x}{x^2}$

Ready to integrate: $(1 - y^{-2})y' = x^{-2} + x^{-1}$

Non-equilibrium solution: $y + \dfrac{1}{y} = \dfrac{-1}{x} + \ln|x| + c$

Equilibria: $y = 1$

Initial value: $y(1) + \dfrac{1}{y(1)} = \dfrac{-1}{1} + \ln|1| + c$

$c = y_0 + \dfrac{1}{y_0} + 1$

36. $y' = \dfrac{(x-1)y^2}{x^3(y^3 + y)}$, $y(0) = y_0$.

37. $2yy' = x(1 - y^2)$

Solution: Divide.

Separated form: $\dfrac{2yy'}{1 - y^2} = x$

Substitution: $u = y^2 - 1$, $du = 2yy'$

Ready to integrate: $\dfrac{du}{u} = x$

Non-equilibrium solution:

$\ln|u| = x^2/2 + c$
$\ln|y^2 - 1| = x^2/2 + c$

Equilibria: $y = 1$, $y = -1$ from $f(x, y) = x\dfrac{1 - y^2}{2y}$

38. $2yy' = x(1 + y^2)$

39. $(1 + x)y' = 1 - y$

Solution: Divide.

Separated form: $\dfrac{y'}{1 - y} = \dfrac{1}{1 + x}$

Substitution: $u = 1 - y$, $du = -dy$

Ready to integrate: $\dfrac{-du}{u} = \dfrac{1}{1 + x}$

Non-equilibrium solution:

$-\ln|u| = \ln|1 + x| + c$
$-\ln|1 - y| = \ln|1 + x| + c$

Equilibria: $y = 1$, from $f(x, y) = \dfrac{1 - y}{1 + x}$

40. $(1 - x)y' = 1 + y$, $y(0) = y_0$.

41. $\tan(x)y' = y$, $y(\pi/2) = y_0$.

Solution: Trig identity $\tan x = \sin x / \cos x$.

Separated form: $\dfrac{y'}{y} = \dfrac{\cos x}{\sin x}$

Substitution: $u = \sin x$, $du = \cos(x)dx$

Ready to integrate: $\dfrac{dy}{y} = \dfrac{du}{u}$

Non-equilibrium solution:

$\ln|y| = \ln|u| + c$
$\ln|y| = \ln|\sin(x)| + c$

Equilibria: $y = 0$, from $f(x, y) = \dfrac{y}{\tan x}$

Initial value: $\ln|y(\pi/2)| = \ln|\sin(\pi/2)| + c$
$\ln|y_0| = \ln|1| + c$
$c = \ln|y_0|$

42. $\tan(x)y' = 1 + y$, $y(\pi/2) = y_0$.

43. $\sqrt{x}y' = \cos^2(y)$, $y(1) = y_0$.

Solution: Trig identity: $\sec^2(y) = 1/\cos^2(y)$

Separated form: $\sec^2(y)y' = x^{-1/2}$

Substitution: $(\tan y)'dy = \sec^2(y)y'dx$

Ready to integrate: $(\tan(y))'dy = x^{-1/2}dx$

Non-equilibrium solution:

$\tan(y) = 2x^{1/2} + c$

Equilibria: $y = (2n+1)\pi/2$, n = any integer, from $f(x,y) = \dfrac{\cos^2 y}{\sqrt{x}}$

Initial value: $\tan(y_0) = 2 + c$

44. $\sqrt{1-x}y' = \sin^2(y)$, $y(0) = y_0$.

45. $\sqrt{x^2-16}yy' = x$, $y(5) = y_0$.

Solution: Separated form: $yy' = \dfrac{x}{\sqrt{x^2-16}}$

Substitution: $u = x^2 - 16$, $du = 2xdx$

$$yy' = \frac{du/2}{\sqrt{u}}$$

Ready to integrate: $ydy = \frac{1}{2}u^{-1/2}du$

Non-equilibrium solution:

$y^2/2 = 2u^{1/2} + c$
$y^2/2 = 2\sqrt{x^2-16} + c$

Equilibria: none, from $f(x,y) = \dfrac{x}{y\sqrt{x^2-16}}$

Initial value: $y_0^2/2 = 2\sqrt{25-16} + c$ or $c = -6 + y_0^2/2$

46. $\sqrt{x^2-1}yy' = x$, $y(2) = y_0$.

47. $y' = x^2(1+y^2)$, $y(0) = 1$.

Solution: Separated form: $\dfrac{y'}{1+y^2} = x^2$

Identity: $(\arctan(y))'dy = \dfrac{y'dx}{1+y^2}$

Ready to integrate: $(\arctan(y))'dy = x^2dx$

Non-equilibrium solution: $\arctan(y) = x^3/3 + c$

Equilibria: none, from $f(x, y) = x^2(1 + y^2)$
Initial value: $\arctan(y(0)) = 0^3/3 + c$ or $c = \arctan(1)$

48. $(1 - x)y' = x(1 + y^2)$, $y(0) = 1$.

Independent of x

Solve the given equation, finding all solutions. See Example 2.11.

49. $y' = \sin y$, $y(0) = y_0$.
Solution: Separated form: $\csc(y)y' = 1$
Answer: $-\csc(y)\cot(y) = x + c$, with $c = -\csc(y_0)\cot(y_0)$

50. $y' = \cos y$, $y(0) = y_0$.

51. $y' = y(1 + \ln y)$, $y(0) = y_0$.

Solution: Separated form: $\dfrac{y'}{y(1 + lny)} = 1$, which makes sense for $y > 0$.
Answer: $\ln(1 + \ln(y) = x + c$, with $c = \ln(1 + \ln(y_0))$

52. $y' = y(2 + \ln y)$, $y(0) = y_0$.

53. $y' = y(y - 1)(y - 2)$, $y(0) = y_0$.

Solution: Separated form: $\dfrac{y'}{y(y-1)(y-2)} = 1$

Answer: $-\ln|y - 1| + (1/2)\ln|y| + (1/2)\ln|y - 2| = x + c$
Initial Value: $c = -\ln|y_0 - 1| + (1/2)\ln|y_0| + (1/2)\ln|y_0 - 2|$

54. $y' = y(y - 1)(y + 1)$, $y(0) = y_0$.

55. $y' = y^2 + 2y + 5$, $y(0) = y_0$.

Solution: Separated form: $\dfrac{y'}{y^2 + 2y + 5} = 1$

Factor: $y^2 + 2y + 5 = 4((y + 1)^2/4 + 1)$, college algebra complete-the-square.
Substitution: $u = (y + 1)/2$, $2du = dy$

$$\frac{2du}{4(u^2+1)} = dx$$
$$\int \frac{2du}{4(u^2+1)} = \int dx$$
$$\tfrac{1}{2}\arctan(u) = x + c_1$$
$$\arctan((y + 1)/2) = 2x + 2c_1$$

Answer: $\arctan((y + 1)/2) = 2x + c$
Initial Value: $c = \arctan((y_0 + 1)/2)$

56. $y' = y^2 + 2y + 7$, $y(0) = y_0$.

Details in the Examples

Collected here are verifications for details in the examples.

57. **(Example 2.7)** The equation $x(1-y)(1+y) = 0$ was solved in the example, but $x = 0$ was ignored, and only $y = -1$ and $y = 1$ were reported. Why?

Solution: Symbol x is the independent variable, which means it is allowed to assume all values. For instance, $x = 1$. Equation $x(1-y)(1+y) = 0$ specializes at $x = 1$ to $(1-y)(1+y) = 0$ with exactly two roots $y = 1$ and $y = -1$.

58. **(Example 2.8)** An absolute value equation $|u| = w$ was replaced by $u = kw$ where $k = \pm 1$. Justify the replacement using the *definition* $|u| = u$ for $u \ge 0$, $|u| = -u$ for $u < 0$.

59. **(Example 2.8)** Verify directly that $y = (1 + y_0)e^{x^3/3} - 1$ solves the initial value problem $y' = x^2(1+y)$, $y(0) = y_0$.

Solution: At $x = 0$, equation $y = (1 + y_0)e^{x^3/3} - 1$ reduces to $y = (1 + y_0)e^0 - 1 = y_0$, because $e^0 = 1$. The IC is verified.

Panel 1: DE Answer Check

$$\begin{aligned} \text{LHS} &= y' \\ &= \left((1+y_0)e^{x^3/3} - 1\right)' \\ &= x^2(1+y_0)e^{x^3/3}. \end{aligned}$$

$$\begin{aligned} \text{RHS} &= y' \\ &= x^2(1+y) \\ &= x^2 + x^2\left((1+y_0)e^{x^3/3} - 1\right) \\ &= x^2(1+y_0)e^{x^3/3} \end{aligned}$$

Then LHS = RHS, which verifies the DE.

60. **(Example 2.9)** The relation $y = 1 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ describes the list $\ldots, 1-\pi, 1, 1+\pi, \ldots$. Write the list for the relation $y = -1 + (2n+1)\frac{\pi}{2}$.

61. **(Example 2.9)** Solve $\sin(u) = 0$ and $\cos(v) = 0$ for u and v. Supply graphs which show why there are infinity many solutions.

Solution: $u = n\pi$ and $v = (2n+1)\pi/2$, $n =$ any integer. Graphs omitted, found in any trig reference, show infinitely many crossings of the two trig functions and the x-axis $y = 0$.

62. **(Example 2.10)** Explain why $y_0/2$ does not equal $\text{Arctan}(\tan(y_0/2))$. Give a calculator example.

63. **(Example 2.10)** Establish the identity $\tan(y/2) = \csc y - \cot y$.

Solution: Let $y = 2u$.

$$\begin{aligned}\csc y - \cot y &= \frac{1}{\sin y} - \frac{\cos y}{\sin y}\\ &= \frac{1-\cos(y)}{\sin(y)}\\ &= \frac{1-\cos(2u)}{\sin(2u)}\end{aligned}$$

Double angle trig formulas:

$\sin(2u) = 2\sin(u)\cos(u), \quad \cos(2u) = 2\cos^2(u) - 1$

$$\begin{aligned}\csc y - \cot y &= \frac{1-\cos(2u)}{\sin(2u)}\\ &= \frac{2-2\cos^2(u)}{2\sin(u)\cos(u)}\\ &= \frac{\sin^2(u)}{\sin(u)\cos(u)}, \qquad \text{used } \cos^2(\theta)+\sin^2(\theta) = 1\\ &= \tan(u)\\ &= \tan(y/2)\end{aligned}$$

64. **(Example 2.11)** Let $y_0 > 0$. Verify that $y = e^{1-(1-\ln y_0)e^{-x}}$ solves

$$y' = y(1-\ln y), \quad y(0) = y_0.$$

2.3 Linear Equations

Integrating Factor Method

Apply the integrating factor method, page 96, to solve the given linear equation. See the examples starting on page 99 for details.

1. $y' + y = e^{-x}$

Solution: Standard Form $y' + py = r$: $p = 1$, $r = e^{-x}$
Integrating Factor: $W(x) = e^{\int p(x)dx} = e^{x}$

Integrating Factor Identity: $\dfrac{(Wy)'}{W} = y' + py$

$$\frac{(e^x y)'}{e^x} = e^{-x}$$
$(e^x y)' = e^{-x}e^{x}$, Multiply by e^x
$\int (e^x y)'\,dx = \int e^{-x}e^{x}dx$, Quadrature
$e^x y = x + c$, Fund. Thm. Calc.
$y = xe^{-x} + ce^{-x}$, Candidate solution

Answer check:

```
# Maple
de:=diff(y(x),x)+y(x)=exp(-x); dsolve(de,y(x));
#  y(x) = (x+_C1)*exp(-x)
```

2. $y' + y = e^{-2x}$

3. $2y' + y = e^{-x}$

Solution: $y(x) = -\mathrm{e}^{-x} + \mathrm{e}^{-x/2}c$

4. $2y' + y = e^{-2x}$

5. $2y' + y = 1$

Solution: $y(x) = 1 + \mathrm{e}^{-x/2}c$

6. $3y' + 2y = 2$

7. $2xy' + y = x$

Solution: $y(x) = x/3 + \dfrac{c}{\sqrt{x}}$

8. $3xy' + y = 3x$

9. $y' + 2y = e^{2x}$

Solution: $y(x) = \left(1/4\,\mathrm{e}^{4\,x} + c\right)\mathrm{e}^{-2\,x}$

10. $2y' + y = 2e^{x/2}$

11. $y' + 2y = e^{-2x}$

Solution: $y(x) = (x + c)\,\mathrm{e}^{-2\,x}$

12. $y' + 4y = e^{-4x}$

13. $2y' + y = e^{-x}$

Solution: $y(x) = -\mathrm{e}^{-x} + \mathrm{e}^{-x/2}c$

14. $2y' + y = e^{-2x}$

15. $4y' + y = 1$

Solution: $y(x) = 1 + \mathrm{e}^{-x/4}c$

16. $4y' + 2y = 3$

17. $2xy' + y = 2x$

Solution: $y(x) = 2/3\,x + \frac{c}{\sqrt{x}}$

18. $3xy' + y = 4x$

19. $y' + 2y = e^{-x}$

Solution: $y(x) = (\mathrm{e}^{x} + c)\,\mathrm{e}^{-2\,x}$

20. $2y' + y = 2e^{-x}$

Superposition

Find a particular solution with fewest terms. See Example 2.15, page 99.

21. $3y' = x$

Solution: Quadrature applies: $y(x) = x^2/6 + c$
Specialize $c = 0$ to find a particular solution with fewest terms.
Then $y_p(x) = x^2/6$.
This linear equation has non-constant coefficients. No shortcut is available.

22. $3y' = 2x$

23. $y' + y = 1$

Solution: $y_p(x) = 1$
The equation has constant coefficients, therefore a shortcut applies: y_p = equilibrium solution = 1.
To find an equilibrium solution, formally replace y' by zero and solve for y. It only works if the coefficients are constant!

24. $y' + 2y = 2$

25. $2y' + y = 1$

Solution: $y_p(x) = 1$
The equation has constant coefficients, therefore a shortcut applies: $y_p =$ equilibrium solution $= 1$.

26. $3y' + 2y = 1$

27. $y' - y = e^x$

Solution: $y = xe^x$.
This linear equation has non-constant coefficients. No shortcut is available. Solve by the linear integrating factor method: $y(x) = (x + c)\,\mathrm{e}^x$ then let $c = 0$.

28. $y' - y = xe^x$

29. $xy' + y = \sin x \ (x > 0)$

Solution: $y = \dfrac{-\cos x}{x}$
This linear equation has non-constant coefficients. No shortcut is available. Solve by the linear integrating factor method: $y(x) = \dfrac{-\cos(x) + c}{x}$ then let $c = 0$.

30. $xy' + y = \cos x \ (x > 0)$

31. $y' + y = x - x^2$

Solution: $y = -x^2 + 3x - 3$
This linear equation has non-constant coefficients. No shortcut is available. Solve by the linear integrating factor method: $y(x) = -x^2 + 3\,x - 3 + \mathrm{e}^{-x}c$ then let $c = 0$.

32. $y' + y = x + x^2$

General Solution

Find y_h and a particular solution y_p. Report the general solution $y = y_h + y_p$. See Example 2.17, page 100.

33. $y' + y = 1$

Solution: The answers: $y_h = ce^{-x}$, $y_p = 1$

To find y_h, solve the homogeneous DE: $y' + y = 0$. The answer is $y = c/W$ where W is the integrating factor. See **Special Equations** in this textbook section. The details:

Standard Homogeneous Form $y' + py = 0$: $p = 1$
Integrating Factor: $W = e^{\int p dx} = e^{\int (1) dx} = e^{x+c}$
As explained in the textbook, take $c = 0$ to simplify the computation, then
$W = e^x, \quad y_h = \dfrac{c}{W} = ce^{-x}$

Method 1: Equilibrium shortcut to find $y_p = 1$.

The equation $y' + y = 1$ has constant coefficients. The method applies, which replaces y' by zero in the equation $y' + y = 1$ to find $y = 1$, the equilibrium solution. In applications, $y = 1$ would be the limit at $x = \infty$ of $y(x)$, referred to as the **steady-state solution**.

Method 2: Find y_h and y_p simultaneously.

The **Integrating Factor Method** will be applied.

Integrating Factor Identity: $\dfrac{(Wy)'}{W}$ replaced $y' + py$
In the present case:
$\dfrac{(Wy)'}{W}$ replaces $y' + y$ in $y' + y = 1$
$\dfrac{(Wy)'}{W} = 1$
$(Wy)' = 1()W \qquad$ Clear fractions.

Quadrature: Integrate across the replacement equation on variable x:

$\int (Wy)' dx = \int (1) W dx$

$Wy = \int (1) e^x dx \qquad$ FTC and equality $W = e^x$.

$y = \dfrac{1}{W} \int (1) e^x dx \qquad$ Divide by W.

$y = e^{-x} \int (1) e^x dx \qquad$ Use $W = e^x$.

$y = e^{-x}(e^x + c) \qquad$ Integral table.

$y = 1 + ce^{-x} \qquad$ Candidate solution.

Isolate $y_p = 1$ by letting $c = 0$. The remaining terms with factor c assemble the homogeneous solution $y_h = ce^{-x}$.

It remains to check the answer. A simple option is a **CAS** like `maple`, `mathematica` or Wolfram Alpha.

```
p:=1; r:=1;# MAPLE
de:=(1)*diff(y(x),x)+(p)*y(x)=r; ANS:=dsolve(de,y(x));
# ANS := y(x) = 1+exp(-x)*_C1
```

34. $xy' + y = 2$

35. $y' + y = x$

Solution: $y(x) = x - 1 + c\mathrm{e}^{-x}$, $y_h(x) = c\mathrm{e}^{-x}$, $y_p(x) = x - 1$

The equilibrium shortcut does not apply. The homogeneous shortcut always applies: $y_h = c/W$, $W =$ the integrating factor. However, it saves no time to use it, because the full integrating factor method computation is required.

36. $xy' + y = 2x$

37. $y' - y = x + 1$

Solution: $y(x) = -x - 2 + c\mathrm{e}^{x}$, $y_h(x) = c\mathrm{e}^{x}$, $y_p(x) = -x - 2$

38. $xy' - y = 2x - 1$

39. $2xy' + y = 2x^2$ $(x > 0)$

Solution: $y(x) = 2/5\,x^2 + \dfrac{c}{\sqrt{x}}$, $y_h(x) = \dfrac{c}{\sqrt{x}}$, $y_p(x) = 2/5\,x^2$

40. $xy' + y = 2x^2$ $(x > 0)$

Classification

Classify as linear or non-linear. Use the test $f(x, y) = f(x, 0) + f_y(x, 0)y$ and a computer algebra system, when available, to check the answer. See Example 2.18, page 101.

41. $y' = 1 + 2y^2$

Solution: Nonlinear.

```
f:=(x,y)->1+2*y^2; # MAPLE
a:=f(x,0); b:=subs(y=0,diff(f(x,y),y));
LHS:=f(x,y);RHS:=a+b*y;
ZERO:=LHS-RHS; # zero for linear DE
```

42. $y' = 1 + 2y^3$

43. $yy' = (1 + x)\ln e^y$

Solution: Linear when the equation makes sense.
For $y = 0$ there is no differential equation defined.
Equation $yy' = (1 + x)\ln e^y$ is identical to $y' = 1 + x$ for $y \neq 0$, because $\ln(e^y) = y$ for all y, and then y cancels. The equation causes issues for any CAS, because of division by zero with definition $f(x, y) = (1 + x)\dfrac{e^y}{y}$.

44. $yy' = (1+x)(\ln e^y)^2$

45. $y' \sec^2 y = 1 + \tan^2 y$

Solution: Linear. Equation $y' \sec^2 y = 1 + \tan^2 y$ is identical to $y' = 1$ because $1 + \tan^2(y) = \sec^2(y)$. A quadrature equation is always linear, in this case $y' = 1$, no test required.

46. $y' = \cos^2(xy) + \sin^2(xy)$

47. $y'(1+y) = xy$

Solution: Nonlinear. Write it as $y' = f(x,y) = \dfrac{xy}{1+y}$. The $a = f(x,0) = 0$, $b = f_y(x,0) =$

```
f:=(x,y)->(x*y)/(1+y); # MAPLE
g:=unapply(diff(f(x,y),y),x);
a:=f(x,0); b:=g(0);
LHS:=f(x,y);RHS:=a+b*y; ZERO:=LHS-RHS; # zero for linear DE
# ZERO := x*y/(y+1) # Must be zero to be linear
```

48. $y' = y(1+y)$

49. $xy' = (x+1)y - xe^{\ln y}$

Solution: Linear. The equation is undefined for $x = 0$. For $x \neq 0$ the equation is the same as $xy' = (x+1)y - xy$ which reduces to $y' = y/x$. This is a homogeneous equation of the form $y' + p(x)y = 0$, all such known to be linear. No test required.

50. $2xy' = (2x+1)y - xye^{-\ln y}$

Shortcuts

Apply theorems for the homogeneous equation $y' + p(x)y = 0$ or for constant coefficient equations $y' + py = r$. Solutions should be done without paper or pencil, then write the answer and check it.

51. $y' - 5y = -1$

Solution: $y_p = 1/5$, $y_h = ce^{5x}$

Equilibrium solution: $y_p = 1/5$, obtained formally by letting $y' = 0$, then solve for y.
Homogeneous solution $y' - 5y = 0$:
$y = c/W$, $W =$ integrating factor $= e^{\int p(x)dx} = e^{-5x}$.
$y_h = ce^{5x}$

```
de:=(1)*diff(y(x),x)+(-5)*y(x) = -1;
ANS:=dsolve(de,y(x));# MAPLE
# ANS := y(x) = 1/5+exp(5*x)*_C1
```

52. $3y' - 5y = -1$

53. $2y' + xy = 0$

Solution: $y_h = ce^{-x^2/4}$, $y_p = 0$.

Homogeneous shortcut: $y = c/W$, $W = e^{\int (x/2)dx} = e^{x^2/4}$

$y_h = ce^{-x^2/4}$
$y_p = 0$ because the equation is homogeneous

54. $3y' - x^2y = 0$

55. $y' = 3x^4y$

Solution: $y_h = ce^{3x^5/5}$, $y_p = 0$

Homogeneous shortcut: $y = c/W$, $W = e^{\int (-3x^4)dx} = e^{-3x^5/5}$

$y_h = ce^{3x^5/5}$
$y_p = 0$ because the equation is homogeneous

56. $y' = (1 + x^2)y$

57. $\pi y' - \pi^2 y = -e^2$

Solution: $y_h = ce^{\pi x}$, $y_p = e^2/\pi^2$

Homogeneous shortcut: $y = c/W$, $W = e^{\int (-\pi)dx} = e^{-\pi x}$

$y_h = ce^{\pi x}$
$y_p = -e^2/(-\pi^2)$ by formally letting $y' = 0$

58. $e^2y' + e^3y = \pi^2$

59. $xy' = (1 + x^2)y$

Solution: $y_h = \dfrac{ce^{x^2/2}}{x}$, $y_p = 0$

Homogeneous shortcut: $y = c_1/W$,

$W = e^{\int (-x^{-1}-x)dx} = e^{-\ln|x|-x^2/2} = |x|e^{-x^2/2}$

$$y_h = \frac{c_1}{|x|e^{-x^2/2}} = \frac{c}{xe^{-x^2/2}} = \frac{ce^{x^2/2}}{x}$$

where $c = \pm c_1$ to eliminate absolute values on $|x|$.

$y_p = 0$ because the equation is homogeneous

60. $e^xy' = (1 + e^{2x})y$

Proofs and Details

61. Prove directly without appeal to Theorem 2.6 that the difference of two solutions of $y' + p(x)y = r(x)$ is a solution of the homogeneous equation $y' + p(x)y = 0$.

Solution: Let $y_1' + p(x)y_1 = r(x)$, $y_2' + p(x)y_2 = r(x)$. Define $y = y_1 - y_2$. To be proved: $y' + p(x)y = 0$.

$$\begin{aligned} y' + p(x)y &= y_1' - y_2' - p(x)(y_1 - y_2) \\ &= y_1' - y_2' + p(x)y_1 - p(x)y_2 \\ &= (y_1' + p(x)y_1) - (y_2' + p(x)y_2) \\ &= (r(x)) - (r(x)) \qquad\qquad = 0 \quad \blacksquare \end{aligned}$$

62. Prove that y_p^* given by equation (2) and $y_p = W^{-1}\int r(x)W(x)dx$ given in the integrating factor method are related by $y_p = y_p^* + y_h$ for some solution y_h of the homogeneous equation.

63. The equation $y' = r$ with r constant can be solved by quadrature, without pencil and paper. Find y.

Solution: $y = rx + c$ by integrating mentally across the DE.
Then $y_h = c$ and $y_p = rx$.

64. The equation $y' = r(x)$ with $r(x)$ continuous can be solved by quadrature. Find a formula for y.

2.4 Undetermined Coefficients

Variation of Parameters I

Report the shortest particular solution given by the formula

$$y_p(x) = \frac{\int rW}{W}, \quad W = e^{\int p(x)dx}$$

1. $y' = x + 1$

Solution: $y_p = x^2/2 + x$

Method 1: Integrate across the equation to obtain $y = x^2/2 + x + c$, then choose $c = 0$ to find the shortest solution. A number of solutions have used this method: it is not wrong, because the exercise does not require use of the Variation of Parameters formula.

Method 2: This is the expected method. Define $p = 0$. $r(x) = x+1$. Then $W = e^{\int p dx} = e^0 = 1$. The formula produces

$$y_p = \frac{1}{W}\int rW\,dx = \frac{1}{1}\int (x+1)(1)dx = x^2/2 + x + c$$

The shortest solution is with $c = 0$. A no-paper-and-pencil answer check is provided by **Method 1**.

2. $y' = 2x - 1$

3. $y' + y = e^{-x}$

Solution: $y_p\,(x) = x\mathrm{e}^{-x}$

Follow exercise 1. Use the formula with
$p(x) = 1$,
$r(x) = e^{-x}$
$W(x) = e^{\int p(x)dx} = e^x$
Then $y_h = ce^{-x}$, $rW = e^{-x}e^x = 1$ and $y_p = xe^{-x}$

4. $y' + y = e^{-2x}$

5. $y' - 2y = 1$

Solution: $y_p\,(x) = -1/2$

Follow exercise 1. Use the formula with
$p(x) = -2$,
$r(x) = 1$
$W(x) = e^{\int p(x)dx} = e^{-2x}$
Then $y_h = ce^{2x}$, $rW = e^{-2x}$ and

$$y_p = \frac{-\frac{1}{2}e^{-2x} + c}{e^{-2x}} = 1/2 \text{ for } c = 0.$$

Alternative Method:
The DE has constant coefficients, therefore $y_p =$ the equilibrium solution, which means $y_p = 1/(-2) = -1/2$.

6. $y' - y = 1$

7. $2y' + y = e^x$

Solution: $y_p\,(x) = 1/3\,\mathrm{e}^x$
Divide by 2 to obtain the standard form
$$y' + \frac{1}{2}y = \frac{1}{2}e^x.$$
Define $p(x) = 1/2$, $r(x) = \frac{1}{2}e^x$. Apply the formula.

8. $2y' + y = e^{-x}$

9. $xy' = x + 1$

Solution: $y_p\,(x) = \ln|x| + x$
The statement requires $x \neq 0$ to make sense. Assume $x > 0$.
The details for $x < 0$ are omitted below, but similar.
Divide by x to obtain the standard form
$$y' = 1 + \frac{1}{x}.$$
Method 1: Solve by quadrature.
Method 2: Define $p(x) = 0$, $r(x) = 1 + 1/x$. Apply the formula.

10. $xy' = 1 - x^2$

Variation of Parameters II

Define $W(t) = e^{\int_{x_0}^{t} p(x)dx}$. Compute
$y_p^*(x) = \frac{\int_{x_0}^{x} r(t)W(t)\,dt}{W(x)}$

11. $y' = x + 1$, $y(0) = 0$

Solution: $y^*\,(x) = 1/2\,x^2 + x$
Exercises 11-20 were solved as exercises 1-10. The exercises evaluate constant c in solution $y_p^*(x)$ from values x_0 and $y_0 = 0$ in initial condition $y^*(x_0) = 0$.
An answer check:
```
a:=1;b:=0;f:=x->x+1;x0:=0;y0:=0; # Maple
de:=a*diff(y(x),x) + b*y(x) = f(x);
ANS:=dsolve([de,y(x0)=y0],y(x));
# ANS := y(x) = (1/2)*x^2+x
```

12. $y' = 2x - 1$, $x_0 = 0$

13. $y' + y = e^{-x}$, $x_0 = 0$

Solution: $y^*(x) = xe^{-x}$

14. $y' + y = e^{-2x}$, $x_0 = 0$

15. $y' - 2y = 1$, $x_0 = 0$

Solution: $y^*(x) = -1/2 + 1/2\,\mathrm{e}^{2\,x}$

16. $y' - y = 1$, $x_0 = 0$

17. $2y' + y = e^x$, $x_0 = 0$

Solution: $y^*(x) = 1/3\,\mathrm{e}^x - 1/3\,\mathrm{e}^{-x/2}$

18. $2y' + y = e^{-x}$, $x_0 = 0$

19. $xy' = x + 2$, $x_0 = 1$

Solution: $y^*(x) = x + 2\,\ln|x| - 1$

20. $xy' = 1 - x^2$, $x_0 = 1$

Euler Solution Atoms

Report the list L of distinct Euler solution atoms found in function $f(x)$. Then $f(x)$ is a sum of constants times the Euler atoms from L.

21. $x + e^x$

Solution: x, e^x
$1, x, x^2, \ldots$ are Euler solution atoms
e^{ax} is an Euler solution atom

22. $1 + 2x + 5e^x$

23. $x(1 + x + 2e^x)$

Solution: x, x^2, xe^x
Constants and signs are stripped because Euler solution atoms have coefficient one.

24. $x^2(2 + x^2) + x^2e^{-x}$

25. $\sin x \cos x + e^x \sin 2x$

Solution: $\sin 2x$, $e^x \sin 2x$
Term $\sin x \cos x$ is a product of two Euler atoms, which is generally not an Euler atom. Trig identity $2\sin x \cos x = \sin 2x$ allows the product to be rewritten as $\frac{1}{2}\sin 2x$, then the constant $\frac{1}{2}$ is stripped to expose the Euler solution atom $\sin 2x$.

26. $\cos^2 x - \sin^2 x + x^2 e^x \cos 2x$

27. $(1 + 2x + 4x^5)e^x e^{-3x} e^{x/2}$

Solution: e^{ax}, xe^{ax}, $x^5 e^{ax}$ where $a = 1 - 3 + 1/2 = -3/2$

28. $(1 + 2x + 4x^5 + e^x \sin 2x)e^{-3x/4} e^{x/2}$

29. $\dfrac{x + e^x}{e^{-2x}} \sin 3x + e^{3x} \cos 3x$

Solution: $xe^{2x} \sin 3x$, $e^{3x} \sin 3x$, $e^{3x} \cos 3x$

Expand the expression as
$(x + e^x)e^{2x} \sin 3x + e^{3x} \cos 3x$, or
$xe^{2x} \sin 3x + e^{3x} \sin 3x + e^{3x} \cos 3x$

30. $\dfrac{x + e^x \sin 2x + x^3}{e^{-2x}} \sin 5x$

Initial Trial Solution

Differentiate repeatedly $f(x)$ and report the list M of distinct Euler solution atoms which appear in f and all its derivatives. Then each of $f, f', \ldots$ is a sum of constants times Euler atoms in M.

31. $12 + 5x^2 + 6x^7$

Solution: 1, x, x^2, x^3, x^4, x^5, $x^4 6$, x^7

The first two terms 12, $5x^2$ merely duplicate Euler atoms found from term $6x^7$.

32. $x^6/x^{-4} + 10x^4/x^{-6}$

33. $x^2 + e^x$

Solution: 1, x, x^2, e^x

34. $x^3 + 5e^{2x}$

35. $(1 + x + x^3)e^x + \cos 2x$

Solution: 1, x, x^2, x^3, $\cos x$, $\sin x$

36. $(x + e^x) \sin x + (x - e^{-x}) \cos 2x$

37. $(x + e^x + \sin 3x + \cos 2x)e^{-2x}$

Solution: e^{-2x}, xe^{-2x}, e^{-x}, $e^{-2x} \cos 3x$, $e^{-2x} \sin 3x$, $e^{-2x} \cos 2x$, $e^{-2x} \sin 2x$

38. $(x^2e^{-x} + 4\cos 3x + 5\sin 2x)e^{-3x}$

39. $(1 + x^2)(\sin x \cos x - \sin 2x)e^{-x}$

Solution: $e^{-x}\sin(2x)$, $e^{-x}\cos(2x)$, $xe^{-x}\sin(2x)$, $xe^{-x}\cos(2x)$, $x^2e^{-x}\sin(2x)$, $x^2e^{-x}\cos(2x)$

Change $\sin x \cos x$ into $\frac{1}{2}\sin 2x$, then
$f = -\frac{1}{2}e^{-x}\sin 2x - \frac{1}{2}x^2e^{-x}\sin 2x$
$f' = -1/2\,\mathrm{e}^{-x}\sin(2\,x) + \mathrm{e}^{-x}\cos(2\,x) - x\mathrm{e}^{-x}\sin(2\,x)$
$\quad +1/2\,x^2\mathrm{e}^{-x}\sin(2\,x) - x^2\mathrm{e}^{-x}\cos(2\,x)$
Derivative f'' introduces one more Euler atom.
Derivatives f''', ... do not generate more Euler atoms.

```
# Maple
F:=x->1/2)exp(-x)*sin (2*x)-1/2* x^2*exp(-x)* sin (2*x);
diff(F(x),x,x);
diff(F(x),x,x,x);
diff(F(x),x,x,x,x);
diff(F(x),x,x,x,x,x);
```

40. $(8 - x^3)(\cos^2 x - \sin^2 x)e^{3x}$

Correction Rule

Given the homogeneous solution y_h and an initial trial solution y, determine the final trial solution according to the correction rule.

41. $y_h(x) = ce^{2x}$, $y = d_1 + d_2x + d_3e^{2x}$

Solution: $y = d_1 + d_2x + d_3xe^{2x}$

Break trial solution y into two Euler atom groups:
Group 1: 1, x
Group 2: e^{2x}

Solution y_h has only one Euler atom: e^{2x}.
Group 1 is unchanged. Group 2 requires multiplication by x. Then
Group 1: 1, x
New Group 2: xe^{2x}

Check: the Euler atoms found in the last two groups do not repeat any Euler atom found in $y_h = ce^{2x}$. The corrected trial solution is a linear combination of the Euler atoms found in Group 1 and New Group 2:

$y = d_1 + d_2x + d_3xe^{2x}$

42. $y_h(x) = ce^{2x}$, $y = d_1 + d_2e^{2x} + d_3xe^{2x}$

43. $y_h(x) = ce^{0x}$, $y = d_1 + d_2x + d_3x^2$

Solution: $y = d_1x + d_2x^2 + d_3x^3$

The Euler atom found in y_h is 1 (same as e^0). There is one group of Euler atoms in y: $1, x, x^2$. Multiply the group by x and test for a conflict with y_h. The new group is x, x^2, x^3 and the corrected trial solution is a linear combination of the Euler atoms in the new group.

44. $y_h(x) = ce^x$, $y = d_1 + d_2x + d_3x^2$

45. $y_h(x) = ce^x$, $y = d_1 \cos x + d_2 \sin x + d_3 e^x$

Solution: $y = d_1 \cos x + d_2 \sin x + d_3 x e^x$

46. $y_h(x) = ce^{2x}$, $y = d_1 e^{2x} \cos x + d_2 e^{2x} \sin x$

47. $y_h(x) = ce^{2x}$, $y = d_1 e^{2x} + d_2 x e^{2x} + d_3 x^2 e^{2x}$

Solution: $y = d_1 x e^{2x} + d_2 x^2 e^{2x} + d_3 x^3 e^{2x}$

48. $y_h(x) = ce^{-2x}$, $y = d_1 e^{-2x} + d_2 x e^{-2x} + d_3 e^{2x} + d_4 x e^{2x}$

49. $y_h(x) = cx^2$, $y = d_1 + d_2x + d_3x^2$

Solution: $y = d_1 x^3 + d_2 x^4 + d_3 x^5$

The group for y is $1, x, x^2$. Three multiplications by x across the group will eliminate conflict with Euler atom x^2 found in y_h.

50. $y_h(x) = cx^3$, $y = d_1 + d_2x + d_3x^2$

Trial Solution

Find the form of the **corrected** trial solution y but do not evaluate the undetermined coefficients.

51. $y' = x^3 + 5 + x^2 e^x (3 + 2x + \sin 2x)$

Solution: $y =$ a linear combination of

x, x^2, x^3, x^4,
e^x, xe^x, x^2e^x, x^3e^x,
$e^x \cos 2x$, $xe^x \cos 2x$, $x^2 e^x \cos 2x$,
$e^x \sin 2x$, $xe^x \sin 2x$, $x^2 e^x \sin 2x$

The homogeneous equation is $y' = 0$ and $y_h = c$ with Euler atom 1. The Euler atoms found from RHS $f(x) = x^3 + 5 + x^2 e^x (3 + 2x + \sin 2x)$ are in four groups:

Group 1: 1, x, x^2, x^3
Group 2: e^x, xe^x, x^2e^x, x^3e^x
Group 3: $e^x \cos 2x$, $xe^x \cos 2x$, $x^2 e^x \cos 2x$
Group 4: $e^x \sin 2x$, $xe^x \sin 2x$, $x^2 e^x \sin 2x$

The Euler atom in y_h conflicts only with Group 1.

Fix Group 1 by multiplying by x:
New Group 1: x, x^2, x^3, x^4
Then the corrected trial solution is a linear combination of New Group 1 and Groups 2,3,4.

52. $y' = x^2 + 5x + 2 + x^3e^x(2 + 3x + 5\cos 4x)$

53. $y' - y = x^3 + 2x + 5 + x^4e^x(2 + 4x + 7\cos 2x)$
Solution: $y =$ a linear combination of

1, x, x^2, x^3
xe^x, x^2e^x, x^3e^x , x^4e^x, x^5e^x, x^6e^x
$e^x\cos 2x$, $xe^x\cos 2x$, $x^2e^x\cos 2x$, $x^3e^x\cos 2x$, $x^4e^x\cos 2x$
$e^x\sin 2x$, $xe^x\sin 2x$, $x^2e^x\sin 2x$, $x^3e^x\sin 2x$, $x^4e^x\sin 2x$

The homogeneous equation is $y' - y = 0$. Then $y_h = ce^x$ with Euler atom e^x. The Euler atoms found from RHS $f(x) = x^3 + 2x + 5 + 2x^4e^x + 4x^5e^x + 7x^4e^x\cos 2x$ are in four groups:

Group 1: 1, x, x^2, x^3
Group 2: e^x, xe^x, x^2e^x, x^3e^x , x^4e^x, x^5e^x
Group 3: $e^x\cos 2x$, $xe^x\cos 2x$, $x^2e^x\cos 2x$, $x^3e^x\cos 2x$, $x^4e^x\cos 2x$
Group 4: $e^x\sin 2x$, $xe^x\sin 2x$, $x^2e^x\sin 2x$, $x^3e^x\sin 2x$, $x^4e^x\sin 2x$

The Euler atom in y_h conflicts only with Group 2.
Multiply by x across Group 2:
New Group 2: xe^x, x^2e^x, x^3e^x , x^4e^x, x^5e^x, x^6e^x

Then the corrected trial solution is a linear combination of New Group 2 and Groups 1,3,4.

54. $y' - y = x^4 + 5x + 2 + x^3e^x(2 + 3x + 5\cos 4x)$

55. $y' - 2y = x^3 + x^2 + x^3e^x(2e^x + 3x + 5\sin 4x)$
Solution: $y =$ a linear combination of

1, x, x^2, x^3
e^{2x}, xe^{2x}, x^2e^{2x}, x^3e^{2x}
e^x, xe^x, x^2e^x, x^3e^x , x^4e^x
$e^x\cos 4x$, $xe^x\cos 4x$, $x^2e^x\cos 4x$, $x^3e^x\cos 4x$
$e^x\sin 4x$, $xe^x\sin 4x$, $x^2e^x\sin 4x$, $x^3e^x\sin 4x$

The homogeneous equation is $y' - 2y = 0$. Then $y_h = ce^{2x}$ with Euler atom e^{2x}. The Euler atoms found from RHS $f(x) = x^3 + x^2 + 2x^3e^{2x} + 3x^4e^x + 5x^3e^x\sin 4x$ are in five groups:

Group 1: 1, x, x^2, x^3
Group 2: e^{2x}, xe^{2x}, x^2e^{2x}, x^3e^{2x}
Group 3: e^x, xe^x, x^2e^x, x^3e^x , x^4e^x

Group 4: $e^x \cos 4x$, $xe^x \cos 4x$, $x^2e^x \cos 4x$, $x^3e^x \cos 4x$
Group 5: $e^x \sin 4x$, $xe^x \sin 4x$, $x^2e^x \sin 4x$, $x^3e^x \sin 4x$

The Euler atom in y_h conflicts only with Group 2.
Multiply by x across Group 2:
New Group 2: xe^{2x}, x^2e^{2x}, x^3e^{2x}, x^4e^{2x}

Then the corrected trial solution is a linear combination of New Group 2 and Groups 1,3,4,5.

56. $y' - 2y = x^3e^{2x} + x^2e^x(3 + 4e^x + 2\cos 2x)$

57. $y' + y = x^2 + 5x + 2 + x^3e^{-x}(6x + 3\sin x + 2\cos x)$

Solution: $y =$ a linear combination of

1, x, x^2
e^{2x}, xe^{2x}, x^2e^{2x}
xe^{-x}, x^2e^{-x}, x^3e^{-x} , x^4e^{-x}, x^4e^{-x}
$e^x \cos x$, $xe^x \cos x$, $x^2e^x \cos x$, $x^3e^x \cos x$
$e^x \sin x$, $xe^x \sin x$, $x^2e^x \sin x$, $x^3e^x \sin x$

58. $y' - 2y = x^5 + 5x^3 + 14 + x^3e^x(5 + 7xe^{-3x})$

59. $2y' + 4y = x^4 + 5x^5 + 2x^8 + x^3e^x(7 + 5xe^x + 5\sin 11x)$

Solution: $y =$ a linear combination of

1, x, x^2, x^3, x^5, x^5, x^6, x^7, x^8
e^x, xe^x, x^2e^x, x^3e^x
e^{2x}, xe^{2x}, x^2e^{2x}, x^3e^{2x}, x^4e^{2x}
$e^x \cos 11x$, $xe^x \cos 11x$, $x^2e^x \cos 11x$, $x^3e^x \cos 11x$
$e^x \sin 11x$, $xe^x \sin 11x$, $x^2e^x \sin 11x$, $x^3e^x \sin 11x$

There is no Euler atom conflict between the homogeneous equation $2y' + 4y = 0$ (Euler atom e^{-2x}) and the Euler atoms found from the RHS of the non-homogeneous equation. No correction rule used.

60. $5y' + y = x^2 + 5x + 2e^{x/5} + x^3e^{x/5}(7 + 9x + 2\sin(9x/2))$

Undetermined Coefficients

Compute a particular solution y_p according to the method of undetermined coefficients. Expected details include:

(1) Initial trial solution
(2) Corrected trial solution
(3) Undetermined coefficient algebraic equations and solution
(4) Formula for y_p, coefficients evaluated

61. $y' + y = x + 1$

Solution: $y_p - x$. The answer can be checked by inspection. Experienced solvers would try to guess the answer first, finding quickly the solution $y = x$. In such simple examples there is no need for the method of undetermined coefficients.

Details:
The homogeneous equation is $y' + y = 0$ with Euler atom e^{-x}.
Euler atoms 1, x are found from the RHS $= x + 1$.
(1) $y_p = d_1 + d_2 x$

(2) No correction rule needed, e^{-2x} does not appear in the list 1, x.

(3) Equations for the undetermined coefficients:
$(d_1 + d_2 x)' + (d_1 + d_2 x) = x + 1$ Substitute $y = d_1 + d_2 x$
$d_2 + (d_1 + d_2 x) = x + 1$
$(d_2 + d_1) + d_2 x = 1 + x$ Prepare to match coefficients
$d_2 + d_1 = 1, \quad d_2 = 1$ Linear algebraic equations found.
$d_1 = 0, \quad d_2 = 1$ Solved by back-substitution.

(4) Report $y_p = d_1 + d_2 x = x$

62. $y' + y = 2x - 1$

63. $y' - y = e^x + e^{-x}$

Solution: $y_p\,(x) = x - 1/2\,\mathrm{e}^{-2\,x}$

```
# Maple answer check
de:=diff(y(x),x)+(-1)*y(x)=exp(x)+exp(-x);
ANS:=dsolve(de,y(x));
# ANS := y(x) = (x-(1/2)*exp(-2*x)+_C1)*exp(x)
```

64. $y' - y = xe^x + e^{-x}$

65. $y' - 2y = 1 + x + e^{2x} + \sin x$

Solution: $y_p\,(x) = -3/4 - x/2 + \mathrm{e}^{2\,x} x - 1/5\,\cos(x) - 2/5\,\sin(x)$

Compute $y_h = ce^{2x}$ from $y' - 2y = 0$. Euler atoms 1, x, e^{2x}, $\sin x$, $\cos x$ are found from the RHS $= 1 + x + e^{2x} + \sin x$. The correction rule is applied to replace e^{2x} by xe^{2x}, then corrected trial solution y is a linear combination of 1, x, xe^{2x}, $\sin x$, $\cos x$. Computer algebra system `maple` is a useful tool to discover algebra and calculus errors on paper.

```
# Maple answer check
de:=de:=diff(y(x),x)+(-2)*y(x)=1+x+exp(2*x)+sin(x);
ANS:=dsolve(de,y(x));
# ANS := -3/4-(1/2)*x+exp(2*x)*x-(1/5)*cos(x)
  -(2/5)*sin(x)+exp(2*x)*_C1
# Discovery of calculus and algebra errors on paper
Trial:=x-> d[1] + d[2]*x+d[3]*exp(2*x)*x+
  d[4]*cos(x)+d[5]*sin(x);
eq1:=diff(Trial(x),x)+(-2)*Trial(x)=1+x+exp(2*x)+sin(x);
```

66. $y' - 2y = 1 + x + xe^{2x} + \cos x$

67. $y' + 2y = xe^{-2x} + x^3$

Solution: $y_p\,(x) = 1/2\,x^3 - 3/4\,x^2 + 3/4\,x - 3/8 + 1/2\,\mathrm{e}^{-2\,x}x^2$

```
# Maple answer check
de:=diff(y(x),x)+(2)*y(x)=x*exp(-2*x)+x^3;
# ANS := (1/2)*x^3-(3/4)*x^2+(3/4)*x
  -3/8+(1/2)*exp(-2*x)*x^2+exp(-2*x)*_C1
```

68. $y' + 2y = (2 + x)e^{-2x} + xe^x$

69. $y' = x^2 + 4 + xe^x(3 + \cos x)$

Solution: $y_p\,(x) = 1/2\,\mathrm{e}^x x \cos(x) - (-x/2 + 1/2)\,\mathrm{e}^x \sin(x) + 3\,\mathrm{e}^x x - 3\,\mathrm{e}^x + 1/3\,x^3 + 4\,x$

```
# Maple answer check
de:=diff(y(x),x)+(0)*y(x)=x^2+4+x*exp(x)*(3+cos(x));
ANS:=dsolve(de,y(x));
# ANS := (1/2)*exp(x)*x*cos(x)-(-(1/2)*x
  +1/2)*exp(x)*sin(x)+3*exp(x)*x-3*exp(x)
  +(1/3)*x^3+4*x+_C1
```

70. $y' = x^2 + 5 + xe^x(2 + \sin x)$

2.5 Linear Applications

Concentration

A lab assistant collects a volume of brine, boils it until only salt crystals remain, then uses a scale to determine the crystal mass or weight.

Find the salt **concentration** of the brine in kilograms per liter.

1. One liter of brine, crystal mass 0.2275 kg

Solution: Answer=0.2275 kg/l.

Concentration is amount/volume. The units are mass: kilograms, volume: liters. This exercise is a check on the definition and the use of proper units.

2. Two liters, crystal mass 0.32665 kg

3. Two liters, crystal mass 15.5 grams

Solution: $(15.5/1000)/2 = 0.00775$ kilograms per liter

4. Five pints, crystals weigh 1/4 lb

5. Eighty cups, crystals weigh 5 lb

Solution: 8.344906553 kilograms per liter

One liter = 4.227 cups. One kilogram is 2.20462 pounds. Let vol=80(1/4.227) = 18.92595221 liters, amt=5/2.20462 = 2.267964547 kilograms. Then amt/vol=18.92595221/2.267964547 = 8.344906553 kilograms per liter.

6. Five gallons, crystals weigh 200 ounces

One-Tank Mixing

Assume one inlet and one outlet. Determine the amount $x(t)$ of salt in the tank at time t. Use the text notation for equation (1).

7. The inlet adds 10 liters per minute with concentration $C_1 = 0.023$ kilograms per liter. The tank contains 110 liters of distilled water. The outlet drains 10 liters per minute.

Solution: $x(t) = 2.53\left(1 - e^{-t/11}\right)$.

Follow the **Pollution** example. Use equation

$$\frac{dx}{dt} = a(t)\,C_1 - b(t)\,\frac{x(t)}{V(t)}$$

Let $a(t) = 10$ liters per minute, $C_1 = 0.023$ kilograms per liter, $b(t) = 10$ liters per minute. The volume is constant: $V(t) = 110$ liters. Because the tank initially has no salt, then $x(0) = 0$. The initial value problem:

$$\frac{dx}{dt} = 10(0.023) - 10\,\frac{x(t)}{110}. \quad x(0) = 0$$

Convert to linear DE standard form using symbols x, y: $y' + \dfrac{10}{110}y = 0.23$, $y(0) = 0$. The constant-equation shortcut solution is $y = y_p + y_h$ where $y_p =$ equilibrium solution, $y_h = c/W$, $W =$ integrating factor for $y' + \dfrac{10}{110}y = 0$. Then $W = e^{\int (1/11)dx} = e^{x/11}$. The equilibrium solution is found from $y' + \dfrac{10}{110}y = 0.23$ by replacing y' by zero, then $y_p = 11(0.23) = 2.53$. The solution is $y = y_p + y_h = 2.53 + c\,e^{-x/11}$. Use $y(0) = 0$ to find $c = -2.53$, then $y = 2.53\left(1 - e^{-x/11}\right)$.

Change symbols $x, y \to t, x$. The solution: $x(t) = 2.53\left(1 - e^{-t/11}\right)$.

```
de:=diff(x(t),t)=10*(0.023) - 10 *x(t)/110;# Maple
ic:=x(0)=0;
dsolve([de,ic],x(t));
# x(t) = 253/100-(253/100)*exp(-(1/11)*t)
```

8. The inlet adds 12 liters per minute with concentration $C_1 = 0.0205$ kilograms per liter. The tank contains 200 liters of distilled water. The outlet drains 12 liters per minute.

9. The inlet adds 10 liters per minute with concentration $C_1 = 0.0375$ kilograms per liter. The tank contains 200 liters of brine in which 3 kilograms of salt is dissolved. The outlet drains 10 liters per minute.

Solution: $x(t) = \dfrac{15}{2} - \dfrac{9}{2}e^{-t/20}$.

Follow exercise 1 above. The initial value problem:

$$\frac{dx}{dt} = 10(0.0375) - 10\,\frac{x(t)}{200}, \quad x(0) = 3$$

```
de:=diff(x(t),t)=10*(0.0375) - 10 *x(t)/200;# Maple
ic:=x(0)=3;
dsolve([de,ic],x(t));
# x(t) = 15/2-(9/2)*exp(-(1/20)*t)
```

10. The inlet adds 12 liters per minute with concentration $C_1 = 0.0375$ kilograms per liter. The tank contains 500 liters of brine in which 7 kilograms of salt is dissolved. The outlet drains 12 liters per minute.

11. The inlet adds 10 liters per minute with concentration $C_1 = 0.1075$ kilograms per liter. The tank contains 1000 liters of brine in which k kilograms of salt is dissolved. The outlet drains 10 liters per minute.

Solution: $x(t) = \dfrac{215}{2} - \left(k - \dfrac{215}{2}\right) e^{-t/100}$.

Follow exercise 1 above. The initial value problem:

$$\frac{dx}{dt} = 10(0.10755) - 10\,\frac{x(t)}{1000}, \quad x(0) = k$$

```
de:=diff(x(t),t)=10*(0.1075) - 10 *x(t)/1000;# Maple
ic:=x(0)=k;
dsolve([de,ic],x(t));
# x(t) = 215/2+exp(-(1/100)*t)*(k-215/2)
```

12. The inlet adds 14 liters per minute with concentration $C_1 = 0.1124$ kilograms per liter. The tank contains 2000 liters of brine in which k kilograms of salt is dissolved. The outlet drains 14 liters per minute.

13. The inlet adds 10 liters per minute with concentration $C_1 = 0.104$ kilograms per liter. The tank contains 100 liters of brine in which 0.25 kilograms of salt is dissolved. The outlet drains 11 liters per minute. Determine additionally the time when the tank is empty.

Solution: $x_p = \dfrac{52}{5} - \dfrac{13}{125}\,t$, $x_h = c\,(100-t)^{11}$, $c = -\dfrac{203}{20}100^{-11}$

The tank drains at time $t = 100$, because the tank drains faster than it fills, drain rate $= 1$ liters per minute.

Follow exercise 1 above. Let $a(t) = 10$, $b(t) = 11$, $C_1 = 0.104$, $V(t) = 100-t$. The initial value problem:

$$\frac{dx}{dt} = 10(0.104) - 11\,\frac{x(t)}{100-t}, \quad x(0) = 0.25$$

Solution requires the linear integrating factor method, due to non-constant coefficients. No shortcut applies.

Let $p(t) = \dfrac{11}{100-t}$
Let $r(t) = 1.04$
Standard linear DE form $x' + px = r$ is verified.
Let $W(t) = e^{\int p(t)dt} = e^{-11\ln|100-t|+c_1}$, then select
$W = (100-t)^{-11}$ for $t = 0$ to 100.

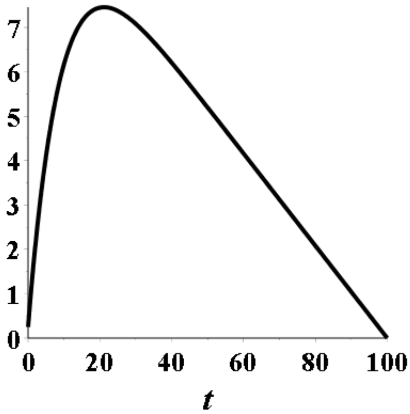
Find x_h and x_p:

$$x_p = \frac{1}{W}\int rW\,dt = \frac{52}{5} - \frac{13}{125}\,t$$
$x_h = c/W = c\,(100-t)^{11}$

It remains to determine c from $x(0) = 0.25$.

$$0.25 = x_h(0) + x_p(0) = c\,100^{11} + \frac{52}{5} - \frac{13}{125}\,(0)$$
$$c = \left(\frac{1}{4} - \frac{52}{5}\right)100^{-11} = -\frac{203}{20000000000000000000000}$$

```
V:=t->100-t; # Tank volume after t minutes
de:=diff(x(t),t)=10*(0.104) - 11 *x(t)/V(t);# Maple
ic:=x(0)=0.25;
dsolve([de,ic],x(t));
# x(t) = 52/5-(13/125)*t
  +(203/20000000000000000000000)*(-100+t)^11
```



The graphic shows that the amount of salt $x(t)$ is zero at time $t = 100$.

14. The inlet adds 16 liters per minute with concentration $C_1 = 0.01114$ kilograms per liter. The tank contains 1000 liters of brine in which 4 kilograms of salt is dissolved. The outlet drains 20 liters per minute. Determine additionally the time when the tank is empty.

15. The inlet adds 10 liters per minute with concentration $C_1 = 0.1$ kilograms per liter. The tank contains 500 liters of brine in which k kilograms of salt is dissolved. The outlet drains 12 liters per minute. Determine additionally the time when the tank is empty.

Solution: $x_p = \frac{1}{W}\int rW\,dt = 50 - \frac{1}{5}\,t$, $x_h = c\,(250-t)^6$, $c = (k-50)\,/250^6$

The tank drains at time $t = 250$, because the tank drains faster than it fills, drain rate = 2 liters per minute.

Follow exercise 1 above. Let $a(t) = 10$, $b(t) = 12$, $C_1 = 0.1$, $V(t) = 500 - 2t$. The initial value problem:

$$\frac{dx}{dt} = 10(0.1) - 12\,\frac{x(t)}{500-2t}, \quad x(0) = k$$

Solution requires the linear integrating factor method, due to non-constant coefficients. No shortcut applies.

Let $p(t) = \dfrac{12}{500-2t} = \dfrac{6}{250-t}$
Let $r(t) = (10)(0.1) = 1$
Standard linear DE form $x' + px = r$ is verified.
Let $W(t) = e^{\int p(t)dt} = e^{-6\ln|250-t|+c_1}$, then select
$W = (250-t)^{-6}$ for $t = 0$ to 250.

Find x_h and x_p:

$x_p = \frac{1}{W}\int rW\,dt = 50 - \frac{1}{5}\,t$
$x_h = c/W = c\,(250-t)^6$
It remains to determine c from $x(0) = k$.
$k = x_h(0) + x_p(0) = c\,250^6 + 50 - \frac{1}{5}\,(0)$
$c = (k-50)\,/250^6$

```
V:=t->500-2*t; # Tank volume after t minutes
de:=diff(x(t),t)=10*(0.1) - 12 *x(t)/V(t);# Maple
ic:=x(0)=k;
dsolve([de,ic],x(t));
# x(t) = 50-(1/5)*t+(-250+t)^6*(k-1/2)/250^6
```

16. The inlet adds 11 liters per minute with concentration $C_1 = 0.0156$ kilograms per liter. The tank contains 700 liters of brine in which k kilograms of salt is dissolved. The outlet drains 12 liters per minute. Determine additionally the time when the tank is empty.

Two-Tank Mixing

Assume brine tanks A and B in Figure 4 have volumes 100 and 200 gallons, respectively. Let $x(t)$ and $y(t)$ denote the number of pounds of salt at time t, respectively, in tanks A and B. Distilled water flows into tank A, then brine flows out of tank A and into tank B, then out of tank B. All flows are at r gallons per minute. Given rate r and initial salt amounts $x(0)$ and $y(0)$, find $x(t)$ and $y(t)$.

17. $r = 4$, $x(0) = 40$, $y(0) = 20$.

Solution: $x(t) = 40\,e^{-4t/100}$, $y(t) = -80\,e^{-4t/100} + 100\,e^{-4t/200}$

The model:

$$\begin{aligned} \frac{dx}{dt} &= a(t)\,C_1 - b(t)\,\frac{x(t)}{V_A(t)}, \\ \frac{dy}{dt} &= b(t)\,\frac{x(t)}{V_A(t)} - c(t)\,\frac{y(t)}{V_B(t)}. \end{aligned}$$

Define tank volumes $A_0 = 100$, $B_0 = 200$. Flow rates are defined by $a(t) = b(t) = c(t) = r$. Given: $V_A = A_0 = 100$, $V_B = B_0 = 200$. Distilled water has no salt: $C_1 = 0$. The initial value problem:

$$\begin{aligned} \frac{dx}{dt} &= (r)(0) - r\,\frac{x(t)}{A_0}, \quad x(0) = 40, \\ \frac{dy}{dt} &= r\,\frac{x(t)}{A_0} - r\,\frac{y(t)}{B_0}, \quad y(0) = 20 \end{aligned}$$

After substitutions and simplifications:

$$\begin{aligned} \frac{dx}{dt} &= - r\,\frac{x(t)}{100}, \quad x(0) = 40, \\ \frac{dy}{dt} &= r\,\frac{x(t)}{100} - r\,\frac{y(t)}{200}, \quad y(0) = 20 \end{aligned}$$

The first equation is homogeneous first order with solution

$$x(t) = x(0)e^{-rt/100} = 40e^{-4t/100}$$

The second equation then becomes

$$\frac{dy}{dt} = \frac{r}{100}\,x(0)e^{-rt/100} - r\,\frac{y(t)}{200} \quad \text{where } r = 4 \text{ and } x(0) = 40$$

The classification is linear first order non-homogeneous with non-constant coefficients. The linear integrating factor method is required to solve it:

$$y(t) = -80\,e^{-4t/100} + 100\,e^{-4t/200}$$

```
de:=diff(y(t),t)=(r/100)*x[0]*exp(-r*t/100) - r*y(t)/200;
ic:=y(0)=y[0];
dsolve([de,ic],y(t));
# y(t) = -2*x[0]*exp(-r*t/100)+(y[0]+2*x[0])*exp(-r*t/200)
```

18. $r = 3$, $x(0) = 10$, $y(0) = 15$.

19. $r = 5$, $x(0) = 20$, $y(0) = 40$.

Solution: $y(t) = -40e^{-t/20} + 80e^{-t/40}$

```
r:=5;x0:=20;y0:=40;
de:=diff(y(t),t)=(r/100)*x0*exp(-r*t/100) - r*y(t)/200;
ic:=y(0)=y0;
dsolve([de,ic],y(t));
# y(t) = -40*exp(-t/20)+80*exp(-t/40)
```

20. $r = 5$, $x(0) = 40$, $y(0) = 30$.

21. $r = 8$, $x(0) = 10$, $y(0) = 12$.

Solution: $y(t) = -20e^{-2t/25} + 32e^{-t/25}$

```
r:=8;x0:=10;y0:=12;
de:=diff(y(t),t)=(r/100)*x0*exp(-r*t/100) - r*y(t)/200;
ic:=y(0)=y0;
dsolve([de,ic],y(t));
# y(t) = -20*exp(-2*t/25)+32*exp(-t/25)
```

22. $r = 8$, $x(0) = 30$, $y(0) = 12$.

23. $r = 9$, $x(0) = 16$, $y(0) = 14$.

Solution: $y(t) = -32e^{-9t/100} + 46e^{-9t/200}$

```
r:=9;x0:=16;y0:=14;
de:=diff(y(t),t)=(r/100)*x0*exp(-r*t/100) - r*y(t)/200;
ic:=y(0)=y0;
dsolve([de,ic],y(t));
# y(t) = (-32*exp(-(9/200)*t)+46)*exp(-(9/200)*t)
```

24. $r = 9$, $x(0) = 22$, $y(0) = 10$.

25. $r = 7$, $x(0) = 6$, $y(0) = 5$.

Solution: $y(t) = -12e^{-7t/100} + 17e^{-7t/200}$

```
r:=7;x0:=6;y0:=5;
de:=diff(y(t),t)=(r/100)*x0*exp(-r*t/100) - r*y(t)/200;
ic:=y(0)=y0;
dsolve([de,ic],y(t));
# y(t) = (-12*exp(-(7/200)*t)+17)*exp(-(7/200)*t)
```

26. $r = 7$, $x(0) = 13$, $y(0) = 26$

Residential Heating

Assume the Newton cooling model for heating and insulation values $1/4 \le k \le 1/2$. Follow Example 2.23, page 116.

27. The office heat goes off at 7PM. It's 74°F inside and 58°F outside overnight. Estimate the office temperature at 10PM, 1AM and 6AM.

Solution: The ranges for 10PM, 1AM and 6AM ($t = 3, 6, 11$):

$t = 3$: $61.57008256 \le k \le 65.55786484$,
$t = 6$: $58.79659309 \le k \le 61.57008256$,
$t = 11$: $58.06538834 \le k \le 59.02284578$

Follow the **Office Heating** example. Newton's law of cooling for linear convection is used:

$$\frac{du}{dt} = k(a(t) - u(t)) + s(t) + f(t)$$

There are no sources, $s(t) = f(t) = 0$. Supplied are values $a(t) = a_0 = 58$ and $u(0) = 74$. Unknown constant k is expected to be in the range of normal insulation: $\frac{1}{2} \le k \le \frac{1}{2}$. Then

$u'(t) + ku(t) = 58k, \quad u(0) = 74$

The constant-coefficient shortcut u = equilibrium + c/W applies, W = integrating factor = e^{kt}. Then

$u(t) = 58 + (74 - 58)e^{-kt}$

The question is answered by finding the max and min of $u(t)$ when $t = 3$, $t = 6$ and $t = 11$ hours, corresponding to times 10PM, 1AM and 6AM. Possible ways to solve the max-min problem are graphing, hand calculation and CAS. The quantities to apply max-min methods are:

$u(3) = 58 + 16\, e^{-3k}$,
$u(6) = 58 + 16\, e^{-6k}$,
$u(11) = 58 + 16\, e^{-11k}$

Computed max-min ranges:

$t = 3$: $\quad 61.57008256 \le k \le 65.55786484$,
$t = 6$: $\quad 58.79659309 \le k \le 61.57008256$,
$t = 11$: $\quad 58.06538834 \le k \le 59.02284578$

Hand computation can use the monotonicity of e^{-kt} to deduce that the max-min is at the endpoint. A calculator will provide the decimal values.

```
u:=t->58+(74-58)*exp(-k*t);# Maple
Krange:=k=1/4 .. 1/2;
F:=t->evalf([minimize(u(t),Krange),maximize(u(t),Krange)]);
F(3);F(6);F(11);
# [61.57008256, 65.55786484]
# [58.79659309, 61.57008256]
# [58.06538834, 59.02284578]
```

28. The office heat goes off at 6:30PM. It's 73°F inside and 55°F outside overnight. Estimate the office temperature at 9PM, 3AM and 7AM.

29. The radiator goes off at 9PM. It's 74°F inside and 58°F outside overnight. Estimate the room temperature at 11PM, 2AM and 6AM.

Solution: The ranges for 10PM, 1AM and 6AM ($t = 3, 6, 11$):

$t = 2$: $\quad 63.88607106 \le k \le 67.70449056$,

$t = 5$: $\quad 59.31335998 \le k \le 62.58407675$,
$t = 9$: $\quad 58.17774394 \le k \le 59.6863875$

The solution from exercise 27 applies directly.

```
u:=t->58+(74-58)*exp(-k*t);# Maple
Krange:=k=1/4 .. 1/2;
F:=t->evalf([minimize(u(t),Krange),maximize(u(t),Krange)]);
F(2);F(5);F(9);
# [63.88607106, 67.70449056]
# [59.31335998, 62.58407675]
# [58.17774394, 59.68638759]
```

30. The radiator goes off at 10PM. It's 72°F inside and 55°F outside overnight. Estimate the room temperature at 2AM, 5AM and 7AM.

31. The office heat goes on in the morning at 6:30AM. It's 57°F inside and 40° to 55°F outside until 11AM. Estimate the office temperature at 8AM, 9AM and 10AM. Assume the furnace provides a five degree temperature rise in 30 minutes with perfect insulation and the thermostat is set for 76°F.

Solution: Estimates:
54 to 65 at 8:00AM,
54 to 66 at 9:00 AM,
54 to 66 at 10:00 AM.

Model: $\dfrac{du}{dt} = k(a(t) - u(t)) + s(t) + k_1(T_0 - u(t))$

Assumptions: No sources: $s(t) = 0$. Thermostat setting: $T_0 = 76$. Assume $0.5 \le k \le 1$, good to poor insulation. Assume $40 \le a(t) \le 55$ for the duration of the analysis. Let $t = 0$ hours correspond to 6:30 AM.

Estimates required: $u(1.5)$, $u(2.5)$, $u(3.5)$ which are temperatures for 8:00, 9:00 and 10:00 AM.

Refined model: $\dfrac{du}{dt} = k(a(t) - u(t)) + k_1(76 - u(t)), \quad u(0) = 57$

Determine constant $k_1 = 0.611$: Assume a five degree temperature rise in 30 minutes with perfect insulation. The refined model uses $k = 0$ to give Newton's cooling equation $w'(t) = k_1(76 - w(t))$, $w(0) = 57$, $w(0.5) = w(0) + 5 = 62$. The constant-coefficient shortcut for standard form $y' + py = q$ gives $w(t) = 76 + ce^{-k_1 t}$. Let $t = 0$ in this equation: $57 = 76 + c$. Solve for $c = -19$, then $w(t) = 76 - 19e^{-k_1 t}$. Substitute $t = 0.5$ and $w(0.5) = 62$. Solve $62 = 76 - 19e^{-k_1/2}$ for $k_1 = -2\ln|14/19| = 0.6107632991$.

Final model:

$$\frac{du}{dt} = k(a(t) - u(t)) + 0.611(76 - u(t)), \quad u(0) = 57$$

$0.5 \le k \le 1, \quad 40 \le a(t) \le 55$

Estimates for $u(1.5)$**,** $u(2.5)$**,** $u(3.5)$:
The worst-case scenarios are $a(t) = 40$ and $a(t) = 55$. Two solution formulas are obtained:

Case $a(t) = 40$:
$$u(t) = \mathrm{e}^{-(k+0.611)t}\left(57 - 4\,\frac{10\,k + 11.609}{k + 0.611}\right) + 4\,\frac{10\,k + 11.609}{k + 0.611}$$
The max and min for $0.5 \le k \le 1$ and
$t = 1.5$: $53.95 \le u \le 59.27$
$t = 2.5$: $53.71 \le u \le 59.62$
$t = 3.5$: $53.67 \le u \le 59.74$

Case $a(t) = 55$:
$$u(t) = \mathrm{e}^{-(k+0.611)t}\left(57 - 4\,\frac{13.750\,k + 11.609}{k + 0.611}\right) + 4\,\frac{13.750\,k + 11.609}{k + 0.611}$$
The max and min for $0.5 \le k \le 1$ and
$t = 1.5$: $62.43 \le u \le 64.75$
$t = 2.5$: $62.86 \le u \le 65.96$
$t = 3.5$: $62.94 \le u \le 66.35$

```
# Assume u0:=57 F inside, heater R=5 F rise after
# RT:=30/60 hours, no sources,
# outside a1:=40 to a2:=55 F, thermostat T0:=76 F
u0:=57;R:=5;RT:=0.5;
a1:=40;a2:=55;T0:=76;
# Estimate k1 from first RT hours
# Assume perfect insulation k=0, outside a1 degrees
# Assume u(RT)=u(0)+R
kk:=0;a:=t->a1;
de:=diff(u(t),t)=kk*(a(t)-u(t))+k1*(T0-u(t));
ic:=u(0)=u0;
ANS:=dsolve([de,ic],u(t));
X:=unapply(rhs(ANS),(t,k1));
kk1:=solve(X(RT,k1)=u0+R,k1);
# Assume hereafter k1 equals kk1 = 0.6107632991
# === worst-case a==a1.
a:=t->a1;
de1:=diff(u(t),t)=k*(a(t)-u(t))+kk1*(T0-u(t));
ic:=u(0)=u0;
ANS:=dsolve([de1,ic],u(t));
X1:=unapply(rhs(ANS),(t,k));
with(Optimization):
# 6:30am is t=0 hours, 8am is T1:=1.5 hours
# 9am is T2:=2.5 hours, 10am is T3:=3.5 hours
# Assume insulation constants k = 0.5 to 1.0
T1:=1.5;T2:=2.5;T3:=3.5;
Minimize(X1(T1,k),k=0.5..1);Maximize(X1(T1,k),k=0.5..1);
Minimize(X1(T2,k),k=0.5..1);Maximize(X1(T2,k),k=0.5..1);
Minimize(X1(T3,k),k=0.5..1);Maximize(X1(T3,k),k=0.5..1);
# T1: 53.949354482610886 to 59.26675808435088
# T2: 53.71006968557396 to 59.62099708075318
# T3: 53.66227611314616 to 59.73765063938258
#plot3d(X1(t,k),t=0..4,k=0.5 .. 1);
# worst-case a==a2.
a:=t->a2;
de2:=diff(u(t),t)=k*(a(t)-u(t))+kk1*(T0-u(t));
ANS:=dsolve([de2,ic],u(t));
X2:=unapply(rhs(ANS),(t,k));
T1:=1.5;T2:=2.5;T3:=3.5;
Minimize(X2(T1,k),k=0.5..1);Maximize(X2(T1,k),k=0.5..1);
Minimize(X2(T2,k),k=0.5..1);Maximize(X2(T2,k),k=0.5..1);
Minimize(X2(T3,k),k=0.5..1);Maximize(X2(T3,k),k=0.5..1);
# T1: 62.430441592457164 to 64.74289674260832
# T2: 62.85639147974006 to 65.9529226338951
# T3: 62.94146862229387 to 66.35139323618871
```

32. The office heat goes on at 6AM. It's 55°F inside and 43° to 53°F outside until 10AM. Estimate the office temperature at 7AM, 8AM and 9AM. Assume the furnace provides a seven degree temperature rise in 45 minutes with perfect insulation and the thermostat is set for 78°F.

33. The hot water heating goes on at 6AM. It's 55°F inside and 50° to 60°F outside until 10AM. Estimate the room temperature at 7:30AM. Assume the radiator provides a four degree temperature rise in 45 minutes with perfect insulation and the thermostat is set for 74°F.

Solution: Estimate: 56 to 62 F. Follow the solution of Exercise 31. Ans check:

```
# Assume u0:=55 F inside, heater R=4 F
# rise after RT:=45/60 hours, no sources,
# outside a1:=50 to a2:=60 F, thermostat T0:=74 F
u0:=55;R:=4;RT:=0.75;a1:=50;a2:=60;T0:=74;
# Estimate k1 from first RT hours
# Assume perfect insulation k=0, outside constant a1 degrees
# Assume u(RT)=u(0)+R
kk:=0;a:=t->a1;
de:=diff(u(t),t)=kk*(a(t)-u(t))+k1*(T0-u(t));
ic:=u(0)=u0;
ANS:=dsolve([de,ic],u(t));
X:=unapply(rhs(ANS),(t,k1));
kk1:=solve(X(RT,k1)=u0+R,k1);
# Assume hereafter k1 equals kk1 = 0.3151850374
a:=t->a1;# worst-case a==a1.
de1:=diff(u(t),t)=k*(a(t)-u(t))+kk1*(T0-u(t));
ic:=u(0)=u0;ANS:=dsolve([de1,ic],u(t));
X1:=unapply(rhs(ANS),(t,k));
with(Optimization):
# 6:00am is t=0 hours, 7:30am is T1:=1.5 hours
# Assume insulation constants k = 0.5 to 1.0
T1:=1.5;
Minimize(X1(T1,k),k=0.5 .. 1);
Maximize(X1(T1,k),k=0.5 .. 1);
# T1: 55.6470897741918  to  58.0195074023563
#plot3d(X1(t,k),t=0..4,k=0.5 .. 1);
a:=t->a2;# worst-case a==a2.
de2:=diff(u(t),t)=k*(a(t)-u(t))+kk1*(T0-u(t));
ANS:=dsolve([de2,ic],u(t));
X2:=unapply(rhs(ANS),(t,k));T1:=1.5;
Minimize(X2(T1,k),k=0.5 .. 1);
Maximize(X2(T1,k),k=0.5 .. 1);
# T1: 62.1931645171822  to  62.3472900101369
```

34. The hot water heating goes on at 5:30AM. It's 54°F inside and 48° to 58°F outside until 9AM. Estimate the room temperature at 7AM. Assume the radiator provides a five degree temperature rise in 45 minutes with perfect insulation and the thermostat is set for 74°F.

35. A portable heater goes on at 7AM. It's 45°F inside and 40° to 46°F outside until 11AM. Estimate the room temperature at 9AM. Assume the heater provides a two degree temperature rise in 30 minutes with perfect insulation and the thermostat is set for 90°F.

Solution: At 9 am it is about 56 F to 62 F.
Follow the solution to Exercise 33.

36. A portable heater goes on at 8AM. It's 40°F inside and 40° to 45°F outside until 11AM. Estimate the room temperature at 10AM. Assume the heater provides a two degree temperature rise in 20 minutes with perfect insulation and the thermostat is set for 90°F.

Evaporative Cooling

Define outside temperature (see Figure 3)

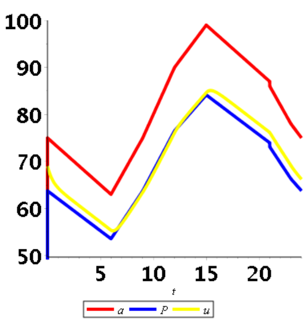
$$a(t) = \begin{cases} 75 - 2\,t & 0 \le t \le 6 \\ 39 + 4\,t & 6 < t \le 9 \\ 30 + 5\,t & 9 < t \le 12 \\ 54 + 3\,t & 12 < t \le 15 \\ 129 - 2\,t & 15 < t \le 21 \\ 170 - 4\,t & 21 < t \le 23 \\ 147 - 3\,t & 23 < t \le 24 \end{cases}.$$

Given k, k_1, $P(t) = wa(t)$ and $u(0) = 69$, then plot $u(t)$, $P(t)$ and $a(t)$ on one graphic.

$$u(t) = u(0)e^{-kt-k_1 t} + (k + wk_1)\int_0^t a(r)e^{(k+k_1)(r-t)}dr.$$

37. $k = 1/4$, $k_1 = 2$, $w = 0.85$

Solution: It is necessary to use a computing workbench or CAS with graphics. The code for `maple` appears below.

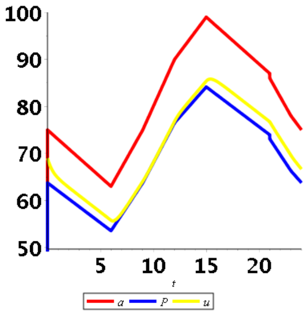


```
w:=0.85;k:=1/4;k1:=2;
F:=< 75-2*t, 39+4*t, 30+5*t, 54+3*t,
     129-2*t, 170-4*t, 147-3*t>;
V:=<0,6,9,12,15,21,23,24>;
N:=ArrayNumElems(F);
for i from 1 to N do
  W[i] := piecewise(V[i] <t and t <= V[i+1],F[i],0);
od;
a:=unapply(sum(W[j],j=1..N),t);
P:=t->w*a(t);
u:=unapply( 69*exp(-k*t-k1*t)+
     (k+w*k1)*int(a(r)*exp((k+k1)*(r-t)),r=0..t),t);
opts:=thickness=4,font=[Courier,bold,20],
color=[red,blue,yellow],legend=[a,P,u];
plot([a(t),P(t),u(t)],t=0..24,50..100,opts);
```

38. $k = 1/4$, $k_1 = 1.8$, $w = 0.85$

39. $k = 3/8$, $k_1 = 2$, $w = 0.85$

Solution: A computing workbench or CAS with graphics is required. The `maple` code in Exercise 37 is used, changes are below.

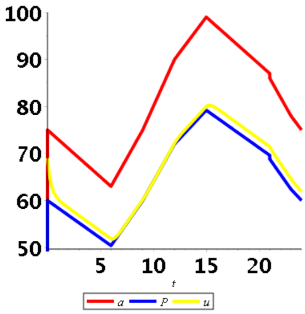



```
# Minor change to Exercise 37
w:=0.85;k:=3/8;k1:=2;
```

40. $k = 3/8$, $k_1 = 2.4$, $w = 0.85$

41. $k = 1/4$, $k_1 = 3$, $w = 0.80$

Solution: A computing workbench or CAS with graphics is required. The `maple` code in Exercise 37 is used, changes are below.

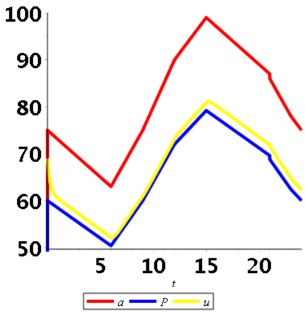



```
# Minor change to Exercise 37
w:=0.80;k:=1/4;k1:=3;
```

42. $k = 1/4$, $k_1 = 4$, $w = 0.80$

43. $k = 1/2$, $k_1 = 4$, $w = 0.80$

Solution: A computing workbench or CAS with graphics is required. The `maple` code in Exercise 37 is used, changes are below.

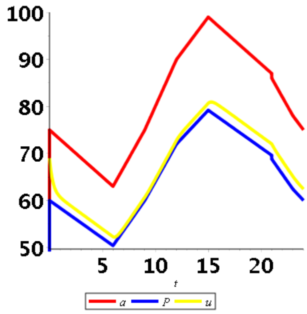



```
# Minor change to Exercise 37
w:=0.80;k:=1/2;k1:=4;
```

44. $k = 1/2$, $k_1 = 5$, $w = 0.80$

45. $k = 3/8$, $k_1 = 3$, $w = 0.80$

Solution: A computing workbench or CAS with graphics is required. The `maple` code in Exercise 37 is used, changes are below.




```
# Minor change to Exercise 37
w:=0.80;k:=3/8;k1:=3;
```

46. $k = 3/8$, $k_1 = 4$, $w = 0.80$

Radioactive Chain

Let A, B and C be the amounts of three radioactive isotopes. Assume A decays into B at rate a, then B decays into C at rate b. Given a, b, $A(0) = A_0$ and $B(0) = B_0$, find formulas for A and B.

47. $a = 2$, $b = 3$, $A_0 = 100$, $B_0 = 10$

Solution: The **Radioactive Chain** Example 2.26, page 119 will be used. Formulas for A and B:

$$A(t) = A_0 e^{-at}, \quad B(t) = B_0 e^{-bt} + aA_0 \frac{e^{-at} - e^{-bt}}{b - a}.$$

Then:

$$A(t) = 100e^{-2t}, \quad B(t) = 200e^{-3t} + 200 \frac{e^{-2t} - e^{-3t}}{3 - 2}.$$

The solution for $B(t)$ is $B(t)$ = homogeneous + particular. The homogeneous solution of $B' = aA - bB$ is the solution $B_h = Ce^{-bt}$ of equation $B_h' = -bB_h$. The particular solution is extracted from the Example.

48. $a = 2$, $b = 3$, $A_0 = 100$, $B_0 = 100$

49. $a = 1$, $b = 4$, $A_0 = 100$, $B_0 = 200$

Solution: Use Exercise 47:

$$A(t) = 100e^{-t}, \quad B(t) = 200e^{-4t} + 1(100)\frac{e^{-t} - e^{-4t}}{4 - 1}.$$

50. $a = 1$, $b = 4$, $A_0 = 300$, $B_0 = 100$

51. $a = 4$, $b = 3$, $A_0 = 100$, $B_0 = 100$

Solution: Use Exercise 47:

$$A(t) = 100e^{-4t}, \quad B(t) = 100e^{-3t} + 4(100)\frac{e^{-4t} - e^{-3t}}{3-4}.$$

52. $a = 4$, $b = 3$, $A_0 = 100$, $B_0 = 200$

53. $a = 6$, $b = 1$, $A_0 = 600$, $B_0 = 100$

Solution: Use Exercise 47:

$$A(t) = 600e^{-6t}, \quad B(t) = 100e^{-3t} + 6(600)\frac{e^{-6t} - e^{-t}}{1-6}.$$

54. $a = 6$, $b = 1$, $A_0 = 500$, $B_0 = 400$

55. $a = 3$, $b = 1$, $A_0 = 100$, $B_0 = 200$

Solution: Use Exercise 47:

$$A(t) = 100e^{-3t}, \quad B(t) = 200e^{-t} + 3(100)\frac{e^{-3t} - e^{-t}}{1-3}.$$

56. $a = 3$, $b = 1$, $A_0 = 400$, $B_0 = 700$

Electric Circuits

In the LR-circuit of Figure 5, assume $E(t) = A\cos wt$ and $I(0) = 0$. Solve for $I(t)$.

57. $A = 100$, $w = 2\pi$, $R = 1$, $L = 2$

Solution: The answer:

$$I(t) = C\,\mathrm{e}^{-t/2} + 100\,\frac{4\,\pi\,\sin(2\,\pi\,t) + \cos(2\,\pi\,t)}{16\,\pi^2 + 1}$$

Electric Circuits Example 2.27 will be used. The current is found from $I(t) =$ homogeneous + particular. The homogeneous solution is the transient current $I_{\mathrm{tr}} = Ce^{-Rt/L}$ for some constant C. Let's use the linear integrating factor method, which finds both the homogeneous solution and a particular solution in one computation.

The Model: $LI'(t) + RI(t) = A\cos wt$

$I'(t) + \frac{R}{L}I(t) = \frac{A}{L}\cos wt$	**Standard Form**
$W = e^{Rt/L}$	**Integrating Factor**

$\dfrac{(W(t)I(t))'}{W(t)} = \frac{A}{L}W(t)\cos wt$	**Quadrature Form**
$(W(t)I(t))' = \frac{A}{L}W(t)\cos(wt)$	**Clear Fraction Left**
$\int(W(t)I(t))'dt = \frac{A}{L}\int W(t)\cos(wt)dt$	**Quadrature Step**
$W(t)I(t) = C + \frac{A}{L}\int e^{-Rt/L}\cos(wt)dt$	**Fund. Theorem of Calculus**

Integral Table:

$$W(t)I(t) = C + 2\,\frac{e^{t/2}\left(4\,\pi\,\sin(2\,\pi\,t) + \cos(2\,\pi\,t)\right)}{16\,\pi^2+1}$$

```
# Maple integration
int(exp(R*t/L)*cos(w*t),t);simplify(%);
# Answer check for L I'(t) + R I(t) = A cos wt
A:=100;w:=2*Pi;R:=1;L:=2;
de:=L*diff(u(t),t) + R*u(t) = A * cos(w*t);
dsolve(de,u(t));
```

58. $A = 100$, $w = 4\pi$, $R = 1$, $L = 2$

59. $A = 100$, $w = 2\pi$, $R = 10$, $L = 1$

Solution: Use the methods in Exercise 57.

$$I(t) = C\,e^{-10\,t} + 50\,\frac{\pi\,\sin(2\,\pi\,t) + 5\,\cos(2\,\pi\,t)}{\pi^2+25}$$

```
A:=100;w:=2*Pi;R:=10;L:=1;
```

60. $A = 100$, $w = 2\pi$, $R = 10$, $L = 2$

61. $A = 5$, $w = 10$, $R = 2$, $L = 3$

Solution: Use the methods in Exercise 57.

$$I(t) = C\ + e^{-2/3\,t} + \frac{5\,\cos(10\,t)}{452} + \frac{75\,\sin(10\,t)}{452}$$

```
A:=5;w:=10;R:=2;L:=3;
```

62. $A = 5$, $w = 4$, $R = 3$, $L = 2$

63. $A = 15$, $w = 2$, $R = 1$, $L = 4$

Solution: Use the methods in Exercise 57.

$$I(t) = C\,e^{-t/4} + \frac{3}{13}\,\cos(2\,t) + \frac{24\,\sin(2\,t)}{13}$$

```
A:=15;w:=2;R:=1;L:=4;
```

64. $A = 20,\ w = 2,\ R = 1,\ L = 3$

65. $A = 25,\ w = 100,\ R = 5,\ L = 15$

Solution: Use the methods in Exercise 57.

$$I(t) = C\,\mathrm{e}^{-t/3} + \frac{5\ \cos(100\,t)}{90001} + \frac{1500\ \sin(100\,t)}{90001}$$

```
A:=25;w:=100;R:=5;L:=15;
```

66. $A = 25,\ w = 50,\ R = 5,\ L = 5$

2.6 Kinetics

Newton's Laws

Review of units and conversions.

1. An object weighs 100 pounds. Find its mass in slugs and kilograms.

Solution: Confusion exists for **lb**, **libre** and **lbf** (pound-force not foot-pound). The pound-force **lbf** is the product of one avoirdupois pound (exactly 0.45359237 kg) and the standard sea level acceleration due to gravity, $g = 9.80665$ m/sec/sec, briefly 1 lbf = 4.448221615 Newtons. Newton's Law $F = ma$ at sea level in a vacuum then gives $4.448221615 = m(9.80665)$ for mass $m = 0.4535923700$ kg. On a kitchen scale with a kg/lb switch some package marked 453 grams will read 453 g or 1 lb, depending on the switch position. This information explains why consumer courses say 1 kg=2.2 lb: it is valid in a vacuum at sea level as a quick way to interpret kg scale values as lb scale values. Most people find an approximate answer: 100 lb on a scale has mass $100/2.2 = 45.45$ kg. Other answers close to 45.45 are also acceptable, one not being better than the other, because the value of g depends on the unknown location on the earth.

The mass in slugs is found directly from Newton's Law $F = ma$ using $F = 100$ lbf and $g = 9.80665$ m/sec/sec or $g = 32.17404856$ ft/sec/sec. Then $100 = 32.17404856\, m$ gives $m = 3.108095017$ slugs.

2. An object has mass 50 kilograms. Find its mass in slugs and its weight in pounds.

3. Convert from **fps** to **mks** systems: position 1000, velocity 10, acceleration 2.

Solution: Answers:
Position = 1000 ft = $1000 * 30.48$ cm = 304.8 m.
Velocity = 10 ft/sec = $10 * 30.48$ cm/sec = $10 * 30.48/100$ m/sec = 3.048 m/sec.
Acceleration = 2 ft/sec/sec = $2*30.48$ cm/sec/sec = $2*30.48/100$ cm/sec/sec = 0.6096 m/sec/sec.

4. Derive $g = \dfrac{Gm}{R^2}$, where m is the mass of the earth and R is its radius.

Velocity and Acceleration

Find the velocity x' and acceleration x''.

5. $x(t) = 16t^2 + 100$

Solution: Answer:
Velocity $= x'(t) = 32t$,
Acceleration $= x''(t) = 32$.

6. $x(t) = 16t^2 + 10t + 100$

7. $x(t) = t^3 + t + 1$

Solution: Answer:
Velocity $= x'(t) = 3t^2 + 1$,
Acceleration $= x''(t) = 6t$.

8. $x(t) = t(t-1)(t-2)$

Free Fall with Constant Gravity

Solve using the model $x''(t) = -g$, $x(0) = x_0$, $x'(0) = v_0$.

9. A brick falls from a tall building, straight down. Find the distance it fell and its speed at three seconds.

Solution: It fell 144 feet and reached 288 ft/sec in 3 seconds.

Model: $x''(t) = -g$, $x(0) = x_0$, $x'(0) = v_0$

Choose coordinates $x = 0$ for the top of the building. Define $v_0 = 0$, the brick falls from rest. Let $g = 32$ ft/sec/sec instead of $g = 32.088$ in system **fps**, because other physical factors have been ignored. Coordinates cause the model to change signs $-g$ to $+g$ because the position vector is $x(t)\vec{\jmath}$, which aligns with the gravity vector $g\vec{\jmath}$. Effectively, x has been replaced by $-x$ in the original model. Then the model becomes

$x''(t) = 32$, $x(0) = 0$, $x'(0) = 0$

A quadrature finds $x(t) = 16t^2$. Then $x(3) = 16(9) = 144$, which means the brick fell 144 feet. The speed at 3 seconds is $x'(3) = (gt)|_{t=3} = 288$ ft/sec.

10. An iron ingot falls from a tall building, straight down. Find the distance it fell and its speed at four seconds.

11. A ball is thrown straight up from the ground with initial velocity 66 feet per second. Find its maximum height.

Solution: It reached 2.0625 feet and then fell back to the ground.

Model: $x''(t) = -g$, $x(0) = x_0$, $x'(0) = v_0$

Choose coordinates $x = 0$ for the ground. Define $v_0 = 66$ ft/sec, the starting velocity. Let $g = 32$ ft/sec/sec instead of $g = 32.088$ in system **fps**, because other physical factors have been ignored. Then the model becomes

$x''(t) = -32$, $x(0) = 0$, $x'(0) = 66$

A quadrature finds $x(t) = -16t^2 + at + b$ for some constants a, b. Initial conditions $x(0) = 0$, $x'(0) = 66$ evaluate $a = 66$, $b = 0$. The maximum height is $x(T) = \max_{t\ge 0} x(t)$, the value T guaranteed by continuity of $x(t)$. Find T from $x'(T) = 0$: $-32T + a = 0$ or $T = a/32 = 66/32 = 2.0625$ feet.

12. A ball is thrown straight up from the ground with initial velocity 88 feet per second. Find its maximum height.

13. An arrow is shot straight up from the ground with initial velocity 23 meters per second. Find the flight time back to the ground.

Solution: The flight time is $T = 0.2$ sec.

Model: $x''(t) = -g$, $x(0) = x_0$, $x'(0) = v_0$

Choose coordinates $x = 0$ for the ground. Define $v_0 = 23$ m/sec, the starting velocity. Let $g = 9.8$ ft/sec/sec, because other physical factors have been ignored. Then the model becomes

$x''(t) = -9.8$, $x(0) = 0$, $x'(0) = 23$

A quadrature finds $x(t) = -4.6t^2 + at + b$ for some constants a, b. Initial conditions $x(0) = 0$, $x'(0) = 23$ evaluate $a = 23$, $b = 0$. The flight time is the first value $T > 0$ with $x(T) = 0$. The equation for T is $0 = -4.6T^2 + 23T$. Then $T = 4.6/23 = 0.2 = 1/5$.

14. An arrow is shot straight up from the ground with initial velocity 44 meters per second. Find the flight time back to the ground.

15. A car travels 140 kilometers per hour. Brakes are applied, with deceleration 10 meters per second per second. Find the distance the car travels before stopping.

Solution: Answer: 75.62 meters.

Assume the car is on a level road traveling with constant velocity 140 km/h. At $t = 0$ the brakes are applied with acceleration $a = -10$ m/sec/sec. Then $x''(t) = -10$ for $t > 0$ and $x'(0) = 140$ km/h. Data units must be changed to meters and seconds. Then $x'(0) = 140 * 1000/3600 = 350/9$ m/sec. Solve by quadrature: $x'(t) = -10t + 350/9$ and $x(t) = -5t^2 + 350t/9 + x(0)$. The car stops when the velocity is zero: $0 = -10t + 350/9$. Then the distance traveled is $x(t) - x(0) = -5t^2 + 350t/9 = -5(35/9)^2 + (350/9)(35/9) = 6125/81 = 75.62$ m.

16. A car travels 120 kilometers per hour. Brakes are applied, with deceleration 40 feet per second per second. Find the distance the car travels before stopping.

17. An arrow is shot straight down from a height of 500 feet, with initial velocity 44 feet per second. Find the flight time to the ground and its impact speed.

Solution: Answer: Flight time 7.13 seconds, impact speed -184.22 ft/sec.

Assume no air resistance. The distance $x(t)$ of the arrow center of mass to the ground has model $x''(t) = -32$ (position vector ground to arrow has direction opposite the gravity vector). Also known is $x(0) = 500$, $x'(0) = 44$ in **fps** units. Solve by quadrature: $x'(t) = -32t + 44$, $x(t) = -16t^2 + 44t + 500$. The arrow impacts the ground at time t satisfying $x(t) = 0$. Solve $-16t^2 + 44t + 500 = 0$ for $t = \frac{1}{8}(11 \pm \sqrt{2121}) = -4.381789470, 7.131789470$. The positive time is relevant: impact at $t = 7.131789470$ seconds. The impact speed is $x'(7.131789470) = -4\sqrt{2121} = -184.2172630$ ft/sec.

18. An arrow is shot straight down from a height of 200 meters, with initial velocity 13 meters per second. Find the flight time to the ground and its impact speed.

Linear Air Resistance

Solve using the linear air resistance model $mx''(t) = -kx'(t) - mg$. An equivalent model is $x'' = -\rho x' - g$, where $\rho = k/m$ is the drag factor.

19. An arrow is shot straight up from the ground with initial velocity 23 meters per second. Find the flight time back to the ground. Assume $\rho = 0.035$.

Solution: Answer: 4.57 seconds.

Assume model $x'' = -\rho x' - g$ with drag factor $\rho = 0.035$. The velocity model is $v' = -0.035v - 9.8$, $v(0) = 23$ with solution given by (see equation 9)

$$v(t) = -\frac{g}{\rho} + \left(v(0) + \frac{g}{\rho}\right) e^{-\rho t},$$
$$x(t) = x(0) - \frac{g}{\rho} t + \frac{1}{\rho}\left(v(0) + \frac{g}{\rho}\right)\left(1 - e^{-\rho t}\right).$$

The flight time to return to the ground is time $T > 0$ with $x(T) = 0$. Because $x(0) = 0$ (launch from the ground), then

$$0 = -\frac{g}{\rho} T + \frac{1}{\rho}\left(v(0) + \frac{g}{\rho}\right)\left(1 - e^{-\rho T}\right)$$

This nonlinear equation is solved graphically for T by plotting two curves on the same xy-axes: $y = \frac{g}{\rho} x$ and $y = \frac{1}{\rho}\left(v(0) + \frac{g}{\rho}\right)(1 - e^{-\rho x})$. Alternatively, a CAS can find T. Both methods require numbers in the equation:

$$0 = -280T + \frac{1}{0.035}(23 + 280)\left(1 - e^{-0.035T}\right)$$

```
Maple answer: T=4.571994605 seconds.
EQ:=-280 *x+(1/0.035)*(23+280)*( 1 - exp(-0.035* x) );
solve(EQ=0,x);
```

20. An arrow is shot straight up from the ground with initial velocity 27 meters per second. Find the maximum height. Assume $\rho = 0.04$.

21. A parcel is dropped from an aircraft at 32,000 feet. It has a parachute that opens automatically after 25 seconds. Assume drag factor $\rho = 0.16$ without the parachute and $\rho = 1.45$ with it. Find the descent time to the ground.

Solution: The descent time to the ground is 1298.59 seconds, about 22 minutes.

The problem requires two models, switching from **Model 1** to **Model 2** at time $t = 25$ seconds. Let $x(t)$ be the distance in feet from the parcel to the ground, $x(0) = 32000$. Let $y(0) = x(25)$, which is the distance in feet to the ground when the parachute opens. For $t > 0$, value $y(t)$ is the parcel distance in feet to the ground. Technical issues: the parcel falls from rest, $x'(0) = 0$, but $y'(0)$ is not zero: it is the speed of the parcel at $t = 25$ seconds.

Flight Time: Let $T > 0$ be the first root of $y(T) = 0$. The flight time to the ground is $25 + T$.

Model 0: $u'' = -0.00u' - 32$, $u(0) = 32000$, $u'(0) = 0$
Assume zero drag force, then compute terminal velocity and flight time: it is used for comparison.

Model 1: $x'' = -0.16x' - 32$, $x(0) = 32000$, $x'(0) = 0$

Model 2: $y'' = -1.45y' - 32$, $y(0) = x(25)$, $y'(0) = x'(25)$

Solve Model 0: $u(t) = -16t^2 + 32000$
The parcel flight time is $T_0 = 44.72$ seconds and the impact speed is $|u'(T_0)| = 1431.1$ ft/sec $= 975.75$ miles/hour. Unrealistic.

Solve Model 1: $x(t) = -1250\,\mathrm{e}^{-\frac{4\,t}{25}} - 200\,t + 33250$
Then $x(25) = 28227.10545$ and $x'(25) = -196.3368722$.

Solve Model 2: $y(t) = 120.1847632\vec{e}^{\,-1.45\,t} - 22.06896552\,t + 28106.92069$
Then $y(T) = 0$ when $T = 1273.594844$ seconds. Flight time $25 + T = 1298.594844$ seconds, about 22 minutes, terminal velocity $y'(T) = -22.06896552$ ft/sec, about 15 mph. This speed is reached for practical purposes after about 5 seconds into flight. The parcel falls from 28,000 feet at 15 mph, taking about 21 minutes.

```
de:=diff(x(t),t,t)=-rho*diff(x(t),t)-g;
de0:=subs(rho=0,g=32,de);
de1:=subs(rho=0.16,g=32,de);
de2:=subs(rho=1.45,g=32,de);
ans0:=dsolve([de0,x(0)=32000,D(x)(0)=0],x(t));
X0:=unapply(rhs(ans0),t);solve(X0(t)=0,t);
T0:=20*sqrt(5); X0(T0);
"X0 terminal velocity"=evalf(D(X0)(T0));
"X0 flight time"=evalf(T0);
ans1:=dsolve([de1,x(0)=32000,D(x)(0)=0],x(t));
X:=unapply(evalf(rhs(ans1)),t);
y0:=X(25);y1:=D(X)(25);
ans2:=dsolve([de2,x(0)=y0,D(x)(0)=y1],x(t));
Y:=unapply(evalf(rhs(ans2)),t);
"Y Time to ground" = solve(Y(t)=0,t);# 1273.594844 sec
"Flight Time" = 25+1273.594844; # 1298.594844 sec
```

22. A first aid kit is dropped from a helicopter at $12,000$ feet. It has a parachute that opens automatically after 15 seconds. Assume drag factor $\rho = 0.12$ without the parachute and $\rho = 1.55$ with it. Find the impact speed with the ground.

23. A motorboat has velocity v satisfying $1100v'(t) = 6000 - 110v$, $v(0) = 0$. Find the maximum speed of the boat.

Solution: The maximum speed should be the maximum of $v(t)$. However, the calculus theory applies to a finite interval and not to interval $0 \le t < \infty$. There is an equilibrium solution found from formally setting $v'(t) = 0$: $0 = 6000 - 110v(t)$ gives $v(t) = 6000/110 = 54.55$. A maximum speed report of about 54 or 55 is a good answer: the boat never travels faster than 54.55.

24. A motorboat has velocity v satisfying $1000v'(t) = 4000 - 90v$, $v(0) = 0$. Find the maximum speed of the boat.

25. A parachutist falls until his speed is 65 miles per hour. He opens the parachute. Assume parachute drag factor $\rho = 1.57$. About how many seconds must elapse before his speed is reduced to within 1% of terminal velocity?

Solution: It takes about 3.76 seconds to reach within 1% of terminal velocity -20.38216561 fps.

Details: Use $\rho = 1.57$ and assume $x(0) = x_0$ is unknown, $x'(0) = -65$ mph. Units must be converted to match $g = 32$, which is in **fps** units. Then $x'(0) = -95.3333$ fps.

Model: $x'' = -1.57x' - 32$, $x(0) = x_0$, $x'(0) = -95.3333$

Solve the Model: Solve first for $v(t) = x'(t)$ in $v' = -1.57v - 32$, then use quadrature to find

$v(t) = -74.95113440\,\mathrm{e}^{-1.57\,t} - 20.38216561$
$x(t) = 47.73957605\,\mathrm{e}^{-1.57\,t} - 20.38216561\,t + x_0 - 47.73957605$

Solve the equation $v(t) = 1.01\,v(\infty)$ which is the equation

$-74.95113440\,\mathrm{e}^{-1.57\,t} - 20.38216561 = 1.01(-20.38216561)$

Then $t = 3.762640919$ seconds.

Why use 1.01 instead of 0.99? Answer: The velocity is negative with magnitude always larger than $|v(\infty)|$.

```
de:=diff(x(t),t,t)=-32-1.57*diff(x(t),t);
ans1:=dsolve([de,x(0)=x0,D(x)(0)=-95.3333],x(t));
X:=unapply(evalf(rhs(ans1)),t);
solve(D(X)(t)=1.01*(-20.38216561),t);
```

26. A parachutist falls until his speed is 120 kilometers per hour. He opens the parachute. Assume drag factor $\rho = 1.51$. About how many seconds must elapse before his speed is reduced to within 2% of terminal velocity?

27. A ball is thrown straight up with initial velocity 35 miles per hour. Find the ascent time and the descent time. Assume drag factor 0.042

Solution: Answers: Rise time = 1.55 seconds, fall time = 1.69 seconds.

Let $x(t)$ be the distance from the ground in feet. Then $x(0) = 0$ and $x'(0) = 35$ mph $= 51.3333$ ft/sec. The rise time T is the first $T > 0$ such that $x'(T) = 0$. The fall time is the second solution S to $x(S) = 0$.

Model: $x'' = -0.042x' - 32$, $x(0) = 0$, $x'(0) = 51.3333$
Then

$v(t) = 813.238062\,\mathrm{e}^{-0.042\,t} - 761.9047619$
$x(t) = -19362.811\,\mathrm{e}^{-0.042\,t} - 761.9047619\,t + 19362.811$

Solve $v(T) = 0$ for $T = 1.552436247$ seconds. Then solve $x(S) = 0$ for $S = 3.139362078$ seconds. The rise time is $T = 1.552436247$ and the fall time is $S - T = 1.586925831$.

```
de:=diff(x(t),t,t)=-32-0.042*diff(x(t),t)
ans1:=dsolve([de,x(0)=0,D(x)(0)=51.3333],x(t));
X:=unapply(evalf(rhs(ans1)),t);
solve(D(X)(T)=0,T); # T = 1.552436247
solve(X(S)=0,S); # S = 3.139362078
S,S-T; # 1.552436247, 1.586925831
```

28. A ball is thrown straight up with initial velocity 60 kilometers per hour. Find the ascent time and the descent time. Assume drag factor 0.042

Linear Ascent and Descent Times

Find the ascent time t_1 and the descent time t_2 for the linear model $x'' = -\rho x' - g$, $x(0) = 0$, $x'(0) = v_0$ where $\rho = k/m$ is the drag factor. Unit system **fps**. Computer algebra system expected.

29. $\rho = 0.01$, $v_0 = 50$

Solution: $t_1 = -100 \ln(64/65)$, $t_2 = 1.558472345$.

```
v0:=50;rho:=1/100;de:=diff(x(t),t,t)=-32-rho*diff(x(t),t);
ans1:=dsolve([de,x(0)=0,D(x)(0)=v0],x(t));
X1:=unapply(rhs(ans1),t);
t1:=solve(diff(X1(t),t)=0,t);
# t1 = -100*ln(64/65)
ans2:=dsolve([de,x(0)=X1(t1),D(x)(0)=0],x(t));
X2:=unapply(rhs(ans2),t);
t2:=solve(X2(t)=0,t);
evalf(t2); # t2 = 1.558472345
```

30. $\rho = 0.015$, $v_0 = 30$

31. $\rho = 0.02$, $v_0 = 50$

Solution: $t_1 = -50 \ln(32/33)$, $t_2 = 1.554528027$.

32. $\rho = 0.018$, $v_0 = 30$

33. $\rho = 0.022$, $v_0 = 50$

Solution: $t_1 = -(500/11) \ln(320/331)$, $t_2 = 1.553748824$.

34. $\rho = 0.025$, $v_0 = 30$

35. $\rho = 1.5$, $v_0 = 50$

Solution: $t_1 = -(2/3) \ln(32/107)$, $t_2 = 1.334352324$.

36. $\rho = 1.55$, $v_0 = 30$

37. $\rho = 1.6$, $v_0 = 50$

Solution: $t_1 = -(5/8) \ln(2/7)$, $t_2 = 1.330114810$.

38. $\rho = 1.65$, $v_0 = 30$

39. $\rho = 1.45$, $v_0 = 50$

Solution: $t_1 = -(20/29)\ln(64/209)$, $t_2 = 1.336698502$.

40. $\rho = 1.48$, $v_0 = 30$

Nonlinear Air Resistance

Assume ascent velocity v_1 satisfies $v_1' = -\rho v_1^2 - g$. Assume descent velocity v_2 satisfies $v_2' = \rho v_2^2 - g$. Motion from the ground $x = 0$. Let t_1 and t_2 be the ascent and descent times, so that $t_1 + t_2$ is the flight time. Let $g = 9.8$, $v_1(0) = v_0$, $v_1(t_1) = v_2(t_1) = 0$, units mks. Define M = maximum height and v_f = impact velocity. Computer algebra system expected.

41. Let $\rho = 0.0012$, $v_0 = 50$. Find t_1, t_2.

Solution: Answers:

$t1 = \frac{50}{21}\sqrt{15}\,\arctan(\frac{1}{7}\sqrt{15}) = 4.6601258,$

$$t2 = \frac{-200\sqrt{15}}{21}\ln\left(\frac{1}{7}\sqrt[4]{2744 - 343\sqrt{15}}\right) = 4.872170$$

Methods follow previous exercises on rise and fall times, the models being replaced by quadratic drag models. The work can be completed by hand using the equations for downward and upward launch in the section on **nonlinear air resistance**. Some details of the hand calculation:

Rise Time: Solve for $t = t_1$ in the equation

$$0 = v(t) = \sqrt{\frac{mg}{k}}\tan\left(\sqrt{\frac{kg}{m}}(c-t)\right),$$

which means $c - t = 0$ and then $t_1 = c$. By the same equation

$$\begin{aligned} 50 &= v_0 \\ &= \sqrt{\frac{mg}{k}}\tan\left(\sqrt{\frac{kg}{m}}(c-0)\right) \\ &= \sqrt{\frac{9.8}{0.0012}}\tan\left(c\sqrt{(9.8)(0.0012)}\right) \end{aligned}$$

Then $0.1084435337\,c = \arctan\left(50/\sqrt{\frac{9.8}{0.0012}}\right)$ gives $t_1 = c = 4.660125809$.

Maximum Height: The height reached on the upward launch is $x(t_1)$. To find the height requires the quadrature result for $x(t)$ obtained from $x'(t) = v(t)$, $x(0) = 0$ (ground launch):

$$x(t) = d + \frac{m}{k}\ln\left|\cos\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right|,$$

where $c = t_1 = 4.660125809$ and d is a constant. Initial data $x(0) = 0$ determines

$$\begin{aligned} d &= -\frac{m}{k} \ln \left| \cos \left(\sqrt{\frac{kg}{m}}(c-0) \right) \right| \\ &= -\frac{1}{0.0012} \ln \left| \cos \left(\sqrt{(0.0012)(9.8)}(4.660125809) \right) \right| \\ &= 111.2761605 \end{aligned}$$

The maximum height of the upward launch is

$$\begin{aligned} x(t_1) &= x(c) \\ &= 111.2761605 + \tfrac{1}{0.0012} \ln|\cos(0)| \\ &= 111.2761605 + \tfrac{1}{0.0012} \ln|1| \\ &= 111.2761605 \end{aligned}$$

Fall Time: Let $y(t)$ be the distance to the ground at time t for the downward motion, differential equation $y'' = -g + \rho\,(y')^2$. Initial data: $y(0) = x(t_1) = 111.2761605$, $y'(0) = x'(t_1) = 0$ (at rest). The textbook solution for $y(t)$ will be used below. Let's solve for $t = t_2$ in the equation

$$\begin{aligned} 0 &= y(t) \\ &= D - \tfrac{m}{k} \ln \left| \cosh \left(\sqrt{\tfrac{kg}{m}}(C-t) \right) \right| \\ &= D - \tfrac{1}{0.0012} \ln \left| \cosh \left(\sqrt{\rho\, g}(C-t) \right) \right| \end{aligned}$$

where uppercase symbols C and D are constants to be determined from physical data in the problem. Constants C, D are found from equations

$$\begin{aligned} 111.2761605 &= y(0) \\ &= D - \frac{1}{0.0012} \ln|\cosh(\sqrt{\rho\, g}(C-0))| \\ 0 &= y'(0) \\ &= \frac{1}{0.0012} \tanh(\sqrt{\rho\, g}(C-0)), \end{aligned}$$

Because $\tanh(u) = 0$ at $u = 0$, then $C = 0$. Because $\cosh(u) = 1$ at $u = 0$ and $\ln(1) = 0$, then $111.2761605 = D$. The problem simplifies to solving for $t = t_2$ in equation

$$0 = D - \frac{1}{0.0012} \ln|\cosh(\sqrt{\rho\, g}(C-t))|$$

Then

$$D = \frac{1}{0.0012} \ln|\cosh(\sqrt{\rho\, g}(0-t))|$$

Because $\cosh(-u) = \cosh(u)$, then

$$\begin{aligned}
(D)(0.0012) &= \ln\left|\cosh\left(\sqrt{\rho\, g}(t)\right)\right| \\
(111.2761605)(0.0012) &= \ln\left|\cosh\left(\sqrt{\rho\, g}(t)\right)\right| \\
0.1335313926 &= \ln\left|\cosh\left(\sqrt{\rho\, g}(t)\right)\right| \\
\mathrm{e}^{0.1335313926} &= \cosh\left(\sqrt{\rho\, g}(t)\right) \\
1.142857143 &= \cosh\left(\sqrt{\rho\, g}(t)\right) \\
\operatorname{arccosh}(1.142857143) &= \sqrt{(0.0012)(9.8)}\, t \\
0.5283553632 &= 0.1084435337\, t \\
t &= \frac{0.5283553632}{0.1084435337} \\
t_2 &= 4.872170292
\end{aligned}$$

```
v0:=50;g:=9.8;rho:=0.0012;
de1:=diff(x(t),t,t)=-g-rho*diff(x(t),t)^2;
ans1:=dsolve([de1,x(0)=0,D(x)(0)=v0],x(t));
X1:=unapply(rhs(ans1),t);
t1:=solve(diff(X1(t),t)=0,t);
evalf(t1); # t1 = 4.660125812
de2:=diff(x(t),t,t)=-g+rho*diff(x(t),t)^2;
ans2:=dsolve([de2,x(0)=X1(t1),D(x)(0)=0],x(t));
X2:=unapply(rhs(ans2),t);
t2:=solve(X2(t)=0,t);
evalf(t2); # t2 = 4.872170280
```

42. Let $\rho = 0.0012$, $v_0 = 30$. Find t_1, t_2.

43. Let $\rho = 0.0015$, $v_0 = 50$. Find t_1, t_2.

Solution: Answers:

$t_1 = \frac{100\ \arctan\left(3/14\,\sqrt{3}\right)\sqrt{3}}{21} = 2.931243230$

$t_2 = \frac{\left(100\ \ln\left(1/14+\frac{3\,\sqrt{669}}{3122}\right)+50\ \ln(223)\right)\sqrt{3}}{21} = 2.994971288$

```
v0:=30;g:=9.8;rho:=0.0015;
de1:=diff(x(t),t,t)=-g-rho*diff(x(t),t)^2;
ans1:=dsolve([de1,x(0)=0,D(x)(0)=v0],x(t));
X1:=unapply(rhs(ans1),t);
t1:=solve(diff(X1(t),t)=0,t);
evalf(t1); # t1 = 2.931243230$
de2:=diff(x(t),t,t)=-g+rho*diff(x(t),t)^2;
ans2:=dsolve([de2,x(0)=X1(t1),D(x)(0)=0],x(t));
X2:=unapply(rhs(ans2),t);
t2:=solve(X2(t)=0,t);
evalf(t2); # t2 = 2.994971288
```

44. Let $\rho = 0.0015$, $v_0 = 30$. Find t_1, t_2.

45. Let $\rho = 0.001$, $v_0 = 50$. Find M, v_f.

Solution: Answers:

$M = 500\ln(123/98) = 113.6084384$ meters

$v_f = -44.63036986$ meters per second

```
v0:=50;g:=9.8;rho:=0.001;
de1:=diff(x(t),t,t)=-g-rho*diff(x(t),t)^2;
ans1:=dsolve([de1,x(0)=0,D(x)(0)=v0],x(t));
X1:=unapply(rhs(ans1),t);
t1:=solve(diff(X1(t),t)=0,t);
evalf(t1); # t1 = 4.724487241
M:=X1(t1); evalf(M);# M = 113.6084384
de2:=diff(x(t),t,t)=-g+rho*diff(x(t),t)^2;
ans2:=dsolve([de2,x(0)=X1(t1),D(x)(0)=0],x(t));
X2:=unapply(rhs(ans2),t);
t2:=solve(X2(t)=0,t);
evalf(t2); # t2 = 4.872170280
vf:=D(X2)(t2);evalf(vf);# vf = -44.63036986
```

46. Let $\rho = 0.001$, $v_0 = 30$. Find M, v_f.

47. Let $\rho = 0.0014$, $v_0 = 50$. Find M, v_f.

Solution: Answers:

$M = (2500/7)\ln(19/14) = 109.0648748$ meters

$v_f = -42.91975375$ meters per second

```
v0:=50;g:=9.8;rho:=0.0014;
de1:=diff(x(t),t,t)=-g-rho*diff(x(t),t)^2;
ans1:=dsolve([de1,x(0)=0,D(x)(0)=v0],x(t));
X1:=unapply(rhs(ans1),t);
t1:=solve(diff(X1(t),t)=0,t);
evalf(t1); # t1 = 4.598757038
M:=X1(t1); evalf(M);# M = 109.0648748
de2:=diff(x(t),t,t)=-g+rho*diff(x(t),t)^2;
ans2:=dsolve([de2,x(0)=X1(t1),D(x)(0)=0],x(t));
X2:=unapply(rhs(ans2),t);
t2:=solve(X2(t)=0,t);
evalf(t2); # t2 = 4.838781478,
vf:=D(X2)(t2);evalf(vf);# vf = -42.91975375
```

48. Let $\rho = 0.0014$, $v_0 = 30$. Find M, v_f.

49. Find t_1, t_2, M and v_f for $\rho = 0.00152$, $v_0 = 60$.

Solution: Answers:

$t_1 = (250/133)\sqrt{19}\,\arctan((6/35)\sqrt{10}) = 5.257981263$ seconds

$t_2 = 5.661141086$ seconds

$M = (6250/19)\ln(1909/1225) = 145.9337831$ meters

$v_f = -48.06360384$ meters per second

```
v0:=60;g:=9.8;rho:=0.00152;
de1:=diff(x(t),t,t)=-g-rho*diff(x(t),t)^2;
ans1:=dsolve([de1,x(0)=0,D(x)(0)=v0],x(t));
X1:=unapply(rhs(ans1),t);
t1:=solve(diff(X1(t),t)=0,t);
evalf(t1); # t1 = 5.257981263
M:=X1(t1); evalf(M);# M = 145.9337831
de2:=diff(x(t),t,t)=-g+rho*diff(x(t),t)^2;
ans2:=dsolve([de2,x(0)=X1(t1),D(x)(0)=0],x(t));
X2:=unapply(rhs(ans2),t);
t2:=solve(X2(t)=0,t);
evalf(t2); # t2 = 5.661141086,
vf:=D(X2)(t2);evalf(vf);# vf = -48.06360384
```

50. Find t_1, t_2, M and v_f for $\rho = 0.00152$, $v_0 = 40$.

Terminal Velocity

Find the terminal velocity for (a) a linear air resistance $a(t) = \rho v(t)$ and (b) a nonlinear air resistance $a(t) = \rho v^2(t)$. Use the model equation $v' = a(t) - g$ and the given drag factor ρ, **mks** units.

51. $\rho = 0.15$

Solution: Answers:

(a) $v_f = 9.8/0.15 = 65.33333333$ meters per second

(b) $v_f = \sqrt{9.8/0.15} = 8.082903768$ meters per second

Models:

(a) $v' = -g - \rho v$. $g = 9.8$, $\rho = 0.15$, $v_f = g/\rho$

(b) $v' = -g - \rho v^2$. $g = 9.8$, $\rho = 0.15$, $v_f = \sqrt{g/\rho}$

```
rho:=0.15;g:=9.8;g/rho;sqrt(g/rho);
```

52. $\rho = 0.155$

53. $\rho = 0.015$

Solution: Answers:

(a) $v_f = 9.8/0.015 = 653.3333333$ meters per second

(b) $v_f = \sqrt{9.8/0.015} = 25.56038602$ meters per second

54. $\rho = 0.017$

55. $\rho = 1.5$

Solution: Answers:

(a) $v_f = 9.8/1.5 = 6.533333333$ meters per second

(b) $v_f = \sqrt{9.8/1.5} = 2.556038602$ meters per second

56. $\rho = 1.55$

57. $\rho = 2.0$

Solution: Answers:

(a) $v_f = 9.8/2.0 = 4.9$ meters per second

(b) $v_f = \sqrt{9.8/2.0} = 2.213594362$ meters per second

58. $\rho = 1.89$

59. $\rho = 0.001$

Solution: Answers:

(a) $v_f = 9.8/0.001 = 9800$ meters per second

(b) $v_f = \sqrt{9.8/0.001} = 98.99494937$ meters per second

60. $\rho = 0.0015$

Parachutes

A skydiver has velocity v_0 and height $5,500$ feet when the parachute opens. Velocity $v(t)$ is given by (a) linear resistance model $v' = -\rho v - g$ or (b) nonlinear resistance downward model $v' = \rho v^2 - g$. Given the drag factor ρ and the parachute-open velocity v_0, compute the elapsed time until the parachutist slows to within 2% of terminal velocity. Then find the flight time from parachute open to the ground. Report two values for (a) and two values for (b).

61. $\rho = 1.446$, $v_0 = -116$ ft/sec.

Solution: Answers:

(a) About 3.7 seconds to reach within 2% of terminal velocity -22.13001383 fps. Flight time 4.1 min.

(b) About 0.33 seconds to reach within 2% of terminal velocity -4.704254864 fps. Flight time 19.5 min.

Sanity Check: The linear model applies below Mach 1 (1115 ft/sec) and the nonlinear model above Mach 1. The skydiver flight is below Mach 1: the nonlinear model is the wrong model to use. Skydivers usually ride the parachute for 4-5 min.

Model 1: $x'' = -\rho x' - g$, $x(0) = x_0$, $x'(0) = v0$

Terminal Velocity Model 1:
$v_f = g/m = 22.13001383$ feet per second.

Time to 2% error Model 1:
Solve the equation $v(t) = -1.02\, v_f$ for $t = 3.704701879$ seconds.

Why use 1.02 instead of 0.98? Answer: The velocity is negative with magnitude always larger than $|v_f|$.

Flight Time Model 1: Solve $x(t) = 0$ for $t = 245.5978129$ sec, 4.1 min.

Model 2: $y'' = \rho|y'|^2 - g$, $y(0) = x_0$, $y'(0) = v0$

Terminal Velocity Model 2:
$v_f = \sqrt{g/m} = 4.704254864$ feet per second.

Time to 2% error Model 2:
Solve the equation $y'(t) = -1.02\, v_f$ for $t = 0.3332647152$ seconds.

Flight Time Model 2:
Solve $y(t) = 0$ for $t = 1168.779217$ sec, 19.8 min.

```
rho:=1.446;v0:=-116;g:=32;x0:=5500;
de1:=diff(x(t),t,t)=-g-rho*diff(x(t),t);
ans1:=dsolve([de1,x(0)=x0,D(x)(0)=v0],x(t));
X1:=unapply(evalf(rhs(ans1)),t);
vf1:=g/rho; # 22.13001383
de2:=diff(x(t),t,t)=-g+rho*diff(x(t),t)^2;
ans2:=dsolve([de2,x(0)=x0,D(x)(0)=v0],x(t));
X2:=unapply(evalf(rhs(ans2)),t);
vf2:=sqrt(g/rho); # 4.704254864
solve(D(X1)(t)= -1.02*vf1,t); # 3.704701879 sec
solve(X1(t)=0,t); # 245.5978129 sec, 4.1 min
solve(D(X2)(t)= -1.02*vf2,t); # 0.3332647152 sec
fsolve(X2(t)= 0,t=0..1200); # 1168.779217 sec, 19.48 min
```

62. $\rho = 1.446$, $v_0 = -84$ ft/sec.

63. $\rho = 1.2$, $v_0 = -116$ ft/sec.

Solution: Answers:
(a) About 4.3 seconds to reach within 2% of terminal velocity -26.66666667 fps. Flight time 3.4 min.

(b) About 0.37 seconds to reach within 2% of terminal velocity -5.163977795 fps. Flight time 17.7 min.

64. $\rho = 1.2$, $v_0 = -84$ ft/sec.

65. $\rho = 1.01$, $v_0 = -120$ ft/sec.

Solution: Answers:
(a) About 4.9 seconds to reach within 2% of terminal velocity -31.68316832 fps. Flight time 2.85 min.

(b) About 0.4 seconds to reach within 2% of terminal velocity -5.628780358 fps. Flight time 16.3 min.

66. $\rho = 1.01$, $v_0 = -60$ ft/sec.

67. $\rho = 0.95$, $v_0 = -10$ ft/sec.

Solution: Answers:
(a) About 3.75 seconds to reach within 2% of terminal velocity -33.68421053 fps. Flight time 2.7 min.

(b) About 0.3 seconds to reach within 2% of terminal velocity -5.803810001 fps. Flight time 15.8 min.

68. $\rho = 0.95$, $v_0 = -5$ ft/sec.

69. $\rho = 0.8$, $v_0 = -66$ ft/sec.

Solution: Answers:

(a) About 4.4 seconds to reach within 2% of terminal velocity -40 fps. Flight time 2.8 min.

(b) About 0.4 seconds to reach within 2% of terminal velocity -6.324555320 fps. Flight time 14.4 min.

70. $\rho = 0.8$, $v_0 = -33$ ft/sec.

Lunar Lander

A lunar lander falls to the moon's surface at v_0 miles per hour. The retrorockets in free space provide a deceleration effect on the lander of a miles per hour per hour. Estimate the retrorocket activation height above the surface which will give the lander zero touch-down velocity. Follow Example 2.30, page 133.

71. $v_0 = -1000$, $a = 18000$

Solution: Answers:

Constant field:

$t_0 = 729.13$ seconds $= 12.15$ minutes, retrorocket activation height $r(0) = 162.98$ kilometers $= 101.27$ miles.

Variable field:

retrorocket activation height $r(0) \approx 136.65$ kilometers or 84.91 miles.

Conversions: Let's use 1 meter = 3.280839895 feet, 1 mile = 1.609344 kilometers, $a = 18000$ mi/h/h $= 2.2352$ m/s/s, $v_0 = -1000$ m/h $= -447.04$ m/s.

Constant Field Model:

$$\begin{aligned} r'(t) &= (a - \mathcal{G})t + v_0, \\ r(t) &= (a - \mathcal{G})t^2/2 + v_0 t + r(0). \end{aligned}$$

Requirements $r'(t_0) = 0$ and $r(t_0) = 0$ give the equations

$$(a - \mathcal{G})t_0 + v_0 = 0, \quad r(0) = -v_0 t_0 - (a - \mathcal{G})t_0^2/2.$$

Evaluation uses mks units: $a = 2.2352$, $v_0 = -447.04$, $\mathcal{G} = 1.621942132$. Solving simultaneously provides the numerical answers

$$t_0 = 728.13 \text{ seconds} = 12.15 \text{ minutes},$$

$$r(0) = 162975.73 \text{ meters} = 101.27 \text{ miles}.$$

Variable Field Model:

$$mr''(t) = ma - \frac{Gm\,m_1}{(R + r(t))^2}, \quad r(t_0) = 0, \quad r'(t_0) = 0, \quad r'(0) = v_0.$$

Multiply the differential equation by $r'(t)/m$ and integrate. Then

$$\frac{(r'(t))^2}{2} = ar(t) + \frac{Gm_1}{R + r(t)} + c, \quad c \equiv -\frac{Gm_1}{R}.$$

We will find $r(0)$, the height above the moon. The equation to solve for $r(0)$ is found by substitution of $t = 0$ into the previous equation:

$$\frac{(r'(0))^2}{2} = a\,r(0) + \frac{G\,m_1}{R + r(0)} - \frac{Gm_1}{R}.$$

After substitution of known values, the quadratic equation for $x = r(0)$ is given by

$$92088.46615 = 2.2352x + \frac{4.9110336 \times 10^{12}}{x + 1740000} - 2.822433103 \times 10^6$$

Solving for the positive root gives $r(0) \approx 136.65$ kilometers or 84.91 miles.

```
# Constant field model
V0:=-1000;A:=18000;
R:=1740*1000;m1:=7.36*10^(22);G:=6.6726*10^(-11);
miles2meters:=(5280*12*2.54/100);meter2feet:=3.280839895;
a:=A*miles2meters/3600/3600;
v0:=(V0*miles2meters)/3600; gm:=G*m1/R^2;
ans1:=dsolve({diff(r(t),t,t)=a-gm,r(0)=r0,D(r)(0)=v0},r(t)):
r1:=unapply(rhs(ans1),t);
t0:=fsolve(diff(r1(t),t)=0,t);
retroHt1:=fsolve(r1(t0)=0,r0=0..infinity);
printf("Constant field: %f minutes, %f miles",
  t0/60,(retroHt1*meter2feet/5280) );
# Variable field model
eq:=(v0)^2/2 = a*x+G*m1/(R+x)-G*m1/R;
retroHt2:=fsolve(eq,x=0..infinity);
printf("Variable field: %f kilometers, %f miles",
  (retroHt2/1000),(retroHt2*meter2feet/5280) );
```

72. $v_0 = -980$, $a = 18000$

73. $v_0 = -1000$, $a = 20000$

Solution: Answers:
Constant field:
$t_0 = 518.93$ seconds $= 8.65$ minutes, retrorocket activation height $r(0) = 115.99$ kilometers $= 72.07$ miles.
Variable field:
retrorocket activation height $r(0) \approx 104.78$ kilometers or 65.11 miles.

74. $v_0 = -1000$, $a = 19000$

75. $v_0 = -900$, $a = 18000$

Solution: Answers:
Constant field:
$t_0 = 656.22$ seconds $= 10.94$ minutes, retrorocket activation height $r(0) =$ 132.01 kilometers $= 82.03$ miles.
Variable field:
retrorocket activation height $r(0) \approx 113.59$ kilometers or 70.58 miles.

76. $v_0 = -900$, $a = 20000$

77. $v_0 = -1100$, $a = 22000$

Solution: Answers:
Constant field:
$t_0 = 443.08$ seconds $= 7.38$ minutes, retrorocket activation height $r(0) =$ 108.94 kilometers $= 67.69$ miles.
Variable field:
retrorocket activation height $r(0) \approx 100.86$ kilometers or 62.67 miles.

78. $v_0 = -1100$, $a = 21000$

79. $v_0 = -800$, $a = 18000$

Solution: Answers:
Constant field:
$t_0 = 498.91$ seconds $= 8.32$ minutes, retrorocket activation height $r(0) =$ 122.67 kilometers $= 76.22$ miles.
Variable field:
retrorocket activation height $r(0) \approx 111.6$ kilometers or 69.34 miles.

80. $v_0 = -800$, $a = 21000$

Escape velocity

Find the escape velocity of the given planet, given the planet's mass m and radius R.

81. **(Planet A)** $m = 3.1 \times 10^{23}$ kilograms, $R = 2.4 \times 10^7$ meters.

Solution: Answer: $v0 = 1312.918505$ meters/sec $= 4726.5$ kilometers/hour.
Model: $v_0 = \sqrt{2gR}$, $gR^2 = Gm$, $G = 6.6726 \times 10^{-11}$

```
# Escape velocity
m:=3.1*10^(23); # kilograms
R:=2.4*10^7; # meters
G:=6.6726*10^(-11);g:=G*m/R^2;v0:=sqrt(2*g*R);
printf("Escape velocity = %f meters/sec = %f kilometers/hour",
  v0,v0*3.6);
```

82. **(Mercury)** $m = 3.18 \times 10^{23}$ kilograms, $R = 2.43 \times 10^6$ meters.

83. **(Venus)** $m = 4.88 \times 10^{24}$ kilograms, $R = 6.06 \times 10^6$ meters.

Solution: Answer: $v0 = 10366.595250$ meters/sec $= 37319.7$ kilometers/hour.

```
m:=4.88*10^(24); # kilograms
R:=6.06*10^6; # meters
G:=6.6726*10^(-11);g:=G*m/R^2;v0:=sqrt(2*g*R);
printf("Escape velocity = %f meters/sec = %f kilometers/hour",
  v0,v0*3.6);
```

84. **(Mars)** $m = 6.42 \times 10^{23}$ kilograms, $R = 3.37 \times 10^6$ meters.

85. **(Neptune)** $m = 1.03 \times 10^{26}$ kilograms, $R = 2.21 \times 10^7$ meters.

Solution: Answer: $v0 = 24939.343610$ meters/sec $= 89781.6$ kilometers/hour.

86. **(Jupiter)** $m = 1.90 \times 10^{27}$ kilograms, $R = 6.99 \times 10^7$ meters.

87. **(Uranus)** $m = 8.68 \times 10^{25}$ kilograms, $R = 2.33 \times 10^7$ meters.

Solution: Answer: $v0 = 22296.897920$ meters/sec $= 80268.8$ kilometers/hour.

88. **(Saturn)** $m = 5.68 \times 10^{26}$ kilograms, $R = 5.85 \times 10^7$ meters.

Lunar Lander Experiments

89. **(Lunar Lander)** Verify that the variable field model for Example 2.30 gives a soft landing flight model in MKS units

$$\begin{aligned} u''(t) &= 2.2352 - \frac{c_1}{(c_2 + u(t))^2}, \\ u(0) &= 127254.1306, \\ u'(0) &= -429.1584, \end{aligned}$$

where $c_1 = 4911033599000$ and $c_2 = 1740000$.

Solution: The model was developed in the text. It remains to evaluate symbols and verify the constants reported. Computer assist is expected.

The flight time calculation uses graphing of a numerical solution on $t = 0$ to $t = 12$ minutes, because the constant field model reported about 12 minutes flight time. The graph then suggests the flight time is between 580 and 650 seconds. A numerical solver finds a flight time of about 625 seconds.

```
# Variable field model
V0:=-960;A:=18000;
R:=1740*1000;m1:=7.36*10^(22);G:=6.6726*10^(-11);
miles2meters:=(5280*12*2.54/100);meter2feet:=3.280839895;
a:=A*miles2meters/3600/3600;
v0:=(V0*miles2meters)/3600;
gm:=G*m1/R^2;
u0:=1.272541306*10^5;# activation height in meters
de:=diff(u(t),t,t)=a-gm*R^2/(R+u(t))^2;
ic:=u(0)=u0,D(u)(0)=v0;
# Find the flight time
ans:=dsolve({de,ic},numeric,output=listprocedure);
uu:=rhs(ans[2]);vv:=rhs(ans[3]);# position, velocity
plot(uu(t),t=0..12*60);
ftime:=fsolve(uu(t)=0,t=590..650);
uu(ftime);# error < 1/10^8
```

90. (Lunar Lander: Numerical Experiment) Using a computer, solve the flight model of the previous exercise. Determine the flight time $t_0 \approx 625.6$ seconds by solving $u(t) = 0$ for t.

Details and Proofs

91. (Linear Rise Time) Using the inequality $e^u > 1 + u$ for $u > 0$, show that the ascent time t_1 in equation (17) satisfies

$$g(1 + \rho t_1) < ge^{\rho t_1} = v_0\rho + g.$$

Conclude that $t_1 < v_0/g$, proving Lemma 2.2.

Solution: Let $u = \rho t_1$ in the inequality $e^u > 1+u$. All symbols are positive, so $u > 0$. Then $e^u > 1 + u$ implies $e^{\rho t_1} > 1 + \rho t_1$. Multiply this inequality by g to prove the result $g(1 + \rho t_1) < ge^{\rho t_1}$.

Equality $ge^{\rho t_1} = v_0\rho + g$ is established using $t_1(\rho, v_0) = \dfrac{1}{\rho} \ln \left| \dfrac{v_0\rho + g}{g} \right|$. The absolute value can be erased: all symbols are positive. Then

$$e^{\rho t_1} = e^{\ln\left(\dfrac{v_0\rho + g}{g}\right)} = \frac{v_0\rho + g}{g},$$

which completes the proof.

92. (Linear Maximum) Verify that Lemma 2.2 plus the inequality $x(t) < -gt^2/2 + v_0 t$ imply $x(t_1) < gv_0^2/2$. Conclude that the maximum for $\rho > 0$ is less than the maximum for $\rho = 0$.

93. **(Linear Rise Time)** Consider the ascent time $t_1(\rho, v_0)$ given by equation (17). Prove that

$$\frac{dt_1}{d\rho} = \frac{\ln \frac{g}{v0\rho+g}}{\rho^2} + \frac{v0}{\rho(v0\rho+g)}.$$

Solution: Arrange equation

$$t_1(\rho, v_0) = \frac{1}{\rho} \ln \left| \frac{v_0\rho + g}{g} \right|$$

in the form

$$\rho t_1 = \ln |v_0\rho + g| - \ln |g|$$

Differentiate across this equation on symbol ρ using $\frac{d}{du} \ln |u| = 1/u$ and the chain rule of calculus. Then

$$t_1 + \rho \frac{d\,t_1}{d\rho} = \frac{v_0}{v_0\rho + g} - 0$$

Use identity $\ln(1/u) = -\ln(u)$ and fraction algebra to arrive at the claimed identity.

94. **(Linear Rise Time)** Consider $dt_1(\rho, v_0)/d\rho$ given in the previous exercise. Let $\rho = gx/v_0$. Show that $dt_1/d\rho < 0$ by considering properties of the function $-(x+1)\ln(x+1) + x$. Then prove Lemma 2.2.

95. **(Compare Rise Times)** The ascent time for nonlinear model $v' = -g - \rho v^2$ is less than the ascent time for linear model $u' = -g - \rho u$. Verify for $\rho = 1$, $g = 32$ ft/sec/sec and initial velocity 50 ft/sec.

Solution: Let t_1, t_2 be the rise times for the linear and nonlinear drag models, respectively. To be shown: $t_2 < t_1$.
The models:

Linear drag:	$u' = -32 - u,$	$u(0) = 50$
Nonlinear drag:	$v' = -32 - v^2,$	$v(0) = 50$

The solutions:

$$u(t) = -\frac{g}{\rho} + \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t} = -32 + (50 + 32)\, e^{-\rho t}$$
$$v(t) = \sqrt{\frac{g}{\rho}} \tan\left(\sqrt{\rho g}(c - t)\right) = \sqrt{32} \tan\left(\sqrt{32}(c - t)\right)$$

Rise times are found by solving $u(t_1) = 0$, $v(t_2) = 0$ for $t_1 = 0.9409833446$ and $t_2 = 0.2577648674$. This verifies $t_2 < t_1$.

```
rho:=1;v0:=50;g:=32;
de1:=diff(v(t),t)=-g-rho*v(t)^2;
ans1:=dsolve([de1,v(0)=v0],v(t));
V1:=unapply(rhs(ans1),t);
t1:=fsolve(V1(t)=0,t=0..1);
de2:=diff(v(t),t)=-g-rho*v(t);
ans2:=dsolve([de2,v(0)=v0],v(t));
V2:=unapply(rhs(ans2),t);
t2:=fsolve(V2(t)=0,t=0..0.5);
```

96. (Compare Fall Times) The descent time for nonlinear model $v' = \rho v^2 - g$, $v(0) = 0$ is greater than the descent time for linear model $u' = -\rho u - g$, $u(0) = 0$. Verify for $\rho = 1$, $g = 32$ ft/sec/sec and maximum heights both 100 feet.

Solution: Let t_1, t_2 be the fall times for the nonlinear and linear drag models, respectively. Each falls at $t = 0$ from maximum height $h_0 = 100$ feet and velocity $v_0 = 0$ feet/second. The `maple` code below finds $t_1 = 17.80020180$ seconds and $t_2 = 4.108568725$ seconds.

```
v0:=0;g:=32;rho:=1;h0:=100;
de1:=diff(x(t),t,t)=-g+rho*diff(x(t),t)^2;
ans1:=dsolve([de1,x(0)=h0,D(x)(0)=v0],x(t));
X1:=unapply(rhs(ans1),t);
t1:=fsolve(X1(t)=0,t=0..20);
de2:=diff(x(t),t,t)=-g-rho*diff(x(t),t);
ans2:=dsolve([de2,x(0)=h0,D(x)(0)=v0],x(t));
X2:=unapply(rhs(ans2),t);
t2:=fsolve(X2(t)=0,t=0..5);
```

2.7 Logistic Equation

Limited Environment

Find the equilibrium solutions and the carrying capacity for each logistic equation.

1. $P' = 0.01(2 - 3P)P$

Solution: Solve $0 = 0.01(2 - 3P)P$ for $P = 0$ and $P = 2/3$. These are the equilibrium solutions. Symbols are $a = 2$, $b = 3$. The carrying capacity is $M = 2/3$.

2. $P' = 0.2P - 3.5P^2$

3. $y' = 0.01(-3 - 2y)y$

Solution: Equilibria $y = 0$ and $y = -3/2$. The symbols in the solution model are $a = -3(0.01)$, $b = 2(0.01)$. Then $M = a/b = -3/2$. A negative number for M has no population interpretation. The limit at infinity of the solution

$$\begin{aligned} y(t) &= \frac{ay(0)}{by(0) + (a - by(0))e^{-at}} \\ &= \frac{-3y(0)}{2y(0) + (-3 - 2y(0))e^{3t}} \end{aligned}$$

is zero, which means the carrying capacity is zero. Every positive population size $y(0)$ gives $\lim_{t\to\infty} y(t) = 0$, the extinction state.

4. $y' = -0.3y - 4y^2$

5. $u' = 30u + 4u^2$

Solution: Factor as $30u + 4u^2 = (30 + 4u)u$, then equilibria are $u = 0$ and $u = -15/2$. Symbols are $a = 30$, $b = -4$. Because $a/b = -15/2$ is negative, the carrying capacity is $M = 0$, extinction.

6. $u' = 10u + 3u^2$

7. $w' = 2(2 - 5w)w$

Solution: Factor as $2(2-5w)w = (4-10w)w$, then equilibria are $w = 0$ and $w = 4/10$. Symbols are $a = 4$, $b = 10$. Because $a/b = 4/10$ is positive, then the carrying capacity is $M = 0.4$.

8. $w' = -2(3 - 7w)w$

9. $Q' = Q^2 - 3(Q-2)Q$

Solution: Expand as $Q^2 - 3Q^2 + 6Q = -2Q^2 + 6Q = (6-2Q)Q$. Equilibria are $Q = 0$ and $Q = 3$. Symbols are $a = 6$, $b = 2$. Because $a/b = 6/2$ is positive, then the carrying capacity is $M = 3$.

10. $Q' = -Q^2 - 2(Q-3)Q$

Spread of a Disease

In each model, find the number of infectives and then the number of susceptibles at $t = 2$ months. Follow Example 2.34, page 143. A calculator is required for approximations.

11. $y' = (5/10 - 3y/100000)y$, $y(0) = 100$.

Solution: Define $a = 5/10$, $b = 3/100000$. Let $M = a/b = 50000/3 = 16666.66667$. We will find the number of infectives $y(2)$ and the number of susceptibles $M - y(2)$.

The logistic formula with $a = 5$, $b = 2$ and $y(0) = 100$ gives

$$y(t) = \frac{50000}{3 + 497e^{-t/2}}.$$

The number of infectives is $y(2) = 269.0543160$. The number of susceptibles is $M - y(2) = 16397.61235$.

12. $y' = (13/10 - 3y/100000)y$, $y(0) = 200$.

13. $y' = (1/2 - 12y/100000)y$, $y(0) = 200$.

Solution: Let $a = 1/2$, $b = 12/100000$, $M = a/b = 4166.666667$. The number of infectives is $y(2) = 502.2333968$. The number of susceptibles is $M - y(2) = 3664.433270$.

14. $y' = (15/10 - 4y/100000)y$, $y(0) = 100$.

15. $P' = (1/5 - 3P/100000)P$, $P(0) = 500$.

Solution: Let $a = 1/5$, $b = 3/100000$, $M = a/b = 6666.666667$. The number of infectives is $y(2) = 719.3768030$. The number of susceptibles is $M - y(2) = 5947.289864$.

16. $P' = (5/10 - 3P/100000)P$, $P(0) = 600$.

17. $10P' = 2P - 5P^2/10000$, $P(0) = 500$.

Solution: Let $a = 1/5$, $b = 5/100000$, $M = a/b = 4000$. The number of infectives is $y(2) = 702.7110198$. The number of susceptibles is $M - y(2) = 3297.288980$.

18. $P' = 3P - 8P^2$, $P(0) = 10$.

Explosion–Extinction

Classify the model as **explosion** or **extinction**.

19. $y' = 2(y - 100)y$, $y(0) = 200$

Solution: Let $M = 100$. Then $y = 0$ and $y = M$ are equilibrium solutions. The sign of $y'(0)$ detects explosion, because $y'(0) = 2(y(0) - M)y(0) = 2(200-M)(200)$ is positive, meaning $y(t)$ increases without bound to infinity.

20. $y' = 2(y - 200)y$, $y(0) = 300$

21. $y' = -100y + 250y^2$, $y(0) = 200$

Solution: Explosion, because $y'(0) = 200(-100 + 250(200)) > 0$.

22. $y' = -50y + 3y^2$, $y(0) = 25$

23. $y' = -60y + 70y^2$, $y(0) = 30$

Solution: Explosion, because $y'(0) = 30(-60 + 70(30)) > 0$.

24. $y' = -540y + 70y^2$, $y(0) = 30$

25. $y' = -16y + 12y^2$, $y(0) = 1$

Solution: Extinction, because $y'(0) = 1(-16 + 12(1)) < 0$.

26. $y' = -8y + 12y^2$, $y(0) = 1/2$

Constant Harvesting

Find the carrying capacity N and the threshold population M.

27. $P' = (3 - 2P)P - 1$

Solution: The carrying capacity is $M = 1$ and the threshold population is $N = 1/2$.

Let $f(P) = (3 - 2P)P - 1$. Solve $f(P) = 0$ for $P = 1/2$, $P = 1$. A shortcut after finding the roots is to declare the larger root to be the carrying capacity and declare the smaller root to be the threshold population. A careful solution can be modeled after the **Constant Harvesting** Example 2.36 page 144 . The shortcut works for quadratic $f(P)$ with two distinct real positive roots.

The carrying capacity $M = \lim_{t\to\infty} P(t)$ is the expected population size found by a biologist estimating or counting the population at some random time. Units could be billions, e.g., expected population size 1 billion and threshold population 1/2 billion.

```
# stability test calculations
F:=P->(3-2*P)*P-1;
L:=[solve(F(P)=0,P)];# L:=[1/2,1], array of roots
D(F)(L[1]);D(F)(L[2]);# find F'(1/2), F'(1)
# M=1 is a funnel/sink by the stability test
```

28. $P' = (4 - 3P)P - 1$

29. $P' = (5 - 4P)P - 1$

Solution: Carrying capacity $M = 1$, threshold population $N = 1/4$.

30. $P' = (6 - 5P)P - 1$

31. $P' = (6 - 3P)P - 1$

Solution: Carrying capacity $M = 1.816496581$, threshold population $N = 0.1835034191$. The roots are $P = 1 \pm \frac{1}{3}\sqrt{6}$.

32. $P' = (6 - 4P)P - 1$

33. $P' = (8 - 5P)P - 2$

Solution: Carrying capacity $M = 1.289897949$, threshold population $N = 0.3101020514$. The roots are $P = \frac{4}{5} \pm \frac{1}{5}\sqrt{6}$.

34. $P' = (8 - 3P)P - 2$

35. $P' = (9 - 4P)P - 2$

Solution: Carrying capacity $M = 2$, threshold population $N = 1/4$.

36. $P' = (10 - P)P - 2$

Variable Harvesting

Re-model the variable harvesting equation as $y' = (a - by)y$ and solve the equation by logistic solution (2) on page 142.

37. $P' = (3 - 2P)P - P$

Solution: The equation is rewritten as $P' = (3 - 2P)P - P = (2 - 2P)P$. This has the form of $y' = (a - by)y$ where $a = b = 2$. Then equation (2) page 142 gives formula

$$P(t) = \frac{2P_0}{2P_0 + (2 - 2P_0)e^{-2t}}$$

which simplifies to

$$P(t) = \frac{P_0}{P_0 + (1 - P_0)e^{-2t}}.$$

38. $P' = (4 - 3P)P - P$

39. $P' = (5 - 4P)P - P$

Solution: The equation is rewritten as $P' = (5 - 4P)P - P = (4 - 2P)P$, which has the form of $y' = (a - by)y$ with $a = 4$, $b = 2$. Then equation (2) page 142 gives formula

$$P(t) = \frac{4P_0}{2P_0 + (4 - 2P_0)e^{-4t}}$$

40. $P' = (6 - 5P)P - P$

41. $P' = (6 - 3P)P - P$

Solution: Because $P' = (6 - 3P)P - P = (5 - 3P)P$ has the form of $y' = (a - by)y$ with $a = 5$, $b = 3$, then equation (2) page 142 gives formula

$$P(t) = \frac{5P_0}{3P_0 + (5 - 3P_0)e^{-5t}}$$

42. $P' = (6 - 4P)P - P$

43. $P' = (8 - 5P)P - 2P$

Solution: $P(t) = \frac{6P_0}{5P_0+(6-5P_0)e^{-6t}}$

44. $P' = (8 - 3P)P - 2P$

45. $P' = (9 - 4P)P - 2P$

Solution: $P(t) = \dfrac{7P_0}{4P_0 + (7 - 4P_0)e^{-7t}}$

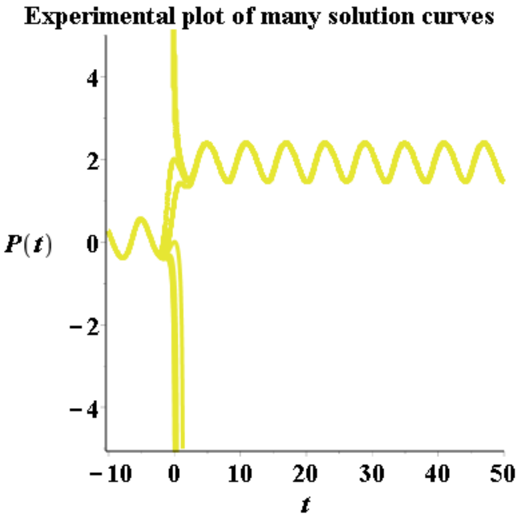
46. $P' = (10 - P)P - 2P$

Restocking

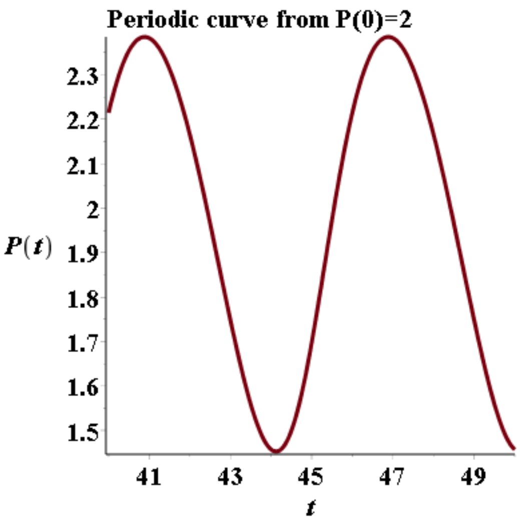
Make a direction field graphic by computer following Example 2.38. Using the graphic, report (a) an estimate for the carrying capacity C and (b) approximations for the amplitude A and period T of a periodic solution which oscillates about $P = C$.

47. $P' = (2 - P)P - \sin(\pi t/3)$

Solution: **Answers**: The period is about 5.5, the amplitude is about 0.9 and the oscillation is approximately about line $P = 1.9$.



The graphic is a computer experiment which selects 15 initial values -7 to 7 and plots the 15 solution curves on one graphic. The plan is to locate a periodic curve and guess its initial value.



The graphic uses guess $P(0) = 2$ to make a single graphic, then extract a section from the graphic to find the amplitude, period and median line $P = C$. The amplitude is decided by cursor probe of maxima and minima. The period is about 5.5 by cursor probe of two adjacent maxima. The median line is $P = 1.9$ by computation from the minima and amplitude.

```
# maple2021
f:=(t,P) -> (2-P)*P-1*sin(1*Pi*t/3);
de:=diff(P(t),t)=f(t,P(t));H:=1;HH:=0.01;
vals:=[seq(H*(i-7),i=0..14)];
a:=-10;b:=50;c:=-5;d:=5;# graph window by experiment
ics:=[seq([P(0)=vals[i]],i=1..nops(vals))];
opts:=font=[courier,16,bold],labelfont=[courier,16,bold],
thickness=3,axes=framed,labels=[t,P(t)]:
pts1:=stepsize=HH,arrows=none,opts,title="Experimental plot";
 # First plot to find P(0)=2 initial value
DEtools[DEplot](de,P(t),t=a..b,P=c..d,ics,opts1);
 # second plot to determine periodic solution
ans:=dsolve([de,P(0)=2],numeric,output=operator);
PP:=rhs(ans[2]);# DE solution P(t)
opts2:=opts,title="Periodic curve from P(0)=2":
plot(PP(x),x=40..50,opts2);
MM:=Optimization[Maximize](PP(x),x=45 .. 49);
 # MM := [2.38568114929172, [x = 46.8843119054944]]
mm:=Optimization[Minimize](PP(x),x=43 .. 45);
 # mm := [1.45203158538608, [x = 44.1213517742313]]
period:=(rhs(MM[2][1])-rhs(mm[2][1]) )*2;# 5.52592026252636
amplitude:=MM[1]-mm[1];# 0.933649563905633
C:=mm[1]+amplitude/2;# 1.91885636733890
```

48. $P' = (2 - P)P - \sin(\pi t/5)$

49. $P' = (2 - P)P - \sin(\pi t/7)$

Solution: **Answers**: The period is about 12.8, the amplitude is about 1.1 and the oscillation is approximately about line $P = 1.9$.

Details follow Exercise 47 using modified computer code. Initial value $P(0) = 0.5$ is selected to make an eventually periodic curve. The curve section to study is on $20 \le t \le 50$. Modified code sections from Exercise 47 are below.

```
MM:=Optimization[Maximize](PP(x),x=30 .. 50);
 # MM := [2.40962353216826, [x = 38.8591015537082]]
mm:=Optimization[Minimize](PP(x),x=30 .. 40);
 # mm := [1.30118248632842, [x = 32.4563671060699]]
 # period := 12.8054688952765
 # amplitude := 1.10844104583984
 # C := 1.85540300924834
```

50. $P' = (2 - P)P - \sin(\pi t/8)$

Richard Function

Ideas of L. von Bertalanffy (1934), A. Pütter (1920) and Verhulst were used by F. J. Richards (1957) to define a sigmoid function $Y(t)$ which generalizes the logistic function. It is suited for data-fitting models, for example forestry, tumor growth and stock-production problems. The Richard function is

$$Y(t) = A + \frac{K - A}{(1 + Qe^{-B(t-M)})^{1/\nu}},$$

where Y = weight, height, size, amount, etc., and t = time.

51. Differentiate for $\alpha > 0$, $\nu > 0$, the specialized Richard function

$$Y(t) = \frac{K}{(1 + Qe^{-\alpha\nu(t-t_0)})^{1/\nu}}$$

to obtain the sigmoid differential equation

$$Y'(t) = \alpha\left(1 - \left(\frac{Y}{K}\right)^{\nu}\right)Y.$$

The relation $Y(t_0) = \frac{K}{(1+Q)^{1/\nu}}$ implies $Q = -1 + \left(\frac{K}{Y(t_0)}\right)^{\nu}$.

Solution: The details expand the left side LHS and right side RHS of the equivalent differential equation

$$\frac{Y'}{\alpha Y} = 1 - \left(\frac{Y}{K}\right)^{\nu}$$

Computer algebra is used to check the computation of Y': see the `maple` code below. Then

$$\text{LHS} = \frac{Q\,\mathrm{e}^{-\alpha\,\nu\,(t-t_0)}}{1 + Q\,\mathrm{e}^{-\alpha\,\nu\,(t-t_0)}}$$

Define $Z = 1 + Q\,\mathrm{e}^{-\alpha\nu(t-t_0)}$. Then $Y = K/Z^{1/\nu}$ and

$$\text{RHS} = 1 - \frac{1}{Z^{1/\nu}}^{\nu} = 1 - \frac{1}{Z}, \qquad \text{LHS} = \frac{Q\,\mathrm{e}^{-\alpha\,\nu\,(t-t_0)}}{1 + Q\,\mathrm{e}^{-\alpha\,\nu\,(t-t_0)}} = \frac{Z-1}{Z}$$

Conclusion: LHS = RHS, which verifies the Richard differential equation.

```
Y:=t->K/(1+Q*exp( -alpha*nu*(t-t0) ) )^(1/nu);
LHS:=simplify(diff(Y(t),t)/Y(t)/alpha);
#LHS:=Q*exp(-alpha*nu*(t-t0))/(1+Q*exp(-alpha*nu*(t-t0)))
```

52. Solve the differential equation $Y'(t) = \alpha\left(1 - \left(\frac{Y}{K}\right)^{\nu}\right)Y$ by means of the substitution $w = (Y/K)^{\nu}$, which gives a familiar logistic equation $w' = \alpha\nu(1-w)w$.

2.8 Science and Engineering Applications

Tank Draining

1. A cylindrical tank 6 feet high with 6-foot diameter is filled with gasoline. In 15 seconds, 5 gallons drain out. Find the drain times for the next 20 gallons and the half-volume.

Solution: The answers are approximately 60.299 seconds and 2227.95 seconds or 37.13 minutes. Why not exactly 1 minute more to drain the next 20 gallons? Because Torricelli's Lemma says droplets fall to the orifice at changing speeds. The fraction 25/1270 of the tank drained in 75 seconds is about 2% of the tank. The half-volume time 37.13 minutes is 5 minutes longer than the guess $(1270/2)/20 = 31.75$ minutes.

Formulas. A USA gallon is defined to be 231 cubic inches, which is 0.133681 cubic feet or 3.785411784 liters. The volume V of a cylindrical tank of radius R and height H is $V = \pi R^2 H$. The area A of a cross-section of this tank at any height y is $A(y) = \pi R^2$. The half-volume of the tank is $\frac{1}{2} V = \frac{1}{2} \pi R^2 H$.

Parameters. Time variable t is in seconds, fluid height variable y is in feet, tank radius $R = 6/2$ feet, tank cross-sectional area $A = 9\pi$, tank height $H = 6$ feet, tank volume $V = \pi R^2 H = 54\pi = 169.6460033$ cubic feet, tank volume at $t = 15$ is $V_0 = V - 5(0.133681) = 168.9775983$ cubic feet, tank height at $t = 15$ is $y_0 = V_0/A(15) = 5.976360008$ feet.

Torricelli's Equation.

$$y'(t) = -k\frac{\sqrt{y(t)}}{A(y(t))} = -k\frac{\sqrt{y(t)}}{\pi R^2}, \quad y(0) = H.$$

The implicit solution:

$$\sqrt{y(t)} + \frac{k\,t}{18\pi} - c = 0$$

Find constant c. Let $t = 0$ and height $y(0) = 6$ in the implicit solution to find $c = \sqrt{6}$. Then the implicit solution is

$$\sqrt{y(t)} + \frac{k\,t}{18\pi} - \sqrt{6} = 0$$

Find k. Height $y(15) = 5.976360008$ and time $t = 15$ seconds are used in the implicit solution to find $k = 0.01820963530$:

$$\sqrt{5.976360008} + \frac{15\,k}{18\pi} - \sqrt{6} = 0$$

Drain Time after 20 more Gallons. Let $V_1 = V - 25(0.133681)$ be the tank volume after 25 gallons drain. The tank height is then $H_1 = V_1/(9\pi)$. Let $y(T_1) = H_1$. Because the first 5 gallons drained in 15 seconds, then $T_1 - 15$ seconds is the drain time for the next 20 gallons. The implicit solution for $t = T_1$ gives equation

$$\sqrt{H_1} + \frac{k\,T_1}{18\pi} - \sqrt{6} = 0$$

with answer $T_1 = 75.29873845$ seconds. Then $T_1 - 15 = 60.29873845$ seconds is the requested drain time.

Drain Time for Half-Volume. Let $V_2 = \frac{1}{2}V$, the half-volume. The half-volume height H_2 satisfies $\pi R^2 H_2 = V_2$, therefore $H_2 = V_2/(9\pi) = 3$. The implicit solution at the half-volume drain time $t = T_2$ gives equation

$$\begin{aligned}\sqrt{H_2} + \tfrac{k\,T_2}{18\pi} - \sqrt{6} = 0 \\ \sqrt{3} + 0.0003220170522\,T_2 - \sqrt{6} = 0\end{aligned}$$

Solve for time $T_2 = 2227.953242$ seconds $= 37.13$ minutes.

```
# Torricelli drain cylindrical tank
R:=6/2;H:=6;V:=Pi*R^2*H;A:=unapply(Pi*R^2,y);
gallons2CubicFeet:=0.133681;
V0:=V-5*gallons2CubicFeet;# Tank vol at 15 sec
H0:=V0/A(15);# Tank height at 15 sec
f:=unapply(-k*sqrt(y)/A(y),y);
de:=diff(y(t),t) = f(y(t));
ans:=dsolve(de,y(t));# implicit solution
c:=solve(subs(t=0,y(0)=H,ans),_C1);
ans1:=subs( _C1=c,ans);
equk:=subs(t=15,y(15)=H0,ans1);
kk:=solve(equk,k);
# Drain time on next 20 gallons
V1:=V-25*0.133681; H1:=V1/9/Pi;
T1:=solve(subs(k=kk,y(t)=H1,ans1),t);
Drain20:=T1-15;
# half-volume time
V2:=V/2;H2:=V2/Pi/R^2;
equT2:=sqrt(H2)+kk*t/18/Pi - sqrt(6)=0;
T2:=solve(equT2,t);T2min:=T2/60;
```

2. A cylindrical tank 4 feet high with 5-foot diameter is filled with gasoline. The half-volume drain time is 11 minutes. Find the drain time for the full volume.

3. A conical tank is filled with water. The tank geometry is a solid of revolution formed from $y = 2x$, $0 \le x \le 5$. The units are in feet. Find the drain time for the tank, given the first 5 gallons drain out in 12 seconds.

Solution: The answer is approximately 703.8 seconds = 11.73 minutes.

The details follow the book's example for a conical tank. The `maple` code is a modification of Exercise 1.

```
# Torricelli drain conical tank
a:=0;b:=5;
A:=y->Pi*(y/2)^2;# y=2x, a <= x <= b
gallons2CubicFeet:=0.133681;
V:=int(A(y),y=a..b);# V = tank volume 32.72492349 ft^3
f:=unapply(-k*sqrt(y)/A(y),y);
de:=diff(y(t),t) = f(y(t));
H:=2*b;# tank height is y=2x at x=b
ans:=dsolve([de,y(0)=H],y(t));# implicit solution
V0:=V-5*gallons2CubicFeet;# Tank vol at 12 sec
solve(int(A(y),y=a..x)=V0,x);# find x0,y0
x0:=4.965723981;# x-value for integral=V0
y0:=2*x0;# y=2x fluid height = 9.931447962
ansk:=subs(t=12,y(12)=y0,ans);
kk:=solve(ansk,k);# kk = 0.1411539155
# Drain time for the whole tank
Y:=unapply(rhs(subs(k=kk,ans)),t);
solve(Y(t)=0,t);# t = 703.8124469 seconds
```

4. A conical tank is filled with oil. The tank geometry is a solid of revolution formed from $y = 3x$, $0 \le x \le 5$. The units are in meters. Find the half-volume drain time for the tank, given the first 5 liters drain out in 10 seconds.

5. A spherical tank of diameter 12 feet is filled with water. Find the drain time for the tank, given the first 5 gallons drain out in 20 seconds.

Solution: A layman guess for the answer is 7.5 hours to drain the tank. The correct answer is about 11.97 hours. The difference in the two times is explained by Torricelli's Lemma: the speed of a droplet through the orifice decreases with decreasing water surface height.

The tank is a solid whose spherical boundary is formed by rotation of a half circle around the y-axis. The orifice is assumed at the origin $x = y = 0$. The tank has diameter $D = 12$ feet and radius $R = 6$ feet. The full circle has equation $x^2 + (y - R)^2 = R^2$. Along the half-circle in the right half-plane $x \ge 0$, variable x is defined by equation $x = \sqrt{R^2 - (R - y)^2}$ for $0 \le y \le 2R$. The cross-sectional area $A(y)$ at height y is

$$A(y) = \pi x^2 = \pi(R^2 - (R - y)^2), \quad 0 \le y \le 2R$$

and the tank volume $V(y)$ at height y is

$$V(y) = \int_0^y A(z)dz = \frac{1}{3}\pi y^3 + 6\pi y^2$$

The differential equation is $y'(t) = f(y(t))$, $f(y) = -k\sqrt{y}/A(y)$, with implicit solution

$$t - \frac{2\pi}{5k} y^{3/2}(y - 20) + c = 0$$

Substitute $y(0) = 12$ to find $c = -\frac{192\pi\sqrt{12}}{5k}$. Then the implicit solution is

$$t - \frac{2\pi}{5k} y^{3/2}(y - 20) - \frac{192\pi\sqrt{12}}{5k} = 0$$

After 20 seconds, 5 gallons drained. Conversion of gallons to cubic feet gives tank volume $V_1 = V(D) - 5(0.133681) = 904.1102794$ cubic feet. Solve $V(y) = V_1$ for $y = 11.81069370$, 12.18733587, -5.998029570. Select $y_1 = 11.81069370$, because the others are outside $0 \le y \le 12$. Check $V(y_1) = V_1$.

Substitute $t = 20$, $y(20) = y_1$ into the implicit solution:

$$20\,k + 417.7053992 - (192/5)\sqrt{12}\pi = 0$$

Then $k = 0.009698715978$ and the implicit solution becomes

$$t - 129.5673639\, y(t)^{3/2}(y(t) - 20) - 12438.46693\sqrt{12} = 0$$

Substitute $y(t) = 0$ to find the drain time:

$$t - 12438.46693\sqrt{12} = 0, \quad t = 43088.11338 \quad \text{seconds.}$$

```
# Torricelli drain spherical tank
DD:=12;R:=DD/2;A:=unapply(Pi*(R^2 - (R-y)^2),y);
V:=unapply(int(A(z),z=0..y),y);
V0:=V(2*R);# Full tank volume in cubic feet
gallons2CubicFeet:=0.133681;
capacity:=V0/gallons2CubicFeet;# gallons in the tank
laymanDrainTimeSecs:=(capacity/5)*20;# Estimate in secs
laymanTimeMin:=laymanDrainTimeSecs/60;# Estimate in minutes
V1:=V(2*R)-5*gallons2CubicFeet;# Tank vol at 20 sec
f:=unapply(-k*sqrt(y)/A(y),y);# RHS of the DE
de:=diff(y(t),t) = f(y(t));
ans:=dsolve(de,y(t));# implicit solution
c:=solve(subs(t=0,y(0)=DD,ans),_C1);
ans1:=subs( _C1=c,ans);
solve(V(y)=V1,y);# Height y=y1 after 20 secs
y1:=11.81069370;# range 0 to 12 required
V(y1)-V1;# Check if zero
equk:=subs(t=20,y(20)=y1,ans1);
kk:=solve(equk,k);
subs(k=kk,ans1);
# Drain time whole tank in seconds
ans2:=subs(y(t)=0,k=kk,ans1);
T:=solve(ans2,t);
```

6. A spherical tank of diameter 9 feet is filled with solvent. Find the half-volume drain time for the tank, given the first gallon drains out in 3 seconds.

7. A hemispherical tank of diameter 16 feet is filled with water. Find the drain time for the tank, given the first 5 gallons drain out in 25 seconds.

Solution: A layman guess for the answer is 11.14 hours to drain the tank. The correct answer is about 15.6 hours.

Details parallel Exercise 5, restricting the range of y to $0 \le y \le R$. The `maple` code in Exercise 5 applies, suitably modified.

8. A hemispherical tank of diameter 10 feet is filled with solvent. Find the half-volume drain time for the tank, given the first gallon drains out in 4 seconds.

9. A parabolic tank is filled with water. The tank geometry is a solid of revolution formed from $y = 2x^2$, $0 \le x \le 2$. The units are in feet. Find the drain time for the tank, given the first 5 gallons drain out in 12 seconds.

Solution: A layman guess for the answer is 15.04 minutes to drain the tank. The correct answer is about 13.34 minutes. A similar tank shape is a saline drip bag in a hospital.

Details use $A(y) = \pi y/2$, $V(y) = \pi y^2/4$, $0 \le y \le H \equiv 8$. The tank capacity is 376 gallons, from which the layman answer is (376/5)(12) seconds. Following Exercise 5, $k = 0.02961387652$ and the drain time is $T = 800.1443397$ seconds.

```
# Torricelli drain parabolic tank
# y=2x^2 on 0 \le x \le 2
A:=unapply(Pi*y/2,y);
H:=2*(2)^2;# y=2x^2 at x=2, tank height
V:=unapply(int(A(z),z=0..y),y);
V0:=V(H);gallons2CubicFeet:=0.133681;
capacity:=V0/gallons2CubicFeet;# gallons in the tank
laymanDrainTimeSecs:=(capacity/5)*12;# Estimate drain time
laymanTimeMin:=laymanDrainTimeSecs/60;
V1:=V(R)-5*gallons2CubicFeet;# Tank vol at 12 sec
f:=unapply(-k*sqrt(y)/A(y),y);
de:=diff(y(t),t) = f(y(t));
ans:=dsolve(de,y(t));# implicit solution
c:=solve(subs(t=0,y(0)=R,ans),_C1);
ans1:=subs( _C1=c,ans);
solve(V(y)=V1,y);H1:=7.919813160;# range 0 to H required
equk:=subs(t=12,y(12)=H1,ans1);
kk:=solve(equk,k);# k = 0.02961387652
ans2:=subs(y(t)=0,k=kk,ans1);# Drain time t for whole tank
T:=solve(ans2,t);
```

10. A parabolic tank is filled with oil. The tank geometry is a solid of revolution formed from $y = 3x^2$, $0 \le x \le 2$. The units are in meters. Find the half-volume drain time for the tank, given the first 4 liters drain out in 16 seconds.

Torricelli's Law and Uniqueness

It it known that Torricelli's law gives a differential equation for which Picard's existence-uniqueness theorem is inapplicable for initial data $y(0) = 0$.

11. Explain why Torricelli's equation $y' = k\sqrt{y}$ plus initial condition $y(0) = 0$ fails to satisfy the hypotheses in Picard's theorem. Cite all failed hypotheses.

Solution: The partial derivative of the RHS of the differential equation in variable y fails to be continuous at $y = 0$. All other hypotheses are satisfied.

12. Consider a typical Torricelli's law equation $y' = k\sqrt{y}$ with initial condition $y(0) = 0$. Argue physically that the depth $y(t)$ of the tank for $t < 0$ can be zero for an arbitrary duration of time t near $t = 0$, even though $y(t)$ is not zero for all t.

13. Display infinitely many solutions $y(t)$ on $-5 \le t \le 5$ of Torricelli's equation $y' = k\sqrt{y}$ such that $y(t)$ is not identically zero but $y(t) = 0$ for $0 \le t \le 1$.

Solution: The solutions correspond to a full tank at an earlier time $t = t_0 < 0$, followed by the tank emptying at time $t = 0$. The tank cross-sectional area in this example is constant. We'll discuss the case $k = 1$ to give the idea of the construction.

One solution $y(t) = (t/2)^2$ of $y' = \sqrt{y}$ can be found by separation of variables, valid for $t > 0$. The differential equation is autonomous, therefore a horizontal translate $z(t) = y(t-d) = (t-d)^2/4$ is a solution of $z' = \sqrt{z}$ with $z(d) = 0$. Define for $-5 < d < 0$ function

$$y_d(t) = \begin{cases} 0 & d \le t \le 5, \\ (t-d)^2/4 & -5 \le t < d. \end{cases}$$

Then $y_d'(t) = \sqrt{y_d(t)}$ for $-5 < t < 5$. Each function y_d models a tank of height $y_d(-5) = (-5-d)^2/4$ which empties at $t = d < 0$ and the tank remains empty until $t = 5$. There are infinitely many functions y_d.

```
# exercise 13
k:=1;f:=unapply(k*sqrt(y)/1,y);
de:=diff(y(t),t) = f(y(t));
ans:=dsolve(de,y(t));# implicit solution
# ans := sqrt(y(t))-(1/2)*t-_C1 = 0
Y:=unapply((t/2)^2,t);# explicit solution
```

14. Does Torricelli's equation $y' = k\sqrt{y}$ plus initial condition $y(0) = 0$ have a solution $y(t)$ defined for $t \ge 0$? Is it unique? Apply Picard's theorem and Peano's theorem, if possible.

Clepsydra: Water Clock Design

A surface of revolution is used to make a container of height h feet for a water clock. An increasing curve $y = f(x)$ on $0 \le x \le 1$ is revolved around the y-axis to make the container shape, e.g., $y = x$ makes a conical tank. Water drains by gravity out of diameter d orifice at $(0,0)$. The tank water level must fall at a constant rate of r inches per hour, important for marking a time scale on the tank. Find d and $f(x)$, given h and r.

15. $h = 5$ feet, $r = 4$ inches/hour. Answers: $f(x) = 5x^4$, $d = 0.05460241726 \approx 3/64$ inch.

Solution: Answers: $f(x) = 5x^4$, $d \approx 1/16$ inch.

Known is $f(x) = cx^4$ for some constant c. Below is a derivation of this fact from Torricelli's Lemma. Constant $c = \left(\dfrac{\pi\, r}{\pi (d/2)^2 \sqrt{2g}}\right)^2$ is in terms of g, h

and r. Units are second, foot, pound. Let $g = 32$ ft/sec/sec.

Define $A(y) = \pi x^2$ where $y = f(x)$ is the curve revolved around the y-axis, $x = 0$ to $x = 1$ feet. Value $h = 5 = f(1)$ is the tank height in feet. Value $r = 4$ is in inches/hour. Let $R = 1$ inch/hour $= \frac{1}{12}$ feet / 3600 seconds $= 1/43200$ ft/sec. Then r equals $4R$ feet/sec. Orifice diameter d feet is to be determined, a small decimal value. Apply Torricelli's model: $A(y)y' = -a\sqrt{2gy}$, $a = \pi(d/2)^2 =$ orifice area. Symbol a is defined in the Torricelli Equation proof, technical details page 148.

Let $A(y) = \pi x^2$, $y' = -rR$, $y = f(x)$ in Torricelli's model to obtain the equation $-\pi\,(r)\,x^2 = -\pi(d/2)^2\sqrt{2g}\sqrt{f(x)}$. Solve for $f(x) = cx^4$ where

$$c = \left(\frac{\pi\,(r)}{\pi(d/2)^2\sqrt{2g}}\right)^2$$

Because $h = f(1) = c$, then d is determined by the equation

$$\frac{r^2}{2g(d/2)^4} = h, \quad \text{or} \quad (d/2)^4 = \frac{r^2}{2gh}$$

Conclusion:

$f(x) = cx^4 = hx^4 = 5x^4$,

$d = 2\left(\dfrac{r^2}{2gh}\right)^{\frac{1}{4}} = 0.00455 \text{ ft} = 0.0546 \text{ in} \approx 3/64 \text{ in.}$

```
# Exercise 15 Clepsydra
AA:=Pi*x^2; # Area of a cross-section
R:=1/12/3600;# unit change inch/hour => ft/sec
a:=Pi*(d/2)^2; # orifice area
DE:=AA*diff(y(t),t)=-a*sqrt(2*g)*sqrt(y(t));
DE1:=subs(diff(y(t),t)=-r,y(t)=Y,g=32,DE);
ff:=unapply(solve(DE1,Y),x);
h:=5;r:=4*R;
d_roots:=solve(ff(1)=h,d);# 4 roots, choose d>0
dd:=evalf(d_roots[1]);# diameter dd feet
dd*12*16;# sixteenths, about 7/128 inch
solve( (d/2)^4=r^2/(2*32.0*h), d);# Equation check
```

16. $h = 4$, $r = 4$

17. $h = 3$, $r = 6$

Solution: Answers: $f(x) = 3x^4$, $d = 0.07598356858 \approx 5/64$ inch.

Follow Exercise 15.

18. $h = 4$, $r = 3$

19. $h = 3$, $r = 2$

Solution: Answers: $f(x) = 3x^4$, $d = 0.04386913378 \approx 3/64$ inch.

Follow Exercise 15.

20. $h = 4$, $r = 1$

Stefan's Law

An unclothed prison inmate is handcuffed to a chair. The inmate's skin temperature is 33° Celsius. Find the number of Joules of heat lost by the inmate's skin after t_0 minutes, given skin area A in square meters, Kelvin room temperature $T_0(r) = C(r/60) + 273.15$ and Celsius room temperature $C(t)$. Variables: t minutes, r seconds. Use equation $\frac{dQ}{dt} = k(T^4 - T_0(t)^4)$ page 149. Assume emissivity $\sigma = 5.6696 \times 10^{-8} K^{-4}$ Watts per square meter, K=degrees Kelvin.

21. $\mathcal{E} = 0.9$, $A = 1.5$, $t_0 = 10$, $C(t) = 24 + 7t/t_0$

Solution: The theory implies that the answer is $Q(t_1)$ where $t_1 = (10)(60)$ is in seconds and $Q' = kT^4 - kT_0^4$. Value $k = 7.65396 \times 10^{-8}$, $T = 33 + 273.15$ degrees K and $T_0(t) = C(t/60) + 273.15$ degrees K. Then

$$Q(t_1) = k \int_0^{t_1} (T^4 - (T_0(t))^4) dt = 28117.35641 \approx 28,117 \quad \text{joules.}$$

```
# Exercise 21 Stefan's Law
t0:=10;t1:=t0*60;T:=33+273.15;
A:=1.5:EE:=0.9:sigma:=5.6696*10^(-8):k:=sigma*A*EE;r:='r';
C:=t->24+(7*t)/t0;# t minutes
T0:=r->C(r/60)+273.15;
dQ:=unapply(k*T^4-k*T0(r)^4,r);
Q1:=int(dQ(t),t=0..t1);# 28117.35641
```

22. $\mathcal{E} = 0.9$, $A = 1.7$, $t_0 = 12$, $C(t) = 21 + 10t/12$

23. $\mathcal{E} = 0.9$, $A = 1.4$, $t_0 = 10$, $C(t) = 15 + 15t/t_0$

Solution: $Q_1 = 48637.89027$ joules

24. $\mathcal{E} = 0.9$, $A = 1.5$, $t_0 = 12$, $C(t) = 15 + 14t/t_0$

On the next two exercises, use a computer algebra system (CAS). Same assumptions as Exercise 21.

25. $\mathcal{E} = 0.8$, $A = 1.4$, $t_0 = 15$, $C(t) = 15 + 15\sin\pi(t - t_0)/12$

Solution: $Q_1 = 108329.3834$ joules.

```
# Exercise 25 Stefan's Law
t0:=15;t1:=t0*60;T:=33+273.15;
A:=1.4:EE:=0.8:sigma:=5.6696*10^(-8):k:=sigma*A*EE;r:='r';
C:=t->15+15*sin(Pi*(t-t0))/12;# t minutes
T0:=r->C(r/60)+273.15;
dQ:=unapply(k*T^4-k*T0(r)^4,r);
Q1:=int(dQ(t),t=0..t1);# 108329.3834
```

26. $\mathcal{E} = 0.8$, $A = 1.4$, $t_0 = 20$, $C(t) = 15 + 14\sin\pi(t - t_0)/12$

Tsunami Wave Shape

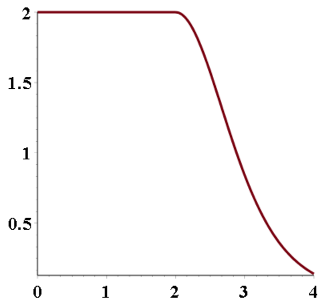
Plot the piecewise solution

$$y(x) = 2 - \begin{cases} 2\tanh^2(x - x_0) & x > x_0, \\ 0 & x \le x_0. \end{cases} \tag{1}$$

See Figure 12 page 155.

27. $x_0 = 2$, $|x - x_0| \le 2$

Solution:




```
# Exercise 27 Tsunami plot
g:=x->2-2*tanh((x-x0))^2;
opts:=thickness=3,font=[courier,18,bold];
f:=x->piecewise(x<x0,2,g(x));
plot(f,x0-2..x0+2,opts);
```

28. $x_0 = 3$, $|x - x_0| \le 4$.

Tsunami Wavefront

Find non-equilibrium solutions for the given differential equation.

29. $(y')^2 = 12y^2 - 10y^3$.

Solution: Factor $12y^2 - 10y^3 = y^2(12 - 10y)$ to find the equilibrium solutions $y = 0$ (sea level) and $y = 12/10$ (water wall). A solution with $y' \geq 0$ satisfies the first order differential equation $y' = y\sqrt{16 - 10y}$ which can be solved by separation of variables:

$$3x + \sqrt{3}\ \mathrm{arctanh}(\frac{1}{6}\sqrt{36 - 30y}) + c_1 = 0. \quad < y < 12/10$$

Solve for y:

$$y = -\frac{6}{5}\ \tanh^2(\sqrt{3}(x + c)) + \frac{6}{5}$$

```
f:=y->y*sqrt(12-10*y);
de:=diff(y(x),x)=f(y(x));
ans:=dsolve(de,y(x));
ans1:=4*subs(y(x)=u,ans);
ans2:=solve(subs(_C1=c,ans1),u);
```

30. $(y')^2 = 13y^2 - 12y^3$.

31. $(y')^2 = 8y^2 - 2y^3$.

Solution: $y = -4\tanh^2(\sqrt{2}(x + c)) + 4$

32. $(y')^2 = 7y^2 - 4y^3$.

Gompertz Tumor Equation

Solve the Gompertz tumor equation $y' = (a - b\ln y)y$.

33. $a = 1$, $b = 1$

Solution: $y(x) = e^{e^{-c-x}+1}$

34. $a = 1$, $b = 2$

35. $a = -1$, $b = 1$

Solution: $y(x) = e^{e^{-c-x}-1}$

36. $a = -1$, $b = 2$

37. $a = 4$, $b = 1$

Solution: $y(x) = e^{e^{-c-x}+4}$

38. $a = 5$, $b = 1$

2.9 Exact Equations and Level Curves

Exactness Test

Test the equality $M_y = N_x$ for the given equation, as written, and report *exact* when true. Do not try to solve the differential equation. See Example 2.43, page 163.

1. $(y - x)dx + (y + x)dy = 0$

Solution: Exact: $M_y - N_x = 1 - 1 = 0$.

2. $(y + x)dx + (x - y)dy = 0$

3. $(y + \sqrt{xy})dx + (-y)dy = 0$

Solution: Not exact: $M_y - N_x = 1 + (1/2)x/\sqrt{x\,y}$.

4. $(y + \sqrt{xy})dx + xydy = 0$

5. $(x^2 + 3y^2)dx + 6xydy = 0$

Solution: Exact: $M_y - N_x = 6y - 6y = 0$.

6. $(y^2 + 3x^2)dx + 2xydy = 0$

7. $(y^3 + x^3)dx + 3xy^2dy = 0$

Solution: Not exact: $M_y - N_x = 3y^2 - 2y^2 = y^2$.

8. $(y^3 + x^3)dx + 2xy^2dy = 0$

9. $2xydx + (x^2 - y^2)dy = 0$

Solution: Exact: $M_y - N_x = 2x - 2x = 0$.

10. $2xydx + (x^2 + y^2)dy = 0$

Conservation Law Test

Test conservation law $U(x, y) = c$ for a solution to $Mdx + Ndy = 0$. See Example 2.44, page 163.

11. $2xydx + (x^2 + 3y^2)dy = 0$,
$x^2y + y^3 = c$

Solution: Let $U = x^2y + y^3$, $M = 2xy$, $N = x^2 + 3y^2$. Then $U_x = 2xy = M$, $U_y = x^2 + 3y^2 = N$. Differentiate across $U(x, y) = c$ implicitly: $U_xdx + U_ydy = (c)' = 0$. Then $Mdx + Ndt = 0$ and $U - c$ is a solution.

12. $2xydx + (x^2 - 3y^2)dy = 0$,
$x^2y - y^3 = c$

13. $(3x^2 + 3y^2)dx + 6xydy = 0$,
$x^3 + 3xy^2 = c$

Solution: Let $U = x^3 + 3xy^2$, $M = 3x^2 + 3y^2$, $N = 6xy$. Then $U_x = 3x^2 + 3y^2 = M$, $U_y = 6xy = N$. Therefore, $U = c$ is a solution.

14. $(x^2 + 3y^2)dx + 6xydy = 0$,
$x^3 + 3xy^2 = c$

15. $(y - 2x)dx + (2y + x)dy = 0$,
$xy - x^2 + y^2 = c$

Solution: Let $U = xy - x^2 + y^2$, $M = y - 2x$, $N = 2y + x$. Then $U_x = y - 2xy = M$, $U_y = x + 2y = N$. Therefore, $U = c$ is a solution.

16. $(y + 2x)dx + (-2y + x)dy = 0$,
$xy + x^2 - y^2 = c$

Exactness Theorem

Find an implicit solution $U(x, y) = c$. See Examples 2.45-2.46, page 163.

17. $(y - 4x)dx + (4y + x)dy = 0$

Solution: The equation has the form $Mdx + Ndy = 0$ where $M = y - 4x$ and $N = 4y + x$. It is exact, by Theorem 2.10, because $M_y = 1$ and $N_x = 1$ are equal.

The method of potentials applies to find the potential $U = x^2y + xy^3 + xy$ as follows.

$U = \int_0^x M(x,y)dx + \int_0^y N(0,y)dy$	Formula for U, Theorem 2.10.
$\quad = \int_0^x (y - 4x)\,dx + \int_0^y (4y + 0)dy$	Insert M and N.
$\quad = xy - 2x^2 + 2y^2$	Evaluate integrals.

Answer check: $U_x = y - 4x + 0 = M$, $U_y = x - 0 + 4y = N$.

```
# Exercise 17 Method of Potentials
M:=(x,y)->y-4*x;N:=(x,y)-> 4*y+x;
A:=diff(M(x,y),y);B:=diff(N(x,y),x);A-B;# Check Exact
M(0,0);N(0,0);# Check (0,0) in domain of M and N
U:=int(M(t,y),t=0..x)+int(N(0,s),s=0..y);
# U := -2*x^2+x*y+2*y^2
```

18. $(y + 4x)dx + (4y + x)dy = 0$

19. $(e^y + e^x)dx + (xe^y)dy = 0$
Solution: $U = -1 + xe^y + e^x$

20. $(e^{2y} + e^x)dx + (2xe^{2y})dy = 0$

21. $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$
Solution: $U = -1 + x + e^{xy} + y^2$

22. $(1 + ye^{-xy})dx + (xe^{-xy} - 4y)dy = 0$

23. $(2x + \arctan y)dx + \dfrac{x}{1+y^2}\,dy = 0$
Solution: $U = x^2 + x\ \arctan(y)$

24. $(2x + \arctan y)dx + \dfrac{x+2y}{1+y^2}\,dy = 0$

25. $\dfrac{2x^5 + 3y^3}{x^4 y}dx - \dfrac{2y^3 + x^5}{x^3 y^2}dy = 0$
Solution: $U = \dfrac{x^5 - y^3}{yx^3}$

```
# Exercise 25 Method of Potentials
# Cannot use (0,0) in the formulas
M:=(x,y)-> (2*x^5+3*y^3)/(x^4*y);
N:=(x,y)-> -(2*y^3+x^5)/(x^3*y^2);
A:=diff(M(x,y),y);B:=diff(N(x,y),x);
simplify(A-B);# Check Exact
a:=1;b:=1;M(a,b);N(a,b);# Domain check
U:=int(M(t,y),t=a..x)+int(N(a,s),s=b..y)
  assuming x::positive, y::positive;
# U := (x^5-y^3)/(y*x^3)
```

26. $\dfrac{2x^4 + y^2}{x^3 y}dx - \dfrac{2x^4 + y^2}{2x^2 y^2}dy = 0$

27. $Mdx + Ndy = 0$, $M = e^x \sin y + \tan y$, $N = e^x \cos y + x \sec^2 y$
Solution: $U = e^x \sin(y) + x \tan(y)$

28. $Mdx + Ndy = 0$, $M = e^x \cos y + \tan y$, $N = -e^x \sin y + x \sec^2 y$

29. $\left(x^2 + \ln y\right) dx + \left(y^3 + x/y\right) dy = 0$
Solution: $U := (1/3)x^3 + x\ln(y) + (1/4)y^4 - 1/4$

30. $\left(x^3 + \ln y\right) dx + \left(y^3 + x/y\right) dy = 0$

2.10 Special equations

Homogeneous-A Equations

Find f such that the equation can be written in the form $y' = f(y/x)$. Solve for y using a computer algebra system.

1. $xy' = y^2/x$

Solution: Answer: $f(u) = u^2$, $y = x/(cx+1)$.

Let $f(u) = u^2$, then $f(y/x) = y^2/x^2 = y'$. Change variable $y \to u$ by equation $u(x) = y(x)/x$. The new equation is $xu' + u = f(u) = u^2$, which is separable: $u' = F(x)G(u)$ with $F(x) = 1/x$, $G(u) = u^2 - u$. Solve by the variables separable method: $u(x) = 1/(cx+1)$, $y(x) = xu(x) = x/(cx+1)$.

2. $x^2y' = x^2 + y^2$

3. $yy' = \dfrac{xy^2}{x^2+y^2}$

Solution: $f(u) = -u^3/(u^2+1)$, $y(x) = \sqrt{x/W(c\,x^2)}$ where W is the Lambert W function.

```
# Exercise 3, Lambert W function
F:=(x,y)->x*y/(x^2+y^2);
de1:=diff(y(x),x)=F(x,y(x));
dsolve(de1,y(x));
f:=(x,u)->simplify(F(x,x*u)-u);
de2:=x*diff(u(x),x)=f(x,u(x));
dsolve(de2,u(x));
?LambertW
```

4. $yy' = \frac{2xy^2}{x^2+y^2}$

5. $y' = \dfrac{1}{x+y}$

Solution: $f(u) = u^2/(1+u)$, $y(x) = e^{W(e^c\,x)} - c$ where W is the Lambert W function.

6. $y' = y/x + x/y$

7. $y' = (1 + y/x)^2$

Solution: $f(u) = u^2 + u + 1$,
$y\,(x) = \frac{1}{6}\,x\left(-\sqrt{3} + 3\,\tan\left(\frac{1}{2}\,(\ln(x) + c)\,\sqrt{3}\right)\right)\sqrt{3}$

8. $y' = 2y/x + x/y$

9. $y' = 3y/x + x/y$
Solution: $f(u) = 3u + 1/u$, $y(x) = \pm\frac{1}{2}\,x\,\sqrt{4cx^4 - 2}$

10. $y' = 4y/x + x/y$

Homogeneous-C Equations

Given $y' = f(x, y)$, decompose $f(x, y) = G(R(x, y))$ where $R(x, y) = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, then convert to Homogeneous-A. Investigate solving $y' = f(x, y)$ by computer.

11. $y' = -\frac{(y+1)x}{y^2+2\,y+1+x^2}$
Solution: Answers: $G(u) = -u/(1 + u^2)$, $R(x, y) = x/(1 + y)$, then let $X = x, Y = y + 1$. The Homogeneous-A equation is $\frac{dY}{dX} = G(X/Y)$. Computer solution:

$$y\,(x) = -1 + \frac{\sqrt{c^2x^4 - cx^2\sqrt{c^2x^4 + 1}}}{c^2x^3 - cx\sqrt{c^2x^4 + 1}}$$

Factor $y^2 + 2\,y + 1 + x^2 = (y + 1)^2 + x^2$, then divide by $(y + 1)^2$ to arrive at $G(u) = -u/(1 + u^2)$, $R(x, y) = x/(1 + y)$. Change variables: $X = x, Y = y + 1$. Then $f(x, y) = G(R(x, y)) = G(R(X, Y - 1)) = G(X/Y)$ and $\frac{dy}{dx} = \frac{dY}{dX}$. The new Homogeneous-A equation is $\frac{dY}{dX} = G(X/Y)$.

```
# Exercise 11, Homogeneous C
infolevel[dsolve]:= 3;# Get classification info
# ?dsolve,algorithms
G:=u->-u/(1+u^2);R:=(x,y)->x/(1+y);
de:=diff(y(x),x)=G(R(x,y(x)));
dsolve(de);# infolevel: homogeneous
# y(x) = -1+sqrt(-(-_C1*x^2+
#  sqrt(_C1^2*x^4+1))*x^2*_C1)/
# (x*(_C1*x^2-sqrt(_C1^2*x^4+1))*_C1)
dsolve(diff(u(X),X)=G(u(X)));# infolevel: separable
# u(X) = exp(-(1/2)*LambertW(exp(-2*X-2*_C1))-X-_C1)
```

12. $y' = 2\,\dfrac{(1 + y)\,x}{x^2 + y^2 + 2\,y + 1}$
Solution: $y(x) = -1 - \frac{1}{2c}\left(1 + \sqrt{4\,c^2\,x^2 + 1}\right)$

13. $y' = \dfrac{(1 + x)\,y}{x^2 + 4\,y^2 + 2\,x + 1}$
Solution: Answers: $G(u) = u/(4 + u^2)$, $R(x, y) = (1 + x)/y$, then let $X = x + 1, Y = y$. The Homogeneous-A equation is $\frac{dY}{dX} = G(X/Y)$. Computer solution in terms of W = Lambert W function:

$$y(x) = -e^{\frac{1}{2}W(\frac{1}{4}e^{2c}(1+x)^2)-c}$$

```
# Exercise 13, Homogeneous C
G:=u->u/(4+u^2);
R:=(x,y)->(1+x)/y;
de:=diff(y(x),x)=G(R(x,y(x)));
dsolve(de);
# y(x) = -exp((1/2)*LambertW((1/4)*(exp(_C1))^2*(1+x)^2)-_C1)
```

14. $y' = \dfrac{1+x}{y+1+x}$

Solution:

$$-\frac{1}{2}\ln\left(-\frac{(x+1)^2-(x+1)\,y(x)-(y(x))^2}{(x+1)^2}\right)$$
$$+\frac{1}{5}\sqrt{5}\,\mathrm{arctanh}\left(\frac{1}{5}\,\frac{(x+1+2\,y(x))\sqrt{5}}{x+1}\right)-\ln(x+1)-c=0$$

15. $y' = \dfrac{1+y}{x+y+1}$

Solution: $G(u) = u/(u+1)$, $R(x,y) = (1+y)/x$, Computer solution: $y(x) = \mathrm{e}^{\mathrm{W}(x\mathrm{e}^c)-c} - 1$ where W is the Lambert W function.

```
# Exercise 15, Homogeneous C
G:=u->u/(1+u);
R:=(x,y)->(1+y)/x;
de:=diff(y(x),x)=G(R(x,y(x)));
dsolve(de);# infolevel: homogeneous
# implicit solution returned
```

16. $x(y+1)y' = x^2 + y^2 + 2y + 1$

Solution: $y(x) = -1 + \sqrt{2\ln(x) + 2\,cx}$

17. $y' = \dfrac{x^2 - y^2 - 2\,y - 1}{(1+y)\,x}$

Solution: $G(u) = u - 1/u$, $R(x,y) = x/(1+y)$, Computer solution: $y(x) = -1 - \dfrac{c}{2x}\sqrt{2 + 2\,x^4/c^2}$

18. $y' = \dfrac{(y+2\,x)^2}{x^2}$

Solution: $y(x) = \dfrac{-3x}{2} + \dfrac{x\sqrt{7}}{2}\tan\left(\dfrac{\sqrt{7}}{2}\ln|x| + \dfrac{c\sqrt{7}}{2}\right)$

19. $y' = \dfrac{x^2 + xy + y^2 + 5\,x + 4\,y + 7}{(x+2)\,(3+y+x)}$

Solution: $G(u) = u+1/(u+1)$, $R(x,y) = (y+1)/(x+2)$, Computer solution: $y(x) = -1 - (x+2)\,(1 + \sqrt{1 + 2\ln(x+2) + 2c})$

20. $y' = -\dfrac{x^2 - xy - y^2 + 5\,x - 5\,y + 5}{(3+x)\,(4+y+x)}$

Solution: $y(x) = -1 - (3+x)(1 + \sqrt{1 - 2\ln(3+x) - 2c})$

Bernoulli's Equation

Identify the exponent n in Bernoulli's equation $y' + p(x)y = q(x)y^n$ and solve for $y(x)$.

21. $y^{-2}y' = 1 + x$

Solution: $n = 2$, $p = 0$, $q = 1 + x$, $y\,(x) = 1/\left(-x - x^2/2 + c\right)$.

Substitution $u = y/y^n = y^{-1}$ gives $u' = -y^{-2}y' = -q = -1 - x$. Quadrature: $u = -x - x^2/2 + c$, $y = 1/u = 1/(-x - x^2/2 + c)$.

```
# Exercise 21, Bernoulli DE
p:=unapply(0,x);
q:=unapply(1+x,x);
n:=2;
de:=diff(y(x),x)=-p(x)*y(x)+q(x)*y(x)^n;
dsolve(de,y(x));
```

22. $yy' = 1 + x$

23. $y^{-2}y' + y^{-1} = 1 + x$

Solution: $n = 2$, $p = 1$, $q = 1 + x$, $y\,(x) = 1/\,(2 + x + c\mathrm{e}^x)$.

Substitution $u = y/y^n = y^{-1}$ gives $u' = -y^{-2}y'$ and then $-u' + pu = q$. The linear integrating factor method applies to $-u' + pu = q$: $u(x) = 2 + x + c\mathrm{e}^x$. Then $y = 1/u = 1/(2 + x + c\mathrm{e}^x)$.

```
# Exercise 23, Bernoulli DE
p:=unapply(1,x);
q:=unapply(1+x,x);
n:=2;
de:=diff(y(x),x)=-p(x)*y(x)+q(x)*y(x)^n;
dsolve(de,y(x));
# Check substitution u=1/y
dsolve(-diff(u(x),x)+p(x)*u(x)=q(x),u(x));
```

24. $yy' + y^2 = 1 + x$

25. $y' + y = y^{1/3}$

Solution: $n = 1/3$, $p = 1$, $q = 1$, $(y(x))^{2/3} - 1 - c\mathrm{e}^{-2x/3} = 0$.

Substitution $u = y/y^n = y^{2/3}$ gives $u' = (2/3)y^{-1/3}y'$ and then $u'/(2/3) + pu = q$. The linear integrating factor method applies to $3u'/2 + pu = q$: $u(x) = 1 + c\mathrm{e}^{-2x/3}$. Then $u = y^{2/3}$ implies $y^{2/3} = 1 + c\mathrm{e}^{-2x/3}$.

```
# Exercise 25, Bernoulli DE
p:=unapply(1,x);
q:=unapply(1,x);
n:=1/3;
de:=diff(y(x),x)=-p(x)*y(x)+q(x)*y(x)^n;
dsolve(de,y(x));
# Check substitution u=y/y^(1/3)=y^(2/3)
dsolve(diff(u(x),x)/(2/3)+p(x)*u(x)=q(x),u(x));
```

26. $y' + y = y^{1/5}$

27. $y' - y = y^{-1/2}$

Solution: $n = 1/3$, $p = 1$, $q = 1$, $(y(x))^{3/2} + 1 - c\,\mathrm{e}^{3x/2} = 0$.

Substitution $u = y/y^n = y^{3/2}$ gives $u' = (3/2)y^{1/2}y'$ and then $u'/(3/2) + pu = q$. The linear integrating factor method applies to $2u'/3 + pu = q$: $u(x) = -1 + c\mathrm{e}^{3x/2}$. Then $u = y^{3/2}$ implies $y^{3/2} = -1 + c\mathrm{e}^{3x/2}$.

```
# Exercise 27, Bernoulli DE
p:=unapply(-1,x);
q:=unapply(1,x);
n:=-1/2;
de:=diff(y(x),x)=-p(x)*y(x)+q(x)*y(x)^n;
dsolve(de,y(x));
# Check substitution u=y/y^(-1/2)=y^(3/2)
dsolve(diff(u(x),x)/(3/2)+p(x)*u(x)=q(x),u(x));
```

28. $y' - y = y^{-1/3}$

29. $yy' + y^2 = e^x$

Solution: Isolate y': $y' + y = e^x y^{-1}$. Then $n = -1$, $p = 1$, $q = e^x$. The substitution is $u = y/y^{-1} = y^2$. Then $u(x) = c\,\mathrm{e}^{-2\,x} + \frac{2}{3}\,\mathrm{e}^x$ and $y^2 = c\,\mathrm{e}^{-2\,x} + \frac{2}{3}\,\mathrm{e}^x$.

```
# Exercise 29, Bernoulli DE
p:=unapply(1,x);
q:=unapply(exp(x),x);
n:=-1;
de:=diff(y(x),x)=-p(x)*y(x)+q(x)*y(x)^n;
dsolve(de,y(x));
# Check substitution u=y/y^(-1)=y^2
dsolve(diff(u(x),x)/(2)+p(x)*u(x)=q(x),u(x));
```

30. $y' + y = e^{2x}y^2$

Integrating Factor $x^a y^b$

Report an implicit solution for the given equation $M dx + N dy = 0$, using an integrating factor $Q = x^a y^b$. Follow Example 2.50, page 169. Computer assist expected.

31. $M = 3xy - 6y^2$, $N = 4x^2 - 15xy$

Solution: Integrating factor xy^3, $y^4(x^3 - 1) - 3y^5(x^2 - 1) - 3y^5 + y^4 = c$, Details follow the example: solve $xy(M_y - N_x) - (ayN - bxM) = xy(-5x + 3y) - ay(4x^2 - 15xy) + bx(3xy - 6y^2) = 0$ for $a = 1$, $b = 3$ by coefficients of $x^i y^j$ equal to zero. Let $M_1 = Mx^a y^b$, $N_1 = Nx^a y^b$ and solve $M_1 dx + N_1 dy = 0$ by the Exactness Theorem.

```
# Exercise 31, Integrating factor x^a*y^b
findIntFactor:=proc(M1,N1)
 local p,q,a,b,Test;
 Test:=(M,N)->x*y*(diff(M,y)-diff(N,x)) - (a*y*N-b*x*M);
 p:=expand(Test(M1,N1));printf("%a",p);
 q:=solve({coeffs(p,[x,y])},{a,b});
 RETURN (q);
end proc;
M1:=3*x*y-6*y^2;N1:=4*x^2-15*x*y;
findIntFactor(M1,N1);# {a = 1, b = 3}
IF:=x^1 * y^3;
M:=unapply(M1*IF,(x,y));
N:=unapply(N1*IF,(x,y));
A:=diff(M(x,y),y);B:=diff(N(x,y),x);
"Exact if zero" = simplify(A-B);# Check Exact
x0:=1;y0:=1;M(x0,y0);N(x0,y0);# Domain check
# Solve Mdx + Ndy=0
U:=int(M(t,y),t=x0..x)+int(N(x0,s),s=y0..y)
assuming x::positive, y::positive;
# U := y^4*(x^3-1)-3*y^5*(x^2-1)-3*y^5+y^4+2
```

32. $M = 3xy - 10y^2$, $N = 4x^2 - 25xy$

33. $M = 2\,y - 12\,xy^2$, $N = 4\,x - 20\,x^2y$

Solution: Integrating factor x^1y^3, solution $-4y^5(x^3-1)+y^4(x^2-1)-4y^5+y^4=c$

34. $M = 2\,y - 21\,xy^2$, $N = 4\,x - 35\,x^2y$

35. $M = 3\,y - 32\,xy^2$, $N = 4\,x - 40\,x^2y$

Solution: Integrating factor xy^3, solution

$-(32/3)y^5(x^3-1)+(3/2)y^4(x^2-1)-8y^5+y^4=c$

36. $M = 3\,y - 20\,xy^2$, $N = 4\,x - 25\,x^2y$

37. $M = 12\,y - 30\,x^2y^2$,
$N = 12\,x - 25\,x^3y$

Solution: Integrating factor x^3y^3, solution

$-(15/2)y^5(x^4-1)+6y^4(x^2-1)-5y^5+3y^4=c$

38. $M = 12\,y + 90\,x^2y^2$,
$N = 12\,x + 75\,x^3y$

39. $M = 15\,y + 90\,xy^2$,
$N = 12\,x + 75\,x^2y$

Solution: Integrating factor x^4y^3, solution

$30y^5(x^3-1)+(15/2)y^4(x^2-1)+15y^5+3y^4=c$

40. $M = 35\,y + 30\,xy^2$,
$N = 28\,x + 25\,x^2y$.

Integrating Factor e^{ax+by}

Report an implicit solution $U(x,y)=c$ for the given equation $Mdx+Ndy=0$ using an integrating factor $Q=e^{ax+by}$. Follow Example 2.51, page 170.

41. $M = e^x + 2e^{2y}$, $N = e^x + 5e^{2y}$

Solution: Integrating factor $\mathrm{e}^{3\,y+2\,x}$, solution $\mathrm{e}^{3\,x+3\,y}+3\,\mathrm{e}^{2\,x+5\,y}=c$.

The test for integrating factor e^{ax+by} is

$$M_y - N_x - aN + bM = 0$$

The plan is to expand the left side and obtain two equations in unknowns a, b by the **sampling method**: substitute values $x=y=0$ in the above equation to get $3-6a+3b=0$, then substitute $x=y=1$ to get $4e^2-e-ae-5ae^2+be+2be^2=0$. Solve the two equations in two unknowns to find

$a = 2$, $b = 3$. Then the integrating factor is e^{2x+3y}. Multiply $Mdx + Ndy$ by the integrating factor and solve by the **Method of Potentials**: $U = (1/3)e^{3x+3y} + e^{2x+5y} - 4/3$. A simplified solution is $e^{3x+3y} + 3e^{2x+5y} = c$.

```
# Exercise 41, Integrating factor exp(a*x+b*y)
findIntFactorExp:=proc(M1,N1)
 local p,q,a,b,Test,eq1,eq2;
 Test:=(M,N)->diff(M,y)-diff(N,x) - a*N+b*M;
 p:=expand(Test(M1,N1));
 eq1:=simplify(subs(x=0,y=0,p));
 eq2:=simplify(subs(x=1,y=1,p));
 q:=solve([eq1,eq2],[a,b]);p:=q[1];
 RETURN (rhs(p[1]),rhs(p[2]));
end proc;
M1:=exp(x)+2*exp(2*y);N1:=exp(x)+5*exp(2*y);
A,B:=findIntFactorExp(M1,N1);# Failed? Modify samples in proc
IF:=subs(a=A,b=B,exp(a*x+b*y));
M:=unapply(simplify(expand(M1*IF)),(x,y));
N:=unapply(simplify(expand(N1*IF)),(x,y));
A:=diff(M(x,y),y);B:=diff(N(x,y),x);
"Exact if zero" = simplify(A-B);# Check Exact
x0:=0;y0:=0;M(x0,y0);N(x0,y0);# Domain check
# Solve Mdx + Ndy=0, method of potentials
U:=int(M(t,y),t=x0..x)+int(N(x0,s),s=y0..y);
# U := (1/3)*exp(3*x+3*y)+exp(2*x+5*y)-4/3
```

42. $M = 3e^x + 2e^y$, $N = 4e^x + 5e^y$

43. $M = 12\,e^x + 2$, $N = 20\,e^x + 5$

Solution: Integrating factor e^{2x+5y}, solution $4e^{3x+5y} + e^{2x+5y} = c$.

44. $M = 12\,e^x + 2\,e^{-y}$, $N = 24\,e^x + 5\,e^{-y}$

45. $M = 12\,e^y + 2\,e^{-x}$, $N = 24\,e^y + 5\,e^{-x}$

Solution: Integrating factor e^{3x+5y}, solution $4\,\mathrm{e}^{6\,y+3\,x} + \mathrm{e}^{2\,x+5\,y} = c$.

46. $M = 12\,e^{-2\,y} + 2\,e^{-x}$, $N = 12\,e^{-2\,y} + 5\,e^{-x}$

47. $M = 16\,e^y + 2\,e^{-2\,x+3\,y}$, $N = 12\,e^y + 5\,e^{-2\,x+3\,y}$

Solution: Integrating factor e^{4x+2y}, solution $4\,\mathrm{e}^{4\,x+3\,y} + \mathrm{e}^{2\,x+5\,y} = c$.

The sampling method changes: use $x = y = 0$ for the first equation and $x = 0$, $y = 1$ for the second equation. Computer code is edited to change the sample values for `eq2`: $x = 0$, $y = 1$. The edit modifies function `findIntFactorExp` in the `maple` text of Exercise 41.

48. $M = 16\,e^{-y} + 2\,e^{-2\,x-3\,y}$, $N = -12\,e^{-y} - 5\,e^{-2\,x-3\,y}$

49. $M = -16 - 2\,e^{2\,x+y}$, $N = 12 + 4\,e^{2\,x+y}$

Solution: Integrating factor $\mathrm{e}^{-4\,x+3\,y}$, solution $\mathrm{e}^{-2\,x+4\,y} + 4\,\mathrm{e}^{-4\,x+3\,y} = c$

50. $M = -16\,e^{-3\,y} - 2\,e^{2\,x}$, $N = 8\,e^{-3\,y} + 5\,e^{2\,x}$

Integrating Factor $Q(x)$

Report an implicit solution $U(x,y) = c$ for the given equation, using an integrating factor $Q = Q(x)$. Follow Example 2.52, page 171.

51. $(x + 2y)dx + (x - x^2)dy = 0$

Solution: Integrating factor $Q = x/(x-1)^3$. Solution

$$-8y + \frac{2\,\ln(x-1)\,x^2 + 6\,x^2 y - 4\,x\ln(x-1)}{x^2 - 2\,x + 1}$$
$$+\frac{5\,x^2 - 16\,xy + 2\,\ln(x-1) - 14\,x + 8\,y + 8}{x^2 - 2\,x + 1} = c$$

The plan for $Q(x)$ a function of x alone: form $\mu = (M_y - N_x)/N$ and then $Q = \int \mu dx$. The new equation $M\mu dx + N\mu dy = 0$ is exact and can be solved by the method of potentials.

```
# Exercise 51, Integrating factor Q(x)
findIntFactorQ:=proc(M1,N1)
 local p,q,mu;
 mu:=(diff(M1,y)-diff(N1,x))/N1;# depends on x only
 p:=expand(mu);printf("mu=%a",p);
 q:=exp(int(p,x));
 if subs(y='Y',q) = q then RETURN (q) fi;
RETURN ("ERROR");
end proc;
M1:=x+2*y;N1:=x-x^2;
IF:=findIntFactorQ(M1,N1);# "ERROR" means no x-only IF
M:=unapply(simplify(expand(M1*IF)),(x,y));
N:=unapply(simplify(expand(N1*IF)),(x,y));
A:=diff(M(x,y),y);B:=diff(N(x,y),x);
"Exact if zero" = simplify(A-B);# Check Exact
x0:=0;y0:=0;M(x0,y0);N(x0,y0);# Domain check
# Solve Mdx + Ndy=0, method of potentials
U:=int(M(t,y),t=x0..x)+int(N(x0,s),s=y0..y);
# U := (1/2)*(2*ln(x-1)*x^2+6*x^2*y-4*x*ln(x-1)
#      +5*x^2-16*x*y+2*ln(x-1)-14*x+8*y+8)/(x^2-2*x+1)-4*y
```

52. $(x + 3y)dx + (x - x^2)dy = 0$

53. $(2x+y)dx+(x-x^2)dy=0$

Solution: Integrating factor $Q=1/(x-1)^2$.
Solution $\frac{2\,x\ln(x-1)+xy-2\,\ln(x-1)+2\,x-2\,y-4}{x-1}-2\,y=c$

54. $(2x+y)dx+(x+x^2)dy=0$

55. $(2x+y)dx+(-x-x^2)dy=0$

Solution: Integrating factor $Q=1/x^2$.
Solution $\frac{1}{3}\,\frac{6\,x\ln(x)-6\,\ln(3)\,x+xy-3\,y}{x}-\frac{4}{3}\,y$

56. $(x+y)dx+(-x-x^2)dy=0$

57. $(x+y)dx+(-x-2x^2)dy=0$

Solution: Integrating factor $Q=1/x^2$.
Solution $\frac{1}{3}\,\frac{3\,x\ln(x)-3\,\ln(3)\,x+xy-3\,y}{x}-\frac{7}{3}\,y$

58. $(x+y)dx+(x+5x^2)dy=0$

59. $(x+y)dx+(3x)dy=0$

Solution: Integrating factor $Q=1/x^{2/3}$.
Solution $-\frac{9}{4}\,\sqrt[3]{3}+3/4\,x^{4/3}+3\,\sqrt[3]{x}y$

60. $(x+y)dx+(7x)dy=0$

Integrating Factor $Q(y)$

61. $(y-y^2)dx+(x+y)dy=0$

Solution: Integrating factor $Q=1/(y-1)^2$.
Solution $-\frac{xy}{y-1}+\frac{\ln(y-1)\,y-\ln(y-1)+y-2}{y-1}=c$

```
# Exercise 61, Integrating factor Q(y)
findIntFactorQ:=proc(M1,N1)
 local p,q,mu;
 mu:=(diff(N1,x)-diff(M1,y))/M1;# depends on y only
 p:=expand(mu);printf("mu=%a",p);
 q:=exp(int(p,y));
 if subs(x='X',q) = q then RETURN (q) fi;
RETURN ("ERROR");
end proc;
M1:=y-y^2;N1:=x+y;
IF:=findIntFactorQ(M1,N1);# "ERROR" means no y-only IF
M:=unapply(simplify(expand(M1*IF)),(x,y));
N:=unapply(simplify(expand(N1*IF)),(x,y));
A:=diff(M(x,y),y);B:=diff(N(x,y),x);
"Exact if zero" = simplify(A-B);# Check Exact
x0:=0;y0:=2;M(x0,y0);N(x0,y0);# Domain check
# Solve Mdx + Ndy=0, method of potentials
U:=int(M(t,y),t=x0..x)+int(N(x0,s),s=y0..y) assuming y>1;
# U := -y*x/(y-1)+(ln(y-1)*y-ln(y-1)+y-2)/(y-1)
```

62. $(y - y^2)dx + (2x + y)dy = 0$

63. $(y - y^2)dx + (2x + 3y)dy = 0$

Solution: Integrating factor $Q = y/(y-1)^3$. Solution

$$-\frac{xy^2}{(y-1)^2} + \frac{3}{2}\,\frac{2\ln(y-1)\,y^2 - 4\ln(y-1)\,y + 5\,y^2 + 2\ln(y-1) - 14\,y + 8}{y^2 - 2\,y + 1} = c$$

64. $(y + y^2)dx + (2x + 3y)dy = 0$

65. $(y + y^2)dx + (x + 3y)dy = 0$

Solution: Integrating factor $Q = 1/(y+1)^2$. Solution

$$\frac{xy}{1+y} + \frac{3\ln(1+y)\,y - 3\ln(3)\,y + 3\ln(1+y) - 3\ln(3) - y + 2}{1+y} = c$$

66. $(y + 5y^2)dx + (x + 3y)dy = 0$

67. $(y + 3y^2)dx + (x + 3y)dy = 0$

Solution: Integrating factor $Q = 1/(3y+1)^2$. Solution

$$\frac{xy}{1+3\,y} + \frac{21\ln(1+3\,y)\,y - 21\ln(7)\,y + 7\ln(1+3\,y) - 7\ln(7) - 3\,y + 6}{21(1+3\,y)}$$

68. $(2y+5y^2)dx+(7x+11y)dy=0$

69. $(2y+5y^2)dx+(x+7y)dy=0$

Solution: Integrating factor $\dfrac{1}{(5\,y+2)^{3/2}\,\sqrt{y}}$, solution

$$\frac{\sqrt{y}x}{\sqrt{5\,y+2}}+\tfrac{2}{25}\,\frac{7\,\sqrt{5}\sqrt{5\,y+2}\,\ln\left(\sqrt{5}\sqrt{5\,y+2}+5\,\sqrt{y}\right)}{\sqrt{5\,y+2}}+$$
$$\tfrac{2}{25}\,\frac{-7\,\sqrt{5}\,\ln\left(\sqrt{5}\sqrt{7}+5\right)\,\sqrt{5\,y+2}}{\sqrt{5\,y+2}}+\tfrac{2}{25}\,\frac{5\,\sqrt{7}\sqrt{5\,y+2}-35\,\sqrt{y}}{\sqrt{5\,y+2}}$$

70. $(3y+5y^3)dx+(7x+9y)dy=0$

Chapter 3

Linear Algebraic Equations No Matrices

Contents

3.1 Systems of Linear Equations

Toolkit

Compute the equivalent system of equations. Definitions of `combo`, `swap` and `mult` on page 177.

1. Given $\left|\begin{array}{rcrcrcr} x & & & + & 2z & = & 1 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$, find the system that results from `combo(2,1,-1)`.

Solution: $\left|\begin{array}{rcrcrcr} & - & y & & & = & -3 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$

2. Given $\left|\begin{array}{ccccccc} x & & & + & 2z & = & 1 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$, find the system that results from `swap(1,2)` followed by `combo(2,1,-1)`.

3. Given $\left|\begin{array}{ccccccc} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$, find the system that results from `combo(1,2,-1)`.

Solution: $\left|\begin{array}{ccccccc} x & & & + & 3z & = & 1 \\ & & y & & & = & 3 \\ & & & & z & = & 1 \end{array}\right|$

4. Given $\left|\begin{array}{ccccccc} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$, find the system that results from `swap(1,2)` followed by `combo(1,2,-1)`.

5. Given $\left|\begin{array}{ccccc} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$, find the system that results from `swap(2,3)`, `combo(2,1,-1)`.

Solution:

$\left|\begin{array}{ccccc} y & + & z & = & 2 \\ y & & & = & 0 \\ 3y & + & 3z & = & 6 \end{array}\right|$ after `swap`

$\left|\begin{array}{ccccc} & & z & = & 2 \\ y & & & = & 0 \\ 3y & + & 3z & = & 6 \end{array}\right|$ after `combo`

6. Given $\left|\begin{array}{ccccc} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$, find the system that results from `mult(2,1/3)`, `combo(1,2,-1)`, `swap(2,3)`, `swap(1,2)`.

Inverse Toolkit

Compute the equivalent system of equations.

7. If $\left|\begin{array}{ccccccc} - & & y & & & = & -3 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$ resulted from `combo(2,1,-1)`, then find the original system.

Solution: $\left|\begin{array}{rcrcrcl} x & & & + & 2z & = & 1 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$ after `combo(2,1,c)` with $c = 1 =$ additive inverse of -1

8. If $\left|\begin{array}{rcrcrcl} & & y & & & = & 3 \\ x & & & + & 2z & = & 1 \\ & & & & z & = & 0 \end{array}\right|$ resulted from `swap(1,2)` followed by `combo(2,1,-1)`, then find the original system.

9. If $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ & & y & - & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$ resulted from `combo(1,2,-1)`, then find the original system.

Solution: $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ x & + & y & & & = & 5 \\ & & & & z & = & 1 \end{array}\right|$

10. If $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$ resulted from `swap(1,2)` followed by `combo(2,1,2)`, then find the original system.

11. If $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$ resulted from `mult(2,-1)`, `swap(2,3)`, `combo(2,1,-1)`, then find the original system.

Solution: Apply inverse operations in reverse order: `combo(2,1,1)`, `swap(2,3)`, `mult(2,1)`.

$\left|\begin{array}{rrcrcl} & 4y & + & 4z & = & 8 \\ - & 3y & + & 3z & = & 6 \\ & y & & & = & 0 \end{array}\right|$ after `combo(2,1,1)`

$\left|\begin{array}{rrcrcl} & 4y & + & 4z & = & 8 \\ & y & & & = & 0 \\ - & 3y & + & 3z & = & 6 \end{array}\right|$ after `swap(2,3)`

$\left|\begin{array}{rrcrcl} & 4y & + & 4z & = & 8 \\ & y & & & = & 0 \\ - & 3y & + & 3z & = & 6 \end{array}\right|$ after `mult(2,1)`

12. If $\left|\begin{array}{rcrcl} 2y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$ resulted from `mult(2,1/3)`, `combo(1,2,-1)`, `swap(2,3)`, `swap(1,2)`, then find the original system.

Planar System

Solve the xy–system and interpret the solution geometrically as

(a) **parallel lines**

(b) **equal lines**

(c) **intersecting lines**.

13. $\left|\begin{array}{rcrcr} x & + & y & = & 1, \\ & & y & = & 1 \end{array}\right|$

Solution: $x = 0$, $y = 1$ intersecting lines

14. $\left|\begin{array}{rcrcr} x & + & y & = & -1 \\ x & & & = & 3 \end{array}\right|$

15. $\left|\begin{array}{rcrcr} x & + & y & = & 1 \\ x & + & 2y & = & 2 \end{array}\right|$

Solution: $x = -1$, $y = 2$, intersecting lines

16. $\left|\begin{array}{rcrcr} x & + & y & = & 1 \\ x & + & 2y & = & 3 \end{array}\right|$

17. $\left|\begin{array}{rcrcr} x & + & y & = & 1 \\ 2x & + & 2y & = & 2 \end{array}\right|$

Solution: Divide the second equation by 2 to get two equal equations. The two lines are actually one line: equal lines.

18. $\left|\begin{array}{rcrcr} 2x & + & y & = & 1 \\ 6x & + & 3y & = & 3 \end{array}\right|$

19. $\left|\begin{array}{rcrcr} x & - & y & = & 1 \\ -x & - & y & = & -1 \end{array}\right|$

Solution: $x = 1$, $y = 0$, intersecting lines

20. $\left|\begin{array}{rcrcr} 2x & - & y & = & 1 \\ x & - & 0.5y & = & 0.5 \end{array}\right|$

21. $\left|\begin{array}{rcrcr} x & + & y & = & 1 \\ x & + & y & = & 2 \end{array}\right|$

Solution: Parallel lines, because equation 2 minus equation 1 is a signal equation $0 = 1$.

22. $\left|\begin{array}{rcrcr} x & - & y & = & 1 \\ x & - & y & = & 0 \end{array}\right|$

System in Space

For each xyz–system:

(a) If no solution, then report **three identical shelves**, **pup tent**, **two parallel shelves** or **book shelf**.

(b) If infinitely many solutions, then report **one shelf**, **open book** or **saw tooth**.

(c) If a unique intersection point, then report the values of x, y and z.

23. $$\left|\begin{array}{rcrcrcl} x & - & y & + & z & = & 2 \\ x & & & & & = & 1 \\ & & y & & & = & 0 \end{array}\right|$$

Solution: Answer: (c) unique intersection $x = 1$, $y = 0$, $z = 1$.

24. $$\left|\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 3 \\ x & & & & & = & 2 \\ & & & & z & = & 1 \end{array}\right|$$

25. $$\left|\begin{array}{rcrcl} x & - & y & = & 2 \\ x & - & y & = & 1 \\ x & - & y & = & 0 \end{array}\right|$$

Solution: Answer: (a) No solution. Three parallel planes $x - y = c$ for $c = 0, 1, 2$. Book shelves.

26. $$\left|\begin{array}{rcrcl} x & + & y & = & 3 \\ x & + & y & = & 2 \\ x & + & y & = & 1 \end{array}\right|$$

27. $$\left|\begin{array}{rcrcrcl} x & + & y & + & z & = & 3 \\ x & + & y & + & z & = & 2 \\ x & + & y & + & z & = & 1 \end{array}\right|$$

Solution: Answer: (a) No solution. Three parallel planes $x + y + z = c$ for $c = 1, 2, 3$. Book shelves.

28. $$\left|\begin{array}{rcrcrcl} x & + & y & + & 2z & = & 2 \\ x & + & y & + & 2z & = & 1 \\ x & + & y & + & 2z & = & 0 \end{array}\right|$$

29. $$\left|\begin{array}{rcrcrcl} x & - & y & + & z & = & 2 \\ 2x & - & 2y & + & 2z & = & 4 \\ & & y & & & = & 0 \end{array}\right|$$

Solution: Answer: (b) Infinitely many solutions. Open book. Two identical planes intersect a second plane $y = 0$.

30. $\left|\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 3 \\ 3x & + & 3y & - & 6z & = & 6 \\ & & & & z & = & 1 \end{array}\right|$

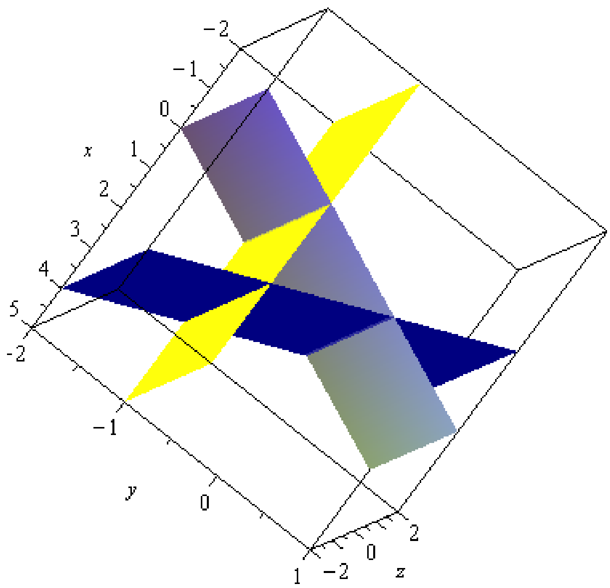
31. $\left|\begin{array}{rcrcrcl} x & - & y & + & z & = & 2 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \end{array}\right|$

Solution: Answer: (b) Infinitely many solutions. One shelf.

32. $\left|\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 3 \\ & & & & 0 & = & 0 \\ & & & & 1 & = & 1 \end{array}\right|$

33. $\left|\begin{array}{rcrcl} x & + & y & = & 2 \\ x & - & y & = & 2 \\ & & y & = & -1 \end{array}\right|$

Solution: Answer: (a) No solution. Three planes intersect pairwise. Pup tent.




```
Exercise 33 pup tent
eqs:=x+y=2,x-y=2,y=-1;
plots[implicitplot3d]({eqs},x=-2..5,y=-2..1,z=-2..2);
```

34. $\left|\begin{array}{rcrcl} x & - & 2z & = & 4 \\ x & + & 2z & = & 0 \\ & & z & = & 2 \end{array}\right|$

35. $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$

Solution: Answer: (b) Open book.

36. $\left|\begin{array}{rcrcl} x & + & 2z & = & 1 \\ 4x & + & 8z & = & 4 \\ & & z & = & 0 \end{array}\right|$

,

3.2 Filmstrips and Toolkit Sequences

Lead and free variables

For each system assume variable list $x_1, \ldots, x_5$. List the lead and free variables.

1. $\left|\begin{array}{r} x_2+3x_3 \quad\quad =0 \\ x_4 \quad =0 \\ 0=0 \end{array}\right|$

Solution: x_2, x_4

2. $\left|\begin{array}{rcl} x_2 & & =0 \\ x_3 & +3x_5 & =0 \\ & x_4+2x_5 & =0 \end{array}\right|$

3. $\left|\begin{array}{r} x_1+3x_3 \quad\quad =0 \\ x_4 \quad =0 \\ 0=0 \end{array}\right|$

Solution: x_1, x_4

4. $\left|\begin{array}{r} x_1+2x_2+3x_3 \quad\quad =0 \\ x_4 \quad =0 \\ 0=0 \end{array}\right|$

5. $\left|\begin{array}{r} x_1+2x_2+3x_3 \quad =0 \\ 0=0 \\ 0=0 \\ 0=0 \end{array}\right|$

Solution: x_1

6. $\left|\begin{array}{r} x_1+x_2 \quad\quad =0 \\ x_3 \quad =0 \\ 0=0 \end{array}\right|$

7. $\left|\begin{array}{r} x_1+x_2+3x_3+5x_4 \quad =0 \\ x_5=0 \\ 0=0 \end{array}\right|$

Solution: x_1, x_5

8. $\left|\begin{array}{rrrr} x_1+2x_2 & & +3x_4+ & 4x_5=0 \\ & x_3+ & x_4+ & x_5=0 \\ & & & 0=0 \end{array}\right|$

9. $$\left|\begin{array}{rrr} x_3 + 2x_4 & & = 0 \\ & x_5 & = 0 \\ & 0 & = 0 \\ & 0 & = 0 \end{array}\right|$$

Solution: x_3, x_5

10. $$\left|\begin{array}{r} x_4 + x_5 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right|$$

11. $$\left|\begin{array}{llrr} x_2 & + 5x_4 & & = 0 \\ & x_3 + 2x_4 & & = 0 \\ & & x_5 & = 0 \\ & & 0 & = 0 \end{array}\right|$$

Solution: x_2, x_3, x_5

12. $$\left|\begin{array}{llllrr} x_1 & & + 3x_3 & & & = 0 \\ & x_2 & & + x_4 & & = 0 \\ & & & & x_5 & = 0 \\ & & & & 0 & = 0 \end{array}\right|$$

Elementary Operations

Consider the 3×3 system

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 2, \\ -2x & + & 3y & + & 4z & = & 0, \\ -3x & + & 5y & + & 7z & = & 3. \end{array}$$

Define symbols **combo**, **swap** and **mult** as in the textbook. Write the 3×3 system which results from each of the following operations.

13. `combo(1,3,-1)`

Solution: Define `combo(s,t,c)` to be the result after adding c times source equation s to target equation t. The operation changes only the target equation. The new system after `combo(1,3,-1)`:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 2, \\ -2x & + & 3y & + & 4z & = & 0, \\ -4x & + & 3y & + & 4z & = & 1. \end{array}$$

14. `combo(2,3,-5)`

15. `combo(3,2,4)`

Solution:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 2, \\ -14x & + & 23y & + & 32z & = & 12, \\ -3x & + & 5y & + & 7z & = & 3. \end{array}$$

16. `combo(2,1,4)`

17. `combo(1,2,-1)`

Solution:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 2, \\ -3x & + & y & + & z & = & -2, \\ -3x & + & 5y & + & 7z & = & 3. \end{array}$$

18. `combo(1,2,-`e^2`)`

19. `mult(1,5)`

Solution: Define `mult(1,5)` to be the result after multiplying equation 1 by 5:

$$\begin{array}{rcrcrcr} 5x & + & 10y & + & 15z & = & 10, \\ -2x & + & 3y & + & 4z & = & 0, \\ -3x & + & 5y & + & 7z & = & 3. \end{array}$$

20. `mult(1,-3)`

21. `mult(2,5)`

Solution:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 2, \\ -10x & + & 15y & + & 20z & = & 0, \\ -3x & + & 5y & + & 7z & = & 3. \end{array}$$

22. `mult(2,-2)`

23. `mult(3,4)`

Solution:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 2, \\ -2x & + & 3y & + & 4z & = & 0, \\ -12x & + & 20y & + & 28z & = & 12. \end{array}$$

24. `mult(3,5)`

25. `mult(2,-`π`)`

Solution:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 2, \\ 2\pi x & + & -3\pi y & + & -4\pi z & = & 0, \\ -3x & + & 5y & + & 7z & = & 3. \end{array}$$

26. `mult(2,`π`)`

27. `mult(1,`e^2`)`

Solution:

$$\begin{array}{rcrcrcr} e^2x & + & 2e^2y & + & 3e^2z & = & 2e^2, \\ -2x & + & 3y & + & 4z & = & 0, \\ -3x & + & 5y & + & 7z & = & 3. \end{array}$$

28. `mult(1,-`e^{-2}`)`

29. `swap(1,3)`

Solution: Define `swap(1,3)` to be the result after swapping equations 1 and 3:

$$\begin{array}{rcrcrcl} -3x & + & 5y & + & 7z & = & 3, \\ -2x & + & 3y & + & 4z & = & 0, \\ x & + & 2y & + & 3z & = & 2. \end{array}$$

30. `swap(1,2)`

31. `swap(2,3)`

Solution:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 2, \\ -3x & + & 5y & + & 7z & = & 3, \\ -2x & + & 3y & + & 4z & = & 0. \end{array}$$

32. `swap(2,1)`

33. `swap(3,2)`

Solution:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 2, \\ -3x & + & 5y & + & 7z & = & 3, \\ -2x & + & 3y & + & 4z & = & 0. \end{array}$$

34. `swap(3,1)`

Unique Solution

Create a toolkit sequence for each system, whose final frame displays the unique solution of the system of equations. Assume variable list order x_1, x_2, x_3, x_4, x_5 and the number of variables is the number of equations.

35. $\left|\begin{array}{rl} x_1+3x_2= & 0 \\ x_2= & -1 \end{array}\right|$

Solution: $\left|\begin{array}{rl} x_1+3x_2= & 0 \\ x_2= & -1 \end{array}\right|$ Frame 1

$\left|\begin{array}{rl} x_1 \quad = & 3 \\ x_2= & -1 \end{array}\right|$ Frame 2, `combo(2,1,-3)`

36. $\left|\begin{array}{rl} x_1+2x_2= & 0 \\ x_2= & -2 \end{array}\right|$

37. $\left|\begin{array}{rl} x_1+3x_2= & 2 \\ x_1- \ x_2= & 1 \end{array}\right|$

Solution: Definition: `combo(s,t,c)` arguments s=source equation, t=target equation, c=multiplier

$\left|\begin{array}{rl} x_1+3x_2= & 2 \\ x_1- \ x_2= & 1 \end{array}\right|$ Frame 1

$\left|\begin{array}{rl} x_1+3x_2= & 2 \\ -4x_2= & -1 \end{array}\right|$ Frame 2, `combo(1,2,-1)`

$\left|\begin{array}{rl} x_1+3x_2= & 2 \\ x_2= & 1/4 \end{array}\right|$ Frame 3, `mult(2,-1/4)`

$\left|\begin{array}{rl} x_1 \quad = & 5/4 \\ x_2= & 1/4 \end{array}\right|$ Frame 4, `combo(2,1,-3)`

38. $\left|\begin{array}{rl} x_1+ \ x_2= & -1 \\ x_1+2x_2= & -2 \end{array}\right|$

39. $\left|\begin{array}{r} x_1+3x_2+2x_3=1 \\ x_2+4x_3=3 \\ 4x_3=4 \end{array}\right|$

Solution:

$\left|\begin{array}{r} x_1+3x_2+2x_3=1 \\ x_2+4x_3=3 \\ 4x_3=4 \end{array}\right|$ Frame 1

$\left|\begin{array}{r} x_1+3x_2+2x_3=1 \\ x_2+4x_3=3 \\ x_3=1 \end{array}\right|$ Frame 2: `mult(3,1/4)`

$$\left|\begin{array}{rcl} x_1 + 3x_2 + 2x_3 &=& 1 \\ x_2 &=& -1 \\ x_3 &=& 1 \end{array}\right| \quad \text{Frame 3: } \texttt{combo(3,2,-4)}$$

$$\left|\begin{array}{rcl} x_1 + 3x_2 &=& -1 \\ x_2 &=& -1 \\ x_3 &=& 1 \end{array}\right| \quad \text{Frame 4: } \texttt{combo(3,1,-2)}$$

$$\left|\begin{array}{rcl} x_1 &=& 2 \\ x_2 &=& -1 \\ x_3 &=& 1 \end{array}\right| \quad \text{Frame 5: } \texttt{combo(2,1,-3)}$$

40. $$\left|\begin{array}{rcl} x_1 &=& 1 \\ 3x_1 + x_2 &=& 0 \\ 2x_1 + 2x_2 + 3x_3 &=& 3 \end{array}\right|$$

41. $$\left|\begin{array}{rcl} x_1 + x_2 + 3x_3 &=& 1 \\ x_2 &=& 2 \\ 3x_3 &=& 0 \end{array}\right|$$

Solution: Reminder: `combo(s,t,c)` arguments s=source equation, t=target equation, c=multiplier

$$\left|\begin{array}{rcl} x_1 + x_2 + 3x_3 &=& 1 \\ x_2 &=& 2 \\ x_3 &=& 0 \end{array}\right| \quad \text{Frame 2: } \texttt{mult(3,1/3)}$$

$$\left|\begin{array}{rcl} x_1 + x_2 &=& 1 \\ x_2 &=& 2 \\ x_3 &=& 0 \end{array}\right| \quad \text{Frame 3: } \texttt{combo(3,1,-3)}$$

$$\left|\begin{array}{rcl} x_1 &=& -1 \\ x_2 &=& 2 \\ x_3 &=& 0 \end{array}\right| \quad \text{Frame 4: } \texttt{combo(2,1,-1)}$$

42. $$\left|\begin{array}{rcl} x_1 + 3x_2 + 2x_3 &=& 1 \\ x_2 &=& 3 \\ 3x_3 &=& 0 \end{array}\right|$$

43. $$\left|\begin{array}{rcl} x_1 &=& 2 \\ x_1 + 2x_2 &=& 1 \\ 2x_1 + 2x_2 + x_3 &=& 0 \\ 3x_1 + 6x_2 + x_3 + 2x_4 &=& 2 \end{array}\right|$$

Solution:

$$\left|\begin{array}{rcl} x_1 &=& 2 \\ 2x_2 &=& -1 \\ 2x_1 + 2x_2 + x_3 &=& 0 \\ 3x_1 + 6x_2 + x_3 + 2x_4 &=& 2 \end{array}\right| \quad \text{Frame 2: } \texttt{combo(1,2,-1)}$$

$$\left|\begin{array}{rcr} x_1 & = & 2 \\ 2x_2 & = & -1 \\ 2x_2 + x_3 & = & -4 \\ 3x_1 + 6x_2 + x_3 + 2x_4 & = & 2 \end{array}\right|$$ Frame 3: `combo(1,3,-2)`

$$\left|\begin{array}{rcr} x_1 & = & 2 \\ 2x_2 & = & -1 \\ 2x_2 + x_3 & = & -4 \\ 6x_2 + x_3 + 2x_4 & = & -4 \end{array}\right|$$ Frame 4: `combo(1,4,-3)`

$$\left|\begin{array}{rcr} x_1 & = & 2 \\ 2x_2 & = & -1 \\ x_3 & = & -3 \\ 6x_2 + x_3 + 2x_4 & = & -4 \end{array}\right|$$ Frame 5: `combo(2,3,-1)`

$$\left|\begin{array}{rcr} x_1 & = & 2 \\ 2x_2 & = & -1 \\ x_3 & = & -3 \\ x_3 + 2x_4 & = & -1 \end{array}\right|$$ Frame 6: `combo(2,4,-3)`

$$\left|\begin{array}{rcr} x_1 & = & 2 \\ 2x_2 & = & -1 \\ x_3 & = & -3 \\ 2x_4 & = & 2 \end{array}\right|$$ Frame 7: `combo(3,4,-1)`

$$\left|\begin{array}{rcr} x_1 & = & 2 \\ x_2 & = & -1/2 \\ x_3 & = & -3 \\ x_4 & = & 1 \end{array}\right|$$ Frame 8: `mult(2,1/2)`, `mult(4,1/2)`

44. $$\left|\begin{array}{rcr} x_1 & = & 3 \\ x_1 - 2x_2 & = & 1 \\ 2x_1 + 2x_2 + x_3 & = & 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 & = & 2 \end{array}\right|$$

45. $$\left|\begin{array}{rcr} x_1 + x_2 & = & 2 \\ x_1 + 2x_2 & = & 1 \\ 2x_1 + 2x_2 + x_3 & = & 0 \\ 3x_1 + 6x_2 + x_3 + 2x_4 & = & 2 \end{array}\right|$$

Solution:

$$\left|\begin{array}{rcr} x_1 + x_2 & = & 2 \\ x_1 + 2x_2 & = & 1 \\ 2x_1 + 2x_2 + x_3 & = & 0 \\ 3x_1 + 6x_2 + x_3 + 2x_4 & = & 2 \end{array}\right|$$ Frame 1

$$\left|\begin{array}{rcr} x_1 + x_2 & = & 2 \\ x_2 & = & -1 \\ 2x_1 + 2x_2 + x_3 & = & 0 \\ 3x_1 + 6x_2 + x_3 + 2x_4 & = & 2 \end{array}\right|$$ Frame 2: `combo(1,2,-1)`

$$\left|\begin{aligned} x_1 + \ \ x_2 \qquad\qquad &= \ \ 2 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= -4 \\ 3x_1 + 6x_2 + x_3 + 2x_4 &= \ \ 2 \end{aligned}\right| \quad \text{Frame 3: } \texttt{combo(1,3,-2)}$$

$$\left|\begin{aligned} x_1 + \ \ x_2 \qquad\qquad &= \ \ 2 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= -4 \\ 3x_2 + x_3 + 2x_4 &= -4 \end{aligned}\right| \quad \text{Frame 4: } \texttt{combo(1,4,-3)}.$$

Variable x_1 has just one occurrence. The next variable to eliminate to just once occurrence is x_2, taken from variable list order x_1, x_2, x_3, x_4.

$$\left|\begin{aligned} x_1 + x_2 \qquad\qquad &= \ \ 2 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= -4 \\ x_3 + 2x_4 &= -1 \end{aligned}\right| \quad \text{Frame 5: } \texttt{combo(2,4,-3)}$$

$$\left|\begin{aligned} x_1 \qquad\qquad\qquad &= \ \ 3 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= -4 \\ x_3 + 2x_4 &= -1 \end{aligned}\right| \quad \text{Frame 6: } \texttt{combo(2,1,-1)}.$$

Variables x_1, x_2 isolated to just one occurrence. Next variable: x_3.

$$\left|\begin{aligned} x_1 + \qquad\qquad\qquad &= \ \ 3 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= -4 \\ 2x_4 &= \ \ 3 \end{aligned}\right| \quad \text{Frame 7: } \texttt{combo(3,4,-1)}.$$

All variables isolated to just one occurrence.

$$\left|\begin{aligned} x_1 + \qquad\qquad\qquad &= \ \ 3 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= -4 \\ x_4 &= 3/2 \end{aligned}\right| \quad \text{Frame 8: } \texttt{mult(4,1/2)}.$$

This is the Reduced Echelon Form of the system of equations, which displays the unique solution.

46. $$\left|\begin{aligned} x_1 - 2x_2 \qquad\qquad\qquad &= 3 \\ x_1 - \ \ x_2 \qquad\qquad\qquad &= 1 \\ 2x_1 + 2x_2 + x_3 \qquad\qquad &= 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 1 \end{aligned}\right|$$

47. $$\left|\begin{aligned} x_1 \qquad\qquad\qquad\qquad\qquad\qquad &= 3 \\ x_1 - \ \ x_2 \qquad\qquad\qquad\qquad\qquad &= 1 \\ 2x_1 + 2x_2 + x_3 \qquad\qquad\qquad\qquad &= 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 \qquad\qquad &= 1 \\ 3x_1 \qquad\quad + x_3 \qquad\quad + 2x_5 &= 1 \end{aligned}\right|$$

Solution: $x_1 = 3$, $x_2 = 2$, $x_3 = -10$, $x_4 = -5/2$, $x_5 = 1$

```
# Maple answer check Ex 47
A:=Matrix
([
  [1,0,0,0,0],
  [1,-1,0,0,0],
  [2,2,1,0,0],
  [3,6,1,4,0],
  [3,0,1,0,2]
]);
b:=<3,1,0,1,1>;
LinearAlgebra[LinearSolve](A,b,free=t);
```

48. $\left|\begin{array}{llllll} x_1 & & & & & = 2 \\ x_1 - & x_2 & & & & = 0 \\ 2x_1 + & 2x_2 + & x_3 & & & = 1 \\ 3x_1 + & 6x_2 + & x_3 + & 3x_4 & & = 1 \\ 3x_1 & & + x_3 & & + 3x_5 & = 1 \end{array}\right|$

49. $\left|\begin{array}{rrrrrr} x_1 - & x_2 + & x_3 - & x_4 + & x_5 = & 0 \\ & 2x_2 - & x_3 + & x_4 - & x_5 = & 0 \\ & & 3x_3 - & x_4 + & x_5 = & 0 \\ & & & 4x_4 - & x_5 = & 0 \\ & & & & 5x_5 = & 20 \end{array}\right|$

Solution: $x_1 = -1$, $x_2 = 1$, $x_3 = -1$, $x_4 = 1$, $x_5 = 4$

```
# Maple answer check Ex 49
A:=Matrix
([
  [1, -1, 1, -1, 1],
  [0, 2, -1, 1, -1],
  [0, 0, 3, -1, 1],
  [0, 0, 0, 4, -1],
  [0, 0, 0, 0, 5]
]);
b:=<0,0,0,0,20>;
LinearAlgebra[LinearSolve](A,b,free=t);
```

50. $\left|\begin{array}{llllll} x_1 - & x_2 & & & & = 3 \\ x_1 - & 2x_2 & & & & = 0 \\ 2x_1 + & 2x_2 + & x_3 & & & = 1 \\ 3x_1 + & 6x_2 + & x_3 + & 3x_4 & & = 1 \\ 3x_1 & & + x_3 & & + x_5 & = 3 \end{array}\right|$

No Solution

Develop a toolkit sequence for each system, whose final frame contains a signal equation (e.g., $0 = 1$), thereby showing that the system has no solution.

51. $\begin{vmatrix} x_1+3x_2=0 \\ x_1+3x_2=1 \end{vmatrix}$

Solution: Parallel lines. Subtract the equations to get signal equation $0 = 1$.

52. $\begin{vmatrix} x_1+2x_2=1 \\ 2x_1+4x_2=2 \end{vmatrix}$

53. $\begin{vmatrix} x_1 + 3x_2 + 2x_3 = 1 \\ x_2 + 4x_3 = 3 \\ x_2 + 4x_3 = 4 \end{vmatrix}$

Solution: Equations 2 and 3 are parallel lines in 3D. Subtract them to get signal equation $0 = 1$.

54. $\begin{vmatrix} x_1 \qquad\qquad = 0 \\ 3x_1 + x_2 + 3x_3 = 1 \\ 2x_1 + 2x_2 + 6x_3 = 0 \end{vmatrix}$

55. $\begin{vmatrix} x_1 + x_2 + 3x_3 = 1 \\ x_2 \qquad = 2 \\ x_1 + 2x_2 + 3x_3 = 2 \end{vmatrix}$

Solution: Subtract equation 2 from equation 3: $x_1 + x_2 + 3x_3 = 0$, which is parallel to equation 1. Subtract it from equation 1 to arrive at signal equation $0 = 1$.

56. $\begin{vmatrix} x_1 + 3x_2 + 2x_3 = 1 \\ x_2 + 2x_3 = 3 \\ x_1 \qquad + 5x_3 = 5 \end{vmatrix}$

57. $\begin{vmatrix} x_1 \qquad\qquad = 2 \\ x_1 + 2x_2 \qquad = 2 \\ x_1 + 2x_2 + x_3 + 2x_4 = 0 \\ x_1 + 6x_2 + x_3 + 2x_4 = 2 \end{vmatrix}$

Solution: The first two equations give $x_1 = 2$, $x_2 = 0$. Then the last two equations become $x_3 + 2x_4 = -2$, $x_3 + 2x_4 = 0$. Subtract them to arrive at signal equation $-2 = 0$.

58. $\begin{vmatrix} x_1 \qquad\qquad = 3 \\ x_1 - 2x_2 \qquad = 1 \\ 2x_1 + 2x_2 + x_3 + 4x_4 = 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 = 2 \end{vmatrix}$

59. $\left|\begin{array}{rl} x_1 \qquad\qquad\qquad\qquad & = 3 \\ x_1 - \ \ x_2 \qquad\qquad\qquad & = 1 \\ 2x_1 + 2x_2 + x_3 \qquad\qquad & = 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 - x_5 & = 1 \\ - 6x_2 - x_3 + 4x_4 + x_5 & = 0 \end{array}\right|$

Solution: Solve the first 3 equations for $x_1 = 3$, $x_2 = 2$, $x_3 = -10$. Substitute into equations 4,5: $4x_4 + x_5 = -10$, $4x_4 + x_5 = 2$. Subtract them to arrive at signal equation $0 = 12$.

60. $\left|\begin{array}{rl} x_1 \qquad\qquad\qquad\qquad & = 3 \\ x_1 - \ \ x_2 \qquad\qquad\qquad & = 1 \\ 3x_1 + 2x_2 + x_3 \qquad\qquad & = 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 - x_5 & = 1 \\ - 6x_2 - x_3 - 4x_4 + x_5 & = 2 \end{array}\right|$

Infinitely Many Solutions

Display a toolkit sequence for each system, whose final frame has this property: *each nonzero equation has a lead variable.* Then apply the **last frame algorithm** to write out the standard general solution of the system. Assume in each system variable list x_1 to x_5.

61. $\left|\begin{array}{rl} x_1 + x_2 + 3x_3 \qquad\quad & = 0 \\ x_2 \qquad\quad + x_4 & = 0 \\ 0 & = 0 \end{array}\right|$

Solution:

$\left|\begin{array}{rl} x_1 \quad + 3x_3 - x_4 & = 0 \\ x_2 \qquad\quad + x_4 & = 0 \\ 0 & = 0 \end{array}\right|$ Frame 2 = Last Frame

The lead variables are x_1, x_2 and the free variables are x_3, x_4, x_5. The last frame algorithm applies:

$$\begin{array}{rcl} x_1 & = & -3x_3 + x_4, \quad \text{isolate lead variables left} \\ x_2 & = & -x_4, \\ x_3 & = & t_1, \quad \text{assign symbols to the free variables} \\ x_4 & = & t_2, \\ x_5 & = & t_3. \end{array}$$

Substitute symbols t_1, t_2, t_3 for free variables on the right side of the lead variable equations.

$$\begin{array}{rcl} x_1 & = & -3t_1 + t_2, \\ x_2 & = & -t_2, \\ x_3 & = & t_1, \\ x_4 & = & t_2, \\ x_5 & = & t_3. \end{array}$$

This is the general solution in terms of invented symbols t_1, t_2, t_3.

62. $\left|\begin{array}{lllll} x_1 & & +x_3 & & =0 \\ x_1 & +x_2 & +x_3 & +3x_5 & =0 \\ & & & x_4+2x_5 & =0 \end{array}\right|$

63. $\left|\begin{array}{lll} x_2+3x_3 & & =0 \\ & x_4 & =0 \\ & & 0=0 \end{array}\right|$

Solution: Lead variables x_2, x_4 and free variables x_1, x_3, x_5. Last frame algorithm:

$$\begin{array}{lll} x_2 &=& -3x_3, \quad \text{isolate lead variables left} \\ x_4 &=& 0, \\ x_1 &=& t_1, \quad \text{assign symbols to the free variables} \\ x_3 &=& t_2, \\ x_5 &=& t_3. \end{array}$$

The general solution:

$$\begin{array}{lll} x_1 &=& t_1, \\ x_2 &=& -3t_2, \\ x_3 &=& t_2, \\ x_4 &=& 0, \\ x_5 &=& t_3. \end{array}$$

64. $\left|\begin{array}{lll} x_1+2x_2+3x_3 & & =0 \\ & x_4 & =0 \\ & & 0=0 \end{array}\right|$

65. $\left|\begin{array}{ll} x_1+2x_2+3x_3 & =0 \\ \qquad x_3+x_4 & 0=0 \end{array}\right|$

Solution: Lead variables x_1, x_3 and free variables x_2, x_4, x_5.

$\left|\begin{array}{lll} x_1+2x_2 & & -3x_4=0 \\ & x_3+x_4 & 0=0 \end{array}\right|$ Last frame.

Last frame algorithm:

$$\begin{array}{lll} x_1 &=& -2x_2+3x_4, \quad \text{isolate lead variables left} \\ x_3 &=& -x_4, \\ x_2 &=& t_1, \quad \text{assign symbols to the free variables} \\ x_4 &=& t_2, \\ x_5 &=& t_3. \end{array}$$

The general solution:

$$\begin{array}{lll} x_1 &=& -2t_1+3t_2, \\ x_2 &=& -t_2, \\ x_3 &=& t_1, \\ x_4 &=& t_2, \\ x_5 &=& t_3. \end{array}$$

66. $\left|\begin{array}{rrrl} x_1 + x_2 & & & = 0 \\ & x_2 + x_3 & & = 0 \\ & & x_3 & 0 = 1 \end{array}\right|$

67. $\left|\begin{array}{r} x_1 + x_2 + 3x_3 + 5x_4 + 2x_5 = 0 \\ x_5 = 0 \end{array}\right|$

Solution: Lead variables x_1, x_5 and free variables x_2, x_3, x_4.

$$\left|\begin{array}{rr} x_1 + x_2 + 3x_3 + 5x_4 & = 0 \\ x_5 & = 0 \end{array}\right| \quad \text{Last Frame.}$$

Last frame algorithm:

$$\begin{array}{rcl} x_1 &=& -x_2 - 3x_3 - 5x_4, \quad \text{isolate lead variables left} \\ x_5 &=& 0, \\ x_2 &=& t_1, \quad \text{assign symbols to the free variables} \\ x_3 &=& t_2, \\ x_4 &=& t_3. \end{array}$$

The general solution:

$$\begin{array}{rcl} x_1 &=& -t_1 - 3t_2 - 5t_3, \\ x_2 &=& t_1, \\ x_3 &=& t_2, \\ x_4 &=& t_3, \\ x_5 &=& 0. \end{array}$$

68. $\left|\begin{array}{r} x_1 + 2x_2 + x_3 + 3x_4 + 4x_5 = 0 \\ x_3 + \ \ x_4 + \ \ x_5 = 0 \end{array}\right|$

69. $\left|\begin{array}{r} x_3 + 2x_4 + \ \ x_5 = 0 \\ 2x_3 + 2x_4 + 2x_5 = 0 \\ x_5 = 0 \end{array}\right|$

Solution: // $\left|\begin{array}{r} x_3 + 2x_4 + x_5 = 0 \\ -\, 2x_4 \qquad = 0 \\ x_5 = 0 \end{array}\right|$ Frame 2: `combo(1,2,-2)`

$$\left|\begin{array}{rrrl} x_3 & & + x_5 & = 0 \\ & -\, 2x_4 & & = 0 \\ & & x_5 & = 0 \end{array}\right| \quad \text{Frame 3: } \texttt{combo(2,1,1)}$$

$$\left|\begin{array}{rrrl} x_3 & & + x_5 & = 0 \\ & x_4 & & = 0 \\ & & x_5 & = 0 \end{array}\right| \quad \text{Frame 4: } \texttt{mult(2,-1/2)}$$

$$\left|\begin{array}{rrrl} x_3 & & & = 0 \\ & x_4 & & = 0 \\ & & x_5 & = 0 \end{array}\right| \quad \text{Last Frame: } \texttt{combo(3,1,-1)}$$

Lead variables x_3, x_4, x_5 and free variables x_1, x_2. Last frame algorithm:

$$\begin{array}{lcll} x_3 & = & 0, & \text{isolate lead variables left} \\ x_4 & = & 0, & \\ x_5 & = & 0, & \\ x_1 & = & t_1, & \text{assign symbols to the free variables} \\ x_2 & = & t_2. & \end{array}$$

The general solution:

$$\begin{array}{lcl} x_1 & = & t_1, \\ x_2 & = & t_2, \\ x_3 & = & 0, \\ x_4 & = & 0, \\ x_5 & = & 0. \end{array}$$

70. $$\left|\begin{array}{r} x_4 + x_5 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right|$$

71. $$\left|\begin{array}{rrrr} x_2 + x_3 + 5x_4 & & = 0 \\ x_3 + 2x_4 & & = 0 \\ & x_5 & = 0 \\ & & 0 = 0 \end{array}\right|$$

Solution:

$$\left|\begin{array}{llll} x_2 & + 3x_4 & & = 0 \\ & x_3 + 2x_4 & & = 0 \\ & & x_5 & = 0 \\ & & & 0 = 0 \end{array}\right| \qquad \text{Last Frame: } \texttt{combo(2,1,-1)}$$

Lead variables x_2, x_3, x_5 and free variables x_1, x_4. Last frame algorithm:

$$\begin{array}{lcll} x_2 & = & -3x_4, & \text{isolate lead variables left} \\ x_3 & = & -2x_4, & \\ x_5 & = & 0, & \\ x_1 & = & t_1, & \text{assign symbols to the free variables} \\ x_4 & = & t_2. & \end{array}$$

The general solution:

$$\begin{array}{lcl} x_1 & = & t_1, \\ x_2 & = & -3t_2, \\ x_3 & = & -2t_2, \\ x_4 & = & t_2, \\ x_5 & = & 0. \end{array}$$

72. $$\left|\begin{array}{lllll} x_1 & & + 3x_3 & & = 0 \\ x_1 + x_2 & & & + x_4 & = 0 \\ & & & x_5 & = 0 \\ & & & & 0 = 0 \end{array}\right|$$

Inverses of Elementary Operations

Given the final frame of a toolkit sequence is

$$\left|\begin{array}{rcrcrcr} 3x & + & 2y & + & 4z & = & 2 \\ x & + & 3y & + & 2z & = & -1 \\ 2x & + & y & + & 5z & = & 0 \end{array}\right|$$

and the given operations, find the original system in the first frame.

73. `combo(1,2,-1)`, `combo(2,3,-3)`, `mult(1,-2)`, `swap(2,3)`.

Solution: Apply to the given system the inverse operations in reverse order: `swap(2,3)`, `mult(1,-1/2)`, `combo(2,3,3)`. The steps:

$$\left|\begin{array}{rcrcrcr} 3x & + & 2y & + & 4z & = & 2 \\ 2x & + & y & + & 5z & = & 0 \\ x & + & 3y & + & 2z & = & -1 \end{array}\right| \quad \texttt{swap(2,3)}$$

$$\left|\begin{array}{rcrcrcr} -3x/2 & - & y & - & 2z & = & -1 \\ 2x & + & y & + & 5z & = & 0 \\ x & + & 3y & + & 2z & = & -1 \end{array}\right| \quad \texttt{mult(1,-1/2)}$$

$$\left|\begin{array}{rcrcrcr} -3x/2 & - & y & - & 2z & = & -1 \\ 2x & + & y & + & 5z & = & 0 \\ 7x & + & 6y & + & 17z & = & -1 \end{array}\right| \quad \texttt{combo(2,3,3)}$$

This is the original system.

74. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,2)`, `swap(3,2)`.

75. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`.

Solution:

$$\left|\begin{array}{rcrcrcr} -3x/2 & - & y & - & 2z & = & -1 \\ x/2 & & & + & 3z & = & -1 \\ 7x & + & 6y & + & 17z & = & -1 \end{array}\right|$$

76. `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`.

77. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`, `swap(2,3)`.

Solution:

$$\left|\begin{array}{rcrcrcr} x/4 & + & 3y/4 & + & z/2 & = & -1/4 \\ 9x/4 & + & 7y/4 & + & 11z/2 & = & -1/4 \\ -3x & - & y & - & 11z & = & 2 \end{array}\right|$$

78. `swap(2,3)`, `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`.

79. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`, `mult(2,3)`.

Solution:
$$\left|\begin{array}{rcrcrcr} x/2 & + & y/4 & + & 5z/4 & = & 0 \\ 5x/6 & + & 5y/4 & + & 23z/12 & = & -1/3 \\ 2x & - & y & + & 2z & = & 3 \end{array}\right|$$

80. `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`, `combo(2,3,-3)`.

,

3.3 General Solution Theory

Classification

Classify the parametric equations as a point, line or plane, then compute as appropriate the tangent to the line or the normal to the plane.

1. $x = 0$, $y = 1$, $z = -2$

Solution: Point.

2. $x = 1$, $y = -1$, $z = 2$

3. $x = t_1$, $y = 1 + t_1$, $z = 0$

Solution: Line. Tangent $= \vec{\imath} + \vec{\jmath}$.

4. $x = 0$, $y = 0$, $z = 1 + t_1$

5. $x = 1 + t_1$, $y = 0$, $z = t_2$

Solution: Plane. The partial derivatives on t_1 and t_2 generate vectors $\vec{\imath}$ and $\vec{k}$. The cross product of these two vectors is $-\vec{\jmath}$ by the right hand rule. The normal vector $\vec{N}$ can also be generated by determinant expansion:

$$\vec{N} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0\vec{\imath} - \vec{\jmath} + 0\vec{k} = -\vec{\jmath}$$

6. $x = t_2 + t_1$, $y = t_2$, $z = t_1$

7. $x = 1$, $y = 1 + t_1$, $z = 1 + t_2$

Solution: Plane. The normal vector:

$$\vec{N} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\vec{\imath} - 0\vec{\jmath} + 0\vec{k} = \vec{\imath}$$

8. $x = t_2 + t_1$, $y = t_1 - t_2$, $z = 0$

9. $x = t_2$, $y = 1 + t_1$, $z = t_1 + t_2$

Solution: Plane.The normal vector:

$$\vec{N} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1\vec{\imath} - 0\vec{\jmath} + (-1)\vec{k} = \vec{\imath} - \vec{k}$$

10. $x = 3t_2 + t_1$, $y = t_1 - t_2$, $z = 2t_1$

Reduced Echelon System

Solve the xyz–system and interpret the solution geometrically.

11. $\left|\begin{array}{rcrcl} & y & + & z & = 1 \\ x & & + & 2z & = 2 \end{array}\right|$

Solution:

$\left|\begin{array}{rcrcl} & y & + & z & = 1 \\ x & & + & 2z & = 2 \end{array}\right|$ Frame 1

$\left|\begin{array}{rcrcl} x & & + & 2z & = 2 \\ & y & + & z & = 1 \end{array}\right|$ Last Frame : `swap(1,2)`

Lead variables x, y, free variable z.

$$\begin{array}{rcll} x & = & 2 - 2z, & \text{isolate lead variables left} \\ y & = & 1 - z, & \\ z & = & t_1. & \text{assign symbols to free variables} \end{array}$$

$$\begin{array}{rcll} x & = & 2 - 2t_1, & \text{replace RHS free variables by symbols} \\ y & = & 1 - t_1, & \text{and report answer in variable list order} \\ z & = & t_1. & \end{array}$$

Geometry: two planes intersect along a line.

```
# Maple answer check
with(LinearAlgebra):
A:=Matrix([[0,1,1,1],[1,0,2,2]]):
ReducedRowEchelonForm(A):
LinearSolve(A,free=t);
# ans=[2-2*t, 1-t, t]
```

12. $\left|\begin{array}{rcrcl} x & & + & z & = 1 \\ & y & + & 2z & = 4 \end{array}\right|$

13. $\left|\begin{array}{rcrcl} & y & + & z & = 1 \\ x & & + & 3z & = 2 \end{array}\right|$

Solution: $x = 2 - 3t$, $y = 1 - t$, $z = t$, two planes intersect in a line.

14. $\left|\begin{array}{rcrcl} x & + & z & = & 1 \\ y & + & z & = & 5 \end{array}\right|$

15. $\left|\begin{array}{rcrcl} x & + & z & = & 1 \\ 2x & + & 2z & = & 2 \end{array}\right|$

Solution: $x = 1 - t_2$, $y = t_1$, $z = t_2$, two equal planes.

16. $\left|\begin{array}{rcrcl} x & + & y & = & 1 \\ 3x & + & 3y & = & 3 \end{array}\right|$

17. $\left|\, x + y + z = 1. \,\right|$

Solution: $x = 1 - t_1 - t_2$, $y = t_1$, $z = t_2$, one plane.

18. $\left|\, x + 2y + 4z = 0. \,\right|$

19. $\left|\begin{array}{rcrcrcl} x & + & y & & & = & 2 \\ & & & & z & = & 1 \end{array}\right|$

Solution: $x = 2 - t$, $y = t$, $z = 1$, two planes intersect in a line.

20. $\left|\begin{array}{rcrcrcl} x & & & + & 4z & = & 0 \\ & & y & & & = & 1 \end{array}\right|$

Homogeneous System

Solve the xyz–system using elimination with variable list order x, y, z.

21. $\left|\begin{array}{rcrcrcl} & & y & + & z & = & 0 \\ 2x & & & + & 2z & = & 0 \end{array}\right|$

Solution: $x = -t_1$, $y = -t_1$, $z = t_1$

```
# Maple answer check
with(LinearAlgebra):
A:=Matrix([[0,1,1],[2,0,2],[0,0,0]]):
LinearSolve(A,Vector([0,0,0]),free=t);
# ans=[-t, -t, t]
```

22. $\left|\begin{array}{rcrcrcl} x & & & + & z & = & 0 \\ & & 2y & + & 2z & = & 0 \end{array}\right|$

23. $\left|\begin{array}{rcrcl} x & + & z & = & 0 \\ & & 2z & = & 0 \end{array}\right|$

Solution: $x = 0$, $y = t$, $z = 0$

24. $\left|\begin{array}{rcrcl} y & + & z & = & 0 \\ y & + & 3z & = & 0 \end{array}\right|$

25. $\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ & & & & 0 & = & 0 \end{array}\right|$

Solution: $x = -2t_1 - 3t_2$, $y = t_1$, $z = t_2$

26. $\left|\begin{array}{rcrcrcr} x & + & 2y & & & = & 0 \\ & & & & 0 & = & 0 \end{array}\right|$

27. $\left|\begin{array}{rcrcrcr} & & y & + & z & = & 0 \\ 2x & & & + & 2z & = & 0 \\ x & & & + & z & = & 0 \end{array}\right|$

Solution: $x = -t$, $y = -t$, $z = t$

28. $\left|\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 0 \\ x & & & + & 2z & = & 0 \\ x & + & y & - & z & = & 0 \end{array}\right|$

29. $\left|\begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ 2x & & & + & 2z & = & 0 \\ x & & & + & z & = & 0 \end{array}\right|$

Solution: $x = t$, $y = 0$, $z = t$

30. $\left|\begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ 2x & & & + & 2z & = & 0 \\ 3x & + & y & + & 3z & = & 0 \end{array}\right|$

Nonhomogeneous 3×3 System

Solve the xyz-system using elimination and variable list order x, y, z.

31. $\left|\begin{array}{rcrcrcr} & & y & & & = & 1 \\ & & & & 2z & = & 2 \end{array}\right|$

Solution: $x = t$, $y = 1$, $z = 1$

```
# Maple answer check
with(LinearAlgebra):
A:=Matrix([[0,1,0],[0,0,2],[0,0,0]]):
LinearSolve(A,Vector([1,2,0]),free=t);
# ans=[t, 1, 1]
```

32. $\left|\begin{array}{rcrcrcr} x & & & & & = & 1 \\ & & & & 2z & = & 2 \end{array}\right|$

33. $\left|\begin{array}{rcrcrcr} & & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ x & & & + & z & = & 1 \end{array}\right|$

Solution: $x = 1 - t$, $y = 1 - t$, $z = t$

34. $\left|\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 1 \\ x & & & + & 2z & = & 2 \\ x & + & y & - & z & = & -1 \end{array}\right|$

35. $\left|\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ x & & & + & z & = & 1 \end{array}\right|$

Solution: $x = 1 - t$, $y = 0$, $z = t$

36. $\left|\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ 3x & + & y & + & 3z & = & 3 \end{array}\right|$

37. $\left|\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 3 \\ 2x & & & + & 2z & = & 2 \\ 4x & + & y & + & 3z & = & 5 \end{array}\right|$

Solution: $x = 1 - t$, $y = 1 + t$, $z = t$

38. $\left|\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 2 \\ 6x & & y & + & 5z & = & 2 \\ 4x & + & y & + & 3z & = & 2 \end{array}\right|$

39. $\left|\begin{array}{rcrcrcr} 6x & + & 2y & + & 6z & = & 10 \\ 6x & & y & + & 6z & = & 11 \\ 4x & + & y & + & 4z & = & 7 \end{array}\right|$

Solution: $x = 2 - t$, $y = -1$, $z = t$

40. $\left|\begin{array}{rcrcrcr} 6x & + & 2y & + & 4z & = & 6 \\ 6x & & y & + & 5z & = & 9 \\ 4x & + & y & + & 3z & = & 5 \end{array}\right|$

Nonhomogeneous 3×4 System

Solve the $yzuv$-system using elimination with variable list order y, z, u, v.

41. $\left|\begin{array}{rcrcrcrcr} y & + & z & + & 4u & + & 8v & = & 10 \\ & & 2z & - & u & + & v & = & 10 \\ 2y & & & - & u & + & 5v & = & 10 \end{array}\right|$

Solution: $y = 5 - 3t$, $z = 5 - t$, $u = -t$, $v = t$

```
# Maple answer check
with(LinearAlgebra):
A:=Matrix([[1,1,4,8],[0,2,-1,1],[2,0,-1,5]]):
LinearSolve(A,Vector([10,10,10]),free=t);
# ans=[5-3*t, 5-t, -t, t]
```

42. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 8v & = 10 \\ & & 2z & - & 2u & + & 2v & = 0 \\ y & + & 3z & + & 2u & + & 5v & = 5 \end{array}\right|$

43. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 8v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 6v & = 1 \end{array}\right|$

Solution: $y = 1 - t$, $z = t$, $u = t$, $v = 0$

44. $\left|\begin{array}{rrrrrrrl} y & + & 3z & + & 4u & + & 8v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 6v & = 1 \end{array}\right|$

45. $\left|\begin{array}{rrrrrrrl} y & + & 3z & + & 4u & + & 8v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 4z & + & 2u & + & 7v & = 1 \end{array}\right|$

Solution: $y = 1 + 19t$, $z = -5t$, $u = -3t$, $v = t$

46. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 9v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 4z & + & 2u & + & 7v & = 1 \end{array}\right|$

47. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 9v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 4z & + & 2u & + & 7v & = 1 \end{array}\right|$

Solution: $y = 1 - 47t$, $z = 6t$, $u = 8t$, $v = t$

48. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 9v & = 10 \\ & & 2z & - & 2u & + & 4v & = 4 \\ y & + & 4z & + & 2u & + & 7v & = 8 \end{array}\right|$

49. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 9v & = 2 \\ & & 2z & - & 2u & + & 4v & = 4 \\ y & + & 3z & + & 5u & + & 13v & = 0 \end{array}\right|$

Solution: $y = 10 - 7t$, $z = -2t$, $u = -2$, $v = t$

50. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 3v & = 2 \\ & & 2z & - & 2u & + & 4v & = 4 \\ y & + & 3z & + & 5u & + & 7v & = 0 \end{array}\right|$

,

3.4 Basis, Dimension, Nullity and Rank

Rank and Nullity

Compute an abbreviated sequence of `combo, swap, mult` steps which finds the value of the rank and nullity.

1. $\left|\begin{array}{rrrrrrrl} x_1 & + & x_2 & + & 4x_3 & + & 8x_4 & = 0 \\ & & 2x_2 & - & x_3 & + & x_4 & = 0 \end{array}\right|$

Solution:

$\left|\begin{array}{rrrrrrrl} x_1 & & & + & 9x_3/2 & + & 15x_4/2 & = 0 \\ & & 2x_2 & - & x_3 & + & x_4 & = 0 \end{array}\right|$ `combo(2,1,-1/2)`

Lead variables x_1, x_2 and free variables x_3, x_4. Rank = 2, nullity = 2.

```
# Maple answer check Ex 1
with(LinearAlgebra):
A:=Matrix([[1,1,4,8],[0,2,-1,1]]);
B:=Vector([0,0]);
ReducedRowEchelonForm(A);
LinearSolve(A,B,free=t);
# ans=[-9*t-12*s, t, 2*t+s, s]
```

2. $\left|\begin{array}{rrrrrl} x_1 & + & x_2 & + & 8x_4 & = 0 \\ & & 2x_2 & + & x_4 & = 0 \end{array}\right|$

3. $\left|\begin{array}{rrrrrrrl} x_1 & + & 2x_2 & + & 4x_3 & + & 9x_4 & = 0 \\ x_1 & + & 8x_2 & + & 2x_3 & + & 7x_4 & = 0 \end{array}\right|$

Solution: Steps: `combo(2,1,-1)`, `combo(1,2,8/6)`. Lead variables x_1, x_2 and free variables x_3, x_4. Rank = 2, nullity = 2.

4. $\left|\begin{array}{rrrrrrrl} x_1 & + & x_2 & + & 4x_3 & + & 11x_4 & = 0 \\ & & 2x_2 & - & 2x_3 & + & 4x_4 & = 0 \end{array}\right|$

Nullspace

Solve using variable order y, z, u, v. Report the values of the **nullity** and **rank** in the equation **nullity**+**rank**=4.

5. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 8v & = 0 \\ & & 2z & - & u & + & v & = 0 \\ 2y & & & - & u & + & 5v & = 0 \end{array}\right|$

Solution: $y = -3t$, $z = -t$, $u = -t$, $v = t$, nullity=1, rank=3.

```
# Maple answer check Ex 5
with(LinearAlgebra):
A:=Matrix([[1,1,4,8],[0,2,-1,1],[2,0,-1,5]]);
B:=Vector([0,0,0]);
LinearSolve(A,B);
n:=ColumnDimension(A);Rank(A); n-Rank(A);
# ans: [-3*t, -t, -t, t]
# n=4, rank=3, nullity=1
```

6. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 8v & = 0 \\ & & 2z & - & 2u & + & 2v & = 0 \\ y & + & 3z & + & 2u & + & 5v & = 0 \end{array}\right|$

7. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 8v & = 0 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 6v & = 0 \end{array}\right|$

Solution: $y = -5t$, $z = t$, $u = t$, $v = 0$, nullity=1, rank=3.

8. $\left|\begin{array}{rrrrrrrl} y & + & 3z & + & 4u & + & 8v & = 0 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 6v & = 0 \end{array}\right|$

9. $\left|\begin{array}{rrrrrrrl} y & + & 3z & + & 4u & + & 8v & = 0 \\ & & 2z & - & 2u & + & 4v & = 0 \end{array}\right|$

Solution: $y = -7t - 2s$, $z = t - 2s$, $u = t$, $v = s$, nullity=2, rank=2.

10. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 9v & = 0 \\ & & 2z & - & 2u & + & 4v & = 0 \end{array}\right|$

11. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 9v & = 0 \\ 3y & + & 4z & + & 2u & + & 5v & = 0 \end{array}\right|$

Solution: $y = -14t - 31s$, $z = 10t + 22s$, $u = t$, $v = s$, nullity=2, rank=2.

12. $\left|\begin{array}{rrrrrrrl} y & + & 2z & + & 4u & + & 9v & = 0 \\ y & + & 8z & + & 2u & + & 7v & = 0 \end{array}\right|$

13. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 4u & + & 11v & = 0 \\ & & 2z & - & 2u & + & 4v & = 0 \end{array}\right|$

Solution: $y = -5t - 9s$, $z = t - 2s$, $u = t$, $v = s$, nullity=2, rank=2.

14. $\left|\begin{array}{rrrrrrrl} y & + & z & + & 5u & + & 11v & = 0 \\ & & 2z & - & 2u & + & 6v & = 0 \end{array}\right|$

Dimension of the nullspace

In the homogeneous systems, assume variable order x, y, z, u, v.

(a) Display an equivalent set of equations in reduced echelon form.

(b) Solve for the general solution and check the answer.

(c) Report the dimension of the nullspace.

15. $\left|\begin{array}{rcrcrcrcrl} x &+& y &+& z &+& 4u &+& 8v &= 0 \\ -x &+& & & 2z &-& 2u &+& 2v &= 0 \\ & & y &-& z &+& 6u &+& 6v &= 0 \end{array}\right|$

Solution:

$\left|\begin{array}{lcrcrcrl} x & & & & & & &= 0 \\ & y & & + & 5u &+& 7v &= 0 \\ & & z &-& u &+& v &= 0 \end{array}\right|$ (a) RREF

(b) $x = 0$, $y = -5t - 7s$, $z = t - s$, $u = t$, $v = s$

(c) Nullity=2.

```
# Maple answer check Ex 15
with(LinearAlgebra):
A:=Matrix([[1,1,1,4,8],[-1,0,2,-2,2],[0,1,-1,6,6]]):
B:=Vector([0,0,0]):
LinearSolve(A,B);
# [0, -5*t1-7*t2, t1-t2, t1, t2]
ReF:=ReducedRowEchelonForm(<A|B>);
ReF.<x,y,z,u,v,-1>;# equations for RREF
# [x, y+5*u+7*v, z-u+v]
```

16. $\left|\begin{array}{rcrcrcrcrl} x &+& y &+& z &+& 4u &+& 8v &= 0 \\ & & & - & 2z &-& u &+& v &= 0 \\ & & 2y & & &-& u &+& 5v &= 0 \end{array}\right|$

17. $\left|\begin{array}{rcrcrcrcrl} & & y &+& z &+& 4u &+& 8v &= 0 \\ x & & &+& 2z &-& 2u &+& 4v &= 0 \\ 2x &+& y &+& 3z &+& 2u &+& 6v &= 0 \end{array}\right|$

Solution:

(a) $x - 6v = 0$, $y + 5u + 3v = 0$, $z - u + 5v = 0$

(b) $x = 6s$, $y = -5t - 3s$, $z = t - 5s$, $u = t$, $v = s$

(c) Nullity=2.

18. $\left|\begin{array}{rcrcrcrcrl} x &+& y &+& 3z &+& 4u &+& 8v &= 0 \\ 2x & & &+& 2z &-& 2u &+& 4v &= 0 \\ x &-& y &+& 3z &+& 2u &+& 12v &= 0 \end{array}\right|$

19. $\left|\begin{array}{rrrrrrrrr} & y & + & 3z & + & 4u & + & 20v & = 0 \\ & & + & 2z & - & 2u & + & 10v & = 0 \\ - & y & + & 3z & + & 2u & + & 30v & = 0 \end{array}\right|$

Solution:

(a) $y = 0,\ z + 8v = 0,\ u + 3v = 0$

(b) $x = t,\ y = 0,\ z = -8s,\ u = -3s,\ v = s$

(c) Nullity=2.

20. $\left|\begin{array}{rrrrrrr} & y & + & 4u & + & 20v & = 0 \\ & & - & 2u & + & 10v & = 0 \\ - & y & + & 2u & + & 30v & = 0 \end{array}\right|$

21. $\left|\begin{array}{rrrrrrrl} x + & y + & z + & 4u & & & = 0 \\ & & - & 2z - & u & & = 0 \\ & 2y & & & - & u+ & = 0 \end{array}\right|$

Solution:

(a) $x + 4u = 0,\ y - u/2 = 0,\ z + u/2 = 0$

(b) $x = 8t,\ y = -t,\ z = t,\ u = -2t,\ v = s$

(c) Nullity=2.

22. $\left|\begin{array}{rrrrrrrr} & + & z & + & 12u & + & 8v & = 0 \\ x & + & 2z & - & 6u & + & 4v & = 0 \\ 2x & + & 3z & + & 6u & + & 6v & = 0 \end{array}\right|$

23. $\left|\begin{array}{rrrrrl} y & + & z & + & 4u & = 0 \\ & & 2z & - & 2u & = 0 \\ y & - & z & + & 6u & = 0 \end{array}\right|$

Solution:

(a) $x + 4u = 0,\ y - u/2 = 0,\ z + u/2 = 0$

(b) $x = t_1,\ y = -5t_2,\ z = t_2,\ u = t_2,\ v = t_3$

(c) Nullity=3.

24. $\left|\begin{array}{rrrrrl} x & + & z & + & 8v & = 0 \\ & - & 2z & + & v & = 0 \\ & & & & 5v & = 0 \end{array}\right|$

Three possibilities with symbols

Assume variables x, y, z. Determine the values of the constants (a, b, c, k, etc) such that the system has (1) *No solution*, (2) *A unique solution* or (3) *Infinitely many solutions.*

25. $\left|\begin{array}{rcrcl} x & + & ky & = & 0 \\ x & + & 2ky & = & 0 \end{array}\right|$

Solution: Use `combo(1,2,-1)`, `combo(2,1,-1)` to arrive at system $x = 0$, $ky = 0$. If $k = 0$, then nullity=1 and there are infinitely many solutions. Otherwise $k \neq 0$, then nullity=0 with unique solution $x = 0$, $y = 0$.

26. $\left|\begin{array}{rcrcl} kx & + & ky & = & 0 \\ x & + & 2ky & = & 0 \end{array}\right|$

27. $\left|\begin{array}{rcrcl} ax & + & by & = & 0 \\ x & + & 2by & = & 0 \end{array}\right|$

Solution: Answer: Infinitely many solutions for $b = 0$, unique solution for $b \neq 0$. A homogeneous system always has a solution $x = y = 0$, so no solution cannot happen.

If $a = 0$, then the system is $by = 0$, $x + 2by = 0$ which is equivalent to $by = 0$, $x = 0$. If $b = 0$, then the system reduces to $0 = 0$, $x = 0$ which has infinitely many solutions $x = 0$, $y = t_1$. If $b \neq 0$, then the system is equivalent to $y = 0$, $x = 0$, a unique solution.

If $a \neq 0$, then the system is equivalent to $x = 0$, $by = 0$. Then $a \neq 0$, $b = 0$ makes infinitely many solutions, while $a \neq 0$, $b \neq 0$ makes for unique solution $x = 0$, $y = 0$.

28. $\left|\begin{array}{rcrcl} bx & + & ay & = & 0 \\ x & + & 2y & = & 0 \end{array}\right|$

29. $\left|\begin{array}{rcrcr} bx & + & ay & = & c \\ x & + & 2y & = & b - c \end{array}\right|$

Solution: Answer: (1) No solution if $a = 2b$, $b \neq 0$ and $c/b \neq c + 2b$. (2) Unique solution for $2b - a \neq 0$. (3) Infinitely many solutions if $a = b = c = 0$ or $a = 2b$, $b \neq 0$ and $c/b = c + 2b$.

Cramer's rule from college algebra detects the unique solution case: determinant $\left|\begin{array}{cc} b & a \\ 1 & 2 \end{array}\right| = 2b - a \neq 0$. Then $2b - a = 0$ is required for either no solution or infinitely many solutions. It can be false that all three possibilities occur.

If $2b - a = 0$, then symbol a is replaced by $2b$ is give system $bx + 2by = c$, $x + 2y = c + 2b$.

If $b \neq 0$, then system $bx + 2by = c$, $x + 2y = c + 2b$ is equivalent to $x + 2y = c/b$, $x + 2y = c + 2b$. This system has no solution if $c/b \neq c + 2b$ (parallel lines) and infinitely many solutions if $c/b = c + 2b$ (equal lines).

If $b = 0$, then system $bx + 2by = c$, $x + 2y = c + 2b$ is equivalent to $0 = c$,

$x+2y=c$, in turn equivalent to the single equation $x+2y=0$ which has infinitely many solutions.

30. $\left|\begin{array}{rcrcr} bx &+& ay &=& 2c \\ x &+& 2y &=& c+a \end{array}\right|$

31. $\left|\begin{array}{rcrcrcr} bx &+& ay &+& z &=& 0 \\ 2bx &+& ay &+& 2z &=& 0 \\ x &+& 2y &+& 2z &=& c \end{array}\right|$

Solution: Case (1): No solution for $a\neq 0$ and $-2b+1=0$ and $c\neq 0$. Case (2): Unique solution for $a\neq 0$ and $b\neq 1/2$. Case (3): Infinitely many solutions for $a\neq 0$ and $-2b+1=0$ and $c=0$, or $a=0$ and any values for b and c.

The unique solution case (2) is determined by a nonzero determinant of coefficients, evaluated using college methods:

$$\left|\begin{array}{ccc} b & a & 1 \\ 2b & a & 2 \\ 1 & 2 & 2 \end{array}\right| = a(-2b+1)$$

The no solution case (1) and infinitely many solution case (3) must assume $a(-2b+1)=0$. Steps applied:

$$\left|\begin{array}{rcrcrcr} bx &+& ay &+& z &=& 0 \\ 2bx &+& ay &+& 2z &=& 0 \\ x &+& 2y &+& 2z &=& c \end{array}\right| \quad \text{Frame 1}$$

$$\left|\begin{array}{rcrcrcr} bx & & &+& z &=& 0 \\ & & &-& ay &=& 0 \\ x &+& 2y &+& 2z &=& c \end{array}\right| \quad \text{Frame 2: } \texttt{combo(1,2-2)}$$

$$\left|\begin{array}{rcrcrcr} &-& 2by &+& (1-2b)z &=& -bc \\ & & &-& ay &=& 0 \\ x &+& 2y &+& 2z &=& c \end{array}\right| \quad \text{Frame 3: } \texttt{combo(3,1,-b)}$$

$$\left|\begin{array}{rcrcrcr} x &+& 2y &+& 2z &=& c \\ & & 2by &+& (2b-1)z &=& -bc \\ & & &-& ay &=& 0 \end{array}\right| \quad \text{Frame 6.}$$

Details Frames 4, 5, 6: `mult(1,-1)`, `swap(2,3)`, `swap(1,2)`.

The unique solution case (2) has 3 lead variables. No solution case (1) is decided by a signal equation. Infinite many solutions case (3) has either 1 or 2 lead variables and no signal equation.

Assume case (1) or case (3) holds, meaning $a(-2b+1)=0$. We examine Frame 6 for lead variables, free variables and signal equations. Variable x is a lead variable. The other lead variable can be y or z, for a total of 2 lead variables:

1. $a \neq 0$ allows division by a to get $y = 0$. Then x, y are lead variables and z is a free variable (signal equation ignored). Substitute $y = 0$ and $(-2b+1) = 0$ into Frame 6 equations: $x + 2z = c$, $0 = -bc$, $y = 0$. Then:

1a. Case (1) occurs for $a \neq 0$, $-2b+1 = 0$, $c \neq 0$ due to signal equation "$0 = -bc$."
1b. Case (3) occurs for $a \neq 0$, $-2b+1 = 0$ and $c = 0$.

2. $a = 0$ and $b \neq 0$ implies y is a lead variable. Frame 6 after substitution of $a = 0$ and division by b becomes $x + 2y + 2z = c$, $2y + (2 - 1/b)z = -c$, $0 = 0$. Eliminate to get $x + (1/b)z = 2c$, $2y + (2 - 1/b)z = -c$, $0 = 0$. Consistent system, one free variable, no signal equation, case (3) infinitely many solutions.

3. $a = 0$ and $b = 0$ implies Frame 6 becomes $x + 2y + 2z = c$, $-z = 0$, $0 = 0$. Then z is a lead variable, y is a free variable, consistent system, no signal equation, case (3) infinitely many solutions.

32. $\left|\begin{array}{rcrcrcr} bx &+& ay &+& z &=& 0 \\ 3bx &+& 2ay &+& 2z &=& 2c, \\ x &+& 2y &+& 2z &=& c \end{array}\right|$

33. $\left|\begin{array}{rcrcrcr} 3x &+& ay &+& z &=& b \\ 2bx &+& ay &+& 2z &=& 0 \\ x &+& 2y &+& 2z &=& c \end{array}\right|$

Solution: Swap equations to put the right hand sides in order $c, b, 0$. Then do operations `combo(1,2,-3)`, `combo(1,3,-2b)` to get equations

$x + 2y + 2z = c$,
$(a-6)y - 5z = b - 3c$,
$(a-4b)y + (2-4b)z = -2bc$

Case : $a - 6 \neq 0$

Then x, y, z are lead variables and the result is case (2): unique solution.

Case: $a - 6 = 0$
Replace $a = 6$ in the preceding equations. Then do operations `combo(1,2,-3)`, `combo(1,3,-2b)` to get equations

$x + 2y = (2b - c)z/5$,
$(6-4b)y = (2bc + 2b - 6c - 4b^2)/5$,
$z = (3c - b)/5$

If $6 - 4b \neq 0$, then there is a unique solution. If $6 - 4b = 0$, then y is a free variable subject to consistency equation $0 = 2bc + 2b - 6c - 4b^2$. If$6 - 4b = 0$ and $0 = 2bc + 2b - 6c - 4b^2$ then case (3) holds: infinitely many solutions. If $6 - 4b = 0$ and $0 \neq 2bc + 2b - 6c - 4b^2$ then case (1) holds: no solution.

34. $$\left|\begin{array}{rcrcrcr} x & + & ay & + & z & = & 2b \\ 3bx & + & 2ay & + & 2z & = & 2c \\ x & + & 2y & + & 2z & = & c \end{array}\right|$$

Three Possibilities

Answer the following questions by using equivalents for the three possibilities in terms of lead and free variables, signal equations, rank and nullity.

35. Does there exist a homogeneous 3×2 system with a unique solution? Give an example or else prove that no such system exists.

Solution: The variable list has 2 unknowns. Let's use x, y. An example: $x = 0$, $y = 0$, $0 = 0$.

36. Does there exist a homogeneous 2×3 system with a unique solution? Either give an example or else prove that no such system exists.

Solution: No such system exists. Proof expected.

37. In a homogeneous 10×10 system, two equations are identical. Prove that the system has a nonzero solution.

Solution: Operation `combo(s,t,c)` applies to replace one of the two equations by $0 = 0$. Therefore, the number of lead variables is at most 9 and there is at least one free variable. A homogenous system always has the zero solution: the no solution case never happens. A unique solution is detected by 10 lead variables. Infinitely many solutions is detected by less than 10 lead variables, or equivalently, at least one free variable. There are infinitely many solutions, hence at least one nonzero solution.

38. In a homogeneous 5×5 system, each equation has a leading variable. Prove that the system has only the zero solution.

39. Suppose given two homogeneous systems A and B, with A having a unique solution and B having infinitely many solutions. Explain why B cannot be obtained from A by a sequence of swap, multiply and combination operations on the equations.

Solution: If B is so obtained from A, then systems A and B must have exactly the same set of solutions. They must both fall into the same classification for the **Three Possibilities**.

40. A 2×3 system cannot have a unique solution. Cite a theorem or explain why.

41. If a 3×3 homogeneous system contains no variables, then what is the general solution?

Solution: All variables that fail to appear are free variables. If the variable list is x, y, z then the general solution is $x = t_1$, $y = t_2$, $z = t_3$ in terms of invented symbols t_1, t_2, t_3.

42. If a 3×3 non-homogeneous solution has a unique solution, then what is the nullity of the homogeneous system?

43. A 7×7 homogeneous system is missing two variables. What is the maximum rank of the system? Give examples for all possible ranks.

Solution: Assume variable list x_1 to x_7. If two are missing then they are free variables so the nullity is at least 2 and the rank is at most 5 (rank + nullity = 7). Examples are systems with 1 to 5 equations of the form $x_1 = 0, \ldots, x_k = 0$ with $k = 1, 2, 3, 4, 5$.

44. Suppose an $n \times n$ system of equations (homogeneous or non-homogeneous) has two solutions. Prove that it has infinitely many solutions.

45. What is the nullity and rank of an $n \times n$ system of homogeneous equations if the system has a unique solution?

Solution: No free variables implies the nullity is zero, the count of the number of free variables. Then rank+nullity=n implies the rank is n.

46. What is the nullity and rank of an $n \times n$ system of non-homogeneous equations if the system has a unique solution?

47. Prove or else disprove by counter-example: A 4×3 nonhomogeneous system cannot have a unique solution.

Solution: Counter-example: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $0 = 0$.

48. Prove or disprove (by example): A 4×3 homogeneous system always has infinitely many solutions.

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3.5 Answer Check, Proofs and Details

Parametric solutions

1. Is there a 2×3 homogeneous system with general solution having 2 parameters t_1, t_2?

Solution: Yes: with variable list x, y, z an example system is $x = 0$, $0 = 0$, $0 = 0$ with general solution $x = 0$, $y = t_1$, $z = t_2$.

2. Is there a 3×3 homogeneous system with general solution having 3 parameters t_1, t_2, t_3?

3. Give an example of a 4×3 homogeneous system with general solution having zero parameters, that is, $x = y = z = 0$ is the only solution.

Solution: Example: $x = 0$, $y = 0$, $z = 0$, $0 = 0$.

4. Give an example of a 4×3 homogeneous system with general solution having exactly one parameter t_1.

5. Give an example of a 4×3 homogeneous system with general solution having exactly two parameters t_1, t_2.

Solution: Example: $x = 0$, $y = 0$, $0 = 0$, $0 = 0$ with general solution $x = 0$, $y = t_1$, $z = t_2$.

6. Give an example of a 4×3 homogeneous system with general solution having exactly three parameters t_1, t_2, t_3.

7. Consider an $n \times n$ homogeneous system with parametric solution having parameters t_1 to t_k. What are the possible values of k?

Solution: The question implies $k \geq 1$. The number of parameters in the nullity and nullity+rank=n. Answer: $1 \leq k \leq n$. The case $k = n$ is special: the system has n free variables, which implies zero lead variables: the equations have no variables, which means each equation must be $0 = 0$!

8. Consider an $n \times m$ homogeneous system with parametric solution having parameters t_1 to t_k. What are the possible values of k?

Answer Checks

Assume variable list x, y, z and parameter t_1. (a) Display the answer check details. (b) Find the rank. (c) Report whether the given solution is a general solution.

9. $\begin{vmatrix} y & & = 1 \\ & 2z & = 2 \end{vmatrix}$
$x = t_1, y = 1, z = 1.$

Solution:
(a) Substitute $x = t_1, y = 1, z = 1$ formally into the equations

$$\begin{vmatrix} y & & = 1 \\ & 2z & = 2 \end{vmatrix}$$

$$\begin{vmatrix} (1) & & = 1 \\ & 2(1) & = 2 \end{vmatrix}$$

The equations are satisfied: $x = t_1, y = 1, z = 1$ is a solution.

(b) Rank=2 because y, z are lead variables.

(c) Perhaps useful is the `maple` code below used to find the exact solution by computer algebra. Expected is a paper and pencil solution with steps using combo, swap, mult steps and the **last frame algorithm**.

$$\begin{vmatrix} y & & = 1 \\ & 2z & = 2 \end{vmatrix} \quad \text{Frame 1}$$

$$\begin{vmatrix} y & & = 1 \\ & z & = 1 \end{vmatrix} \quad \text{Frame 2: } \texttt{mult(2,1/2)}$$

Last frame algorithm:

$x = t_1,$
$y = 1,$
$z = 1$

No further steps needed, variables are in list order and the right sides involve only constants and invented symbols. This is the general solution, which matches the supplied solution.

```
with(LinearAlgebra):
A:=Matrix([[0,1,0],[0,0,2],[0,0,0]]):
B:=Vector([1,2,0]):
LinearSolve(A,B);
# x = t, y = 1, z = 1
```

10. $\begin{vmatrix} x & & = 1 \\ & 2z & = 2 \end{vmatrix}$
$x = 1, y = t_1, z = 1.$

11. $\begin{vmatrix} & y & + & z & = 1 \\ 2x & & + & 2z & = 2 \\ x & & + & z & = 1 \end{vmatrix}$
$x = 0, y = 0, z = 1.$

Solution:

(a) The steps for verifying a solution:

$$\left|\begin{array}{rrrrrrr} & & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ x & & & + & z & = & 1 \end{array}\right| \quad \text{Given system}$$

$$\left|\begin{array}{rrrrrrr} & & 0 & + & 1 & = & 1 \\ 2(0) & & & + & 2(1) & = & 2 \\ 0 & & & + & 1 & = & 1 \end{array}\right| \quad \text{Substitute } x=0, y=0, z=1$$

The three equations are valid, so $x = 0, y = 0, z = 1$ is a solution.

(b) Rank=2, lead variables x, y.

(c) Not the general solution. Combo, swap, mult steps find general solution $x = 1 - t_1$, $y = 1 - t_1$, $z = t_1$. The solution from `maple` is the same.

12. $\left|\begin{array}{rrrrrrr} 2x & + & y & + & z & = & 1 \\ x & & & + & 2z & = & 2 \\ x & + & y & - & z & = & -1 \end{array}\right|$
$x = 2, y = -3, z = 0$.

13. $\left|\begin{array}{rrrrrrr} x & + & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ x & & & + & z & = & 1 \end{array}\right|$
$x = 1 - t_1, y = 0, z = t_1$.

Solution:
(a) Substitute $x = 1 - t_1, y = 0, z = t_1$:

$$\left|\begin{array}{rrrrrrr} (1-t_1) & + & 0 & + & t_1 & = & 1 \\ 2(1-t_1) & & & + & 2t_1 & = & 2 \\ 1-t_1 & & & + & t_1 & = & 1 \end{array}\right|$$

The three equations are valid, so $x = 1 - t_1, y = 0, z = t_1$ is a solution.

(b) Rank=2, lead variables x, y.

(c) Yes, it is the general solution. Checked in `maple`.

14. $\left|\begin{array}{rrrrrrr} x & + & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ 3x & + & y & + & 3z & = & 3 \end{array}\right|$
$x = 1 - t_1, y = 0, z = t_1$.

Failure of Answer Checks

Find the unique solution for $\epsilon > 0$. Discuss how a machine might translate the system to obtain infinitely many solutions.

15. $x + \epsilon y = 1$, $x - \epsilon y = 1$

Solution: Answer: $x = 2$, $y = 1/\epsilon$. If ϵ translates to zero on the machine, then both equations are $x = 1$ and y is absent, a free variable, then there are infinitely many solutions $x = 1$, $y = t_1$.

16. $x + y = 1$, $x + (1 + \epsilon)y = 1 + \epsilon$

17. $x + \epsilon y = 10\epsilon$, $x - \epsilon y = 10\epsilon$

Solution: Answer: $x = 20\epsilon$, $y = 10$. If ϵ translates to zero on the machine, then both equations are $x = 0$ and y is absent, a free variable, then there are infinitely many solutions $x = 0$, $y = t_1$. Machine answer checks using floating point engines may fail on this example, whereas computer algebra systems will not make an error.

18. $x + y = 1 + \epsilon$, $x + (1 + \epsilon)y = 1 + 11\epsilon$

Minimal Parametric Solutions

For each given system, determine if the expression is a minimal general solution.

19. $$\left|\begin{array}{rrrrl} y \;+ & z \;+ & 4u \;+ & 8v & = 0 \\ & 2z \;- & u \;+ & v & = 0 \\ 2y & - & u \;+ & 5v & = 0 \end{array}\right|$$
$y = -3t_1, z = -t_1,$
$u = -t_1, v = t_1.$

Solution: The answer given is checked as a solution, computer algebra system expected. The given solution is minimal because the rank is 3 and the nullity is 1. It would not be minimal if the number of parameters differed from the number of free variables. The nullity equals the number of free variables and the rank equals the number of lead variables.

20. $$\left|\begin{array}{rrrrl} y \;+ & z \;+ & 4u \;+ & 8v & = 0 \\ & 2z \;- & 2u \;+ & 2v & = 0 \\ y \;- & z \;+ & 6u \;+ & 6v & = 0 \end{array}\right|$$
$y = -5t_1 - 7t_2, z = t_1 - t_2,$
$u = t_1, v = t_2.$

21. $$\left|\begin{array}{rrrrl} y \;+ & z \;+ & 4u \;+ & 8v & = 0 \\ & 2z \;- & 2u \;+ & 4v & = 0 \\ y \;+ & 3z \;+ & 2u \;+ & 6v & = 0 \end{array}\right|$$
$y = -5t_1 + 5t_2, z = t_1 - t_2,$
$u = t_1 - t_2, v = 0.$

Solution: First, check the expression:

$$\left|\begin{array}{rrrl} (-5t_1 + 5t_2) \;+ & (t_1 - t_2) \;+ & 4(t_1 - t_2) \;+ & 8(0) = 0 \\ & 2(t_1 - t_2) \;- & 2(t_1 - t_2) \;+ & 4(0) = 0 \\ (-5t_1 + 5t_2) \;+ & 3(t_1 - t_2) \;+ & 2(t_1 - t_2) \;+ & 6(0) = 0 \end{array}\right|$$

The three equations are valid, so the given expression $y = -5t_1 + 5t_2, z = t_1 - t_2,\ u = t_1 - t_2, v = 0$. is a solution for all values of symbols $t_1,\ t_2$. A computer algebra system like `maple` reports the rank is 3, nullity 1 with solution

$$y = -5t_1, \quad z = t_1, \quad u = t_1, \quad v = 0.$$

The expression is not a minimal solution, because it has one extra parameter.

22. $\left|\begin{array}{rcrcrcrl} y & + & 3z & + & 4u & + & 8v & = 0 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 12v & = 0 \end{array}\right|$
$y = 5t_1 + 4t_2, z = -3t_1 - 6t_2,$
$u = -t_1 - 2t_2, v = t_1 + 2t_2.$

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Chapter 4

Numerical Methods with Applications

Contents

4.1 Solving $y' = F(x)$ Numerically

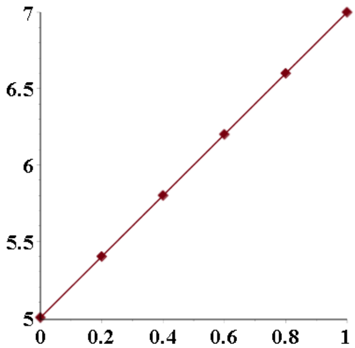
Connect-the-Dots

Make a numerical table of 6 rows and a connect-the-dots graphic for exercises 1-10.

1. $y = 2x + 5$, $x = 0$ to $x = 1$

Solution:

x	y
0.0	5.0
0.1	5.8
0.2	6.2
0.3	6.6
0.4	7.0
0.5	7.4



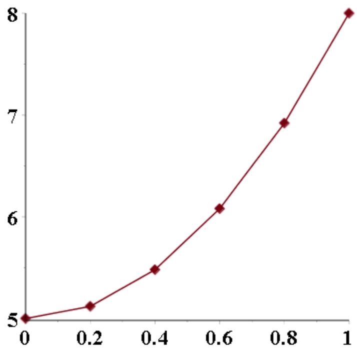

```
# Maple: compute table values
 f:=x->2*x+5;N:=6;a:=0;b:=1.0;h:=(b-a)/(N-1);
 vals:=seq(f(a+h*k),k=0..N-1);evalf(vals,2);
 # 5.0, 5.4, 5.8, 6.2, 6.6, 7.0, 7.4
# Maple: connect-the-dots graphic
 Dots:=[seq([a+h*k,f(a+h*k)],k=0..N-1)];
 plot(Dots,style=pointline);
```

2. $y = 3x + 5$, $x = 0$ to $x = 2$

3. $y = 2x^2 + 5$, $x = 0$ to $x = 1$

Solution:

x	y
0.0	5.0
0.1	5.12
0.2	5.48
0.3	6.08
0.4	6.92
0.5	8.00

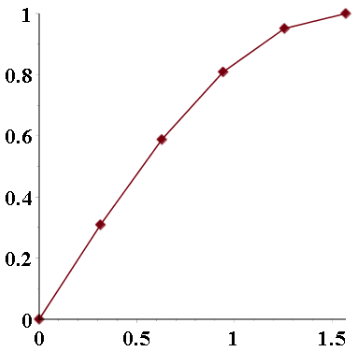


4. $y = 3x^2 + 5$, $x = 0$ to $x = 2$

5. $y = \sin x$, $x = 0$ to $x = \pi/2$

Solution:

x	y
0.0	0.000
0.1	0.309
0.2	0.588
0.3	0.809
0.4	0.952
0.5	1.00

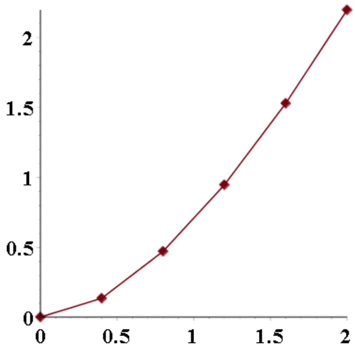


6. $y = \sin 2x$, $x = 0$ to $x = \pi/4$

7. $y = x \ln |1 + x|$, $x = 0$ to $x = 2$

Solution:

x	y
0.0	0.000
0.1	0.134
0.2	0.470
0.3	0.946
0.4	1.530
0.5	2.20

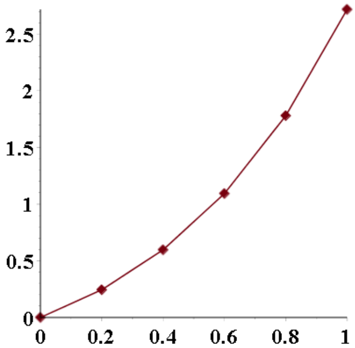


8. $y = x \ln|1 + 2x|$, $x = 0$ to $x = 1$

9. $y = xe^x$, $x = 0$ to $x = 1$

Solution:

x	y
0.0	0.000
0.1	0.244
0.2	0.597
0.3	1.093
0.4	1.781
0.5	2.718



10. $y = x^2e^x$, $x = 0$ to $x = 1/2$

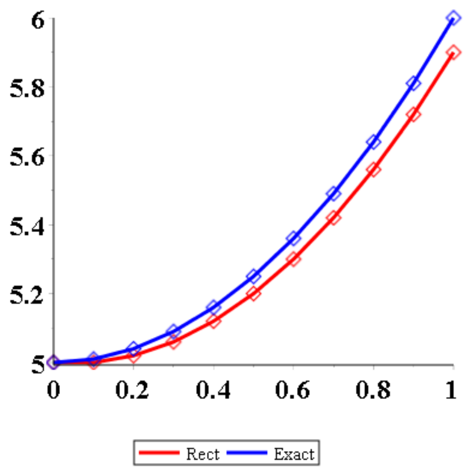
Rectangular Rule

Apply the rectangular rule to make an xy-table for $y(x)$ with 11 rows, $h = 0.1$. Graph the approximate solution and the exact solution. Follow example 4.1.

11. $y' = 2x$, $y(0) = 5$.

Solution: Let $F(x) = 2x$. The exact solution of $Y' = F(x)$, $Y(0) = 5$ is $Y(x) = x^2 + 5$ by the method of quadrature.

x	y-RECT	y-EXACT
0.0	5.00	5.00
0.1	5.00	5.01
0.2	5.02	5.04
0.3	5.06	5.09
0.4	5.12	5.16
0.5	5.20	5.25
0.6	5.30	5.36
0.7	5.42	5.49
0.8	5.56	5.64
0.9	5.72	5.81
1.0	5.90	6.00



The y-RECT value is found from $y(x+h) = y(x) + \int_x^{x+h} F(u)du \approx y(x) + hF(x)$. Then $y(0.1) = y(0) + \int_0^{0.1} F(u)du \approx 5 + 0.1F(0)$. Values for the first row of the table :

$x = 0, \quad y\text{-RECT} = 5, \quad y\text{-EXACT} = 5$

The second row values:

$x = 0.1$, y-RECT $= 5 + 0.1F(0) = 5$, y-EXACT $= Y(0.1) = 5.01$

The third row values:

$x = 0.2$, y-RECT $= 5 + 0.1F(0.1) = 5.02$, y-EXACT $= Y(0.2) = 5.04$

The fourth row values:

$x = 0.3$, y-RECT $= 5.02+0.1F(0.2) = 5.06$, y-EXACT $= Y(0.3) = 5.09$

The fifth and later row values follow the same pattern:

> y-RECT = (previous row y-RECT value) + $0.1F$(this row x-value).

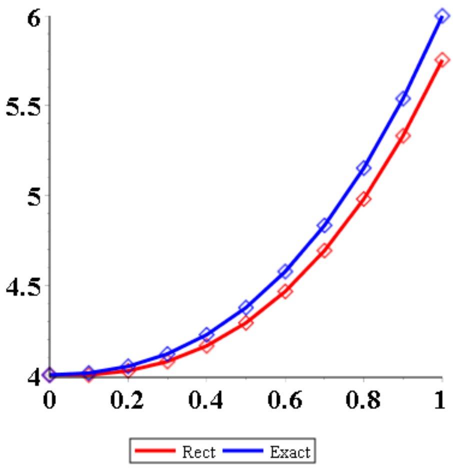
```
# Maple: Exact solution
 x:='x';y:='y'; X:='X';Y:='Y';
 F:=x->2*x;de:=diff(y(x),x)=F(x);y0:=5;x0:=0;
 ans:=dsolve([de,y(0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# Y(x)=x^2+5
# Maple: table rectangular rule and exact solution
 N:=11;a:=0;b:=1.0;h:=0.1;
 rect:=x -> h*F(x-h);# rectangular rule
 DotsRECT:=[x0,y0];DotsEXACT:=[x0,y0];Y:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k; Y:= Y+rect(X);
   DotsRECT:=DotsRECT,[X,Y];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
 DotsRECT;DotsEXACT;# table values
# Maple: Two connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
       symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsRECT],[DotsEXACT]],opts,
      color=[red,blue],legend=["Rect","Exact"]);
```

12. $y' = 3x^2$, $y(0) = 5$.

13. $y' = 3x^2 + 2x$, $y(0) = 4$.

Solution: Let $F(x) = 3x^2 + 2x$. The exact solution of $Y' = F(x)$, $Y(0) = 4$ is $Y(x) = x^3 + x^2 + 4$ by the method of quadrature.

x	$y-$RECT	$y-$EXACT
0.0	4.000	4.000
0.1	4.000	4.011
0.2	4.023	4.048
0.3	4.075	4.117
0.4	4.162	4.224
0.5	4.290	4.375
0.6	4.465	4.576
0.7	4.693	4.833
0.8	4.980	5.152
0.9	5.332	5.539
1	5.755	6.000

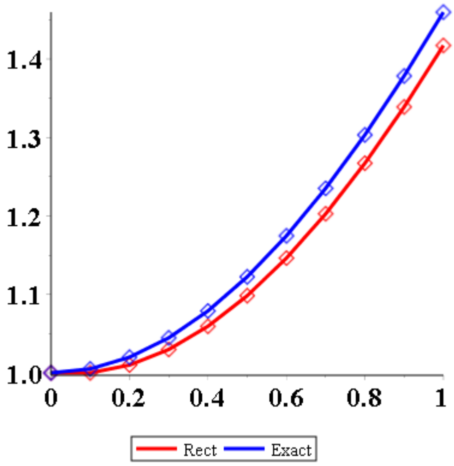


14. $y' = 3x^2 + 4x^3$, $y(0) = 4$.

15. $y' = \sin x$, $y(0) = 1$.

Solution: Let $F(x) = \sin(x)$. The exact solution of $Y' = F(x)$, $Y(0) = 1$ is $Y(x) = 2 - \cos(x)$ by the method of quadrature.

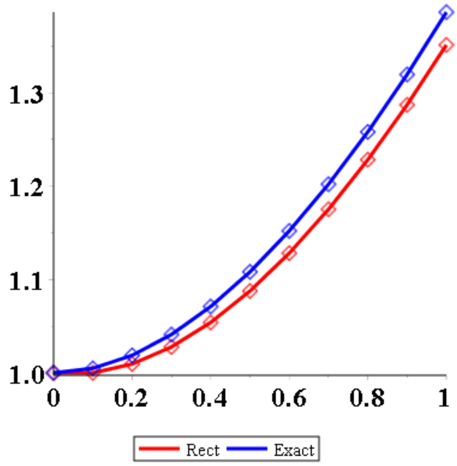
x	y – RECT	y – EXACT
0	1	1
0.1	1.00000	1.00500
0.2	1.00998	1.01993
0.3	1.02985	1.04466
0.4	1.0594	1.07894
0.5	1.09834	1.12242
0.6	1.14629	1.17466
0.7	1.20275	1.23516
0.8	1.26717	1.30329
0.9	1.33891	1.37839
1	1.41724	1.45970



16. $y' = 2\sin 2x$, $y(0) = 1$.

17. $y' = \ln(1 + x)$, $y(0) = 1$. Exact $(1 + x)\ln|1 + x| + 1 - x$.
Solution: Let $F(x) = \ln(1+x)$. The exact solution of $Y' = F(x)$, $Y(0) = 1$ is $Y(x) = (1_x)\ln(1 + x) - x + 1$ by the method of quadrature.

x	$y-$RECT	$y-$EXACT
0	1	1
0.1	1	1.00484
0.2	1.00953	1.01879
0.3	1.02776	1.04107
0.4	1.054	1.07106
0.5	1.08765	1.1082
0.6	1.12819	1.15201
0.7	1.17519	1.20207
0.8	1.22826	1.25802
0.9	1.28704	1.31952
1	1.35122	1.38629



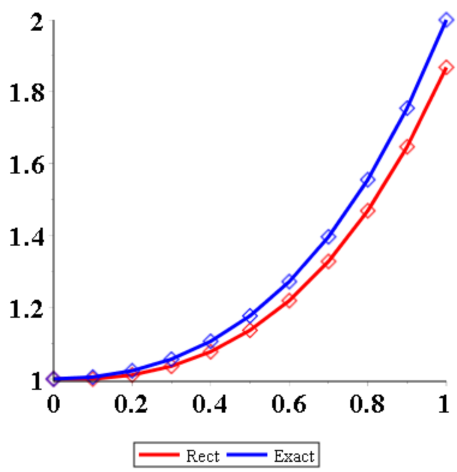
18. $y' = 2\ln(1+2x)$, $y(0) = 1$. Exact $(1+2x)\ln|1+2x| + 1 - 2x$.

19. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

Solution:

Let $F(x) = x\,e^x$. The exact solution of $Y' = F(x)$, $Y(0) = 1$ is $Y(x) = xe^x - e^x + 21$ by the method of quadrature. The details require integration by parts.

x	$y-$RECT	$y-$EXACT
0	1	1
0.1	1	1.00535
0.2	1.01105	1.02288
0.3	1.03548	1.0551
0.4	1.07598	1.10491
0.5	1.13565	1.17564
0.6	1.21808	1.27115
0.7	1.32741	1.39587
0.8	1.46837	1.55489
0.9	1.64642	1.75404
1	1.86778	2



20. $y' = 2x^2e^{2x}$, $y(0) = 4$. Exact $2\,x^2e^x - 4\,xe^x + 4\,e^x$.

Trapezoidal Rule

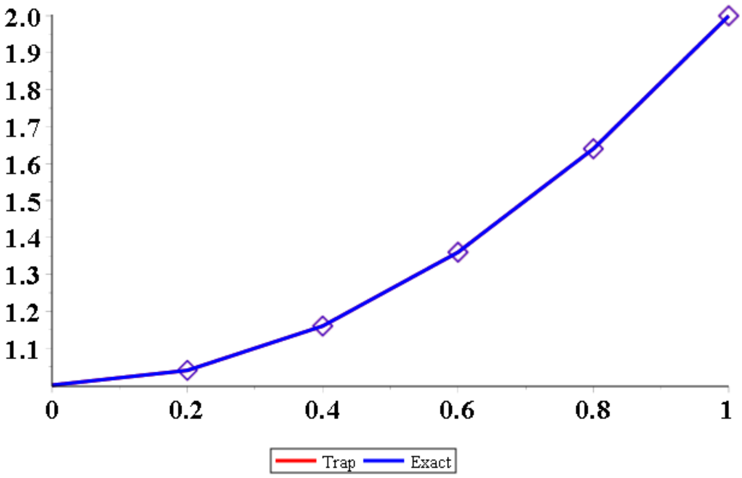
Apply the trapezoidal rule to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow example 4.2.

21. $y' = 2x$, $y(0) = 1$.

Solution: Let $F(x) = 2x$. The exact solution of $Y' = F(x)$, $Y(0) = 1$ is $Y(x) = x^2 + 1$ by the method of quadrature.

x	y-TRAP	y-EXACT
0	1	1
0.2	1.04	1.04
0.4	1.16	1.16
0.6	1.36	1.36
0.8	1.64	1.64
1	2	2

The graphic shows only the blue curve because the red and blue curves are identical.



The y-TRAP value is found from $y(x+h) = y(x) + \int_x^{x+h} F(u)du \approx y(x) + 0.5h(F(x) + F(x+h))$. Then $y(0.1) = y(0) + \int_0^{0.1} F(u)du \approx 1 + 0.1(F(0) + F(0.1))/2$. Values for the first row of the table :

$x = 0, \quad y\text{-TRAP} = 1, \quad y\text{-EXACT} = 1$

The second row values:

$x = 0.1, \quad y\text{-TRAP} = 1 + 0.1(F(0) + F(0.1))/2 = 1.01, \quad y\text{-EXACT} = 1.01$

The third row values:

$x = 0.2, \quad y\text{-TRAP} = 1.01 + 0.1(F(0.1) + F(0.2))/2 = 1.04, \quad y\text{-EXACT} = 1.04$

The fourth and later row values follow the same pattern:

$$\begin{aligned} &y\text{-TRAP} = (\text{previous row } y\text{-TRAP value}) + \\ &\quad 0.1\left(F(\text{this row } x\text{-value}) + F(h + (\text{this row } x\text{-value})\right)/2. \end{aligned}$$

```
# Maple: Exact solution
 x:='x';y:='y'; X:='X';Y:='Y';
 F:=x->2*x;de:=diff(y(x),x)=F(x);y0:=1;x0:=0;
 ans:=dsolve([de,y(0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# Y(x)=x^2+1
# Maple: table trapzoidal rule and exact solution
 N:=6;a:=0;b:=1.0;h:=0.2;
 trap:=x -> 0.5*h*(F(x-h)+F(x));# trapezoidal rule
 DotsTRAP:=[x0,y0];DotsEXACT:=[x0,y0];Y:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k; Y:= Y+trap(X);
   DotsTRAP:=DotsTRAP,[X,Y];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
DotsTRAP;DotsEXACT; # table values are the same
# Maple: Two connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
       symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsTRAP],[DotsEXACT]],opts,
      color=[red,blue],legend=["Trap","Exact"]);
 # only the blue plot is visible: duplicate data
```

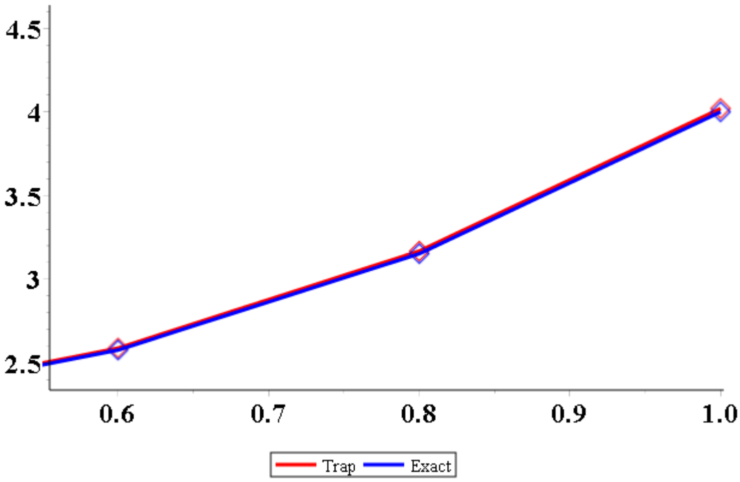
22. $y' = 3x^2$, $y(0) = 1$.

23. $y' = 3x^2 + 2x$, $y(0) = 2$.

Solution: Let $F(x) = 3x^2 + 2x$. The exact solution of $Y' = F(x)$, $Y(0) = 2$ is $Y(x) = x^3 + x^2 + 2$ by the method of quadrature.

x	y-TRAP	y-EXACT
0	2	2
0.2	2.052	2.048
0.4	2.232	2.224
0.6	2.588	2.576
0.8	3.168	3.152
1	4.02	4

The graphic shows a limited range because the red and blue curves are nearly identical.



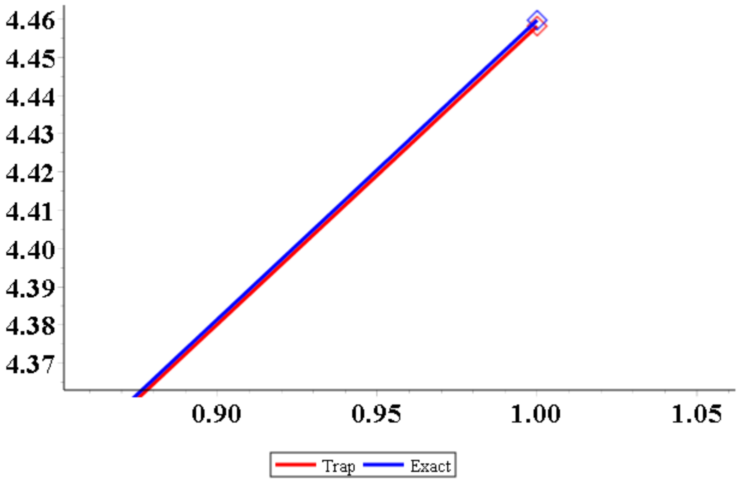
24. $y' = 3x^2 + 4x^3$, $y(0) = 2$.

25. $y' = \sin x$, $y(0) = 4$.

Solution: Let $F(x) = 3x^2 + 2x$. The exact solution of $Y' = F(x)$, $Y(0) = 2$ is $Y(x) = x^3 + x^2 + 2$ by the method of quadrature.

x	y-TRAP	y-EXACT
0	4	4
0.2	4.01987	4.01993
0.4	4.07868	4.07894
0.6	4.17408	4.17466
0.8	4.30228	4.30329
1	4.45816	4.4597

The graphic shows a limited range because the red and blue curves are nearly identical.



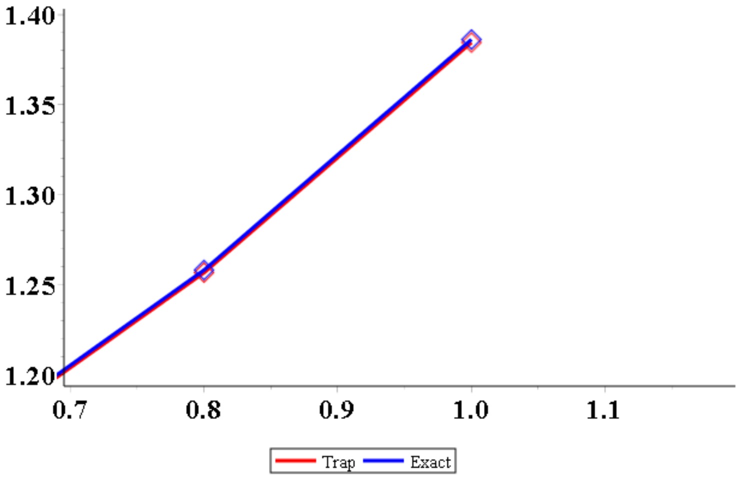
26. $y' = 2\sin 2x$, $y(0) = 4$.

27. $y' = \ln(1+x)$, $y(0) = 1$. Exact $(1+x)\ln|1+x| + 1 - x$.

Solution: Let $F(x) = \ln(1+x)$. The exact solution of $Y' = F(x)$, $Y(0) = 2$ is $Y(x) = (1+x)\ln|1+x| + 1 - x$ by the method of quadrature, using integration by parts.

x	y-TRAP	y-EXACT
0	1	1
0.2	1.01823	1.01879
0.4	1.07011	1.07106
0.6	1.15076	1.15201
0.8	1.25654	1.25802
1	1.38463	1.38629

The graphic shows a limited range because the red and blue curves are nearly identical.



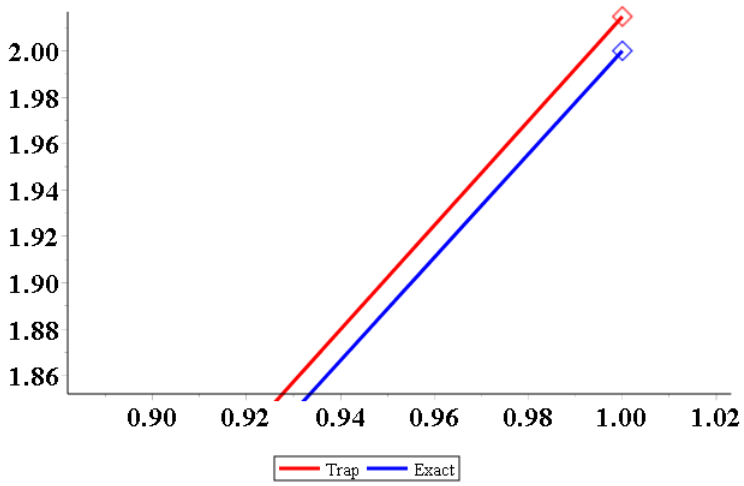
28. $y' = 2\ln(1+2x)$, $y(0) = 1$. Exact $(1+2x)\ln|1+2x| + 1 - 2x$.

29. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

Solution: Let $F(x) = xe^x$. The exact solution of $Y' = F(x)$, $Y(0) = 2$ is $Y(x) = xe^x - e^x + 2$ by the method of quadrature, using integration by parts.

x	y-TRAP	y-EXACT
0	1	1
0.2	1.02443	1.02288
0.4	1.10853	1.10491
0.6	1.27753	1.27115
0.8	1.5649	1.55489
1	2.01477	2

The graphic shows a limited range because the red and blue curves are nearly identical.



30. $y' = 2x^2e^{2x}$, $y(0) = 4$. Exact $2\,x^2e^x - 4\,xe^x + 4\,e^x$.

Simpson Rule

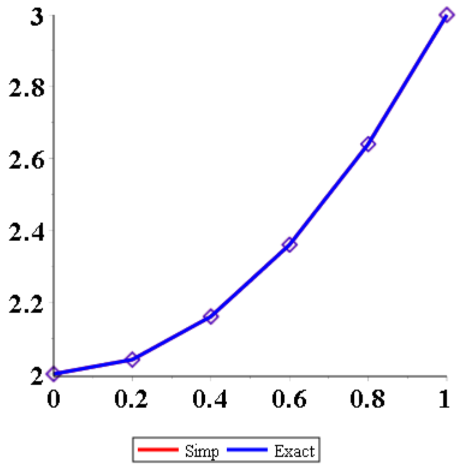
Apply Simpson's rule to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow example 4.3.

31. $y' = 2x$, $y(0) = 2$.

Solution: Let $F(x) = 2x$. The exact solution of $Y' = F(x)$, $Y(0) = 2$ is $Y(x) = x^2 + 2$ by the method of quadrature.

x	y-SIMP	y-EXACT
0	2	2
0.2	2.04	2.04
0.4	2.16	2.16
0.6	2.36	2.36
0.8	2.64	2.64
1	3	3

The graphic shows only the blue curve because the red and blue curves are identical.



The y-SIMP value is found from $y(x+h) = y(x) + \int_x^{x+h} F(u)du \approx y(x) + h(F(x) + 4F(x+h/2) + F(x+h))/6$. Then $y(0.2) = y(0) + \int_0^{0.2} F(u)du \approx 2 + 0.2(F(0) + 4F(0.1) + F(0.2))/6$. Values for the first row of the table :

$x = 0$, y-SIMP $= 2$, y-EXACT $= 2$

The second row values:

$x = 0.2$, y-SIMP $= 2 + 0.2(F(0) + 4F(0.1) + F(0.2))/6 = 2.04$, y-EXACT $= 2.04$

The third row values:

$x = 0.4$, y-SIMP $= 2.04 + 0.2(F(0.2) + 4F(0.3) + F(0.4))/6 = 2.168$, y-EXACT $= 2.168$

The fourth and later row values follow the same pattern:

$$\begin{aligned} &y\text{-SIMP} = (\text{previous row } y\text{-SIMP value}) + \\ &\quad 0.2(F(\text{current } x\text{-value} - 0.2) + 4F((\text{current } x\text{-value} - 0.1) + \\ &\quad F(\text{current } x\text{-value}))/6. \end{aligned}$$

The values obtained for the Simpson's rule solution and the exact solution are identical. This is no accident: it is known that Simpson's rule is **exact** for $F(x)$ equal to a polynomial of degree 3 or less.

```
# Maple: Exact solution
 x:='x';y:='y'; X:='X';Y:='Y';
 F:=x->2*x;de:=diff(y(x),x)=F(x);y0:=2;x0:=0;
 ans:=dsolve([de,y(0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# Y(x)=x^2+1
# Maple: Simpson rule solution
 N:=6;a:=0;b:=1.0;h:=0.2;
 simp:=x -> h*(F(x-h)+4*F(x-h/2)+F(x))/6;# Simpson rule
 DotsSIMP:=[x0,y0];DotsEXACT:=[x0,y0];Y:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k; Y:= Y+simp(X);
   DotsSIMP:=DotsSIMP,[X,Y];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
 DotsSIMP;DotsEXACT; # table values
# Maple: Two connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
       symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsSIMP],[DotsEXACT]],opts,
      color=[red,blue],legend=["Simp","Exact"]);
 # only the blue plot is visible: duplicate data
```

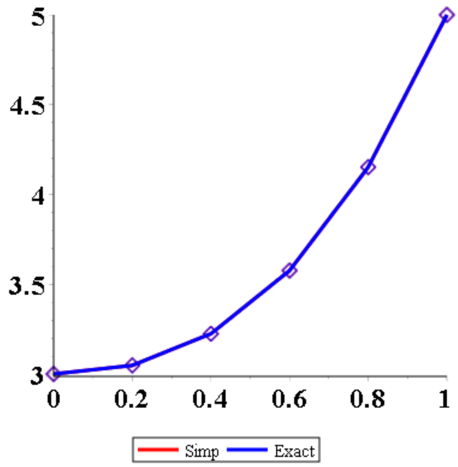
32. $y' = 3x^2$, $y(0) = 2$.

33. $y' = 3x^2 + 2x$, $y(0) = 3$.

Solution: Let $F(x) = 2x$. The exact solution of $Y' = F(x)$, $Y(0) = 2$ is $Y(x) = x^3 + x^2 + 3$ by the method of quadrature.

x	y-SIMP	y-EXACT
0	3	3
0.2	3.048	3.048
0.4	3.224	3.224
0.6	3.576	3.576
0.8	4.152	4.152
1	5	5

The graphic shows only the blue curve because the red and blue curves are identical.



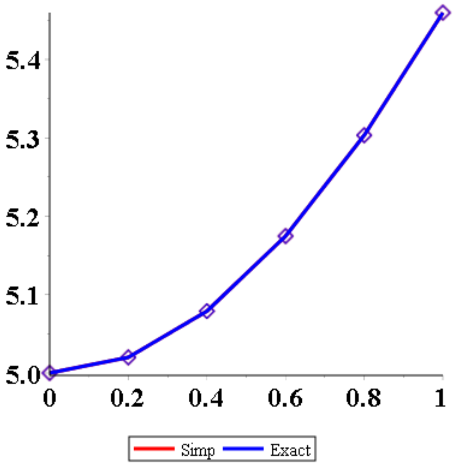
34. $y' = 3x^2 + 4x^3$, $y(0) = 3$.

35. $y' = \sin x$, $y(0) = 5$.

Solution: Let $F(x) = \sin(x)$. The exact solution of $Y' = F(x)$, $Y(0) = 5$ is $Y(x) = 6 - \cos(x)$ by the method of quadrature.

x	y-SIMP	y-EXACT
0	5	5
0.2	5.01993	5.01993
0.4	5.07894	5.07894
0.6	5.17466	5.17466
0.8	5.30329	5.30329
1	5.4597	5.4597

The graphic shows only the blue curve because the red and blue curves are identical.



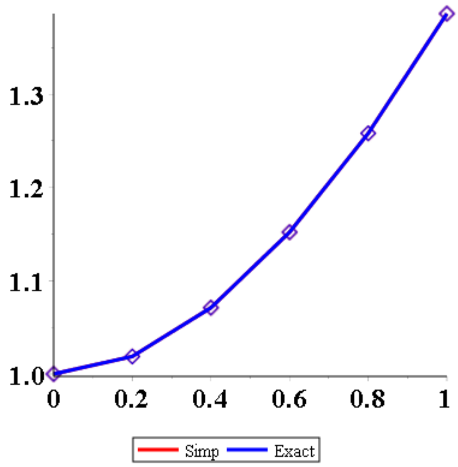
36. $y' = 2\sin 2x$, $y(0) = 5$.

37. $y' = \ln(1+x)$, $y(0) = 1$. Exact $(1+x)\ln|1+x| + 1 - x$.

Solution: Let $F(x) = \ln(1+x)$. The exact solution of $Y' = F(x)$, $Y(0) = 1$ is $(1+x)\ln|1+x| + 1 - x$ by the method of quadrature, using integration by parts.

x	y-SIMP	y-EXACT
0	1	1
0.2	1.01879	1.01879
0.4	1.07106	1.07106
0.6	1.152	1.15201
0.8	1.25802	1.25802
1	1.38629	1.38629

The graphic shows only the blue curve because the red and blue curves are essentially identical.



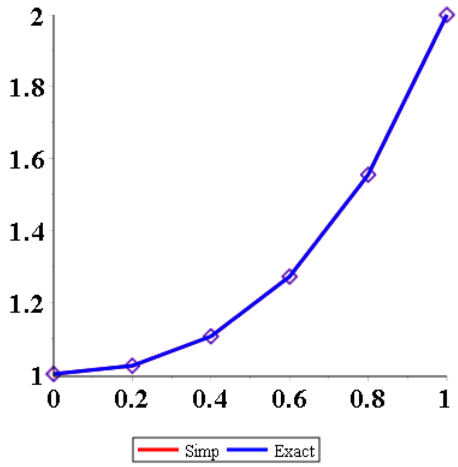
38. $y' = 2\ln(1+2x)$, $y(0) = 1$. Exact $(1+2x)\ln|1+2x| + 1 - 2x$.

39. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

Solution: Let $F(x) = \ln(1+x)$. The exact solution of $Y' = F(x)$, $Y(0) = 1$ is $xe^x - e^x + 2$ by the method of quadrature, using integration by parts.

x	y-SIMP	y-EXACT
0	1	1
0.2	1.02288	1.02288
0.4	1.10491	1.10491
0.6	1.27115	1.27115
0.8	1.55489	1.55489
1	2	2

The graphic shows only the blue curve because the red and blue curves are identical.



40. $y' = 2x^2e^{2x}$, $y(0) = 4$. Exact $2\,x^2e^x - 4\,xe^x + 4\,e^x$.

Simpson's Rule

The following exercises use formulas and techniques found in the proof on page 234 and in Example 4.4, page 233.

41. Verify with Simpson's rule (5) for cubic polynomials the equality $\int_1^2(x^3 + 16x^2 + 4)dx = 541/12$.

Solution: Simpson's rule is exact for $Q(x)$ a polynomial of degree 3 or less: $\int_a^b Q(x)dx = (b-a)(Q(a)+4Q((a+b)/2)+Q(b))/6$. Let $Q(x) = x^3+16x^2+4$, $a = 1$, $b = 2$ and evaluate $(b-a)(Q(a)+4Q((a+b)/2)+Q(b))/6 = 541/12$.

```
# Maple: Simpson's Rule
 a:='a';b:='b';x:='x';
 SimpRule:=(a,b,Q)->(b-a)*(Q(a)+4*Q((a+b)/2)+Q(b))/6;
 QQ:=x->x^3+16*x^2+4;
 SimpRule(1,2,QQ);# 541/12
 int(QQ(x),x=1..2); # answer check
```

42. Verify with Simpson's rule (5) for cubic polynomials the equality $\int_1^2(x^3 + x + 14)dx = 77/4$.

43. Let $f(x)$ satisfy $f(0) = 1$, $f(1/2) = 6/5$, $f(1) = 3/4$. Apply Simpson's rule with one division to verify that $\int_0^1 f(x)dx \approx 131/120$.

Solution: Let $a = 0$, $b = 1$. Evaluate:

$(b-a)(f(a) + 4f(a+b)/2) + f(b))/6 = 131/120$

```
# Maple: Simpson's Rule data version
 a:='a';b:='b';
 SimpRuleData:=(a,b,f1,f2,f3)->(b-a)*(f1+4*f2+f3)/6;
 SimpRuleData(0,1,1,6/5,3/4); # 131/120
```

44. Let $f(x)$ satisfy $f(0) = -1$, $f(1/2) = 1$, $f(1) = 2$. Apply Simpson's rule with one division to verify that $\int_0^1 f(x)dx \approx 5/6$.

45. Verify Simpson's equality (5), assuming $Q(x) = 1$ and $Q(x) = x$.

Solution: **Part I**. Verify for $Q(x) = 1$:
LHS $= \int_a^b Q(x)dx = \int_a^b 1dx = b - a$,
RHS $= (b-a)(Q(a)+4A((a+b)/2)+Q(b))/6 = (b-a)(1+4+1)/6 = b-a$.
Then LHS = RHS, identity verified.

Part II. Verify for $Q(x) = x$:
LHS $= \int_a^b Q(x)dx = \int_a^b xdx = b^2/2 - a^2/2$,
RHS $= (b - a)(Q(a) + 4A((a + b)/2) + Q(b))/6 =$
$(b - a)(a + 4(b + a)/2 + b)/6 = (b - a)(3a + 3b)/6 = b^2/2 - a^2/2$.
Then LHS = RHS, identity verified.

46. Verify Simpson's equality (5), assuming $Q(x) = x^2$. Use college algebra identity $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$.

Quadratic Interpolation

The following exercises use formulas and techniques from the proof on page 234.

47. Verify directly that the quadratic polynomial $y = x(7 - 4x)$ goes through the points $(0, 0)$, $(1, 3)$, $(2, -2)$.

Solution: Details:

$$\begin{aligned}
y(0) &= x(7-4x)|_{x=0} = 0(7-0) = 0 \quad (0,0) \text{ verified}\\
y(1) &= x(7-4x)|_{x=1} = 1(7-4(1)) = 3 \quad (1,3) \text{ verified}\\
y(2) &= x(7-4x)|_{x=2} = 2(7-4(2)) = -2 \quad (2,-2) \text{ verified}
\end{aligned}$$

48. Verify directly that the quadratic polynomial $y = x(8 - 5x)$ goes through the points $(0, 0)$, $(1, 3)$, $(2, -4)$.

49. Compute the quadratic interpolation polynomial $Q(x)$ which goes through the points $(0, 1)$, $(0.5, 1.2)$, $(1, 0.75)$.

Solution: Details:
Let $Q(x) = a+bx+cx^2$. Plan: determine a, b, c by linear algebra. Equations:

$$\begin{aligned}
a + bx + cx^2\big|_{x=0} &= 1\\
a + bx + cx^2\big|_{x=0.5} &= 1.2
\end{aligned}$$

$a + bx + cx^2\big|_{x=1} = 0.75$
$a + 0 + 0 = 1$
$a + b/2 + c/4 = 1.2$
$a + b + c = 0.75$

Solve by computer: $a = 1$, $b = 1.05$, $c = -1.3$.

```
# Maple: Solve system of equations
 eqs:={a+0+0 = 1, a+b/2+c/4= 1.2, a+b+c = 0.75};
 # braces { ... } needed!
 solve(eqs,[a,b,c]);# brackets preserve order a,b,c
 # {a = 1., b = 1.05, c = -1.3}
# Maple: Answer check
 Y := X->1 + (1.05)*X + (-1.3)*X^2;
 Y(0),Y(0.5),Y(1); # 1., 1.2, .75
```

50. Compute the quadratic interpolation polynomial $Q(x)$ which goes through the points $(0, -1)$, $(0.5, 1)$, $(1, 2)$.

51. Verify the remaining cases in Lemma 4.1, page 235.

Solution: Given Y_0, Y_1 and Y_2, define $y_1 = Y_1 - Y_0$, $y_2 = \frac{1}{2}(Y_2 - Y_0)$, $A = y_2 - y_1$, $B = 2y_1 - y_2$ and $x = 2(X - a)/(b - a)$. Formula $y = x(Ax + B)$ will be tested to go through the given data points $(0, 0)$, $(1, y_1)$. The details:

$x(AX + B)\big|_{x=0} = 0(A(0) + B) = 0$ Verified $(0, 0)$.
$x(AX + B)\big|_{x=1} = 1(A(1) + B) = y_2 - y_1 + 2y_1 - y_2 = y_1$ Verified $(1, y_1)$.

■

52. Verify the remaining cases in Lemma 4.2, page 235.

4.2 Solving $y' = f(x, y)$ Numerically

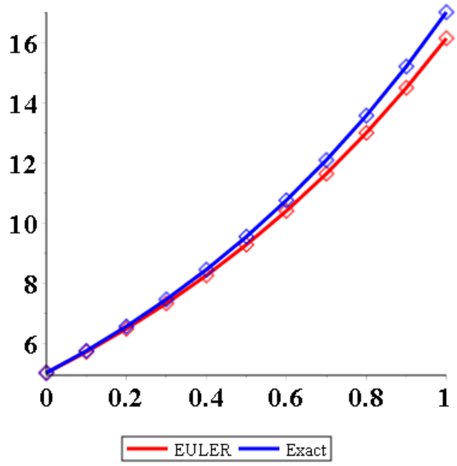
Euler's Method

Apply Euler's method to make an xy-table for $y(x)$ with 11 rows and step size $h = 0.1$. Graph the approximate solution and the exact solution. Follow Example 4.5.

1. $y' = 2 + y$, $y(0) = 5$. Exact $y(x) = -2 + 7e^x$.

Solution: The exact answer for $y' = 2+y$, $y(0) = 5$ is $y(x) = -2+7e^x$, found by the linear integrating factor method. The constant coefficient shortcut applies: $y = y_p + y_h$, $y_p = -2 =$ equilibrium solution, $y_h = c/W$, $W =$ integrating factor $= e^{-x}$.

x	y-EULER	y-EXACT
0	5	5
0.1	5.7	5.7362
0.2	6.47	6.54982
0.3	7.317	7.44901
0.4	8.2487	8.44277
0.5	9.27357	9.54105
0.6	10.4009	10.7548
0.7	11.641	12.0963
0.8	13.0051	13.5788
0.9	14.5056	15.2172
1	16.1562	17.028



Let $F(x, y) = 2 + y$. The y-EULER value is found from $y(x + h) = y(x) + \int_x^{x+h} F(u, y(u))du \approx y(x) + hF(x, y(x))$. Then $y(0.1) = y(0) + \int_0^{0.1} F(u)du \approx 5 + 0.1F(0, y(0)) = 5 + 0.1(2 + 5) = 5.7$. Values for the first row of the table :

$x = 0$, y-EULER $= 5$, y-EXACT $= 5$

The second row values:

$x = 0.1$, y-EULER $= 5 + 0.1F(0, 5) = 5 + 0.1(2 + 5) = 5.7$, y-EXACT $= Y(0.1) = 5.7362$

The third row values:

$x = 0.2$, y-EULER $= 5.7 + 0.1F(0.1, 5.7) \approx 6.47$, y-EXACT $= Y(0.2) = 6.54982$

The fourth row values:

$x = 0.3$, y-EULER $= 6.47 + 0.1F(0.2, 6.47) \approx 7.317$, y-EXACT $= Y(0.3) = 7.449019$

The fifth and later row values follow the same pattern:

y-EULER = (previous row y-EULER value) +
$0.1F$(previous row x-value, previous row y-EULER value).

An online check in WolframAlpha: use input
`y'=2+y, y(0)=5 by Euler's method h=0.1 t=0 to 1`.

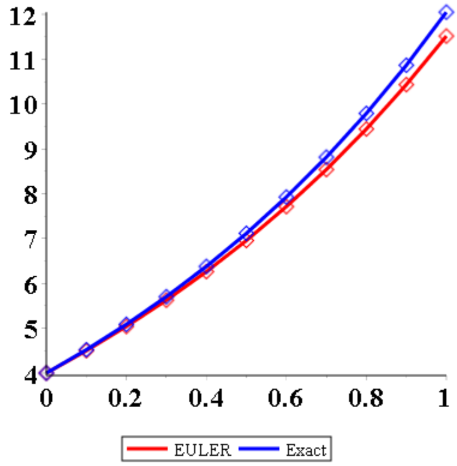
```
# Maple: Exact solution
 F:=(x,y)->2+y;de:=diff(y(x),x)=F(x,y(x));y0:=5;x0:=0;
 ans:=dsolve([de,y(x0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# EY(x)=-2+7*exp(x)
# Maple: Euler's method
 N:=11;h:=0.1;
 EULER:=(x,y) -> h*F(x,y);# Euler algorithm
 DotsEULER:=[x0,y0];DotsEXACT:=[x0,y0];Y:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k; Y:= Y+EULER(X-h,Y);
   DotsEULER:=DotsEULER,[X,Y];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
 DotsEULER;DotsEXACT;# answers
# Maple: Two connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
 symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsEULER],[DotsEXACT]],opts,
      color=[red,blue],legend=["EULER","Exact"]);
```

2. $y' = 3 + y$, $y(0) = 5$. Exact $y(x) = -3 + 8e^x$.

3. $y' = e^{-x} + y$, $y(0) = 4$. Exact $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$.

Solution: The exact answer for $y' = e^{-x} + y$, $y(0) = 5$ is $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^{x}$, found by the linear integrating factor method. No shortcut applies.

x	y-EULER	y-EXACT
0	4	4
0.1	4.5	4.52085
0.2	5.04048	5.08695
0.3	5.62641	5.70396
0.4	6.26313	6.37805
0.5	6.95647	7.11598
0.6	7.71277	7.92513
0.7	8.53893	8.81359
0.8	9.44248	9.79027
0.9	10.4317	10.8649
1	11.5155	12.0483

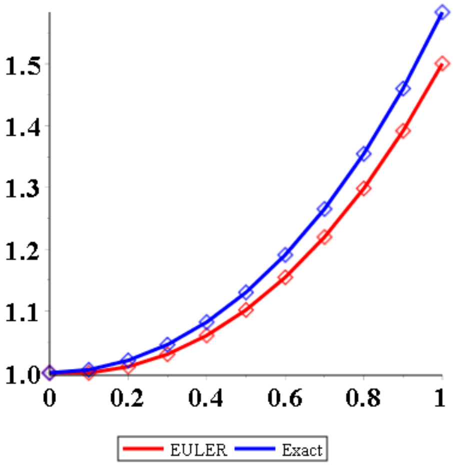


4. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact $y(x) = -e^{-2x} + 5e^{x}$.

5. $y' = y \sin x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos x}$.

Solution: The exact answer for $y' = y\sin(x)$, $y(0) = 1$ is $y(x) = e^{1-\cos x}$, found by the variables separable method.

x	y-EULER	y-EXACT
0	1	1
0.1	1	1.00501
0.2	1.00998	1.02013
0.3	1.03005	1.04568
0.4	1.06049	1.08214
0.5	1.10179	1.13023
0.6	1.15461	1.19085
0.7	1.2198	1.26511
0.8	1.29838	1.35431
0.9	1.39152	1.45993
1	1.50053	1.5836



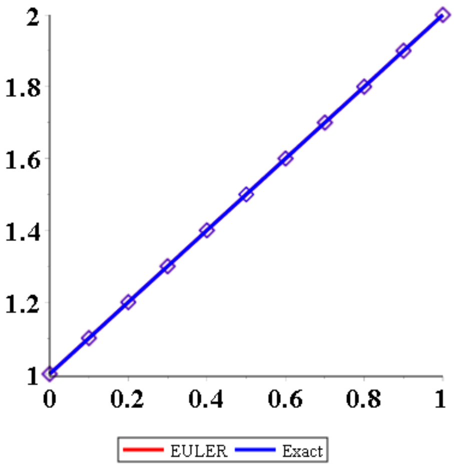
6. $y' = 2y \sin 2x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos 2x}$.

7. $y' = y/(1 + x)$, $y(0) = 1$. Exact $y(x) = 1 + x$.

Solution: The exact answer for $y' = y/(1 + x)$, $y(0) = 1$ is $y(x) = 1 + x$, found by the variables separable method.

x	y-EULER	y-EXACT
0	1.000000	1.000000
0.1	1.100000	1.100000
0.2	1.200000	1.200000
0.3	1.300000	1.300000
0.4	1.400000	1.400000
0.5	1.500000	1.500000
0.6	1.600000	1.600000
0.7	1.700000	1.700000
0.8	1.800000	1.800000
0.9	1.900000	1.900000
1	2.000000	2.000000

The graphic shows only the exact solution (blue) because the two data sets are identical.



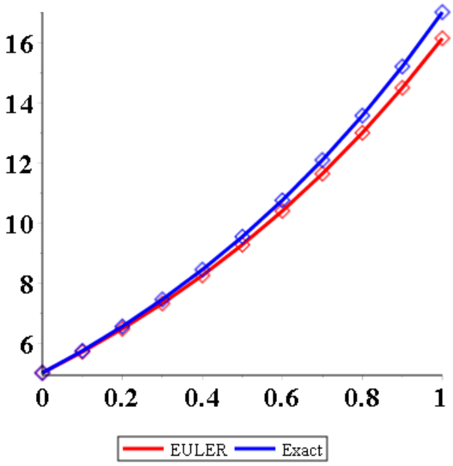
8. $y' = y(x)/(1+2x)$, $y(0) = 1$. Exact $y(x) = \sqrt{1+2x}$.

9. $y' = yxe^x$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = 1 + (x-1)e^x$.

Solution: The exact answer for $y' = xye^x$, $y(0) = 1$ is $y(x) = e^{1+(x-1)e^x}$, found by the variables separable method, using integration by parts for $\int xe^x dx$.

x	y-EULER	y-EXACT
0	1.000000	1.000000
0.1	1.000000	1.005360
0.2	1.011052	1.023141
0.3	1.035750	1.056645
0.4	1.077693	1.110605
0.5	1.142002	1.192008
0.6	1.236145	1.311475
0.7	1.371289	1.485682
0.8	1.564589	1.741753
0.9	1.843154	2.125569
1	2.251162	2.718282

The graphic shows significant errors, caused by the exponential factor.



10. $y' = 2y(x^2 + x)e^{2x}$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = x^2e^{2x}$.

Heun's Method

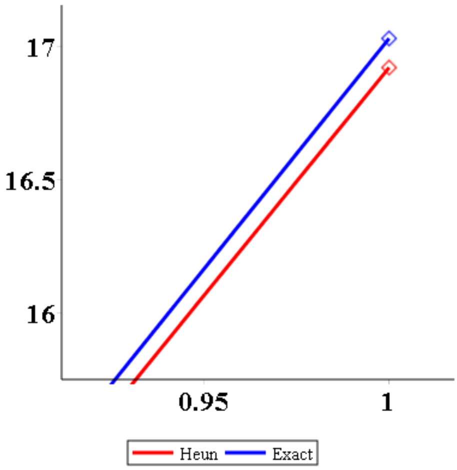
Apply Heun's method to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow Example 4.6.

11. $y' = 2 + y$, $y(0) = 5$. Exact $y(x) = -2 + 7e^x$.

Solution: The exact answer for $y' = 2+y$, $y(0) = 5$ is $y(x) = -2+7e^x$, found by the linear integrating factor method. The constant coefficient shortcut applies: $y = y_p + y_h$, $y_p = -2$ = equilibrium solution, $y_h = c/W$, W = integrating factor = e^{-x}.

x	y-HEUN	y-EXACT
0	5.000000	5.000000
0.2	6.540000	6.549819
0.4	8.418800	8.442773
0.6	10.710936	10.754832
0.8	13.507342	13.578786
1	16.918957	17.027973

The graphic below was zoomed to show detail, because the table values are close.



Let $F(x, y) = 2 + y$. The y-HEUN value is found from $y(x + h) = y(x) + \int_x^{x+h} F(u, y(u))du \approx y(x) + h(F(x, y(x)) + F(x + h, y(x + h))/2$, using the Trapezoidal Rule. Value $y(x + h) \approx y(x) + hF(x, y(x))$ by Euler's Method. For instance, $y(0.2) \approx y(0) + 0.2F(0, y(0)) = 5 + 0.2(2 + 5) = 6.4$ by Euler's method. Then $y(0.2) = y(0) + \int_0^{0.2} F(u, y(u))du \approx y(0) + 0.2(F(0, y(0)) + F(0.2, y(0.2))/2 = 5 + 0.2((2 + 5) + (2 + 6.4))/2 = 6.54$.

Values for the first row of the table :

$x = 0$, $\quad y$-HEUN $= 5$, $\quad y$-EXACT $= 5$

The second row values:

$x = 0.2$, $\quad y_1 = 5 + 0.2F(0, 5) = 6.4$, y-HEUN $= 5 + 0.2(F(0, 5) + F(0.2, y_1))/2 = 6.54$, $\quad y$-EXACT $= Y(0.2) = 6.549819306$

The third row values:

$x = 0.4$, $\quad y_1 = 6.54 + 0.2F(0, 6.54) = 8.248$, y-HEUN $= 6.54 + 0.2(F(0.2, 6.54) + F(0.4, 8.248))/2 = 8.4188$, $\quad y$-EXACT $= Y(0.2) = 8.44277289$

The fourth and later row values follow the same pattern, described precisely in the `maple` code below. An online check in WolframAlpha: use input `y'=2+y, y(0)=5 by Heun's method h=0.2 t=0 to 1`.

```
# Maple: Exact solution
 F:=(x,y)->2+y;de:=diff(y(x),x)=F(x,y(x));y0:=5;x0:=0;
 ans:=dsolve([de,y(x0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# EY(x)=-2+7*exp(x)
# Maple: Heun's method
 HEUN:=proc(x,y)
  local y1,y2;
  y1:=y+h*F(x,y);
  y2:=0.5*h*(F(x,y)+F(x+h,y1));
  RETURN (y2);
 end proc;
 DotsHEUN:=[x0,y0];DotsEXACT:=[x0,y0];Y:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k; Y:= Y+HEUN(X-h,Y);
   DotsHEUN:=DotsHEUN,[X,Y];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
 DotsHEUN;DotsEXACT; # answers
# Maple: Two connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
       symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsHEUN],[DotsEXACT]],opts,
      color=[red,blue],legend=["Heun","Exact"]);
```

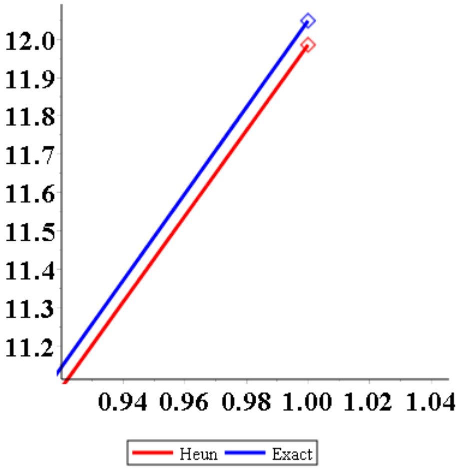
12. $y' = 3 + y$, $y(0) = 5$. Exact $y(x) = -3 + 8e^x$.

13. $y' = e^{-x} + y$, $y(0) = 4$. Exact $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$.

Solution: The exact answer for $y' = e^{-x}+y$, $y(0) = 5$ is $y(x) = -\frac{1}{2}e^{-x}+\frac{9}{2}e^x$, found by the linear integrating factor method. No shortcut applies.

x	y-HEUN	y-EXACT
0	4.000000	4.000000
0.2	5.081873	5.086947
0.4	6.365165	6.378051
0.6	7.900821	7.925129
0.8	9.749792	9.790270
1	11.985453	12.048328

The graphic below was zoomed to show detail.



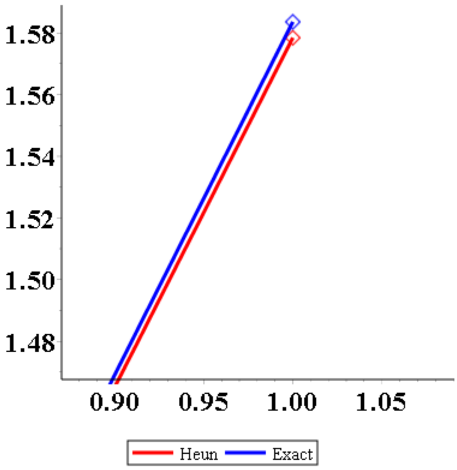
14. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact $y(x) = -e^{-2x} + 5e^x$.

15. $y' = y \sin x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos x}$.

Solution: The exact answer for $y' = y\sin(x)$, $y(0) = 1$ is $y(x) = e^{1-\cos x}$, found by the variables separable method.

x	y-HEUN	y-EXACT
0	1.000000	1.000000
0.2	1.019867	1.020133
0.4	1.081422	1.082138
0.6	1.189352	1.190846
0.8	1.351462	1.354312
1	1.578447	1.583595

The graphic below was zoomed to show detail.



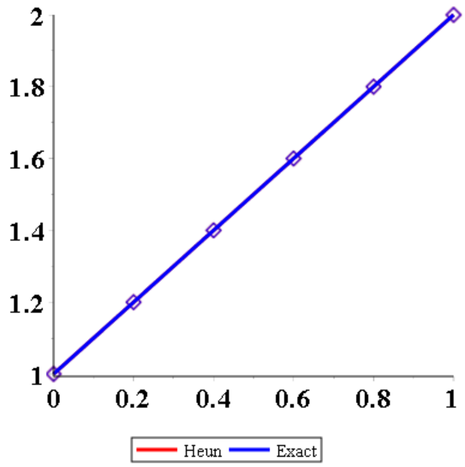
16. $y' = 2y \sin 2x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos 2x}$.

17. $y' = y/(1 + x)$, $y(0) = 1$. Exact $y(x) = 1 + x$.

Solution: The exact answer for $y' = y/(1 + x)$, $y(0) = 1$ is $y(x) = 1 + x$, found by the variables separable method.

x	y-HEUN	y-EXACT
0	1.000000	1.000000
0.2	1.200000	1.200000
0.4	1.400000	1.400000
0.6	1.600000	1.600000
0.8	1.800000	1.800000
1	2.000000	2.000000

The graphic shows only the exact solution (blue) because the two data sets match to 6 digits.



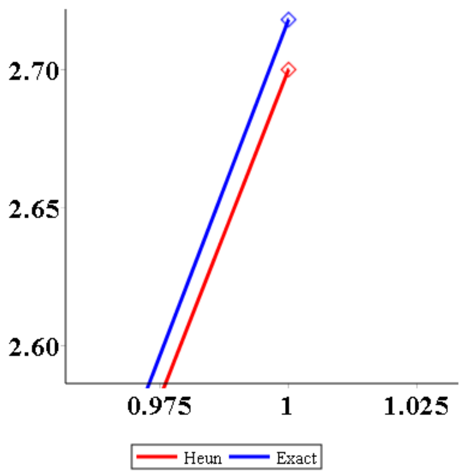
18. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact $y(x) = \sqrt{1 + 2x}$.

19. $y' = yxe^x$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = 1 + (x - 1)e^x$.

Solution: The exact answer for $y' = xye^x$, $y(0) = 1$ is $y(x) = e^{1+(x-1)e^x}$, found by the variables separable method, using integration by parts on $\int xe^x dx$.

x	y-HEUN	y-EXACT
0	1.000000	1.000000
0.2	1.024428	1.023141
0.4	1.113570	1.110605
0.6	1.316293	1.311475
0.8	1.745800	1.741753
1	2.700169	2.718282

The graphic was zoomed to show detail.



20. $y' = 2y(x^2 + x)e^{2x}$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = x^2e^{2x}$.

RK4 Method

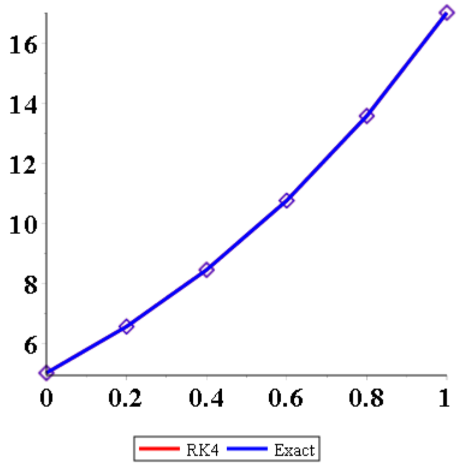
Apply the Runge-Kutta method (RK4) to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow Example 4.7.

21. $y' = 2 + y$, $y(0) = 5$. Exact $y(x) = -2 + 7e^x$.

Solution: The exact answer for $y' = 2+y$, $y(0) = 5$ is $y(x) = -2+7e^x$, found by the linear integrating factor method. The constant coefficient shortcut applies: $y = y_p+y_h$, $y_p = -2 =$ equilibrium solution, $y_h = c/W$, integrating factor $W = e^{-x}$.

x	y-RK4	y-EXACT
0	5.000000	5.000000
0.2	6.549800	6.549819
0.4	8.442726	8.442773
0.6	10.754745	10.754832
0.8	13.578646	13.578786
1	17.027758	17.027973

The graphic shows only the exact curve (blue) because the two data sets agree to 3-digit accuracy.



Let $F(x, y) = 2 + y$. The y-RK4 value is found from the 5-line algorithm

$k_1 = hF(x, y);$
$k_2 = hF(x + h/2, y + k_1/2);$
$k_3 = hF(x + h/2, y + k_2/2);$
$k_4 = hF(x + h, y + k_3);$
$y(x + h) = y(x) + (k_1 + 2k_2 + 2k_3 + k_4)/6;$

The computation by hand calculator is lengthy. Some check points are supplied:

Values for the first row of the table :

$x = 0$, y-RK4 $= 5$, y-EXACT $= 5$

The second row values:

$x = 0.2$, $k_1 = 1.4$, $k_2 = 1.54$, $k_3 = 1.554$, $k_4 = 1.7108$, y-RK4 $= 6.5498$, y-EXACT $= Y(0.2) = 6.549819306$

The third row values:

$x = 0.4$, $k_1 = 1.709960$, $k_2 = 1.880956$, $k_3 = 1.898056$, $k_4 = 2.089571$, y-RK4 $= 8.442726$, y-EXACT $= Y(0.4) = 8.44277289$

The fourth and later row values follow the same pattern, each row depending only on the answer from the previous row.

An online check in WolframAlpha: use input

`y'=2+y, y(0)=5 by runge kutta method h=0.2 t=0 to 1`.

WolframAlpha numerical answers disagreed on date 9.2021 with online RK4 calculators. The WolframAlpha algorithm below computes values in agreement with the table above:

$$y'(t) = f(t, y) = y(t) + 2,\ y(0) = 5$$
$$y_{n+1} = y_n + h\left(\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}\right)$$
$$t_{n+1} = t_n + h$$
$$k_1 = f(t_n, y_n)$$
$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h\,k_1}{2}\right)$$
$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h\,k_2}{2}\right)$$
$$k_4 = f(t_n + h, y_n + h\,k_3)$$

where $y_0 = 5$

$t_0 = 0$

$h = 0.2$

$n = 0,\ldots,2$

```
# Maple: Exact solution
 F:=(x,y)->2+y;de:=diff(y(x),x)=F(x,y(x));y0:=5;x0:=0;
 ans:=dsolve([de,y(x0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# EY(x)=-2+7*exp(x)
# Maple: RK4 method
 RK4:=proc(x,y)
  local k1,k2,k3,k4,Z;
  k1:=h*F(x,y);
  k2:=h*F(x+h/2,y+k1/2);
  k3:=h*F(x+h/2,y+k2/2);
  k4:=h*F(x+h,y+k3);
  Z:=(k1+2*k2+2*k3+k4)/6;
  RETURN (Z);
 end proc;
 DotsRK4:=[x0,y0];DotsEXACT:=[x0,y0];Y:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k; Y:= Y+RK4(X-h,Y);
   DotsRK4:=DotsRK4,[X,Y];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
 DotsRK4;DotsEXACT; # answers
# Maple: Two connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
       symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsRK4],[DotsEXACT]],opts,
      color=[red,blue],legend=["RK4","Exact"]);
```

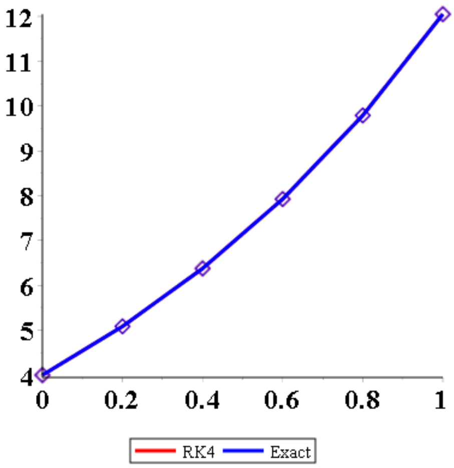
22. $y' = 3 + y$, $y(0) = 5$. Exact $y(x) = -3 + 8e^x$.

23. $y' = e^{-x} + y$, $y(0) = 4$. Exact $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$.

Solution: The exact answer for $y' = e^{-x}+y$, $y(0) = 5$ is $y(x) = -\frac{1}{2}e^{-x}+\frac{9}{2}e^x$, found by the linear integrating factor method. No shortcut applies.

x	y-RK4	y-EXACT
0	4.000000	4.000000
0.2	5.086937	5.086947
0.4	6.378026	6.378051
0.6	7.925081	7.925129
0.8	9.790190	9.790270
1	12.048205	12.048328

The graphic shows only the exact (blue) curve, because the table values agree to 4 digits.



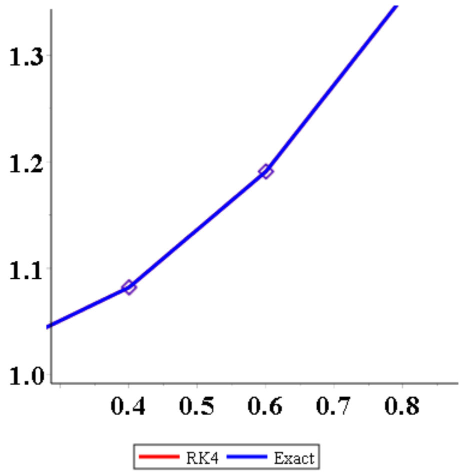
24. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact $y(x) = -e^{-2x} + 5e^x$.

25. $y' = y \sin x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos x}$.

Solution: The exact answer for $y' = y\sin(x)$, $y(0) = 1$ is $y(x) = e^{1-\cos x}$, found by the variables separable method.

x	y-RK4	y-EXACT
0	1.000000	1.000000
0.2	1.020133	1.020133
0.4	1.082138	1.082138
0.6	1.190846	1.190846
0.8	1.354311	1.354312
1	1.583593	1.583595

The graphic shows exact (blue) only because the data matches to 4 digits.



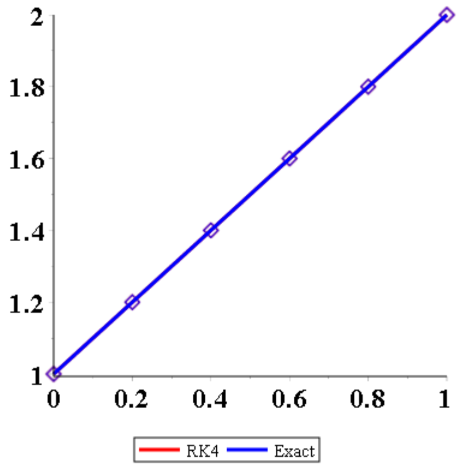
26. $y' = 2y\sin 2x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos 2x}$.

27. $y' = y/(1 + x)$, $y(0) = 1$. Exact $y(x) = 1 + x$.

Solution: The exact answer for $y' = y/(1 + x)$, $y(0) = 1$ is $y(x) = 1 + x$, found by the variables separable method.

x	y-RK4	y-EXACT
0	1.000000	1.000000
0.2	1.200000	1.200000
0.4	1.400000	1.400000
0.6	1.600000	1.600000
0.8	1.800000	1.800000
1	2.000000	2.000000

The graphic shows only the exact solution (blue) because the two data sets match to 4-digit accuracy.



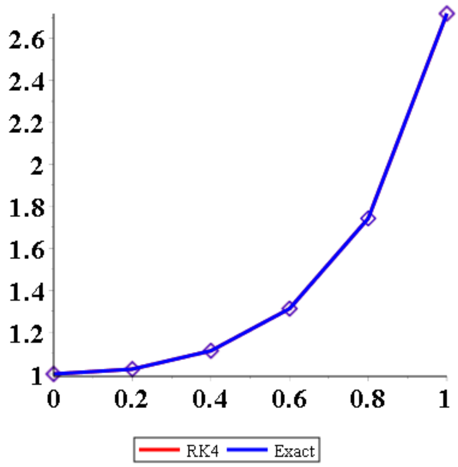
28. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact $y(x) = \sqrt{1 + 2x}$.

29. $y' = yxe^x$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = 1 + (x - 1)e^x$.

Solution: The exact answer for $y' = xye^x$, $y(0) = 1$ is $y(x) = e^{1+(x-1)e^x}$, found by the variables separable method, using integration by parts on $\int xe^x dx$.

x	y-RK4	y-EXACT
0	1.000000	1.000000
0.2	1.023142	1.023141
0.4	1.110605	1.110605
0.6	1.311471	1.311475
0.8	1.741709	1.741753
1	2.717842	2.718282

The graphic shows only the exact solution (blue) because the two data sets match to 4-digit accuracy.



30. $y' = 2y(x^2 + x)e^{2x}$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = x^2e^{2x}$.

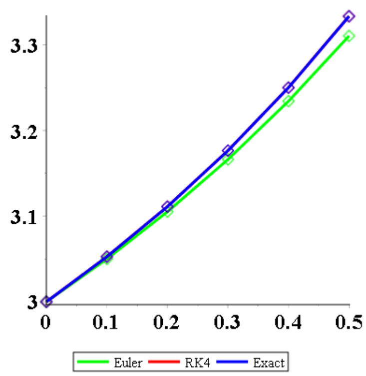
Euler and RK4 Methods

Apply the Euler method and the Runge-Kutta method (RK4) to make a table with 6 rows and step size $h = 0.1$. The table columns are x, y_1, y_2, y where y_1 is the Euler approximation, y_2 is the RK4 approximation and y is the exact solution. Graph y_1, y_2, y.

31. $y' = \frac{1}{2}(y-2)^2$, $y(0) = 3$. Exact $y(x) = \dfrac{2x-6}{x-2}$.

Solution: The exact answer for $y' = \frac{1}{2}(y-2)^2$, $y(0) = 3$ is $y(x) = 2\,\dfrac{x-3}{x-2}$, found by the variables separable method.

The graphic shows only Euler (green) and the exact solution (blue) because RK4 and EXACT data sets match to 4-digit accuracy.



x	y-EULER	y-RK4	y-EXACT
0.0	3.0000000000	3.0000000000	3.0000000000
0.1	3.0500000000	3.0526315630	3.0526315780
0.2	3.1051250000	3.1111110710	3.1111111120
0.3	3.1661900630	3.1764705130	3.1764705880
0.4	3.2341900260	3.2499998710	3.2500000000
0.5	3.3103512770	3.3333331230	3.3333333340

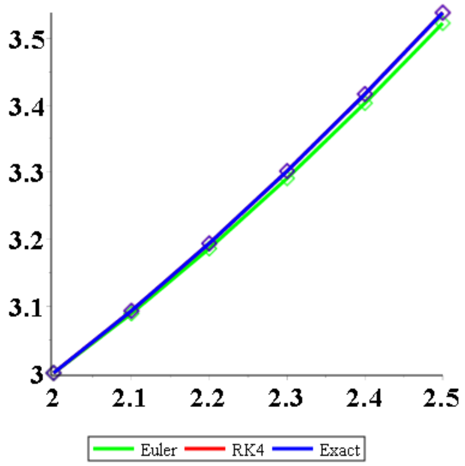
```
# Maple: Exact solution
 F:=(x,y)->(y-2)^2/2;de:=diff(y(x),x)=F(x,y(x));
 y0:=3;x0:=0;
 ans:=dsolve([de,y(x0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# EY(x)=(2*x-6)/(x-2)
# Maple: Euler's method and RK4 method
 N:=6;h:=0.1;
 EULER:=(x,y) -> h*F(x,y);# Euler algorithm
 RK4:=proc(x,y)# RK4 algorithm
  local k1,k2,k3,k4,Z;
  k1:=h*F(x,y);
  k2:=h*F(x+h/2,y+k1/2);
  k3:=h*F(x+h/2,y+k2/2);
  k4:=h*F(x+h,y+k3);
  Z:=(k1+2*k2+2*k3+k4)/6;
  RETURN (Z);
 end proc;
 DotsEULER:=[x0,y0];DotsRK4:=[x0,y0];DotsEXACT:=[x0,y0];
 Z:=y0;Y:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+EULER(X-h,Z);Y:= Y+RK4(X-h,Y);
   DotsEULER:=DotsEULER,[X,Z];
   DotsRK4:=DotsRK4,[X,Y];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
 DotsEULER;DotsRK4;DotsEXACT; # answers
# Maple: Three connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
       symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsEULER],[DotsRK4],[DotsEXACT]],opts,
      color=[green,red,blue],legend=["Euler","RK4","Exact"]);
```

32. $y' = \frac{1}{2}(y-3)^2$, $y(0) = 4$. Exact $y(x) = \dfrac{3x-8}{x-2}$.

33. $y' = x^3/y^2$, $y(2) = 3$. Exact $y(x) = \frac{1}{2}\sqrt[3]{6x^4 + 120}$.

Solution: The exact answer for $y' = x^3/y^2$, $y(2) = 3$ is $y(x) = \frac{1}{2}(6x^4 + 120)^{1/3}$, found by the variables separable method.

The graphic shows only Euler (green) and the exact solution (blue) because RK4 and EXACT data sets match to 4-digit accuracy.



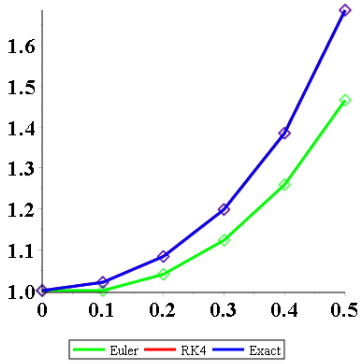
x	y-EULER	y-RK4	y-EXACT
2.0	3.0000000000	3.0000000000	3.0000000000
2.1	3.0888888890	3.0928756410	3.0928755920
2.2	3.1859518000	3.1935156080	3.1935155140
2.3	3.2908552180	3.3015627530	3.3015626210
2.4	3.4032033760	3.4166200600	3.4166198950
2.5	3.5225631310	3.5382706770	3.5382704850

34. $y' = x^5/y^2$, $y(2) = 3$. Exact $y(x) = \frac{1}{2}\sqrt[3]{4x^6 - 40}$.

35. $y' = 2x(1 + y^2)$, $y(0) = 1$. Exact $y(x) = \tan(x^2 + \pi/4)$.

Solution: The exact answer for $y' = x^3/y^2$, $y(0) = 1$ is $y(x) = \tan(x^2 + \pi/4)$, found by the variables separable method.

The graphic shows only Euler (green) and the exact solution (blue) because RK4 and EXACT data sets match to 4-digit accuracy.



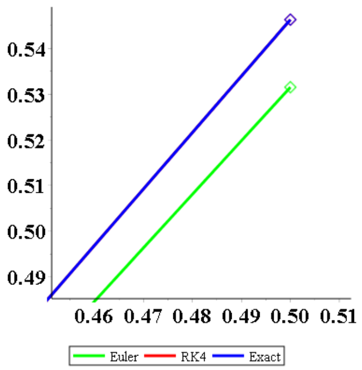
x	y-EULER	y-RK4	y-EXACT
0	1.0000000000	1.0000000000	1.0000000000
0.1	1.0000000000	1.0202030340	1.0202027010
0.2	1.0400000000	1.0833811120	1.0833796610
0.3	1.1232640000	1.1983950950	1.1983911490
0.4	1.2589673210	1.3848782010	1.3848688490
0.5	1.4657672180	1.6858165600	1.6857964190

36. $y' = 3y^{2/3}$, $y(0) = 1$. Exact $y(x) = (x + 1)^3$.

37. $y' = 1 + y^2$, $y(0) = 0$. Exact $y(x) = \tan x$.

Solution: The exact answer for $y' = 1 + y^2$, $y(0) = 0$ is $y(x) = \tan(x)$, found by the variables separable method.

The graphic shows only Euler (green) and the exact solution (blue) because RK4 and EXACT data sets match to 4-digit accuracy. The graphic has been zoomed to show detail.



x	y-EULER	y-RK4	y-EXACT
0	1.0000000000	1.0000000000	1.0000000000
0.1	1.0000000000	1.0202030340	1.0202027010
0.2	1.0400000000	1.0833811120	1.0833796610
0.3	1.1232640000	1.1983950950	1.1983911490
0.4	1.2589673210	1.3848782010	1.3848688490
0.5	1.4657672180	1.6858165600	1.6857964190

38. $y' = 1 + y^2$, $y(0) = 1$. Exact $y(x) = \tan(x + \pi/4)$.

4.3 Error in Numerical Methods

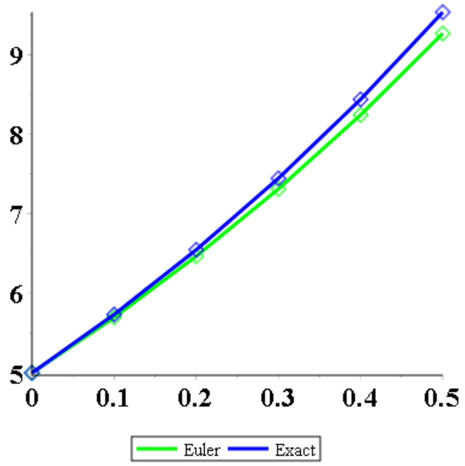
Cumulative Error

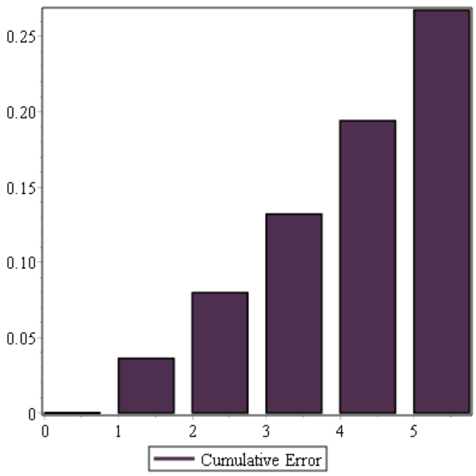
Make a table of 6 lines which has four columns x, y_1, y, E. Symbols y_1 and y are the approximate and exact solutions while $E = |y - y_1|$ is the cumulative error. Find y_1 using Euler's method in steps $h = 0.1$.

1. $y' = 2 + y$, $y(0) = 5$. Exact solution $y(x) = -2 + 7e^x$.

Solution: The exact answer for $y' = 2+y$, $y(2) = 5$ is $y(x) = -2+7e^x$, found by the linear integrating factor shortcut for constant-coefficient equations.

The first graphic shows Euler (green), exact solution (blue). The second graphic is a bar chart for the cumulative error.





x	y-EULER	y-EXACT	y-Cumulative Error
0	5.0000000000	5.0000000000	0.0000000000
0.1	5.7000000000	5.7361964260	0.0361964260
0.2	6.4700000000	6.5498193060	0.0798193060
0.3	7.3170000000	7.4490116560	0.1320116560
0.4	8.2487000000	8.4427728900	0.1940728900
0.5	9.2735700000	9.5410489000	0.2674789000

The cumulative error is |EULER - EXACT|. On calculators without absolute value, remove the sign of the answer in column 4.

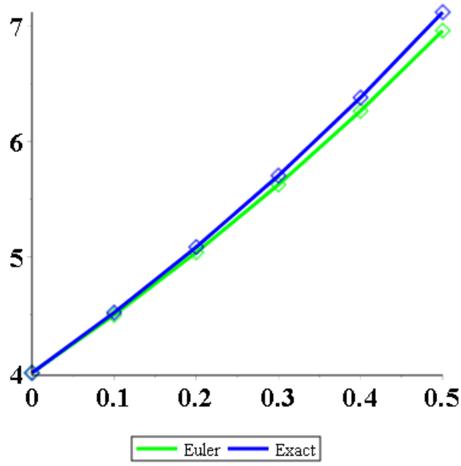
```
# Cumulative Error Exercise 1
# Maple: Exact solution
 F:=(x,y)->2+y;de:=diff(y(x),x)=F(x,y(x));y0:=5;x0:=0;
 ans:=dsolve([de,y(x0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# EY(x)=-2+7*exp(x)
# Numerical solution
 N:=6;h:=0.1;# 6 rows and stepsize
 EULER:=(x,y) -> h*F(x,y);# Euler algorithm
 DotsEULER:=[x0,y0];DotsEXACT:=[x0,y0];Z:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+EULER(X-h,Z);
   DotsEULER:=DotsEULER,[X,Z];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
 DotsEULER;DotsEXACT; # answers
# Compute cumulative error = |EULER-EXACT|
  cErr:=k->abs(DotsEULER[k][2]-DotsEXACT[k][2]);
  cumulativeError:=seq(cErr(k),k=1..N);
# Maple: Two connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
       symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsEULER],[DotsEXACT]],
      opts,color=[green,blue,red],
      legend=["Euler","Exact"]);
# Maple: Bar chart cumulative error
 Statistics[ColumnGraph](<cumulativeError>,
   color=violet,legend=["Cumulative Error"]);
```

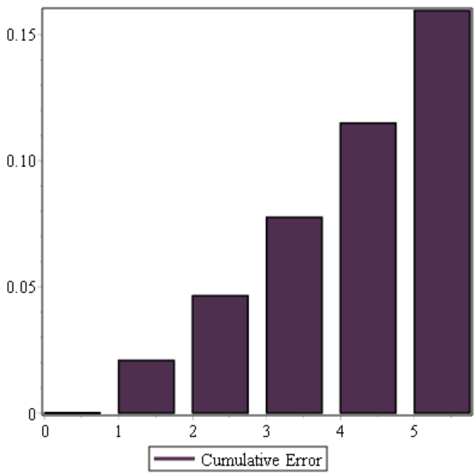
2. $y' = 3 + y$, $y(0) = 5$. Exact solution $y(x) = -3 + 8e^x$.

3. $y' = e^{-x} + y$, $y(0) = 4$. Exact solution $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$.

Solution: The exact answer for $y' = e^{-x} + y$, $y(0) = 4$ is $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$, found by the linear integrating factor method.

The first graphic shows Euler (green) and the exact solution (blue). The second graphic is a bar chart for cumulative error.





x	y-EULER	y-EXACT	y-Cumulative Error
0	4.0000000000	4.0000000000	0.0000000000
0.1	4.5000000000	4.5208504220	0.0208504220
0.2	5.0404837420	5.0869470350	0.0464632930
0.3	5.6264051920	5.7039555260	0.0775503340
0.4	6.2631275330	6.3780511180	0.1149235850
0.5	6.9564722910	7.1159803890	0.1595080980

4. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact solution $y(x) = -e^{-2x} + 5e^x$.

Local Error

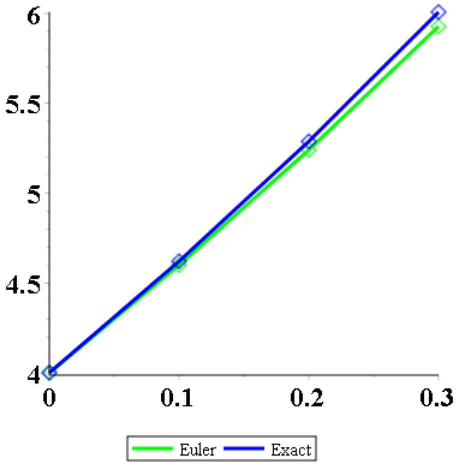
Make a table of 4 lines which has four columns x, y_1, y, E. Symbols y_1 and

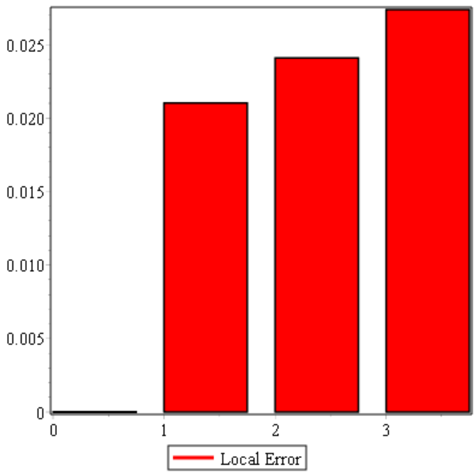
y are the approximate and exact solutions while E is the local error. Find y_1 using Euler's method in steps $h = 0.1$. The general solution in each exercise is the solution for $y(0) = c$.

5. $y' = 2 + y$, $y(0) = 5$. General solution $y(x) = -2 + (2 + c)e^x$.

Solution: The exact answer for $y' = 2+y$, $y(2) = 5$ is $y(x) = -2+7e^x$, found by the linear integrating factor shortcut for constant-coefficient equations.

The first graphic shows Euler (green) and exact (blue) solutions. The second graphic is a bar chart for local error at x-values $0, 01., 0.2, 0.3$.





x	y-EULER	y-EXACT	y-Local Error
0	5.0000000000	5.0000000000	0.0000000000
0.1	5.7000000000	5.7361964260	0.0361964260
0.2	6.4700000000	6.5498193060	0.0398160690
0.3	7.3170000000	7.4490116560	0.0437976770

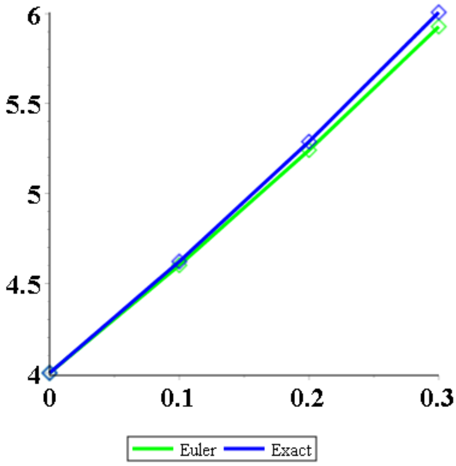
```
# Local Error Exercise 5
# Maple: Exact solution
 F:=(x,y)->2+y;de:=diff(y(x),x)=F(x,y(x));y0:=5;x0:=0;
 ans:=dsolve([de,y(x0)=y0],y(x));
 EY:=unapply(rhs(ans),x);# EY(x)=-2+7*exp(x)
# Numerical solution
 N:=4;h:=0.1;# 4 rows and stepsize
 EULER:=(x,y) -> h*F(x,y);# Euler algorithm
 DotsEULER:=[x0,y0];DotsEXACT:=[x0,y0];Z:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+EULER(X-h,Z);
   DotsEULER:=DotsEULER,[X,Z];
   DotsEXACT:=DotsEXACT,[X,EY(X)];
 od:
 DotsEULER;DotsEXACT; # answers
# Compute IVP solutions at 0.1 to 0.3
 for k from 1 to N-1 do #
  X:=DotsEULER[k][1];# x
  Y:=DotsEULER[k][2];# y-EULER
  ansLocal:=dsolve([de,y(X)=Y],y(x)):
  ELocal:=unapply(rhs(ansLocal),x);
  Ivp[k]:=evalf(ELocal(X+h));# y-value for next node
 od:
# Compute local error = |EULER-(IVP-value)|
 lErr:=k->abs(DotsEULER[k][2]-Ivp[k-1]);
 localError:=0,seq(lErr(k),k=2..N);
# Maple: Two connect-the-dots curves on 1 graphic
 opts:=style=pointline,font=[courier,18,bold],
       symbol=diamond,symbolsize=24,thickness=3;
 plot([[DotsEULER],[DotsEXACT]],opts,
      color=[green,blue],legend=["Euler","Exact"]);
# Maple: Bar Chart local error
 Statistics[ColumnGraph](<localError>,color=red,
   legend=["Local Error"]);
```

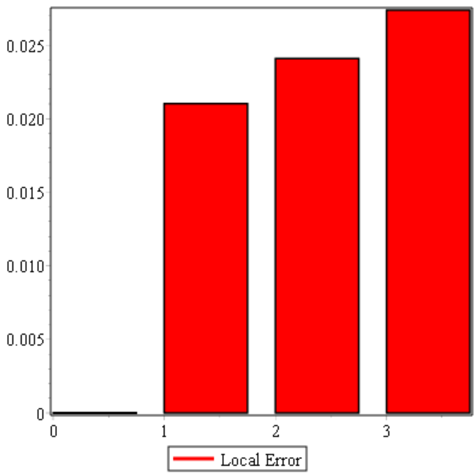
6. $y' = 3 + y$, $y(0) = 5$. General solution $y(x) = -3 + (3 + c)e^x$.

7. $y' = 2e^{-x} + y$, $y(0) = 4$. General solution $y(x) = -e^{-x} + (1 + c)e^x$.

Solution: The exact answer for $y' = 2e^{-x} + y$, $y(0) = 4$ is $y(x) = -e^{-x} + 5e^{x}$, found by the linear integrating factor method.

The first graphic shows Euler (green) and exact (blue) solutions. The second graphic is a bar chart for local error at x-values $0, 01., 0.2, 0.3$.





x	y-EULER	y-EXACT	y-Local Error
0	4.0000000000	4.0000000000	0.0000000000
0.1	4.6000000000	4.6210171720	0.0210171720
0.2	5.2409674840	5.2882830370	0.0240879850
0.3	5.9288103830	6.0084758190	0.0273736610

8. $y' = 3e^{-2x} + y$, $y(0) = 4$. General solution $y(x) = -e^{-2x} + (1 + c)e^{x}$.

Roundoff Error

Compute the roundoff error for $y = 5a + 4b$.

9. Assume 3-digit precision. Let $a = 0.0001$ and $b = 0.0003$.

Solution: In 3-digit precision: $\hat{a} = 0.000$, $\hat{b} = 0.000$. Then $y = 5a + 4b = 0.0005 + 0.0012 = 0.0017$ while $\hat{y} = 5\hat{a} + 4\hat{b} = 0.000$. The roundoff error is $y - \hat{y} = 0.0017 - 0.000 = 0.0017$. Roundoff error can be a positive or negative number, or zero. Some key examples and rounding methods can be found at

https://en.wikipedia.org/wiki/Round-off_error

```
# Roundoff, Exercise 9
Round:=(x,n)->evalf( round(x*10^(n))/10.0^n );
a:=0.0001;b:=0.0003;
ahat:=Round(a,3);bhat:=Round(b,3);
y:=5*a+4*b;yhat:=5*ahat+4*bhat;
rErr:=y-yhat;# Roundoff error 0.0017
```

10. Assume 3-digit precision. Let $a = 0.0002$ and $b = 0.0001$.

11. Assume 5-digit precision. Let $a = 0.000007$ and $b = 0.000003$.

Solution: In 5-digit precision: $\hat{a} = 0.00001$, $\hat{b} = 0.00000$. Then $y = 5a+4b = 0.000035 + 0.000012 = 0.000047$ while $\hat{y} = 5\hat{a} + 4\hat{b} = 0.00005$. The roundoff error is $y - \hat{y} = 0.000047 - 0.00005 = -0.000003$.

12. Assume 5-digit precision. Let $a = 0.000005$ and $b = 0.000001$.

Truncation Error

Find the truncation error.

13. Truncate $x = 1.123456789$ to 3 digits right of the decimal point.

14. Truncate $x = 1.123456789$ to 4 digits right of the decimal point.

Solution: Answer: 1.1234

```
# Truncation, Exercise 13
Truncate:=(x,n)->evalf( trunc(x*10^(n))/10.0^n );
X:=1.123456789;
Xtrunc:=Truncate(X,4);# Xtrunc = 1.1234
```

15. Truncate $x = 1.017171717$ to 7 digits right of the decimal point.

Solution: Answer: 1.0171717

16. Truncate $x = 1.03939393939$ to 9 digits right of the decimal point.

Guessing the Step Size

Do a numerical experiment using the given method to estimate the number of steps needed to generate a numerical solution with 2-digit accuracy on $0 \le x \le 1$. The number reported, if increased, should not change the 2-digit accuracy.

17. $y' = 2 + y$, $y(0) = 5$. Exact solution $y(x) = -2 + 7e^x$. Euler's method.

Solution: The answer: about 5800 data points gives 2-digit accuracy. This numerical project requires a CAS or Numerical Workbench.
A practical experiment is to evaluate Euler estimates at x-values $x_0 = 0$, $x_1, \ldots, x_M = 1$, then compare Euler values to the Exact solution values for 2-digit agreement. Once a step size h is found that appears to work, then increase the step size and repeat the experiment. There is no precise answer possible for M, only an estimate.

```
# Guessing the stepsize, Exercise 17
# Maple: Exact solution
F:=(x,y)->2+y;de:=diff(y(x),x)=F(x,y(x));y0:=5;x0:=0;
ans:=dsolve([de,y(x0)=y0],y(x));
EY:=unapply(rhs(ans),x);# EY(x)=-2+7*exp(x)
M:=5800;# M steps
h:=1.0/M; # step size.
N:=M+1; # table rows
EULER:=(x,y) -> h*F(x,y);# Euler algorithm
approx:=EULER;# or HEUN, RK4
vals:=y0: Z:=y0;
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+approx(X-h,Z);
   vals:=vals,Z;
 od:
 maxERR:=0;W:=vals:
 for k from 1 to N do
  X:= x0 + h*k;
  Z:=abs(W[k]-EY(X));# cumulative error
  maxERR:=max(maxERR,Z);
 od:# colon=no echo
 printf("MaxERR=%10f, h =%10f\n",maxERR,h);
 # MaxERR=  0.004921, h =  0.000172
```

18. $y' = 3 + y$, $y(0) = 5$. Exact solution $y(x) = -3 + 8e^x$. Euler's method

19. $y' = e^{-x} + y$, $y(0) = 4$. Exact solution $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$. Euler's method

Solution: The answer: about 3700 data points gives 2-digit accuracy.

20. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact solution $y(x) = -e^{-2x} + 5e^{x}$. Euler's method.

21. $y' = y/(1 + x)$, $y(0) = 1$. Exact solution $y(x) = 1 + x$. Euler's method.

Solution: The answer: about 201 data points gives 2-digit accuracy.

22. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact solution $y(x) = \sqrt{1 + 2x}$. Euler's method.

23. $y' = 2 + y$, $y(0) = 5$. Exact solution $y(x) = -2 + 7e^{x}$. Heun's method.

Solution: The answer: about 3810 data points gives 2-digit accuracy.

24. $y' = 3 + y$, $y(0) = 5$. Exact solution $y(x) = -3 + 8e^{x}$. Heun's method

25. $y' = e^{-x} + y$, $y(0) = 4$. Exact solution $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^{x}$. Heun's method

Solution: The answer: about 2485 data points gives 2-digit accuracy.

26. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact solution $y(x) = -e^{-2x} + 5e^{x}$. Heun's method.

27. $y' = y/(1 + x)$, $y(0) = 1$. Exact solution $y(x) = 1 + x$. Heun's method.

Solution: The answer: about 201 data points gives 2-digit accuracy.

28. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact solution $y(x) = \sqrt{1 + 2x}$. Heun's method.

29. $y' = 2 + y$, $y(0) = 5$. Exact solution $y(x) = -2 + 7e^{x}$. RK4 method.

Solution: The answer: about 3810 data points gives 2-digit accuracy.

30. $y' = 3 + y$, $y(0) = 5$. Exact solution $y(x) = -3 + 8e^{x}$. RK4 method

31. $y' = e^{-x} + y$, $y(0) = 4$. Exact solution $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^{x}$. RK4 method

Solution: The answer: about 2485 data points gives 2-digit accuracy.

32. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact solution $y(x) = -e^{-2x} + 5e^{x}$. RK4 method.

33. $y' = y/(1 + x)$, $y(0) = 1$. Exact solution $y(x) = 1 + x$. RK4 method.

Solution: The answer: about 201 data points gives 2-digit accuracy.

34. $y' = y(x)/(1+2x)$, $y(0) = 1$. Exact solution $y(x) = \sqrt{1 + 2x}$. RK4 method.

4.4 Computing π, $\ln 2$ and e

Computing π

Compute $\pi = y(1)$ from the initial value problem $y' = 4/(1+x^2)$, $y(0) = 0$, using the given method. Number 3.14159 with 3-digit precision is the rounded number 3.142.

1. Use the Rectangular integration rule. Determine the number of steps for 3-digit precision.

Solution: About 1102 steps, $h = 1/1102$.

```
# RECT 3-digit precision, Exercise 1
 F:=x->4/(1+x^2);x0:=0;y0:=0;
 precision:=3;EXACT:=Pi; # Default is 10 digits
 Round:=(x,n)->evalf( round(x*10^(n))/10.0^n ):
 RECT:=x -> h*F(x);# Rectangular rule algorithm
 M:=1102;h:=1.0/M;N:=M+1; # N rows, stepsize h
 approx:=y0;Z:=y0; ALGORITHM:=RECT:
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+ALGORITHM(X-h);
   approx:=approx,Z;
 od:
 PiApprox:=Round(approx[N],precision);
 PiExact:=Round(EXACT,precision);
 ERR:=abs(PiExact-PiApprox);
 printf("ERR=%10f, 3-digit Pi=%10f, h=1/%a\n",ERR,PiApprox,M);
# ERR=  0.000000, 3-digit Pi=  3.142000, h=1/1102
```

2. Use the Rectangular integration rule. Determine the number of steps for 4-digit precision.

Solution: More than 2180 steps.

3. Use the Trapezoidal integration rule. Determine the number of steps for 3-digit precision.

Solution: About 43 steps, $h = 1/43 = 0.02325581395$.

```
# TRAP 3-digit precision, Exercise 3
 F:=x->4/(1+x^2);x0:=0;y0:=0;
 precision:=3;EXACT:=Pi; # Default is 10 digits
 Round:=(x,n)->evalf( round(x*10^(n))/10.0^n ):
 RECT:=x -> h*F(x);# Rectangular rule algorithm
 TRAP:=x->h*(F(x)+F(x+h))/2;
 M:=43;h:=1.0/M;N:=M+1; # N rows, stepsize h
 approx:=y0;Z:=y0; ALGORITHM:=TRAP:
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+ALGORITHM(X-h);
   approx:=approx,Z;
 od:
 PiApprox:=Round(approx[N],precision);
 PiExact:=Round(EXACT,precision);
 ERR:=abs(PiExact-PiApprox);
 printf("ERR=%10f, PiApprox=%10f, h=1/%a\n",ERR,PiApprox,M);
# ERR=  0.000000, PiApprox=  3.142000, h=1/43
```

4. Use the Trapezoidal integration rule. Determine the number of steps for 4-digit precision.

5. Use Simpson's rule. Determine the number of steps for 5-digit precision.

Solution: About 3 steps, $h = 1/3 = 0.3333333333$.

```
# SIMP 5-digit precision, Exercise 5
 F:=x->4/(1+x^2);x0:=0;y0:=0;
 precision:=5;EXACT:=Pi; # Default is 10 digits
 Round:=(x,n)->evalf( round(x*10^(n))/10.0^n ):
 RECT:=x -> h*F(x);# Rectangular rule algorithm
 TRAP:=x->h*(F(x)+F(x+h))/2;
 SIMP:=x ->( h*(F(x)+4*F(x+h/2)+F(x+h))/6 );
 M:=3;h:=1.0/M;N:=M+1; # N rows, stepsize h
 approx:=y0;Z:=y0; ALGORITHM:=SIMP:
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+ALGORITHM(X-h);
   approx:=approx,Z;
 od:
 PiApprox:=Round(approx[N],precision);
 PiExact:=Round(EXACT,precision);
 ERR:=abs(PiExact-PiApprox);
 printf("ERR=%10f, PiApprox=%10f, h=1/%a\n",ERR,PiApprox,M);
# ERR=  0.000000, PiApprox=  3.141590, h=1/3
```

6. Use Simpson's rule. Determine the number of steps for 6-digit precision.

7. Use a computer algebra system library routine for RK4. Report the step size used and the number of steps for 5-digit precision.

Solution: WolframAlpha:
Number of steps: 26. Step size: $1/26 = 0.03846153846$.

```
URL: https://www.wolframalpha.com/
Input:
Runge-Kutta method, dy/dx = 4/(1+x^2),
  y(0) = 0, from 0 to 1, h = 1/26
```

8. Use a numerical workbench library routine for RK4. Report the step size used and the number of steps for 5-digit precision.

Solution: MATLAB:
No online input, like WolframAlpha. Write your own code.

```
URL of code source for RK4:
 https://www.mathworks.com/matlabcentral/
         answers/460395-runge-kutta-4th-order-method
```

Computing $\ln(2)$

Compute $\ln(2) = y(1)$ from the initial value problem $y' = 1/(1+x)$, $y(0) = 0$, using the given method.

9. Use the Rectangular integration rule. Determine the number of steps for 3-digit precision.

Solution: About 709 steps, $h = 1/709 = 0.001410437236$.

```
# RECT 3-digit precision, Exercise 9
 F:=x->1/(1+x);x0:=0;y0:=0;
 precision:=3;EXACT:=ln(2); # Default is 10 digits
 Round:=(x,n)->evalf( round(x*10^n)/10.0^n ):
 RECT:=x -> h*F(x);# Rectangular rule algorithm
 M:=709;h:=1.0/M;N:=M+1; # N rows, stepsize h
 approx:=y0;Z:=y0; ALGORITHM:=RECT:
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+ALGORITHM(X-h);
   approx:=approx,Z;
 od:
 ln2Approx:=Round(approx[N],precision);
 ln2Exact:=Round(EXACT,precision);
 ERR:=abs(ln2Exact-ln2Approx);
 printf("ERR=%10f, %a-digit ln(2)=%10f, h=1/%a\n",
         ERR,precision,ln2Approx,M);
# ERR=  0.000000, 3-digit ln(2)=  0.693000, h=1/709
```

10. Use the Rectangular integration rule. Determine the number of steps for 4-digit precision.

11. Use the Trapezoidal integration rule. Determine the number of steps for 5-digit precision.

Solution: About 90 steps, $h = 1/90 = 0.01111111111$.

```
# TRAP 5-digit precision, Exercise 11
 F:=x->1/(1+x);x0:=0;y0:=0;
 precision:=5;EXACT:=ln(2); # Default is 10 digits
 Round:=(x,n)->evalf( round(x*10^n)/10.0^n ):
 TRAP:=x->h*(F(x)+F(x+h))/2;
 M:=90;h:=1.0/M;N:=M+1; # N rows, stepsize h
 approx:=y0;Z:=y0; ALGORITHM:=TRAP:
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+ALGORITHM(X-h);
   approx:=approx,Z;
 od:
 ln2Approx:=Round(approx[N],precision);
 ln2Exact:=Round(EXACT,precision);
 ERR:=abs(ln2Exact-ln2Approx);
 printf("ERR=%10f, %a-digit ln(2)=%10f, h=1/%a\n",
         ERR,precision,ln2Approx,M);
# ERR=  0.000000, 5-digit ln(2)=  0.693150, h=1/90
```

12. Use the Trapezoidal integration rule. Determine the number of steps for 6-digit precision.

13. Use Simpson's rule. Determine the number of steps for 5-digit precision.

Solution: About 4 steps, $h = 1/4 = 0.25$.

```
# SIMP 5-digit precision, Exercise 13
 F:=x->1/(1+x);x0:=0;y0:=0;
 precision:=5;EXACT:=ln(2); # Default is 10 digits
 Round:=(x,n)->evalf( round(x*10^n)/10.0^n ):
 TRAP:=x->h*(F(x)+F(x+h))/2;
 SIMP:=x ->( h*(F(x)+4*F(x+h/2)+F(x+h))/6 );
 M:=4;h:=1.0/M;N:=M+1; # N rows, stepsize h
 approx:=y0;Z:=y0; ALGORITHM:=SIMP:
 for k from 1 to N-1 do
   X:= x0 + h*k;Z:=Z+ALGORITHM(X-h);
   approx:=approx,Z;
 od:
 ln2Approx:=Round(approx[N],precision);
 ln2Exact:=Round(EXACT,precision);
 ERR:=abs(ln2Exact-ln2Approx);
 printf("ERR=%10f, %a-digit ln(2)=%10f, h=1/%a\n",
         ERR,precision,ln2Approx,M);
# ERR=  0.000000, 5-digit ln(2)=  0.693150, h=1/4
```

14. Use Simpson's rule. Determine the number of steps for 6-digit precision.

15. Use a computer algebra system library routine for RK4. Report the step size used and the number of steps for 5-digit precision.

Solution: MAPLE:
Estimate: $\ln(2) \approx 0.693147180561166$, error 0.0. The default step size for this problem is 0.005. The engine `dsolve` is used with options found from `maple` help: `?dsolve,classical` from the HELP Menu. The Runge-Kutta 4 method is called `rk4` in `maple` but the method is called `classical[rk4]`.
MATHEMATICA:
Number of steps: 13. Step size: $1/13 = 0.07692307692$.

```
# MAPLE, Exercise 15
 F:=(x,y)->1/(1+x);x0:=0;y0:=0;
 EXACT:=ln(2); # Default is 10 digits
 sys:=[diff(y(x),x)=F(x,y(x)),y(x0)=y0]:
 ans:=dsolve(sys,numeric,method=classical[rk4]);
 ln2Approx:=rhs(ans(1)[2]); # ln2Approx = 0.693147180561166,
 ERR:=abs(evalf(ln2Approx-EXACT,14));
# ERR = 1.21591625656947*10^(-12)
# MATHEMATICA
URL: https://www.wolframalpha.com/
Input:
Runge-Kutta method, dy/dx = 1/(1+x),
  y(0) = 0, from 0 to 1, h = 1/13
```

16. Use a numerical workbench library routine for RK4. Report the step size used and the number of steps for 5-digit precision.

Solution: MATLAB:
No online input, like WolframAlpha. Write your own code using the cited Mathworks download. To use Matlab in 2021, a license is required for the desktop app or a 30-day free trial for the online Matlab workbench.

```
URL of Matlab code source for RK4:
 https://www.mathworks.com/matlabcentral/
 fileexchange/29851-runge-kutta-4th-order-ode
 # MAPLE: Numeric, RK4 method
 F:=(x,y) -> y; x0:=0;y0:=1;
 EXACT:=exp(1); # Default is 10 digits
 sys:=[diff(y(x),x)=F(x,y(x)),y(x0)=y0]:
 ans:=dsolve(sys,numeric,method=classical[rk4]);
 eApprox:=rhs(ans(1)[2]); # eApprox = 2.71827054469638,
 ERR:=abs(evalf(eApprox-EXACT,14));
 # ERR = 0.000112837626158324
```

Computing e

Compute $e = y(1)$ from the initial value problem $y' = y$, $y(0) = 1$, using the given computer library routines. Report the approximate number of digits of precision attained with a computer algebra system or numerical workbench.

17. Improved Euler method, also known as Heun's method.

Solution:
MAPLE:
Estimate: $e \approx 2.71827054469638$, error 0.0001128, default step size 0.005. The engine `dsolve` is used with options. The options can be found from the `maple` help menu: `?dsolve,classical`. The Improved Euler method is called `heun` in `maple` but method = `classical[heunform]` or equivalently, method = `classical[rk2]`.

MATHEMATICA:
Number of steps: 10. Step size: 1/10. Estimate: $e \approx 2.71408$, error 0.0040098.

```
# MAPLE: Numeric, Heun's method, Exercise 17
 F:=(x,y) -> y; x0:=0;y0:=1;
 EXACT:=exp(1); # Default is 10 digits
 sys:=[diff(y(x),x)=F(x,y(x)),y(x0)=y0]:
 ans:=dsolve(sys,numeric,method=classical[heunform]);
 eApprox:=rhs(ans(1)[2]); # eApprox = 2.71827054469638,
 ERR:=abs(evalf(eApprox-EXACT,14)); # ERR = 0.0001128
# MATHEMATICA
 URL: https://www.wolframalpha.com/
 Input:
   Heun method, dy/dx = y, y(0) = 1, from 0 to 1
```

18. RK4 method.

19. RKF45 method.

Solution:
MAPLE:
Number of steps: adaptive. Step size: adaptive. Estimate: $e \approx 2.71828133411964$, error 0.000000494.

MATHEMATICA:
Number of steps: 11. Step size: 1/11. Estimate: $e \approx 2.71828$, error 0.000000198.

```
# Runge-Kutta-Fehlberg RKF45, Exercise 19
# MAPLE
 F:=(x,y) -> y; x0:=0;y0:=1;
 EXACT:=exp(1); # Default is 10 digits
 sys:=[diff(y(x),x)=F(x,y(x)),y(x0)=y0]:
 ans:=dsolve(sys,numeric,method=rkf45);
 eApprox:=rhs(ans(1)[2]);
 ERR:=abs(evalf(eApprox-EXACT,14)); # ERR = 0.000000494
# MATHEMATICA
 URL: https://www.wolframalpha.com/
 Input:
  runge-kutta-fehlberg method, dy/dx = y, y(0) = 1, from 0 to 1
```

20. Adams-Moulton method.

Solution: The `maple` method is called `abmoulton`, using modified code from exercise 17. Literature citations might use **the Adams-Bashforth-Moulton method**. See also

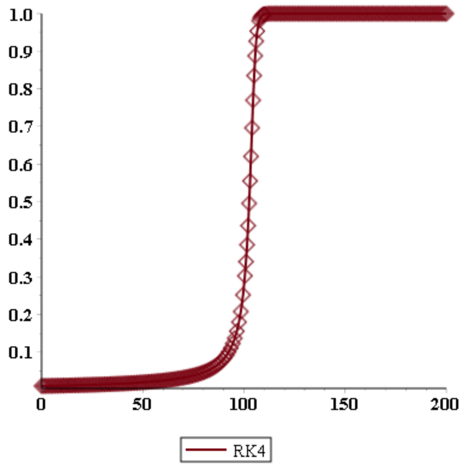
`https://en.wikipedia.org/wiki/Linear_multistep_method`

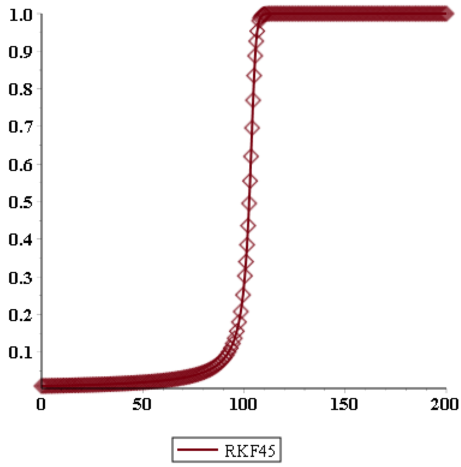
Stiff Differential Equation

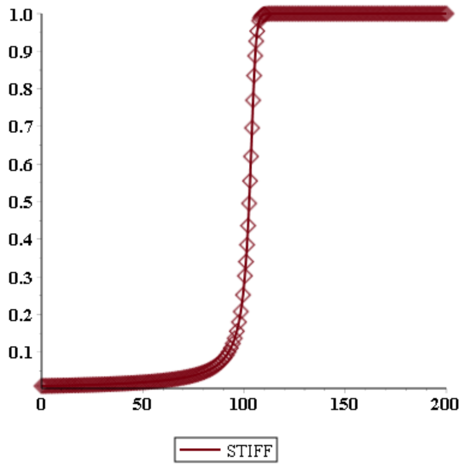
The flame propagation equation $y' = y^2(1-y)$ is known to be **stiff** for small initial conditions $y(0) > 0$. Use classical rk4, then Runge-Kutta-Fehlberg rkf45 and finally a stiff solver to compute and plot the solution $y(t)$ in each case. Expect rk4 to fail, no matter the step size. Both rkf45 and a stiff solver will produce about the same plot, but at different speeds. Reference: `matlab` author Cleve Moler, blogs.mathworks.com 2014.

21. $y(0) = 0.01$

Solution: Classical RK4 does not improve the plot by using a smaller step-size. The other two plots are nearly identical: an increasing curve which at $x = 100$ quickly rises to $y = 1$ and stays there. All plots use a large number of data points (diamonds).








```
# MAPLE: Numeric, RK4 method, Exercise 21
 y0:=0.01;x0:=0;F:=(x,y) -> (1-y)*y^2;
 de:=diff(y(t),t)=F(t,y(t));
 sys:=[diff(y(x),x)=F(x,y(x)),y(x0)=y0]:
 ansRK4:=dsolve(sys,numeric,
   method=classical[rk4],stepsize=1/10);
 yRK4:=x -> rhs(ansRK4(x)[2]); # Runge-Kutta 4
 ansRKF45:=dsolve(sys,numeric,method=rkf45);
 yRKF45:=x -> rhs(ansRKF45(x)[2]); # Runge-Kutta-Fehlberg 45
 ansSTIFF:=dsolve(sys,numeric,stiff=true);
 yStiff:=x -> rhs(ansSTIFF(x)[2]); # Default stiff solver
 opts:=style=pointline,font=[courier,12,bold],
       symbol=diamond,symbolsize=24,thickness=2;
 plot(yRK4,x0..x0+2/y0,opts,legend=["RK4"]);
 plot(yRKF45,x0..x0+2/y0,opts,legend=["RKF45"]);
 plot(yStiff,x0..x0+2/y0,opts,legend=["STIFF"]);
```

22. $y(0) = 0.005$

23. $y(0) = 0.001$

Solution: Classical RK4 improves the plot with stepsize $= 1/2$; increasing the stepsize eventually fails. The other two plots are nearly identical: an increasing curve which at $x = 1000$ quickly rises to $y = 1$ and stays there. Plots not shown because they are no different in shape from those in Exercise 21.

24. $y(0) = 0.0001$

Solution: Classical RK4 fails. The other two plots are nearly identical: an increasing curve which at $x = 10000$ quickly rises to $y = 1$ and stays there.

4.5 Earth to the Moon

Critical Altitude r^*

The symbol r^* is the altitude $r(t)$ at which gravitational effects of the moon take over, causing the projectile to fall to the moon.

1. Justify from the differential equation that $r''(t) = 0$ at $r^* = r(t)$ implies the first relation in (2):

$$\frac{Gm_2}{(R_2 - R_1 - r^*)^2} - \frac{Gm_1}{(R_1 + r^*)^2} = 0.$$

Solution: Insert $r''(t) = 0$ and $r^* = r(t)$ into the Jules Verne differential equation, then:

$$0 = -\frac{Gm_1}{(R_1 + r^*)^2} + \frac{Gm_2}{(R_2 - R_1 - r^*)^2}$$

Re-arrange:

$$\frac{Gm_2}{(R_2 - R_1 - r^*)^2} - \frac{Gm_1}{(R_1 + r^*)^2} = 0$$

2. Solve symbolically the relation of the previous exercise for r^*, to obtain the second equation of (2):

$$r^* = \frac{R_2}{1 + \sqrt{m_2/m_1}} - R_1.$$

Solution: The solution r^* is obtained by conversion to a quadratic equation, then solve by the quadratic formula. The trick: use $a^2 - b^2 = (a-b)(a+b)$ where $a^2 = m_1$, $b^2 = m_2$. Expected details omitted.

```
# MAPLE: Answer check Exercise 2
R1:='R1':R2:='R2':m1:='m1':m2:='m2':G:='G':
w:=r -> G*m2/(R2-R1-r)^2 - G*m1/(R1+r)^2;
rStar:=R2/( 1+sqrt(m2/m1) )-R1;
"w(rStar)" = simplify(w(rStar)); # w(rStar) = 0
```

3. Use the previous exercise and values for the constants R_1, R_2, m_1, m_2 to obtain the approximation

$$r^* = 339,649,780 \text{ meters}.$$

Solution: Assume: `maple` values

```
G:=6.6726e-11:  m1:=5.975e24:  m2:=7.36e22:
R1:=6.378e6:  R2:=3.844e8:
```

$$
\begin{aligned}
r^* &= \frac{R_2}{1+\sqrt{m_2/m_1}} - R_1 \\
&= \frac{384400000}{1+\sqrt{(7.36)10^{22}/((5.975)10^{24})}} - 6378000 \\
&\approx 339,620,820 \text{ meters}
\end{aligned}
$$

4. Determine the effect on r^* for a one percent error in measurement m_2. Replace m_2 by $0.99m_2$ and $1.01m_2$ in the formula for r^* and report the two estimated critical altitudes.

Escape Velocity v_0^*

The symbol v_0^* is the velocity $r'(0)$ such that $\lim_{t\to\infty} r(t) = \infty$, but smaller launch velocities will cause the projectile to fall back to the earth. Throughout, define

$$F(r) = \frac{Gm_1}{R_1 + r} + \frac{Gm_2}{R_2 - R_1 - r}.$$

5. Let $v_0 = r'(0)$, $r^* = r(t_0)$. Derive the formula

$$\frac{1}{2}\,(r'(t_0))^2 = F(r^*) - F(0) + \frac{1}{2}\,v_0^2$$

which appears in the proof details.

Solution: Following the technical details, multiply differential equation $r''(t) = -\frac{Gm_1}{(R_1+r(t))^2} + \frac{Gm_2}{(R_2-R_1-r(t))^2}$ by $r'(t)$ and integrate:

$$\int_0^{t_0} r'(t)r''(t)dt = -\int_0^{t_0} \frac{Gm_1 r'(t)dt}{(R_1+r(t))^2} + \int_0^{t_0} \frac{Gm_2 r'(t)dt}{(R_2-R_1-r(t))^2}$$

Then LHS $= (r'(t_0))^2/2 - (r'(0))^2/2 = r'(t_0)^2/2 - v_0^2/2$ because $r'(0) = v_0$. Similarly

$$\text{RHS} = \left.\frac{Gm_1}{R_1+r(t)} + \frac{Gm_2}{R_2-R_1-r(t)}\right|_{t=0}^{t=t_0}$$

simplifies to

$$\text{RHS} = F(r^*) - F(0)$$

Then LHS = RHS becomes

$$r'(t_0)^2/2 - v_0^2/2 = F(r^*) - F(0)$$

which is the claimed identity. ■

6. Verify using the previous exercise that $r'(t_0) = 0$ implies

$$v_0^* = \sqrt{2(F(0) - F(r^*))}.$$

7. Verify by hand calculation that $v_0^* \approx 11067.31016$ meters per second.

Solution: Let $F(r) = \dfrac{Gm_1}{R_1 + r} + \dfrac{Gm_2}{R_2 - R_1 - r}$. Use Exercise 6:

$$\begin{aligned} v_0^* &= \sqrt{2(F(0) - F(r^*))} \\ &= \sqrt{2\left(\frac{Gm_1}{R_1} + \frac{Gm_2}{R_2 - R_1} - \frac{Gm_1}{R_1 + r^*} - \frac{Gm_2}{R_2 - R_1 - r^*}\right)} \end{aligned}$$

The constants are

```
G:=6.6726e-11:  m1:=5.975e24:  m2:=7.36e22:
R1:=6.378e6:  R2:=3.844e8:  rstar:=339620820
```

and then by calculator $v_0^* \approx 11067.32755$ meters per second.

8. Argue by mathematical proof that $F(r)$ is not minimized at the endpoints of the interval $0 \le r \le R$.

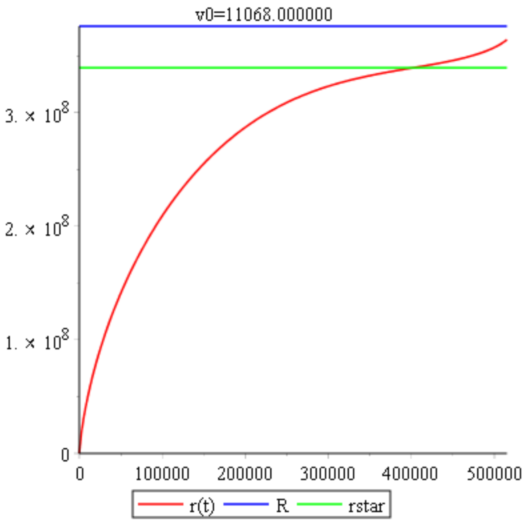
Numerical Experiments

Assume values given in the text for physical constants. Perform the given experiment with numerical software on initial value problem (1), page 260. The cases when $v_0 > v_0^*$ escape the earth, while the others fall back to earth.

9. RKF45 solver, $v_0 = 11068$, $T = 515000$. Plot the solution on $0 \le t \le T$.

Solution: Code results:

```
v0=11068, T=515000.00
Moon at distance R=376282000.00 (blue)
Acceleration=0 at r=rstar (green)
```



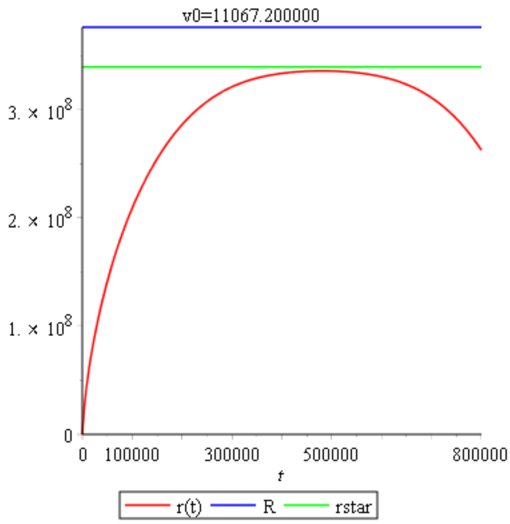

```
# MAPLE: Numeric, RKF45 method, Exercise 9
 G:=6.6726e-11: m1:=5.975e24: m2:=7.36e22:
 R1:=6.378e6: R2:=3.844e8: R3:=1.74e6:
 R:=R2-R1-R3:
 ans:=[solve(-G*m1/(r+R1)^2 + G*m2/(R2-R1-r)^2=0,r)]:
 rstar:=ans[1];
 FF:=r->G*m1/(R1+r)+G*m2/(R2-R1-r):
 v0star:=sqrt(2*(FF(0)-FF(rstar)));# v0star=11067.31016
report:=proc() #rMAX,tMAX) # print maximum + time
 printf("v0=%a, T=%.2f\n",v0,T);
 printf("Moon at distance R=%.2f (blue)\n",R);
 printf("Acceleration=0 at r=rstar (green)\n");
end proc;
makePlot:=proc() local opt;global T,Y,R,rstar;
 opt:=legend=["r(t)","R","rstar"],
 color=[red,blue,green],title=sprintf("v0=%f",v0);
 plot([Y(t),R,rstar],t=0..T,opt);
end proc:
v0:=11068;T:=515000;
ic:=r(0)=0,D(r)(0)=v0:
de:=diff(r(t),t,t)=-G*m1/(r(t)+R1)^2+G*m2/(R2-R1-r(t))^2:
NS:=numeric,method=rkf45,output=listprocedure:
p:=dsolve([de,ic],r(t),NS):Y:=eval(r(t),p):
DY:=eval(diff(r(t),t),p):
makePlot();report();
```

10. Stiff solver, $v_0 = 11068$, $T = 515000$. Plot the solution on $0 \le t \le T$.

11. RKF45 solver, $v_0 = 11067.2$, $T = 800000$. Plot the solution on $0 \leq t \leq T$.

Solution: Results:

```
v0=11067.2, T=800000.00
Moon at distance R=376282000.00 (blue)
Acceleration=0 at r=rstar (green)
```

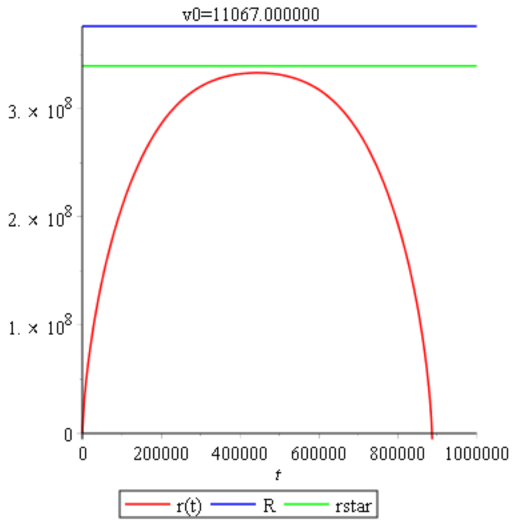


12. Stiff solver, $v_0 = 11067.2$, $T = 800000$. Plot the solution on $0 \leq t \leq T$.

13. RKF45 solver, $v_0 = 11067$, $T = 1000000$. Plot the solution on $0 \leq t \leq T$.

Solution: Results:

```
v0=11067.2, T=1000000.00
Moon at distance R=376282000.00 (blue)
Acceleration=0 at r=rstar (green)
```

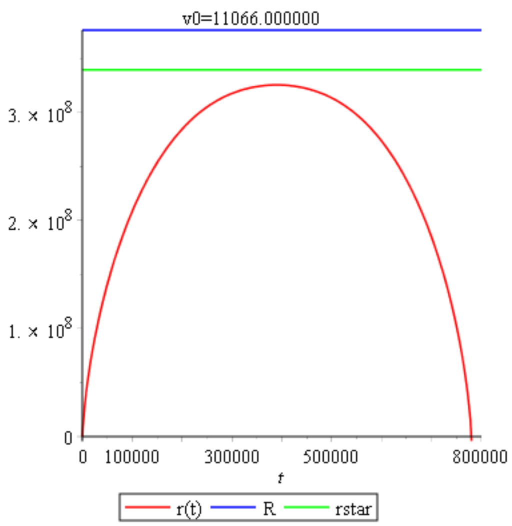


14. Stiff solver, $v_0 = 11067$, $T = 1000000$. Plot the solution on $0 \le t \le T$.

15. RKF45 solver, $v_0 = 11066$, $T = 800000$. Plot the solution on $0 \le t \le T$.

Solution: Results:

```
v0=11066, T=800000.00
Moon at distance R=376282000.00 (blue)
Acceleration=0 at r=rstar (green)
```

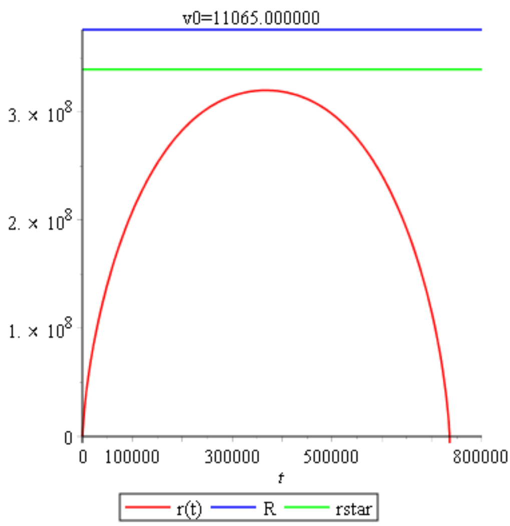


16. Stiff solver, $v_0 = 11066$, $T = 800000$. Plot the solution on $0 \le t \le T$.

17. RKF45 solver, $v_0 = 11065$. Find a suitable value T which shows that the projectile falls back to earth, then plot the solution on $0 \le t \le T$.

Solution: Results:

```
v0=11065, T=800000.00
Moon at distance R=376282000.00 (blue)
Acceleration=0 at r=rstar (green)
```

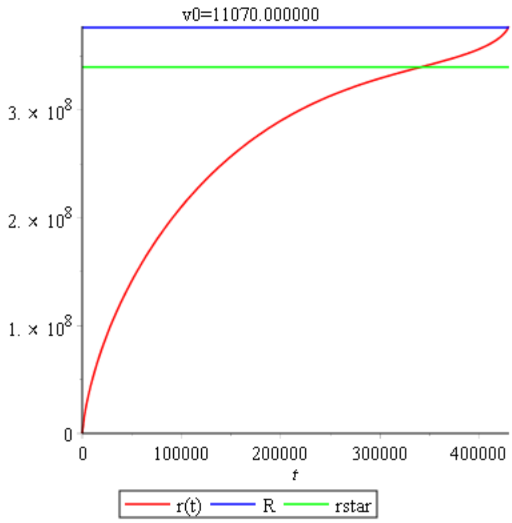


18. Stiff solver, $v_0 = 11065$. Find a suitable value T which shows that the projectile falls back to earth, then plot the solution on $0 \le t \le T$.

19. RKF45 solver, $v_0 = 11070$. Find a suitable value T which shows that the projectile falls to the moon, then plot the solution on $0 \le t \le T$.

Solution: Results:

```
v0=11070, T=430000
Moon at distance R=376282000.00 (blue)
Acceleration=0 at r=rstar (green)
```



20. Stiff solver, $v_0 = 11070$. Find a suitable value T which shows that the projectile falls to the moon, then plot the solution on $0 \le t \le T$.

4.6 Skydiving

Terminal Velocity

Assume force $F(v) = av + bv^2 + cv^3$ and $g = 32$, $m = 160/g$. Using computer assist, find the terminal velocity v_∞ from the velocity model $v' = g - \frac{1}{m}F(v)$, $v(0) = 0$.

1. $a = 0$, $b = 0$ and $c = 0.0002$.

Solution: The equilibrium solution is $v = 92.83177667$.

```
# MAPLE: Terminal velocity, Exercise 1
 F:=v->a*v+b*v^2+c*v^3;
 H:=v->subs(m=160/g,g=32,a=0,b=0,c=0.0002,G(v));
 solve(H(v)=0,v);
 # 92.83, -46.42+80.3*I, -46.42-80.39*I
```

2. $a = 0$, $b = 0$ and $c = 0.00015$.

3. $a = 0$, $b = 0.0007$ and $c = 0.00009$.

Solution: $v = 118.6034740$.

4. $a = 0$, $b = 0.0007$ and $c = 0.000095$.

5. $a = 0.009$, $b = 0.0008$ and $c = 0.00015$.

Solution: $v = 100.2350541$.

6. $a = 0.009$, $b = 0.00075$ and $c = 0.00015$.

7. $a = 0.009$, $b = 0.0007$ and $c = 0.00009$.

Solution: $v = 118.3342112$.

8. $a = 0.009$, $b = 0.00077$ and $c = 0.00009$.

9. $a = 0.009$, $b = 0.0007$ and $c = 0$.

Solution: $v = 471.7060907$ because $v'(0) = g > 0$.

```
# MAPLE: Terminal velocity, Exercise 9
 F:=v->a*v+b*v^2+c*v^3;
 H:=v->subs(m=160/g,g=32,a=0.009,b=0.0007,c=0.0,G(v));
 solve(H(v)=0,v);
 # -484.5632335, 471.7060907
 p:=dsolve([diff(v(t),t)=H(v(t)),v(0)=0],v(t));
 limit(rhs(p),t=infinity);
 # 45/7+(5/7)*sqrt(448081) = 471.7060907
```

10. $a = 0.009$, $b = 0.00077$ and $c = 0$.

Numerical Experiment

Let $F(v) = av + bv^2 + cv^3$ and $g = 32$. Consider the skydiver problem $mv'(t) = mg - F(v)$ and constants m, a, b, c supplied below. Using computer assist, apply a numerical method to produce a table for the elapsed time t, the velocity $v(t)$ and the distance $x(t)$. The table must end at $x(t) \approx 10000$ feet, which determines the flight time.

11. $m = 160/g$, $a = 0$, $b = 0$ and $c = 0.0002$.

Solution: A possible table:

t	X(t)	V(t)
0.00	0.00	0.00
0.50	4.00	15.98
1.00	15.94	31.68
1.50	35.52	46.43
2.00	62.07	59.39
2.50	94.50	69.90
3.00	131.52	77.76
3.50	171.86	83.24
4.00	214.46	86.87
4.50	258.52	89.19
5.00	303.50	90.62
5.50	349.05	91.50
6.00	394.94	92.03
6.50	441.05	92.35
7.00	487.28	92.55
7.50	533.58	92.66
8.00	579.93	92.73
8.50	626.31	92.77
9.00	672.70	92.80
9.50	719.10	92.81
10.00	765.51	92.82
10.50	811.92	92.82
11.00	858.33	92.83
11.50	904.75	92.83
12.00	951.16	92.83
12.50	997.58	92.83
13.00	1043.99	92.83

```
# Maple: Numerical experiment, skydiving Exercise 11
dive:=proc(w,a,b,c,n)
 global f,X,V,p,inc;
 local de1,de2,ic,fmt,opts;
 f:=unapply(32 - (32/w)*(a*v+b*v^2+c*v^3),v);
 de1:=diff(x(t),t)=v(t); de2:=diff(v(t),t)=f(v(t));
 ic:=x(0)=0,v(0)=0;opts:=numeric,output=listprocedure:
 p:=dsolve({de1,de2,ic},[x(t),v(t)],opts);
 X:=eval(x(t),p); V:=eval(v(t),p);
 fmt:="%10.2f  %10.2f  %10.2f\n";
 seq(printf(fmt,inc*t,X(inc*t),V(inc*t)),t=0..n);
end proc:
inc:=0.5;dive(160,0.0,0.0,0.0002,26);
```

12. $m = 160/g$, $a = 0$, $b = 0$ and $c = 0.00015$.

13. $m = 130/g$, $a = 0$, $b = 0.0007$ and $c = 0.00009$.

Solution: Code:

`inc:=0.4;dive(130,0.0,0.0007,0.00009,28);`

Last line of the table:

11.20 1005.21 110.47

14. $m = 130/g$, $a = 0$, $b = 0.0007$ and $c = 0.000095$.

15. $m = 180/g$, $a = 0.009$, $b = 0.0008$ and $c = 0.00015$.

Solution: Code:

`inc:=0.4;dive(180,0.009,0.0008,0.00015,29);`

Last line of the table:

11.60 1003.08 104.32

16. $m = 180/g$, $a = 0.009$, $b = 0.00075$ and $c = 0.00015$.

17. $m = 170/g$, $a = 0.009$, $b = 0.0007$ and $c = 0.00009$.

Solution: Code:

`inc:=0.4;dive(170,0.009,0.00077,0.00009,27);`

Last line of the table:

10.80 1024.89 120.45

18. $m = 170/g$, $a = 0.009$, $b = 0.00077$ and $c = 0.00009$.

19. $m = 200/g$, $a = 0.009$, $b = 0.0007$ and $c = 0$.

Solution: Code:

```
inc:=0.4;dive(200,0.009,0.0007,0.0,21);
```

Last line of the table:

8.40 1080.24 246.84

20. $m = 200/g$, $a = 0.009$, $b = 0.00077$ and $c = 0$.

Flight Time

Let $F(v) = av + bv^2 + cv^3$ and $g = 32$. Consider the skydiver problem $mv'(t) = mg - F(v)$ with constants m, a, b, c supplied below. Using computer assist, apply a numerical method to find accurate values for the flight time to 10,000 feet and the time required to reach terminal velocity.

21. $mg = 160$, $a = 0.0095$, $b = 0.0007$ and $c = 0.000092$.

Solution: Reaches 10,000 feet in 85.4 seconds. Terminal velocity = 117.5 ft/sec.

```
# Maple: Flight time, Exercise 21
skydiveIvp:=proc(w,a,b,c)
 global f,X,V,p;
 local de1,de2,ic,fmt,opts;
 f:=unapply(32 - (32/w)*(a*v+b*v^2+c*v^3),v);
 de1:=diff(x(t),t)=v(t); de2:=diff(v(t),t)=f(v(t));
 ic:=x(0)=0,v(0)=0;opts:=numeric,output=listprocedure:
 p:=dsolve({de1,de2,ic},[x(t),v(t)],opts);
 X:=eval(x(t),p); V:=eval(v(t),p);
end proc:
skydiveIvp(160,0.0095,0.0007,0.000092);# define X,V,f
plot(X,0..100);# Locate approx root = 80
x1:=fsolve(X(t)=10000,t=80);# 87.35197951
v1:=fsolve(f(v)=0,v);# 117.4934273
```

22. $mg = 160$, $a = 0.0097$, $b = 0.00075$ and $c = 0.000095$.

23. $mg = 240$, $a = 0.0092$, $b = 0.0007$ and $c = 0$.

Solution: Reaches 10,000 feet in 29.2 seconds. Terminal velocity = 579 ft/sec.

```
# Maple: Flight time, Exercise 23
skydiveIvp(240,0.0092,0.0007,0.0);# define X,V,f
plot(X,0..100);# Locate approx root = 30
x1:=fsolve(X(t)=10000,t=30);# 29.15860533
v1:=fsolve(f(v)=0,v);# 579.0054891
```

24. $mg = 240$, $a = 0.0095$, $b = 0.00075$ and $c = 0$.

Ejected Baggage

Baggage of 45 pounds is dropped from a hovercraft at $15,000$ feet. Assume air resistance force $F(v) = av + bv^2 + cv^3$, $g = 32$ and $mg = 45$. Using computer assist, find accurate values for the flight time to the ground and the terminal velocity. Estimate the time required to reach 99.95% of terminal velocity.

25. $a = 0.0095$, $b = 0.0007$, $c = 0.00009$

Solution: Flight time to ground: 197.7 seconds. Terminal velocity: 76.4 ft/sec. Time to reach 99.95% of terminal velocity: 7.5 seconds.

```
# Maple: Ejected Baggage, Exercise 25
skydiveIvp:=proc(w,a,b,c)
 global f,X,V,p;
 local de1,de2,ic,opts;
 f:=unapply(32 - (32/w)*(a*v+b*v^2+c*v^3),v);
 de1:=diff(x(t),t)=v(t); de2:=diff(v(t),t)=f(v(t));
 ic:=x(0)=0,v(0)=0;opts:=numeric,output=listprocedure:
 p:=dsolve({de1,de2,ic},[x(t),v(t)],opts);
 X:=eval(x(t),p); V:=eval(v(t),p);
end proc:
skydiveIvp(45,0.0095,0.0007,0.00009);# define X,V,f
plot(X(t),t=0..250);# Locate approx root = 180
x1:=fsolve(X(t)=15000,t=180);# 197.7216521
v1:=fsolve(f(v)=0,v);# 76.43153427
plot(V(t),t=0..20);# Locate approx root = 10
fsolve(V(t)=99.95*v1/100,t=10);# 7.455104385
```

26. $a = 0.0097$, $b = 0.00075$, $c = 0.00009$

27. $a = 0.0099$, $b = 0.0007$, $c = 0.00009$

Solution: Flight time to ground: 197.8 seconds. Terminal velocity: 76.4 ft/sec. Time to reach 99.95% of terminal velocity: 7.5 seconds.

```
# Maple: Ejected Baggage, Exercise 27
skydiveIvp(45,0.0099,0.0007,0.00009);# define X,V,f
plot(X(t),t=0..250);# Locate approx root = 180
x1:=fsolve(X(t)=15000,t=200);# 197.7679961
v1:=fsolve(f(v)=0,v);# 76.41348454
plot(V(t),t=0..20);# Locate approx root = 8
fsolve(V(t)=99.95*v1/100,t=8);# 7.456210409
```

28. $a = 0.0099$, $b = 0.00075$, $c = 0.00009$

4.7 Lunar Lander

Lunar Lander Constant Field

Find the retrorocket activation time T and the activation height $x(T)$. Assume the constant gravitational field model. Units are miles/hour and miles/hour per hour.

1. $v_0 = 1210$, $A = 30020$.

Solution: $T = 2.418387742$ min, $x(T) = 24.38540973$ miles

```
# Maple: Constant field, Exercise 1
v0:=1210; A:=30020.0;
X:=t->-A*t^2/2+v0*t;
T:=(v0/A): (T*60.0).'min',X(T).'miles';
# 2.418387742 min, 24.38540973 miles
A1:=A*2.54*12*5280/100/3600/3600; # mks units
v1:=v0*12*2.54*5280/100/3600;     # mks units
evalf(convert(X(T),units,miles,meters));
# 39244.51283 meters
```

2. $v_0 = 1200$, $A = 30100$.

3. $v_0 = 1300$, $A = 32000$.

Solution: $T = 2.437500000$ min, $x(T) = 26.40625000$ miles

4. $v_0 = 1350$, $A = 32000$.

5. $v_0 = 1500$, $A = 45000$.

Solution: $T = 2$ min, $x(T) = 25$ miles

6. $v_0 = 1550$, $A = 45000$.

7. $v_0 = 1600$, $A = 53000$.

Solution: $T = 1.811320755$ min, $x(T) = 24.15094340$ miles

8. $v_0 = 1650$, $A = 53000$.

9. $v_0 = 1400$, $A = 40000$.

Solution: $T = 2.1$ min, $x(T) = 24.5$ miles

10. $v_0 = 1450$, $A = 40000$.

Lunar Lander Variable Field

Find the retrorocket activation time T and the activation height $x(T)$. Assume the variable gravitational field model and *mks* units.

11. $v_0 = 540.92$, $g_1 = 5.277$.

Solution: Activation height = 24.61 miles, activation time = 2.449 minutes

```
# Maple: Variable field, Exercise 11
 v0:=540.92; g0:=G*M/R^2: g1:=5.277;
 M:=7.35* 10^(22);R:=1.74* 10^6;G:=6.6726* 10^(-11);
eq:= -(v0^2/2) + g1*H + G*M/(R+H) - G*M/R=0:
HH:=[solve(eq,H)][1];  # HH := 39612.87725 meters
de:=diff(x(t),t,t) = -g1 + M*G/(R+HH-x(t))^2;
ic:= x(0)=0, D(x)(0)=v0;
p:=dsolve({de,ic},x(t),numeric):
X:=t->evalf(rhs(p(t)[2])):
V:=t-> evalf(rhs(p(t)[3])):
plot('V(t)',t=0..300);# Locate zero of x' approx t=145
TT1:=fsolve('V(t)'=0,t=145): TT:=TT1/60:
TT1.'seconds', TT.'minutes';
  # 146.9421397 seconds, 2.449035662 minutes
X(TT1).'meters', ((X(TT1)*100/2.54)/12/5280).'miles';
  # 39612.8828293276 meters, 24.6143042301382 miles
```

12. $v_0 = 536.45$, $g_1 = 5.288$.

13. $v_0 = 581.15$, $g_1 = 5.517$.

Solution: Activation height = 26.66 miles, activation time = 2.47 minutes

14. $v_0 = 603.504$, $g_1 = 5.5115$.

15. $v_0 = 625.86$, $g_1 = 5.59$.

Solution: Activation height = 30.32 miles, activation time = 2.61 minutes

16. $v_0 = 603.504$, $g_1 = 5.59$.

17. $v_0 = 581.15$, $g_1 = 5.59$.

Solution: Activation height = 26.18 miles, activation time = 2.42 minutes

18. $v_0 = 670.56$, $g_1 = 6.59$.

19. $v_0 = 670.56$, $g_1 = 6.83$.

Solution: Activation height = 26.61 miles, activation time = 2.13 minutes

20. $v_0 = 715.26$, $g_1 = 7.83$.

Distinguishing Models

The constant field model (**1**) page 272 and the variable field model (**2**) page 273 are verified here to be distinct, by example. Find the retrorocket activation times T_1, T_2 and the activation heights $x_1(T_1)$, $x_2(T_2)$ for the two models (**1**), (**2**). Relations $A = g_1 - g_0$ and $g_0 = GM/R^2$ apply to compute g_1 for the variable field model.

21. $v_0 = 1200$ mph, $A = 10000$ mph/h. Answer: 72, 66.91 miles.

Solution:
Constant field: 7.2 minutes, 72 miles.
Variable field: 6.85 minutes, 66.91 miles.

```
# Maple: Constant field, book example
v0_CFM:=1200: A_CFM:=10000: # Constant field model values
X:=t->-A_CFM*t^2/2+v0_CFM*t;
T:=(v0_CFM/A_CFM): (T*60.0).'minutes',X(T).'miles';
# 7.2 minutes, 72 miles
# Maple: Variable field, Exercise 21
v0_CFM:=1200: A_CFM:=10000:
cf:=1*5280*12*2.54/100/3600; # mi/h to m/s
v0:=v0_CFM*cf; A:=A_CFM*cf/3600;
g0:=G*M/R^2: g1:=A+g0;
eq:= -(v0^2/2) + g1*H + G*M/(R+H) - G*M/R=0:
HH:=[solve(eq,H)][1];# 107685.7059
de:=diff(x(t),t,t) = -g1 + M*G/(R+HH-x(t))^2;
ic:= x(0)=0, D(x)(0)=v0;
p:=dsolve({de,ic},x(t),numeric):
X:=t->evalf(rhs(p(t)[2])):
V:=t-> evalf(rhs(p(t)[3])):
plot('V(t)',t=0..500);# Locate zero of x' approx t=410
TT1:=fsolve('V(t)'=0,t=410): TT:=TT1/60:
TT1.'seconds', TT.'minutes';
X(TT1).'meters', ((X(TT1)*100/2.54)/12/5280).'miles';
# 6.85 min, 66.91 miles
```

22. $v_0 = 1200$ mph, $A = 12000$ mph/h. Answer: 60, 56.9 miles.

23. $v_0 = 1300$ mph, $A = 10000$ mph/h. Answer: 84.5, 74.23 miles.

Solution:
Constant field: 7.8 minutes, 84.5 miles.
Variable field: 5.79 minutes, 74.23 miles.

24. $v_0 = 1300$ mph, $A = 12000$ mph/h. Answer: 76.82, 71.55 miles.

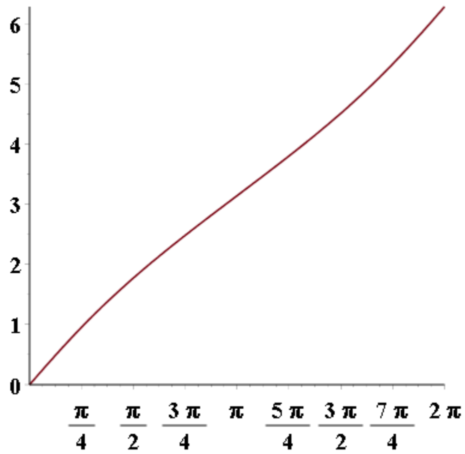
4.8 Comets

Eccentric Anomaly for the Planets

Make a plot of the eccentric anomaly $E(M)$ on $0 \leq M \leq 2\pi$.

1. Mercury, $e = 0.2056$

Solution:

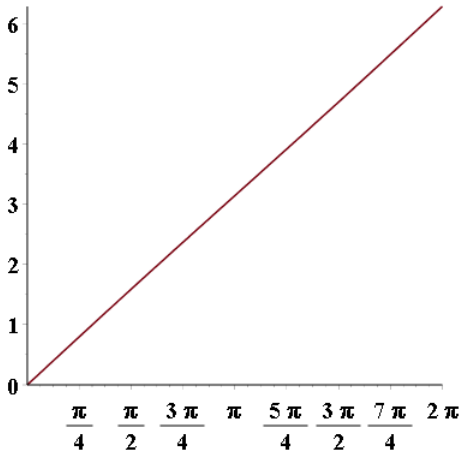



```
# Eccentric anomoly Mercury, Exercise 1
e:=0.2056:de:=diff(y(x),x)=1/(1-e*cos(y(x))); ic:=y(0)=0;
p:=dsolve({de,ic},numeric,output=listprocedure);
EE := eval(y(x),p):
plot(EE,0..2*Pi);
```

2. Venus, $e = 0.0068$

3. Earth, $e = 0.0167$

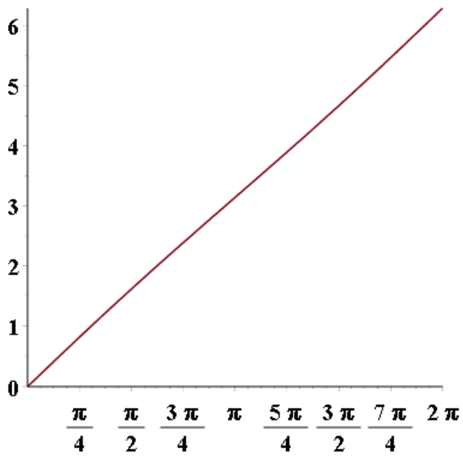
Solution:



4. Mars, $e = 0.0934$

5. Jupiter, $e = 0.0483$

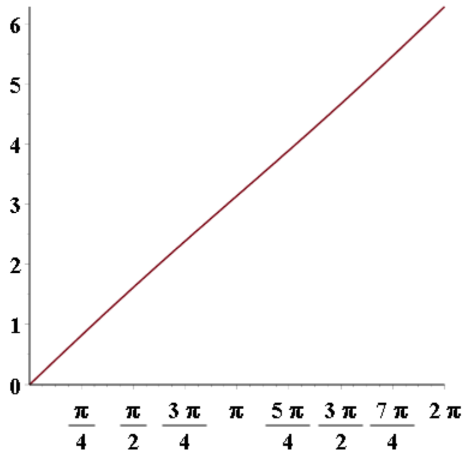
Solution:



6. Saturn, $e = 0.0560$

7. Uranus, $e = 0.0461$

Solution:



8. Neptune, $e = 0.0097$

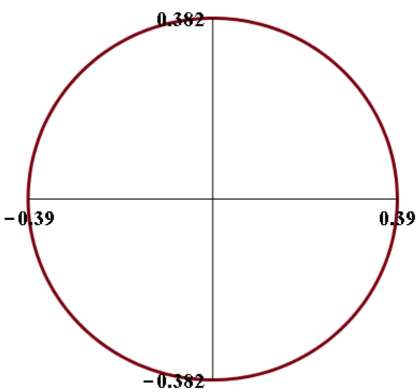
Elliptic Path of the Planets

Make a plot of the elliptic path of each planet, using constrained scaling with the given major semi-axis A (in astronomical units AU). The equations:

$$\begin{aligned} x(M) &= A\cos(E(M)), \\ y(M) &= A\sqrt{1-e^2}\sin(E(M)) \end{aligned}$$

9. Mercury, $e = 0.2056$, $A = 0.39$

Solution:

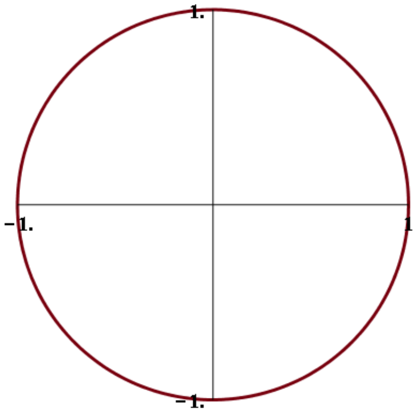


```
# Elliptic Path of the Planets, Exercise 9
e:=0.2056:A:=0.39:EE := unapply(RootOf(_Z-M-e*sin(_Z)),M);
Ex:=A*cos(EE(M)):Ey:=A*sqrt(1-e^2)*sin(EE(M)):
opt:=font=[courier,bold,16],thickness=3,tickmarks=[2,2],
scaling=constrained;
plot([Ex,Ey,M=0..2*Pi],opt);
```

10. Venus, $e = 0.0068$, $A = 0.72$

11. Earth, $e = 0.0167$, $A = 1$

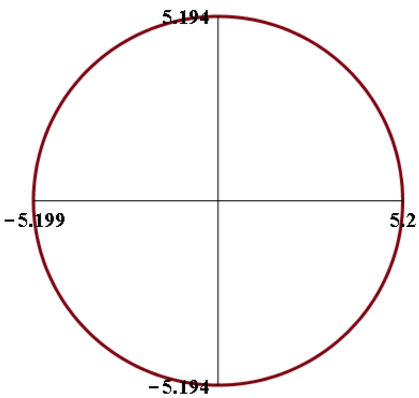
Solution:



12. Mars, $e = 0.0934$, $A = 1.52$

13. Jupiter, $e = 0.0483$, $A = 5.20$

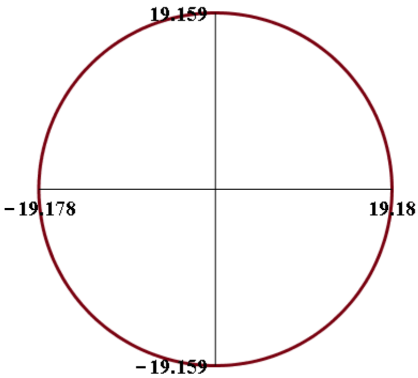
Solution:



14. Saturn, $e = 0.0560$, $A = 9.54$

15. Uranus, $e = 0.0461$, $A = 19.18$

Solution:



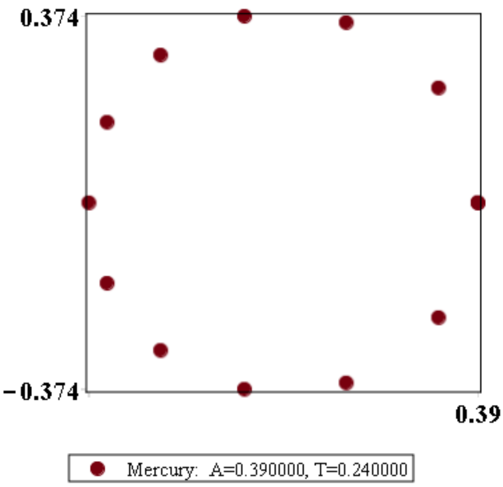
16. Neptune $e = 0.0097$, $A = 30.06$

Planet Positions

Make a plot with at least 8 planet positions displayed. Use constrained scaling with major semi-axis A in the plot. Display the given major semi-axis A and period T in the legend.

17. Mercury, $e = 0.2056$, $A = 0.39$ AU, $T = 0.24$ earth-years

Solution:

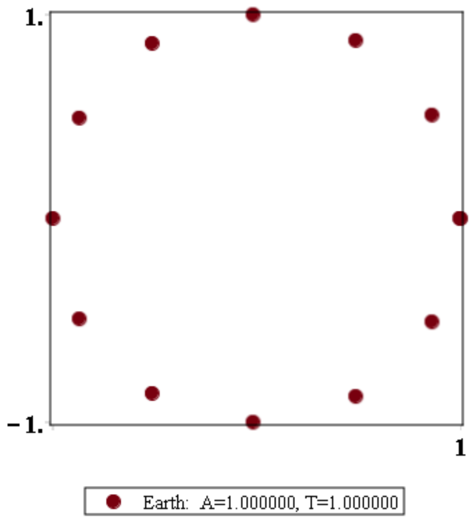


```
# Planet Positions, Exercise 17
e:=0.2056:A:=0.39:T:=0.24:planet:="Mercury":
de:=diff(y(x),x)=1/(1-e*cos(y(x))); ic:=y(0)=0;
p:=dsolve({de,ic},numeric,output=listprocedure);
EE := eval(y(x),p):
Ex:=unapply(A*cos(EE(M)),M):
Ey:=unapply(A*sqrt(1-e^2)*sin(EE(M)),M):
opts:=font=[courier,bold,16],thickness=3,
tickmarks=[2,2],scaling=constrained,axes=boxed,
symbol=solidcircle,style=point,symbolsize=22,
legend=sprintf("%s:  A=%f, T=%f",planet,A,T);;
snapshots:=seq([Ex(2*n*Pi/12),Ey(2*n*Pi/12)],n=0..12):
plot([snapshots],opts);
```

18. Venus, $e = 0.0068$, $A = 0.72$ AU, $T = 0.62$ earth-years

19. Earth, $e = 0.0167$, $A = 1$ AU, $T = 1$ earth-years

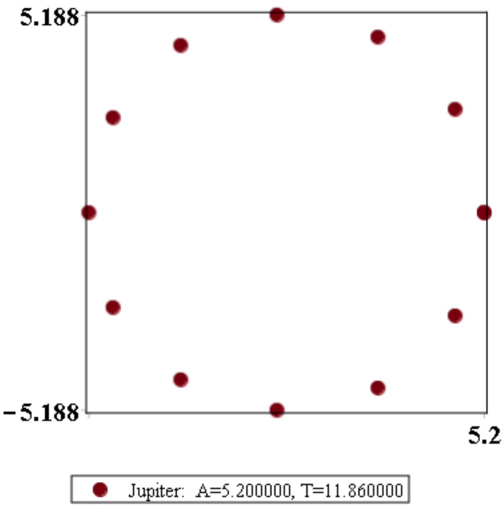
Solution:



20. Mars, $e = 0.0934$, $A = 1.52$ AU, $T = 1.88$ earth-years

21. Jupiter, $e = 0.0483$, $A = 5.20$ AU, $T = 11.86$ earth-years

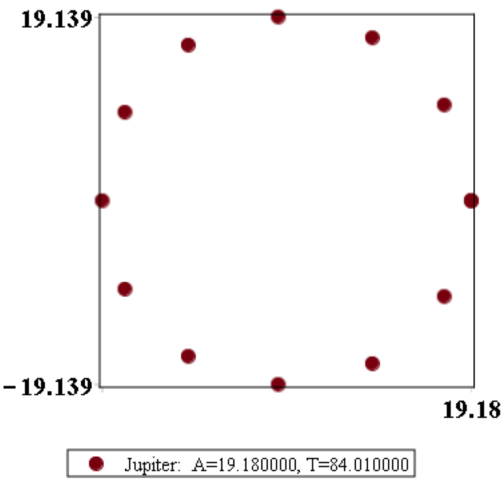
Solution:



22. Saturn, $e = 0.0560$, $A = 9.54$ AU, $T = 29.46$ earth-years

23. Uranus, $e = 0.0461$, $A = 19.18$ AU, $T = 84.01$ earth-years

Solution:



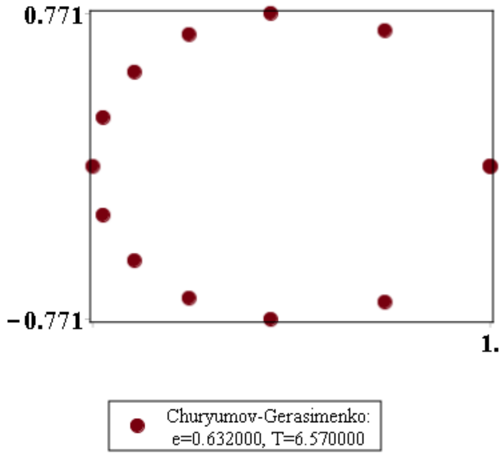
24. Neptune $e = 0.0097$, $A = 30.06$ AU, $T = 164.8$ earth-years

Comet Positions

Make a plot with at least 8 comet positions displayed. Use constrained scaling with major-semiaxis 1 in the plot. Display the given eccentricity e and period T in the legend.

25. Churyumov-Gerasimenko orbits the sun every 6.57 earth-years. Discovered in 1969. Eccentricity $e = 0.632$.

Solution:

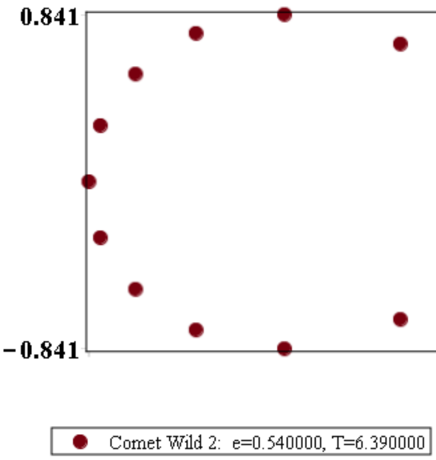



```
# Comet Positions, Exercise 25
e:=0.632:T:=6.57:comet:="Churyumov-Gerasimenko":
de:=diff(y(x),x)=1/(1-e*cos(y(x))); ic:=y(0)=0;
p:=dsolve({de,ic},numeric,output=listprocedure);
EE := eval(y(x),p):
Ex:=unapply(cos(EE(M)),M):
Ey:=unapply(sqrt(1-e^2)*sin(EE(M)),M):
opts:=font=[courier,bold,16],thickness=3,tickmarks=[2,2],
scaling=constrained,axes=boxed,symbol=solidcircle,
style=point,symbolsize=22,
legend=sprintf("%s:  e=%f, T=%f",comet,e,T);;
snapshots:=seq([Ex(2*n*Pi/12),Ey(2*n*Pi/12)],n=0..12):
plot([snapshots],opts);
```

26. Comet Wirtanen was the original target of the Rosetta space mission. This comet was discovered in 1948. The comet orbits the sun once every 5.46 earth-years. Eccentricity $e = 0.652$.

27. Comet Wild 2 was discovered in 1978. The comet orbits the sun once every 6.39 earth-years. Eccentricity $e = 0.540$.

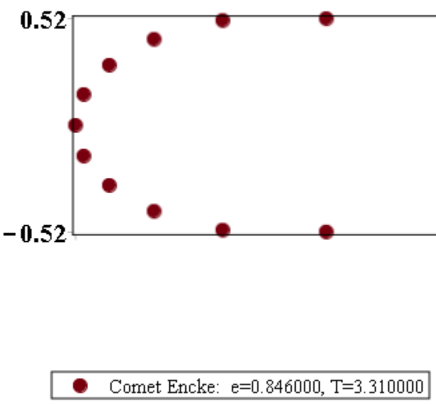
Solution:



28. Comet Biela was discovered in 1772. It orbits the sun every 6.62 earth-years. Eccentricity $e = 0.756$.

29. Comet Encke was discovered in 1786. It orbits the sun each 3.31 earth-years. Eccentricity $e = 0.846$.

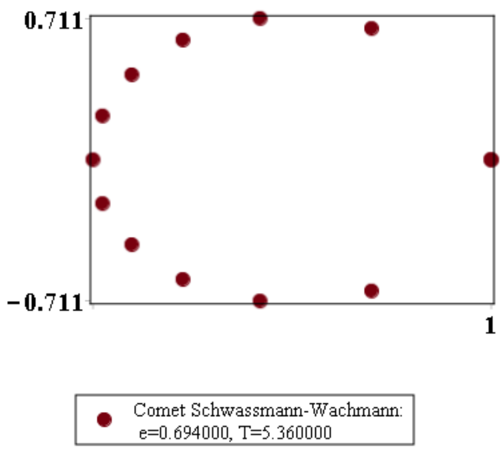
Solution:



30. Comet Giacobini-Zinner, discovered in 1900, orbits the sun each 6.59 earth-years. Eccentricity $e = 0.708$.

31. Comet Schwassmann-Wachmann, discovered in 1930, orbits the sun every 5.36 earth-years. Eccentricity $e = 0.694$.

Solution:



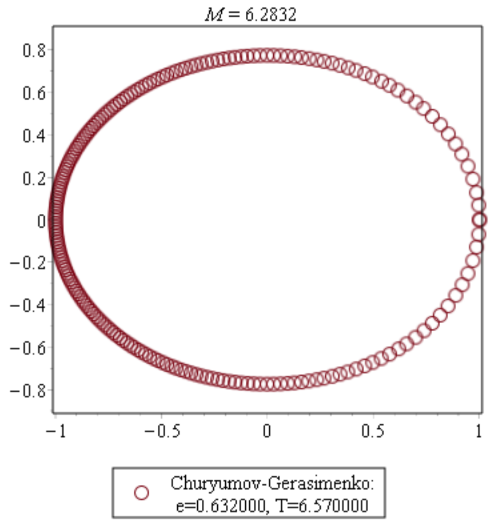
32. Comet Swift-Tuttle was discovered in 1862. It orbits the sun each 120 earth-years. Eccentricity $e = 0.960$.

Comet Animations

Make an animation plot of comet positions. Use constrained scaling with major-semiaxis 1 in the plot. Display the given period T and eccentricity e in the legend.

33. Comet Churyumov-Gerasimenko
$T = 6.57$, $e = 0.632$.

Solution:

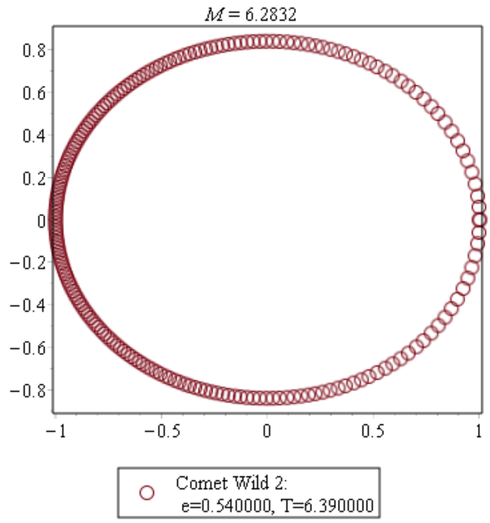



```
# Comet animation, Exercise 33
e:=0.632:T:=6.57:comet:="Churyumov-Gerasimenko":
de:=diff(y(x),x)=1/(1-e*cos(y(x))): ic:=y(0)=0:
p:=dsolve({de,ic},numeric,output=listprocedure):
EE := eval(y(x),p):
xt:=cos(EE(M)):yt:=sqrt(1-e^2)*sin(EE(M)):
opts:=view=[-1..1,-0.9..0.9],frames=2,axes=none,
scaling=constrained,axes=boxed,style=point,
symbolsize=22,symbol=circle,thickness=3,
legend=sprintf("%s: \n e=%f, T=%f",comet,e,T);
plots[animatecurve]([xt,yt,M=0..2*Pi],opts);
```

34. Comet Wirtanen
$T = 5.46$, $e = 0.652$.

35. Comet Wild 2
$T = 6.39$, $e = 0.540$.

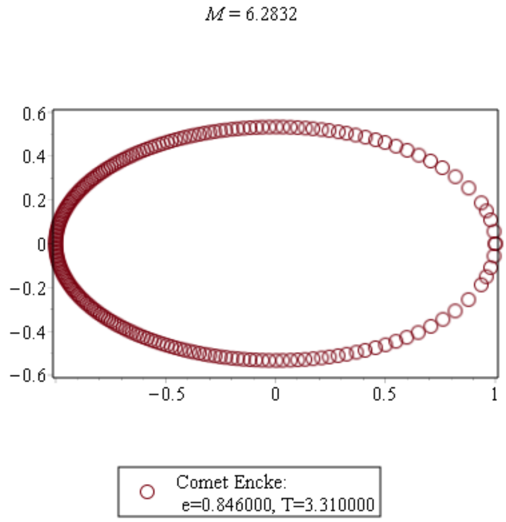
Solution:



36. Comet Biela
$T = 6.62$, $e = 0.756$.

37. Comet Encke
$T = 3.31$, $e = 0.846$.

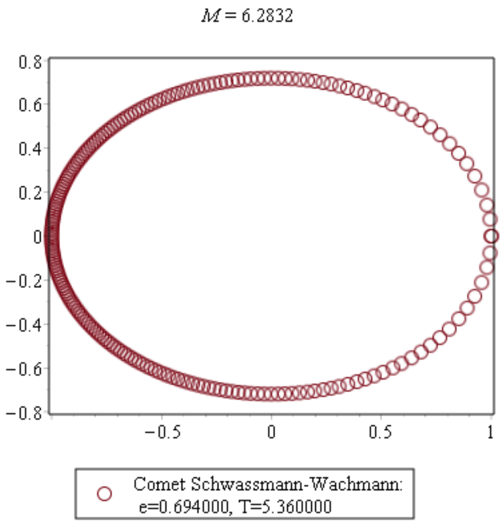
Solution:



38. Comet Giacobini-Zinner
$T = 6.59$, $e = 0.708$.

39. Comet Schwassmann-Wachmann
$T = 5.36$, $e = 0.694$.

Solution:



40. Comet Swift-Tuttle
$T = 120$, $e = 0.960$.

4.9 Fish Farming

Constant Logistic Harvesting

The model

$$x'(t) = kx(t)(M - x(t)) - h$$

can be converted to the logistic model

$$y'(t) = (a - by(t))y(t)$$

by a change of variables. Find the change of variables $y = x + c$ for the following pairs of equations.

1. $x' = -3x^2 + 8x - 5$,
$y' = (2 - 3y)y$

Solution: A way to find $y = x - 1$ is to factor $-3x^2 + 8x - 5 = (-3x + 5)(x - 1)$ and then choose $y = x - 1$. After enough experience with finding changes of variables, this will become the preferred method.

A general technique for finding the change of variables is to substitute $x = y - c$ into the differential equation. Then

$$\begin{aligned} y' &= x' + 0 \\ &= -3x^2 + 8x - 5 \\ &= -3(y - c)^2 + 11(y - c) - 14 \\ &= -3y^2 + (6c + 8)y + (-3c^2 - 11c - 14). \end{aligned}$$

Equation $y' = (2 - 3y)y$ holds provided:

$$\begin{aligned} &6c + 8 = 2, \\ &-3c^2 - 11c - 14 = 0. \end{aligned}$$

Equation $6c + 8 = 2$ gives $c = -1$. Equation $-3c^2 - 11c - 14 = 0$ holds for $c = -1$. **Conclusion**: $y = x + c = x - 1$.

2. $x' = -2x^2 + 11x - 14$,
$y' = (3 - 2y)y$

3. $x' = -5x^2 - 19x - 18$,
$y' = (1 - 5y)y$

Solution: Factor $-5x^2 - 19x - 18 = -(5x + 9)(x + 2)$, then let $y = x + 2$ to get $y' = x' = -(5x + 9)y = -(5y - 10 + 9)y = (1 - 5y)y$.

4. $x' = -x^2 + 3x + 4$,
$y' = (5 - y)y$

Periodic Logistic Harvesting

The periodic harvesting model

$$x'(t) = 0.8x(t)\left(1 - \frac{x(t)}{780500}\right) - H(t)$$

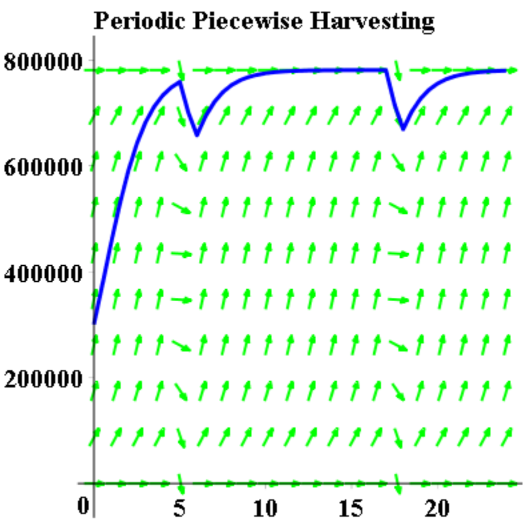
is considered with H defined by

$$H(t) = \begin{cases} 0 & 0 < t < 5, \\ H_0 & 5 < t < 6, \\ 0 & 6 < t < 17, \\ H_0 & 17 < t < 18, \\ 0 & 18 < t < 24. \end{cases}$$

This project makes as computer graph of the solution on $0 < t < 24$ for various values of H_0 and $x(0)$. See Figures 17 and 18 and the corresponding examples.

5. $H_0 = 156100$, $P(0) = 300000$

Solution:

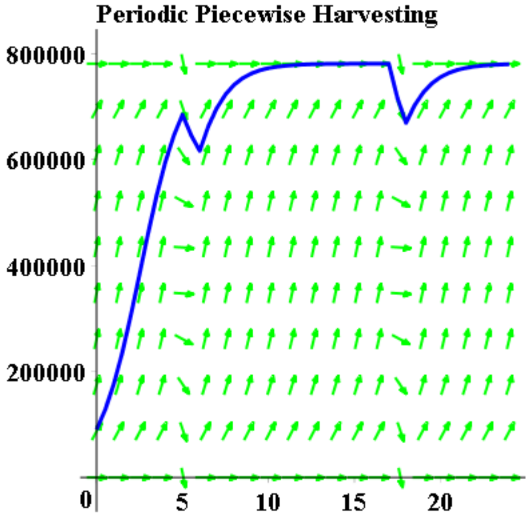



```
# Periodic Logistic Harvesting, Exercise 5
de:=diff(x(t),t)=r*(1-x(t)/M)*x(t)-H(t);
r:=0.8:M:=780500:H0:=156100:x0:=300000:
H:=t->H0*piecewise(t<5,0,t<6,1,t<12+5,0,t<12+6,1,0);
with(DEtools):DEplot(de,x(t),t=0..24,x=0..M,
 [[x(0)=x0]],arrows=smalltwo,color=green,
 dirfield=[20,10],linecolor=blue,font=[courier,bold,16],
 title="Periodic Piecewise Harvesting",tickmarks=[6,6]);
```

6. $H_0 = 156100$, $P(0) = 800000$

7. $H_0 = 800100$, $P(0) = 90000$

Solution:



8. $H_0 = 800100$, $P(0) = 100000$

von Bertalanffy Equation

Karl Ludwig von Bertalanffy (1901-1972) derived in 1938 the equation

$$\frac{dL}{dt} = r_B(L_\infty - L(t))$$

from simple physiological arguments. It is a widely used growth curve, especially important in fisheries studies. The symbols:

t	time,
$L(t)$	length,
r_B	growth rate,
L_∞	expected length for zero growth.

9. Solve $\frac{dL}{dt} = 2(10 - L)$, $L(0) = 0$. The answer is the length in inches of a fish over time, with final adult size 10 inches.

Solution:

Model: $x' + px = q$ with p, q constant has shortcut solution $x = x_p + x_h$ where x_p is the equilibrium solution and $x_h = c/W$, $W =$ integrating factor.

Then $L = 10 + c/W$, $W = e^{\int 2dt} = e^{2t}$. Solve for c: $0 = L(0) = 10 + c/e^0$.

Answer: $L(t) = 10 - 10/e^{2t}$. Symbol $L_\infty = 10$ = equilibrium solution. Symbol $r_B = 2$ = growth rate.

10. Solve von Bertalanffy's equation to obtain the algebraic model

$$L(t) = L_\infty \left(1 - e^{-r_B(t-t_0)}\right).$$

11. Assume von Bertalanffy's model. Suppose field data $L(0) = 0$, $L(1) = 5$, $L(2) = 7$. Display details using Exercise 10 to arrive for $t_0 = 0$ at values $L_\infty = 25/3$ and $r_B = \ln(5/2)$.

Solution:

Model: $L(t) = L_\infty\,(1 - e^{-r_B\,t})$ because $t_0 = 0$.
Then $L(0) = 0$ holds. To satisfy the other two data items $L(1) = 5$, $L(2) = 7$ requires values for L_∞, r_B satisfying the nonlinear system of equations

$$\begin{aligned} L_\infty \left(1 - e^{-r_B\,(1)}\right) &= 5, \\ L_\infty \left(1 - e^{-r_B\,(2)}\right) &= 7. \end{aligned}$$

A computer algebra system is a reliable tool to solve these equations, giving $L_\infty = 25/3$ and $r_B = \ln(5/2)$. Rule $\ln(1/u) = -\ln(u)$ converts the `maple` answer.

```
# Maple: Bertalanffy's model with field data
eq1:=L * (1-exp(-r_B) ) = 5;
eq2:=L *(1-exp(-2*r_B)) = 7;
solve([eq1,eq2],[L,r_B]); # L = 25/3, r_B = -ln(2/5)
```

12. Assume von Bertalanffy's model with field data $L(0) = 0$, $L(1) = 10$, $L(2) = 13$. Find the expected length L_∞ of the fish.

Chapter 5

Linear Algebra

Contents

5.1 Vectors and Matrices

Fixed vectors

Perform the indicated operation(s).

1. $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Solution: $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

2. $\begin{pmatrix} 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

3. $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

Solution: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

4. $\begin{pmatrix} 2 \\ -2 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$

5. $2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Solution: $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

6. $3\begin{pmatrix} 2 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

7. $5\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

Solution: $\begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}$

8. $3\begin{pmatrix} 2 \\ -2 \\ 9 \end{pmatrix} - 5\begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$

9. $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

Solution: $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$

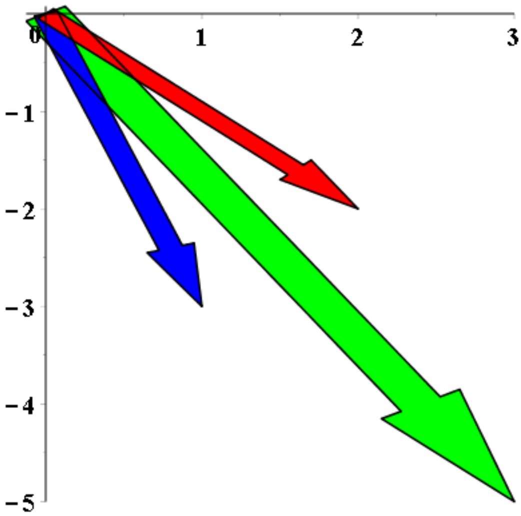
10. $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

Parallelogram Rule

Determine the resultant vector in two ways: (a) the parallelogram rule, and (b) fixed vector addition.

11. $\begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Solution: $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$




```
# Maple: Parallelogram rule
with(VectorCalculus):opts:=font=[courier,bold,16];
A:=<2,-2>;B:=<1,-3>;
PlotVector([A,B,A+B], color = [red, blue,green],opts);
```

12. $(2\vec{\imath} - 2\vec{\jmath}) + (\vec{\imath} - 3\vec{\jmath})$

13. $\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$

Solution: $\begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix}$

14. $(2\vec{\imath} - 2\vec{\jmath} + 3\vec{k}) + (\vec{\imath} - 3\vec{\jmath} - \vec{k})$

Toolkit

Let V be the data set of all fixed 2-vectors, $V = \mathcal{R}^2$. Define addition and scalar multiplication componentwise. Verify the following toolkit rules by direct computation.

15. (Commutative)
$\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$

Solution:

$$\vec{X} + \vec{Y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
$$= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$
$$\vec{Y} + \vec{X} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} y_1 + x_1 \\ y_2 + x_2 \end{pmatrix}$$

Commutativity of addition of real numbers implies the result. ■

16. (Associative)
$\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{Y} + \vec{X}) + \vec{Z}$

17. (Zero)
Vector $\vec{0}$ is defined and $\vec{0} + \vec{X} = \vec{X}$

Solution: Define $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then:

$$\vec{0} + \vec{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 + x_1 \\ 0 + x_2 \end{pmatrix}$$
$$= \vec{X} \quad ■$$

18. (Negative)
Vector $-\vec{X}$ is defined and
$\vec{X} + (-\vec{X}) = \vec{0}$

19. (Distributive I)
$k(\vec{X} + \vec{Y}) = k\vec{X} + k\vec{Y}$

Solution: The plan: expand both LHS and RHS of the identity and show they are equal.

$$\text{LHS} = k(\vec{X} + \vec{Y})$$
$$= k\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$$
$$= k\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$
$$= \begin{pmatrix} kx_1 + ky_1 \\ kx_2 + ky_2 \end{pmatrix}$$

$$\begin{aligned}\text{RHS} &= k\vec{X} + k\vec{Y} \\ &= k\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + k\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} kx_1 \\ kx_2 \end{pmatrix} + \begin{pmatrix} ky_1 \\ ky_2 \end{pmatrix} \\ &= \begin{pmatrix} kx_1 + ky_1 \\ kx_2 + ky_2 \end{pmatrix}\end{aligned}$$

Therefore, LHS = RHS by the definition of vector equality: components are equal. ■

20. (Distributive II)
$(k_1 + k_2)\vec{X} = k_1\vec{X} + k_2\vec{X}$

21. (Distributive III)
$k_1(k_2\vec{X}) = (k_1k_2)\vec{X}$

Solution: Plan: expand both LHS and RHS of the identity and show they are equal.

$$\begin{aligned}\text{LHS} &= k_1\left(k_2\vec{X}\right) \\ &= k_1\left(k_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \\ &= k_1\begin{pmatrix} k_2x_1 \\ k_2x_2 \end{pmatrix} \\ &= \begin{pmatrix} k_1k_2x_1 \\ k_1k_2x_2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= (k_1k_2)\vec{X} \\ &= (k_1k_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} k_1k_2x_1 \\ k_1k_2x_2 \end{pmatrix}\end{aligned}$$

Therefore, LHS = RHS by the definition of vector equality: components are equal. ■

22. (Identity)
$1\vec{X} = \vec{X}$

Subspaces

Verify that the given restriction equation defines a subspace S of $V = \mathcal{R}^3$. Use Theorem 5.2, page 300.

23. $z = 0$

Solution: The equation $z = 0$ is a homogeneous linear equation. The Theorem applies: S is a subspace of V.

24. $y = 0$

25. $x + z = 0$

Solution: The equation $x + z = 0$ is a homogeneous linear equation. The Theorem applies: S is a subspace of V.

26. $2x + y + z = 0$

27. $x = 2y + 3z$

Solution: The equation $x = 2y + 3z$ is a homogeneous linear equation $x - 2y - 3z = 0$. The Theorem applies: S is a subspace of V.

28. $x = 0$, $z = x$

29. $z = 0$, $x + y = 0$

Solution: Equations $z = 0$, $x + y = 0$ are homogeneous linear equations. The Theorem applies: S is a subspace of V.

30. $x = 3z - y$, $2x = z$

31. $x + y + z = 0$, $x + y = 0$

Solution: Equations $x + y + z = 0$, $x + y = 0$ are homogeneous linear equations. The Theorem applies: S is a subspace of V.

32. $x + y - z = 0$, $x - z = y$

Test S Not a Subspace

Test the following restriction equations for $V = \mathcal{R}^3$ and show that the corresponding subset S is not a subspace of V. Use Theorem 5.4 page 301 .

33. $x = 1$

Solution: Vector $\vec{0}$ is given by the equations $x = 0$, $y = 0$, $z = 0$. If $\vec{0}$ is in S, then equation $x = 1$ allows substitution of $x = 0$, resulting in the false equation $0 = 1$. Therefore, $\vec{0}$ is not in S. Theorem 5.4 applies: S is not a subspace of V. ■

34. $x + z = 1$

35. $xz = 2$

Solution: Vector $\vec{0}$ is given by the equations $x = 0$, $y = 0$, $z = 0$. If $\vec{0}$ is in S, then equation $xz = 2$ allows substitution of $x = 0$, resulting in the false equation $0 = 2$. Therefore, $\vec{0}$ is not in S. Theorem 5.4 applies: S is not a subspace of V. ■

36. $xz + y = 1$

37. $xz + y = 0$

Solution: Equation $xz + y = 0$ is nonlinear but homogeneous, therefore (1) of Theorem 5.4 does not apply. Both (2) or (3) in Theorem 5.4 will be tested instead of (1). Both (2) and (3) hold, but only one of them is required. Let's verify (2) by selecting a vector $\vec{A}$ in S for which $-\vec{A}$ violates the equation $xz + y = 0$.

Choose $\vec{A} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Then $xz + y = (1)(-1) + 1 = 0$ and $\vec{A}$ is verified to belong to S. Vector $-\vec{A} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ fails to belong to S because $xz + y = (-1)(-1) + 1 = 2 \neq 0$. Then (2) in Theorem 5.4 holds. Conclusion: S is not a subspace. ■

38. $xyz = 0$

39. $z \geq 0$

Solution: The violation is from (2) in Theorem 5.4. Choose $A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in S. Then $-A = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ fails $z \geq 0$ because $z = -1$. ■

40. $x \geq 0$ and $y \geq 0$

41. Octant I

Solution: Octant 1 is defined by $x \geq 0$. The proof parallels Exercise 39. ■

42. The interior of the unit sphere

Dot Product

Find the dot product of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

43. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

Solution: $\vec{a} \cdot \vec{b} = 2$

44. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

45. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

Solution: $\vec{a} \cdot \vec{b} = 2$

46. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

47. $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are in $\mathcal{R}^{169}$, $\vec{\mathbf{a}}$ has all 169 components 1 and $\vec{\mathbf{b}}$ has all components -1, except four, which all equal 5.

Solution: $\vec{a} \cdot \vec{b} = 169 - 4 + (5)(4) = 185$

48. $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are in $\mathcal{R}^{200}$, $\vec{\mathbf{a}}$ has all 200 components -1 and $\vec{\mathbf{b}}$ has all components -1 except three, which are zero.

Length of a Vector

Find the length of the vector $\vec{\mathbf{v}}$.

49. $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Solution: $\sqrt{1^2 + (-1)^2} = \sqrt{2}$

50. $\vec{\mathbf{v}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

51. $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Solution: $\sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

52. $\vec{\mathbf{v}} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

Shadow Projection

Find the shadow projection $d = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}}/|\vec{\mathbf{b}}|$.

53. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

Solution: $d = 1$

54. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

55. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

Solution: $d = 2/\sqrt{5}$

56. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

Projections and Reflections

Let L denote a line through the origin with unit direction $\vec{\mathbf{u}}$.

The **projection** of vector $\vec{\mathbf{x}}$ onto L is $P(\vec{\mathbf{x}}) = d\vec{\mathbf{u}}$, where $d = \vec{\mathbf{x}} \cdot \vec{\mathbf{u}}$ is the shadow projection.

The **reflection** of vector $\vec{\mathbf{x}}$ across L is $R(\vec{\mathbf{x}}) = 2d\vec{\mathbf{u}} - \vec{\mathbf{x}}$ (a generalized complex conjugate).

57. Let $\vec{\mathbf{u}}$ be the direction of the x-axis in the plane. Establish that $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$ are sides of a right triangle and P duplicates the complex conjugate operation $z \to \bar{z}$. Include a figure.

Solution: A right triangle with sides $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ has third side $\vec{\mathbf{b}} - \vec{\mathbf{a}}$. The right angle condition is verified by the Pythagorean identity $|\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2 = |\vec{\mathbf{b}} - \vec{\mathbf{a}}|^2$. $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$, which is equivalent to $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$. Let $\vec{a} = P(\vec{\mathbf{x}}) = d\vec{\mathbf{u}}$ and $\vec{\mathbf{b}} = R(\vec{\mathbf{x}}) - P(\vec{\mathbf{x}}) = d\vec{\mathbf{u}} - \vec{\mathbf{x}}$. Then:

$$\begin{aligned}\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} &= d\vec{\mathbf{u}} \cdot (d\vec{\mathbf{u}} - \vec{\mathbf{x}}) \\ &= d^2(\vec{\mathbf{u}} \cdot \vec{\mathbf{u}}) - d\vec{\mathbf{u}} \cdot \vec{\mathbf{x}} \\ &= d^2 - d^2 = 0.\end{aligned}$$

This proves that $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$ are sides of a right triangle.

Complex conjugation is duplicated by the reflection $R(\vec{\mathbf{x}})$ provided $\vec{\mathbf{u}}$ is along the x-axis, which means $\vec{\mathbf{u}} = \vec{\imath}$. Then for $\vec{\mathbf{x}} = x_1\vec{\imath} + x_2\vec{\jmath}$:

$$\begin{aligned}R(\vec{\mathbf{x}}) &= 2d\vec{\mathbf{u}} - \vec{\mathbf{x}} \\ &= 2(\vec{\mathbf{x}} \cdot \vec{\mathbf{u}})\vec{\imath} - \vec{\mathbf{x}}\end{aligned}$$

$$= 2x_1 \imath - x_1\vec{\imath} - x_2\vec{\jmath}$$
$$= x_1 \imath - x_2\vec{\jmath}$$

Let $z = x_1 + x_2 i$, $i = \sqrt{-1}$. Then $\overline{z} = x_1 - x_2 i$ is the complex conjugate of z. Complex numbers correspond to vectors by the mapping $x_1 + x_2 i \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Therefore, $z \mapsto R(\vec{\mathbf{x}})$, showing that reflections duplicate complex conjugation in the special case when L is the x-axis. ■

58. Let $\vec{\mathbf{u}}$ be any direction in the plane. Establish that $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$ are sides of a right triangle. Draw a suitable figure, which includes $\vec{\mathbf{x}}$.

59. Let $\vec{\mathbf{u}}$ be the direction of $2\vec{\imath} + \vec{\jmath}$. Define $\vec{\mathbf{x}} = 4\vec{\imath} + 3\vec{\jmath}$. Compute the vectors $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$.

Solution: Let's use fixed vectors for the computations:

$\vec{\mathbf{x}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\vec{\mathbf{u}} = c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ where $c = \frac{1}{\sqrt{5}}$

Then:

$$\begin{aligned} d &= \vec{\mathbf{x}} \cdot \vec{\mathbf{u}} \\ &= c\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ &= 11c \end{aligned}$$

$$dc = 11c^2 = 11/5$$

$$\begin{aligned} P(\vec{\mathbf{x}}) &= d\vec{\mathbf{u}} = dc\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 22/5 \\ 11/5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} R(\vec{\mathbf{x}}) &= 2d\vec{\mathbf{u}} - \vec{\mathbf{x}} = 2dc\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 4dc - 4 \\ 2dc - 3 \end{pmatrix} \\ &= \begin{pmatrix} 44/5 - 4 \\ 22/5 - 3 \end{pmatrix} \\ &= \begin{pmatrix} 24/5 \\ 7/5 \end{pmatrix} \end{aligned}$$

```
# Projections and refections, Exercise 59
X:=<4,3>;u:=c*<2,1>;c:=1/sqrt(5);
d:=X.u; P:=d*u; R:=2*d*u-X;
# P = [22/5, 11/5], R = [24/5, 7/5]
```

60. Let $\vec{\mathbf{u}}$ be the direction of $\vec{\imath} + 2\vec{\jmath}$. Define $\vec{\mathbf{x}} = 3\vec{\imath} + 5\vec{\jmath}$. Compute the vectors $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$.

Angle

Find the angle θ between the given vectors.

61. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

Solution: We will use identity $\cos(\theta) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} / \|\vec{\mathbf{a}}\| \, \|\vec{\mathbf{b}}\|$

$\|\vec{\mathbf{a}}\| = \sqrt{2}$, $\|\vec{\mathbf{b}}\| = \sqrt{4} = 2$, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 2$, $\cos(\theta) = 1/\sqrt{2} = \cos(\pi/4)$. Then $\theta = \pi/4$ is the acute angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

62. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

63. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

Solution: $\|\vec{\mathbf{a}}\| = \sqrt{2}$, $\|\vec{\mathbf{b}}\| = \sqrt{5}$, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = -3$, $\cos(\theta) = -3/\sqrt{10}$. Then $\theta = \cos^{-1}(-3/\sqrt{10}) = 2.819842099$ radians is the acute angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

64. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

65. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$.

Solution: $\|\vec{\mathbf{a}}\| = \sqrt{2}$, $\|\vec{\mathbf{b}}\| = \sqrt{6}$, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 2$, $\cos(\theta) = 2/\sqrt{12} = 1/\sqrt{3}$. Then $\theta = \cos^{-1}(2/\sqrt{12}) = 0.9553166184$ radians is the acute angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

66. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$.

67. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Solution: $\|\vec{\mathbf{a}}\| = \sqrt{6}$, $\|\vec{\mathbf{b}}\| = \sqrt{4+4+1} = 3$, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 6$, $\cos(\theta) = 6/\sqrt{54}$. Then $\theta = \cos^{-1}(6/\sqrt{54}) = 0.6154797085$ radians is the acute angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

68. $\vec{\mathbf{a}} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Matrix Multiply

Find the given matrix product or else explain why it does not exist.

69. $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Solution: $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

70. $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

71. $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution: $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

72. $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

73. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

Solution: $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

74. $\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

75. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

Solution: $\begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix}$

76. $\begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

77. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution: $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

```
# Maple: Answer check Exercise 77
A:=Matrix([[1 , 1 , 1],[1 ,-1 ,1],[1 , 0 , 0]]);
B:=Matrix([[1 , 0 , 0],[0 ,-1 ,0],[0 , 0 , 1]]);
A.B; # Matrix([[1, -1, 1], [1, 1, 1], [1, 0, 0]])
```

78. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

79. $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

Solution: $\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$

80. $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

81. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Solution: $\begin{pmatrix} 0 & 2 \\ 1 & 1 \\ -1 & 3 \end{pmatrix}$

82. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

Matrix Classification

Classify as square, non-square, upper triangular, lower triangular, scalar, diagonal, symmetric, non-symmetric. Cite as many terms as apply.

83. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Solution: square, upper triangular, lower triangular, diagonal, symmetric

84. $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

85. $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

Solution: square, non-symmetric

86. $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$

87. $\begin{pmatrix} 1 & 3 & 4 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: square, non-symmetric

88. $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

89. $\begin{pmatrix} 1 & 3 & 4 \\ 3 & 2 & 0 \\ 4 & 0 & 3 \end{pmatrix}$

Solution: square, symmetric

90. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

91. $\begin{pmatrix} i & 0 \\ 0 & 2i \end{pmatrix}$

Solution: square, upper triangular, lower triangular, diagonal, symmetric

92. $\begin{pmatrix} i & 3 \\ 3 & 2i \end{pmatrix}$

Digital Photographs

Assume integer 24-bit color encoding $x = r + (256)g + (65536)b$, which means r units **red**, g units **green** and b units **blue**. Given matrix $X = R + 256G + 65536B$, find the red, green and blue color separation matrices R, G, B. Computer assist expected.

93. $X = \begin{pmatrix} 514 & 3 \\ 131843 & 197125 \end{pmatrix}$

Solution: $R = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $G = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$

```
# Digital Photographs, Exercise 93
with(LinearAlgebra:-Modular):
X := Matrix([[514, 3], [131843, 197125]]);
R1:=Mod(2^16,X,integer);
R:=Mod(2^8,R1,integer);# R:=Matrix([[2,3],[3,5]]);
G:=(R1-R)/2^8;# G:=Matrix([[2,0],[3,2]]);
B:=(X-R-G*2^8)/2^16;# B:=Matrix([[0,0],[2,3]]);
ZERO:=R+256*G+65536*B-X;# ZERO:=Matrix([[0,0],[0,0]]);
```

94. $X = \begin{pmatrix} 514 & 3 \\ 131331 & 66049 \end{pmatrix}$

95. $X = \begin{pmatrix} 513 & 7 \\ 131333 & 66057 \end{pmatrix}$

Solution:

```
# Digital Photographs, Exercise 95
with(LinearAlgebra:-Modular):
X := Matrix([[513, 7], [131333, 66057]]);
R1:=Mod(2^16,X,integer);
R:=Mod(2^8,R1,integer);# R:=Matrix([[1,7],[5,9]]);
G:=(R1-R)/2^8;# G:=Matrix([[2,0],[1,2]]);
B:=(X-R-G*2^8)/2^16;# B:=Matrix([[0,0],[2,1]]);
ZERO:=R+256*G+65536*B-X;# ZERO:=Matrix([[0,0],[0,0]]);
```

96. $X = \begin{pmatrix} 257 & 7 \\ 131101 & 66057 \end{pmatrix}$

97. $X = \begin{pmatrix} 257 & 17 \\ 131101 & 265 \end{pmatrix}$

Solution:

```
# Digital Photographs, Exercise 95
with(LinearAlgebra:-Modular):
X := Matrix([[257, 17], [131101, 265]]);
R1:=Mod(2^16,X,integer);
R:=Mod(2^8,R1,integer);# R:=Matrix([[1,17],[29,9]]);
G:=(R1-R)/2^8;# G:=Matrix([[1,0],[0,1]]);
B:=(X-R-G*2^8)/2^16;# B:=Matrix([[0,0],[2,0]]);
ZERO:=R+256*G+65536*B-X;# ZERO:=Matrix([[0,0],[0,0]]);
```

98. $X = \begin{pmatrix} 65537 & 269 \\ 65829 & 261 \end{pmatrix}$

99. $X = \begin{pmatrix} 65538 & 65803 \\ 65833 & 7 \end{pmatrix}$

Solution:

```
# Digital Photographs, Exercise 95
with(LinearAlgebra:-Modular):
X := Matrix([[65538, 65803], [65833, 7]]);
R1:=Mod(2^16,X,integer);
R:=Mod(2^8,R1,integer);# R:=Matrix([[2,11],[41,7]]);
G:=(R1-R)/2^8;# G:=Matrix([[0,1],[1,0]]);
B:=(X-R-G*2^8)/2^16;# B:=Matrix([[1,1],[1,0]]);
ZERO:=R+256*G+65536*B-X;# ZERO:=Matrix([[0,0],[0,0]]);
```

100. $X = \begin{pmatrix} 259 & 65805 \\ 299 & 5 \end{pmatrix}$

Matrix Properties

Verify the result.

101. Let C be an $m \times n$ matrix. Let $\vec{X}$ be column i of the $n \times n$ identity I. Define $\vec{Y} = C\vec{X}$. Verify that $\vec{Y}$ is column i of C.

Solution: To prove: the entries of $\vec{\mathbf{Y}}$ are $c_{1i}, \ldots, c_{ni}$. Matrix multiply defines the entries of $C\vec{\mathbf{X}}$ to be $\sum_{j=1}^n c_{1j}x_j, \ldots, \sum_{j=1}^n c_{nj}x_j$. Because $x_j = 0$ except for $x_i = 1$, then the entries of $C\vec{\mathbf{X}}$ are $c1i, \ldots, c_{ni}$, which matches the entries of column i of matrix C. ■

102. Let A and C be an $m \times n$ matrices such that $AC = \mathbf{0}$. Verify that each column $\vec{Y}$ of C satisfies $A\vec{Y} = \vec{\mathbf{0}}$.

103. Let A be a 2×3 matrix and let $\vec{Y}_1$, $\vec{Y}_2$, $\vec{Y}_3$ be column vectors packaged into a 3×3 matrix C. Assume each column vector $\vec{Y}_i$ satisfies the equation $A\vec{Y}_i = \vec{\mathbf{0}}$, $1 \le i \le 3$. Show that $AC = \mathbf{0}$.

Solution: Let matrix $A = (a_{ij})$ be 2×3. Let matrix $C = (c_{ij})$ be 3×3. To prove: $AC = \mathbf{0}$ provided the columns $\vec{\mathbf{Y}}$ of C satisfy $A\vec{\mathbf{Y}} = \vec{\mathbf{0}}$.

Exercise 101 implies that $AC\vec{\mathbf{X}}$ is column i of AC, provided $\vec{\mathbf{X}}$ is column i of the 3×3 identity matrix. The same result implies $C\vec{\mathbf{X}} = \vec{\mathbf{Y}}$. then: $AC\vec{\mathbf{X}} = A\vec{\mathbf{Y}} = \vec{\mathbf{0}}$. The result: the columns of AC are the zero vector. ■

104. Let A be an $m \times n$ matrix and let $\vec{Y}_1, \ldots, \vec{Y}_n$ be column vectors packaged into an $n \times n$ matrix C. Assume each column vector $\vec{Y}_i$ satisfies the equation $A\vec{Y}_i = \vec{\mathbf{0}}$, $1 \le i \le n$. Show that $AC = \mathbf{0}$.

Triangular Matrices

Verify the result.

105. The product of two upper triangular 2×2 matrices is upper triangular.

Solution: Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$. Then:

$AB = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ac + bf \\ 0 & cf \end{pmatrix}$, which is upper triangular. ■

106. The product of two upper triangular $n \times n$ matrices is upper triangular.

107. The product of two triangular 2×2 matrices is not necessarily triangular.

Solution: An example is required. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then:

$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, which is not triangular. ■

108. The product of two lower triangular $n \times n$ matrices is upper triangular.

109. The product of two lower triangular 2×2 matrices is lower triangular.

Solution: Let $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, $B = \begin{pmatrix} d & 0 \\ e & f \end{pmatrix}$. Then:

$AB = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} ad & ac + bf \\ 0 & cf \end{pmatrix}$, which is lower triangular. ■

An alternative proof uses transposes: $(AB)^T = B^T A^T$ is the product of upper triangular matrices, therefore $AB)^T$ is upper triangular by Exercise 105. Because the transpose swaps rows and columns then AB is lower triangular. ■

110. The only 3×3 matrices which are both upper and lower triangular are the 3×3 diagonal matrices.

Matrix Multiply Properties

Verify the result.

111. The associative law $A(BC) = (AB)C$ holds for matrix multiplication.

Sketch: Expand $L = A(BC)$ entry L_{ij} according to matrix multiply rules. Expand $R = (AB)C$ entry R_{ij} the same way. Show $L_{ij} = R_{ij}$.

Solution: Let $A = (a_{ij})$, $B = (b_{jk})$, $C = (c_{km})$. Then

$BC = (d_{jm})$ where $d_{jm} = \sum_k b_{jk} c_{km}$
$AB = (e_{ik})$ where $e_{ik} = \sum_j a_{ij} b_{jk}$

Then

$A(BC) = (f_{im})$ where

$f_{im} = \sum_j a_{ij} d_{jm}$
$\quad = \sum_j a_{ij} \sum_k b_{jk} c_{km}$
$\quad = \sum_j \sum_k a_{ij} b_{jk} c_{km}$

and

$(AB)C = (g_{im})$ where

$g_{im} = \sum_k e_{ik} c_{km}$
$\quad = \sum_k \sum_j a_{ij} b_{jk} c_{km}$
$\quad = \sum_j \sum_k a_{ij} b_{jk} c_{km}$

The last equality holds by changing the order of summation. Then $A(BC) = (AB)C$ by equality of matrices. ■

112. The distributive law $A(B + C) = AB + AC$ holds for matrices.
Sketch: Expand $L = A(B + C)$ entry L_{ij} according to matrix multiply rules. Expand $R = AB + AC$ entry R_{ij} the same way. Show $L_{ij} = \sum_{k=1}^n a_{ik}(b_{kj} + c_{kj})$ and $R_{ij} = \sum_{k=1}^n a_{ik} b_{kj} + a_{ik} c_{kj}$. Then $L_{ij} = R_{ij}$.

113. For any matrix A the transpose formula $(A^T)^T = A$ holds.
Sketch: Expand $L = (A^T)^T$ entry L_{ij} according to matrix transpose rules. Then $L_{ij} = a_{ij}$.

Solution: Let $L = (A^T)^T = (L_{ij})$, $A = (a_{ij})$ and $A^T = (b_{ij})$. Then $b_{ij} = a_{ji}$. Because $L = B^T$, then $L_{ij} = b_{ji} = a_{ij}$. Equality of matrices implies $L = A$. ■

114. For matrices A, B the transpose formula $(A + B)^T = A^T + B^T$ holds.
Sketch: Expand $L = (A + B)^T$ entry L_{ij} according to matrix transpose rules. Repeat for entry R_{ij} of $R = A^T + B^T$. Show $L_{ij} = R_{ij}$.

115. For matrices A, B the transpose formula $(AB)^T = B^T A^T$ holds.
Sketch: Expand $L = (AB)^T$ entry L_{ij} according to matrix multiply and transpose rules. Repeat for entry R_{ij} of $R = B^T A^T$. Show $L_{ij} = R_{ij}$.

Solution: Let $L = (AB)^T = (L_{ij})$ and $R = B^T A^T = (R_{ij})$. To prove: $L = R$. The proof is completed by proving that $L_{ij} = R_{ij}$. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = B^T = (b_{ji})$, $D = A^T = (a_{ji})$. Then:

$AB = (e_{ij})$ where $e_{ij} = \sum_k a_{ik} b_{kj}$
$B^T A^T = (f_{ij})$ where $f_{ij} = \sum_k c_{ik} d_{kj}$

Compare:

$e_{ij} = \sum_k a_{ik} b_{kj}$
$\quad = \sum_k d_{ki} c_{jk}$

$= \sum_k c_{jk} d_{ki}$
$= f_{ji}$

Therefore,

$(AB)^T = (e_{ij})^T = (f_{ij}) = B^T A^T$. ■

116. For a matrix A and constant k, the transpose formula $(kA)^T = kA^T$ holds.

Invertible Matrices

Verify the result.

117. There are infinitely many 2×2 matrices A, B such that $AB = 0$

Solution: Let $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ for all possible values of a, b. Then

$AB = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$
$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

■

118. The zero matrix is not invertible.

119. The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ is not invertible.

Solution: Assume A has an inverse a matrix B: $AB = BA = I$. Then $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some constants a, b, c, d and

$I = AB$
$= \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$= \begin{pmatrix} a + 2c & b + 2d \\ 0 & 0 \end{pmatrix}$

Matrix equality implies entries match. Then $a + 2c = 0$, $b + 2d = 0$, $0 = 0$, $1 = 0$. The false equation $0 = 1$ is a contradiction to the assumption that A has an inverse. ■

120. The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is invertible.

121. The matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ satisfy

$$AB = BA = (ad - bc)I.$$

Solution:

$$
\begin{aligned}
AB &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \begin{pmatrix} ad-bc & -ab+ab \\ dc-cd & ad-bc \end{pmatrix} \\
&= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\
&= (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= (ad-bc)I
\end{aligned}
$$

Let $a_1 = d$, $b_1 = -b$, $c_1 = -c$, $d_1 = a$ (case sensitive) and apply the results above:

$$
\begin{aligned}
BA &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \\
&= (a_1d_1 - b_1c_1)I \\
&= (ad-bc)I
\end{aligned}
$$

Then: $AB = BA = (ad-bc)I$. ■

122. If $AB = 0$, then one of A or B is not invertible.

Symmetric Matrices

Verify the result.

123. The product of two symmetric $n \times n$ matrices A, B such that $AB = BA$ is symmetric.

Solution: $(AB)^T = B^TA^T = BA = AB$. ■

124. The product of two symmetric 2×2 matrices may not be symmetric.

125. If A is symmetric, then so is A^{-1}.
Sketch: Let $B = A^{-1}$. Compute B^T using transpose rules.

Solution: Assume A has inverse $B = A^{-1}$. Then $AB = BA = I$ and $A^T = A$. To prove: $B^T = B$.

First $I = I^T = (AB)^T = B^TA^T = B^TA$. Similarly, $AB^T = I$. Then $B^TA = AB^T = I$ and B^T is the (unique) inverse of A, i.e., $B^T = A^{-1} = B$. ■

126. If B is an $m \times n$ matrix and $A = B^TB$, then A is $n \times n$ symmetric.
Sketch: Compute A^T using transpose rules.

5.2 Matrix Equations

Identify RREF

Mark the matrices which pass the RREF Test, page 324 . Explain the failures.

1. $\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: RREF. Each nonzero row has a leading one. Above and below a leading one are zeros. Variable list $=x_1, x_2, x_3, x_4, x_5$. The lead variables are x_2, x_4. Nonzero rows appear in lead variable order x_2, x_4 Zero rows appear last.

2. $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

Solution: FAIL: not an RREF. The issue is row 3. It is an RREF after swapping rows 2 and 3. Lead variables are $x_1, x_2.x_3$. The rows violate lead variable order: x_1, x_3, x_2.

4. $\begin{pmatrix} 1 & 1 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Lead and Free Variables

For each matrix A, assume a homogeneous system $A\vec{X} = \vec{\mathbf{0}}$ with variable list $x_1, \ldots, x_n$. List the lead and free variables. Then report the rank and nullity of matrix A.

5. $\begin{pmatrix} 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: Lead: x_2, x_4. Free: x_1, x_3, x_5. Rank $= 2 =$ number of lead variables. Nullity $= 3 =$ number of free variables.

6. $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

7. $\begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: Lead: x_2, x_4. Free: x_1, x_3. Rank = 2 = number of lead variables. Nullity = 2 = number of free variables.

8. $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: Lead: x_1. Free: x_2, x_3. Rank = 1 = number of lead variables. Nullity = 2 = number of free variables.

10. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: Lead: x_1, x_5. Free: x_2, x_3, x_4. Rank = 2 = number of lead variables. Nullity = 3 = number of free variables.

12. $\begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

13. $\begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: Lead: x_3, x_5. Free: x_1, x_2, x_4. Rank = 2 = number of lead variables. Nullity = 3 = number of free variables.

14. $\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

15. $\begin{pmatrix} 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: Lead: x_2, x_3, x_5. Free: x_1, x_4. Rank $= 3 =$ number of lead variables. Nullity $= 2 =$ number of free variables.

16. $\begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Elementary Matrices

Write the 3×3 elementary matrix E and its inverse E^{-1} for each of the following operations, defined on page 323.

17. `combo(1,3,-1)`

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply `combo(1,3,-1)`

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, change entry 3,1 to additive inverse $-(-1) = 1$

Inverse rule for `combo(s,t,c)`: replace entry t, s by $-c$.

18. `combo(2,3,-5)`

19. `combo(3,2,4)`

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply `combo(3,2,4)`

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{pmatrix}$, change entry 2,3 to additive inverse $-(4) = -4$

20. `combo(2,1,4)`

21. `combo(1,2,-1)`

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply `combo(1,2-1)`

$E = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, change entry 2,1 to additive inverse $-(-1) = 1$

22. `combo(1,2,-`e^2`)`

23. `mult(1,5)`

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply `mult(1,5)`

$E = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, multiply I row 1 by 1/5

24. `mult(1,-3)`

25. `mult(2,5)`

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply `mult(2,5)`

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, multiply I row 2 by 1/5

26. `mult(2,-2)`

27. `mult(3,4)`

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply `mult(3,4)`

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1/4 \end{pmatrix}$, multiply I row 3 by 1/4

28. `mult(3,5)`

29. mult(2,-π)

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply mult(2,-$\pi$)

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\pi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\pi & 0 \\ 1 & 0 & 1 \end{pmatrix}$, multiply I row 2 by $-1/\pi$

30. mult(1,e^2)

31. `swap(1,3)`

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply `swap(1,3)`

$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$E^{-1} = E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, a swap is its own inverse

32. `swap(1,2)`

33. `swap(2,3)`

Solution:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, apply `swap(2,3)`

$$E = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}$$

$$E^{-1} = E = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}, \text{ a swap is its own inverse}$$

34. `swap(2,1)`

35. `swap(3,2)`

Solution:

$$I = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \text{ apply } \texttt{swap(3,2)}$$

$$E = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}$$

$$E^{-1} = E = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}, \text{ a swap is its own inverse}$$

36. `swap(3,1)`

Elementary Matrix Multiply

For each given matrix B_1, perform the toolkit operation (`combo, swap, mult`) to obtain the result B_2. Then compute the elementary matrix E for the identical toolkit operation. Finally, verify the matrix multiply equation $B_2 = EB_1$.

37. $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$, `mult(2,1/3)`.

Solution:

$$B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ after } \texttt{mult(2,1/3)}$$

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \text{ which is } I \text{ after } \texttt{mult(2,1/3)}$$

$$EB_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ which equals } B_2$$

38. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, `mult(1,3)`.

39. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, `combo(3,2,-1)`.

Solution:

$B_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

$B_2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ after `combo(3,2,-1)`

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ after `combo(3,2,-1)`

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

40. $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, `combo(2,1,-3)`.

41. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, `swap(2,3)`.

Solution:

$B_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

$B_2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ after `swap(2,3)`

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ after `swap(2,3)`

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

42. $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, `swap(1,2)`.

Inverse Row Operations

Given the final frame B of a sequence starting with matrix A, and the given operations, find matrix A. Do not use matrix multiply.

43. $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, operations

`combo(1,2,-1)`, `combo(2,3,-3)`, `mult(1,-2)`, `swap(2,3)`.

Solution: The inverse operations in reverse order are:

`swap(2,3)`, `mult(1,-1/2)`, `combo(2,3,3)`, `combo(1,2,1)`

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, given frame B

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, after `swap(2,3)`

$\begin{pmatrix} -1/2 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, after `mult(1,-1/2)`

$\begin{pmatrix} -1/2 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, after `combo(2,3,3)`

$\begin{pmatrix} -1/2 & -1/2 & 0 \\ -1/2 & -1/2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, after `combo(1,2,1)`

This is matrix A.

44. $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, operations

`combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,2)`, `swap(3,2)`.

45. $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, operations

`combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`.

Solution: The inverse operations in reverse order are:

`swap(1,3)`, `mult(1,1/4)`, `combo(2,3,-3)`, `combo(1,2,1)`

$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, given frame B

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$, after `swap(1,3)`

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$, after `mult(1,1/4)`

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & -2 & -7 \end{pmatrix}, \text{ after } \texttt{combo(2,3,-3)}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & -2 & -7 \end{pmatrix}, \text{ after } \texttt{combo(1,2,1)}$$

This is matrix A.

46. $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, operations
`combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`.

Elementary Matrix Products

Given the first frame B_1 of a sequence and elementary matrix operations E_1, E_2, E_3, find matrices $F = E_3E_2E_1$ and $B_4 = FB_1$. Hint: Compute $\left\langle B_4 | F \right\rangle$ from toolkit operations on $\left\langle B_1 | I \right\rangle$.

47. $B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, operations
`combo(1,2,-1)`, `combo(2,3,-3)`, `mult(1,-2)`.

Solution:

$$\left\langle B_1 | I \right\rangle = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$
$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \text{ after } \texttt{combo(1,2,-1)}$$
$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 1 & 0 \\ 3 & 0 & -6 & 3 & -3 & 1 \end{array}\right) \text{ after } \texttt{combo(2,3,-3)}$$
$$= \left(\begin{array}{ccc|ccc} -2 & -2 & 0 & -2 & 0 & 0 \\ -1 & 0 & 2 & -1 & 1 & 0 \\ 3 & 0 & -6 & 3 & -3 & 1 \end{array}\right) \text{ after } \texttt{mult(1,-2)}$$

Then:

$$B_4 = \begin{pmatrix} -2 & -2 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & -6 \end{pmatrix}, \quad F = \begin{pmatrix} -2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix}$$

48. $B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `combo(2,3,3)`, `swap(3,2)`.

49. $B_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, operations
`combo(1,2,-1)`, `mult(1,4)`, `swap(1,3)`.

Solution:
$$B_4 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 4 & 4 & 8 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

50. $B_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, operations
`combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`.

Miscellany

51. Justify with English sentences why all possible 2×2 matrices in reduced row-echelon form must look like

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $*$ denotes an arbitrary number.

Solution: (1) If there are no leading ones then all rows are zeros.
(2) If there is one leading one then it occurs in column 1 or column 2, resulting in the second and third forms.
(3) If there are two leading ones then one is in column 1 and the other is in column 2. The order of variables is preserved, so the leading one in column one must be in row one. All other entries in a column with a leading one must be zero.

52. Display all possible 3×3 matrices in reduced row-echelon form. Besides the zero matrix and the identity matrix, please report five other forms, most containing symbol $*$ representing an arbitrary number.

53. Determine all possible 4×4 matrices in reduced row-echelon form.

Solution: No leading ones: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

One leading one: $\begin{pmatrix}1 * * *\\0\,0\,0\,0\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}0\,1 * *\\0\,0\,0\,0\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}0\,0\,1 *\\0\,0\,0\,0\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}0\,0\,0\,1\\0\,0\,0\,0\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$

Two leading ones: $\begin{pmatrix}1\,0 * *\\0\,1 * *\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}1 * 0 *\\0\,0\,1 *\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}1 * * 0\\0 * * 1\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}0\,1\,0 *\\0\,0\,1 *\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}0\,1 * 0\\0\,0 * 1\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}0\,0\,1\,0\\0\,0\,0\,1\\0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}$,

Three leading ones: $\begin{pmatrix}1\,0\,0 *\\0\,1\,0 *\\0\,0\,1 *\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}1 * 0\,0\\0\,0\,1\,0\\0\,0\,0\,1\\0\,0\,0\,0\end{pmatrix}$, $\begin{pmatrix}0\,1\,0\,0\\0\,0\,1\,0\\0\,0\,0\,1\\0\,0\,0\,0\end{pmatrix}$,

Four leading ones: $\begin{pmatrix}1\,0\,0\,0\\0\,1\,0\,0\\0\,0\,1\,0\\0\,0\,0\,1\end{pmatrix}$

54. Display a 6×6 matrix in reduced row-echelon form with rank 4 and only entries of zero and one.

55. Display a 5×5 matrix in reduced row-echelon form with nullity 2 having entries of zero, one and two, but no other entries.

Solution: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

56. Display the rank and nullity of any $n \times n$ elementary matrix.

57. Let $F = \left\langle C|D \right\rangle$ and let E be a square matrix with row dimension matching F. Display the details for the equality

$$EF = \left\langle EC|ED \right\rangle.$$

Solution: Matrix multiply of $k \times n$ matrix M against $n \times m$ matrix N is defined by the identity

$$MN = \left\langle M\,\mathbf{col}(N,1)|\cdots|M\,\mathbf{col}(N,n) \right\rangle$$

Assume C is $k \times n$, D is $n \times m$, F is $k \times (n+m)$. Then:
$EF = \left\langle E\,\mathbf{col}(F,1)|\cdots|E\,\mathbf{col}(F,n+m) \right\rangle$

$$EF = \left\langle E\,\mathbf{col}(C,1)|\cdots|E\,\mathbf{col}(C,n)|E\,\mathbf{col}(D,1)|\cdots|E\,\mathbf{col}(D,m)\right\rangle$$
$$EF = \left\langle EC|ED\right\rangle. \quad \blacksquare$$

58. Let $F = \left\langle C|D\right\rangle$ and let E_1, E_2 be $n \times n$ matrices with n equal to the row dimension of F. Display the details for the equality

$$E_2E_1F = \left\langle E_2E_1C|E_2E_1D\right\rangle.$$

59. Assume matrix A is invertible. Display details explaining why $\mathbf{rref}(\left\langle A|I\right\rangle)$ equals the matrix $\left\langle R|E\right\rangle$, where matrix $R = \mathbf{rref}(A)$ and matrix $E = E_k \cdots E_1$. Symbols E_i are elementary matrices in toolkit steps taking matrix A into reduced row-echelon form. Suggestion: Use the preceding exercises.

Solution: Write $R = \mathbf{rref}(A) = E_n \cdots E_1A$ with elementary matrices $E_1, \ldots, E_n$ representing the `combo, swap, mult` steps. Apply Exercise 57 to obtain

$$b \equiv E_n \cdots E_1\left\langle A|I\right\rangle = \left\langle E_n \cdots E_1A|E_n \cdots E_1I\right\rangle = \left\langle R|B\right\rangle$$

It remains to explain why matrix B equals $\mathbf{rref}(\left\langle A|I\right\rangle)$.

Because A is square $k \times k$, then $B = \left\langle R|E\right\rangle$ where $E = E_n \cdots E_1$ is invertible $k \times k$. Leading ones of B occur in the first k columns. Above and below the leading ones are zeros. Each leading one is in a column of the $k \times k$ identity I and these columns appear in natural order of I. There are no other rows to consider, so B is in reduced echelon form: $B = \mathbf{rref}(\left\langle A|I\right\rangle)$. $\blacksquare$

60. Assume E_1, E_2 are elementary matrices in toolkit steps taking A into reduced row-echelon form. Prove that $A^{-1} = E_2E_1$. In words, A^{-1} is found by doing the same toolkit steps to the identity matrix.

61. Assume matrix A is invertible and $E_1, \ldots, E_k$ are elementary matrices in toolkit steps taking A into reduced row-echelon form. Prove that $A^{-1} = E_k \cdots E_1$.

Solution: Let $E = E_k \cdots E_1$, an invertible matrix. Equation $E_k \cdots E_1A = \mathbf{rref}(A)$ means $EA = I$. By basic invertibility theorems, E is the inverse of A.

62. Assume A, B are 2×2 matrices. Assume A is invertible and $\mathbf{rref}(\left\langle A|B\right\rangle) = \left\langle I|D\right\rangle$. Explain why the first column $\vec{x}$ of D is the unique solution of $A\vec{x} = \vec{b}$, where $\vec{b}$ is the first column of B.

63. Assume A, B are $n \times n$ matrices with A invertible. Explain how to solve the matrix equation $AX = B$ for matrix X using the augmented matrix of A, B.

Solution: Multiply $AX = B$ by the inverse of A. Then $X = A^{-1}B$. Exercise 61 provides $A^{-1} = E_n \cdots E_1$ in terms of elementary matrices $E_1 \ldots, E_n$. Exercises 57 and 59 apply:

$$E\left\langle A|B\right\rangle = \left\langle \mathbf{rref}(A)|EB\right\rangle = \mathbf{rref}(\left\langle A|B\right\rangle)$$

Because $X = A^{-1}B = EB$, then row-reduction of the augmented matrix of A and B has X in the last n columns.

5.3 Determinants and Cramer's Rule

Determinant Notation

Write formulae for x and y as quotients of 2×2 determinants. Do not evaluate the determinants!

1. $\begin{pmatrix} 1 & -1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 \\ 3 \end{pmatrix}$

Solution: $x = \dfrac{\begin{vmatrix} -10 & -1 \\ 3 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}}, \; y = \dfrac{\begin{vmatrix} 1 & -10 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}}$

2. $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \end{pmatrix}$

3. $\begin{pmatrix} 0 & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \end{pmatrix}$

Solution: $x = \dfrac{\begin{vmatrix} -1 & -1 \\ 10 & 5 \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ 2 & 5 \end{vmatrix}}, \; y = \dfrac{\begin{vmatrix} 0 & -1 \\ 2 & 10 \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ 2 & 5 \end{vmatrix}}$

4. $\begin{pmatrix} 0 & -3 \\ 3 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Sarrus' 2×2 rule

Evaluate $\det(A)$.

5. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Solution: $\det(A) = 3$

6. $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

7. $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

Solution: $\det(A) = 7$

8. $A = \begin{pmatrix} 5a & 1 \\ -1 & 2a \end{pmatrix}$

Sarrus' rule 3×3

Evaluate $\det(A)$.

9. $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

Solution: $\det(A) = -1$

10. $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

11. $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Solution: $\det(A) = -1$

12. $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{pmatrix}$

Inverse of a 2×2 Matrix

Define matrix A and its adjugate C:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

13. Verify $AC = |A| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution:

$$\begin{aligned} AC &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ad - bc & -ab + ba \\ cd - dc & -cb + da \end{pmatrix} \\ &= \begin{pmatrix} |A| & 0 \\ 0 & |A| \end{pmatrix} \\ &= |A| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

14. Display the details of the argument that $|A| \neq 0$ implies A^{-1} exists and $A^{-1} = \dfrac{C}{|A|}$.

15. Show that A^{-1} exists implies $|A| \neq 0$. Suggestion: Assume not, then $AB = BA = I$ for some matrix B and also $|A| = 0$. Find a contradiction using $AC = |A|I$ from Exercise 13.

Solution: Assume A^{-1} exists but $|A| = 0$. Then Exercise 13 gives $AC = |A|I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Multiply by A^{-1}: $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $a = b = c = d = 0$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The definition of inverse gives $AA^{-1} = I$ which implies $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction. ■

16. Calculate the inverse of $\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$ using the formula developed in these exercises.

Unique Solution of a 2×2 System

Solve $A\vec{\mathbf{X}} = \vec{\mathbf{b}}$ for $\vec{\mathbf{X}}$ using Cramer's rule for 2×2 systems.

17. $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Solution: $$x = \frac{\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}} = 2,\ y = \frac{\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}} = -1$$

18. $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

19. $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$

Solution: $$x = \frac{\begin{vmatrix} -4 & 0 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}} = -8/4 = -2,\ y = \frac{\begin{vmatrix} 2 & -4 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}} = 12/4 = 3$$

20. $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} -10 \\ 10 \end{pmatrix}$

Definition of Determinant

21. Let A be 3×3 with zero first row. Use the college algebra definition of determinant to show that $\det(A) = 0$.

22. Let A be 3×3 with equal first and second row. Use the college algebra definition of determinant to show that $\det(A) = 0$.

23. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Use the college algebra definition of determinant to verify that $|A| = ad - bc$.

Solution: The college algebra definition of $|A|$ for a 2×2 matrix A involves two permutations: $\Sigma_1 = (1, 2)$ and $\Sigma_2 = (2, 1)$. Then $\mathbf{parity}(\Sigma_1) = 0$, $\mathbf{parity}(\Sigma_2) = 1$ by counting the swaps needed to rearrange the permutation in natural order $(1, 2)$. By the college algebra definition:

$$\begin{aligned} |A| &= (-1)^{\mathbf{parity}(\Sigma_1)} a_{11}a_{22} + (-1)^{\mathbf{parity}(\Sigma_2)} a_{12}a_{21} \\ &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

Substitute $a_{11} = a$, $a_{12} = b$, $a_{21} = c$, $a_{22} = d$. Then

$$\begin{aligned} |A| &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}a_{22} - a_{12}a_{21} \\ &= ad - bc \end{aligned}$$

The college algebra definition reduces to Sarrus' 2×2 rule $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. ■

24. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Use the college algebra definition of determinant to verify that the determinant of A equals

$$\begin{aligned} &a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} \\ &+a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} \\ &-a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{aligned}$$

Four Properties

Evaluate $\det(A)$ using the four properties for determinants, page 345.

25. $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Solution: $|A| = -1$

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} \quad \text{Combination Rule: } \texttt{combo(2,3,-1)}$$

$|A| = (-1)\begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$ Swap Rule: `swap(1,2)`

$|A| = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ Swap Rule: `swap(2,3)`

$|A| = (-1)(-1)(1)(-1)(1)$ Triangular Rule

$|A| = -1$

26. $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

27. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Solution: $|A| = 1$

$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$ Combination Rule: `combo(1,2,-1)`

$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ Combination Rule: `combo(1,3,-1)`

$|A| = (-1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix}$ Swap Rule: `swap(2,3)`

$|A| = (-1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix}$ Combination Rule: `combo(2,3,-2)`

$|A| = (-1)(1)(1)(-1)$ Triangular Rule

$|A| = 1$

28. $A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

29. $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}$

Solution: $|A| = 1$

$$|A| = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{vmatrix}$$

$$|A| = (-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{vmatrix} \text{ Swap Rule: } \texttt{swap(1,2)}$$

$$|A| = (-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{vmatrix} \text{ Combination Rule: } \texttt{combo(1,3,-1)}$$

$$|A| = (-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -3 & -1 & 2 \end{vmatrix} \text{ Combination Rule: } \texttt{combo(1,4,-2)}$$

$$|A| = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -1 & 2 \end{vmatrix} \text{ Swap Rule: } \texttt{swap(2,3)}$$

$$|A| = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{vmatrix} \text{ Combination Rule: } \texttt{combo(2,4,-3)}$$

$$|A| = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \text{ Combination Rule: } \texttt{combo(3,4,1)}$$

$|A| = (-1)(-1)(1)(-1)(1)(-1)$ Triangular Rule

$|A| = 1$

30. $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

31. $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

Solution: $|A| = 5$

$$|A| = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

$$|A| = (-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} \text{ Swap Rule: } \texttt{swap(1,2)}$$

$$|A| = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & -2 & 0 \\ 0 & 0 & 1 & 2 \end{vmatrix} \text{ Swap Rule: } \texttt{swap(2,3)}$$

$$|A| = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix} \text{ Combination Rule: } \texttt{combo(2,3,3)}$$

$$|A| = (-1)(-1)(-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{vmatrix} \text{ Swap Rule: } \texttt{swap(3,4)}$$

$$|A| = (-1)(-1)(-1)\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{vmatrix} \text{ Combination Rule: } \texttt{combo(3,4,-4)}$$

$$|A| = (-1)(-1)(-1)(1)(1)(1)(-5) \text{ Triangular Rule}$$
$$|A| = 5$$

32. $A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$

Elementary Matrices and the Four Rules

Find $\det(A)$.

33. A is 3×3 and obtained from the identity matrix I by three row swaps.

Solution: $|A| = -1$

34. A is 7×7, obtained from I by swapping rows 1 and 2, then rows 4 and 1, then rows 1 and 3.

35. A is obtained from the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ by swapping rows 1 and 3, then two row combinations.

Solution: $|A| = (-1)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (-1)(1) = -1$

36. A is obtained from the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ by two row combinations, then multiply row 2 by -5.

More Determinant Rules

Cite the determinant rule that verifies $\det(A) = 0$. **Never** expand $\det(A)$! See page 347.

37. $A = \begin{pmatrix} -1 & 5 & 1 \\ 2 & -4 & -4 \\ 1 & 1 & -3 \end{pmatrix}$

Solution: Dependent rows. Add rows 1 and 2 to get row 3.

38. $A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -4 \\ 1 & 1 & -3 \end{pmatrix}$

39. $A = \begin{pmatrix} 4 & -8 & -8 \\ 2 & -4 & -4 \\ 1 & 1 & -3 \end{pmatrix}$

Solution: Common factor. Row 2 times 2 equals row 1.

40. $A = \begin{pmatrix} -1 & 5 & 0 \\ 2 & -4 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

41. $A = \begin{pmatrix} -1 & 5 & 3 \\ 2 & -4 & 0 \\ 1 & 1 & 3 \end{pmatrix}$

Solution: Dependent rows. Row 1 plus row 2 equals row 3.

42. $A = \begin{pmatrix} -1 & 5 & 4 \\ 2 & -4 & -2 \\ 1 & 1 & 2 \end{pmatrix}$

Cofactor Expansion and College Algebra

Evaluate the determinant with an efficient cofactor expansion.

43. $\begin{vmatrix} 2 & 5 & 1 \\ 2 & 0 & -4 \\ 1 & 0 & 0 \end{vmatrix}$

Solution: Expand along row 3: $|A| = (+1)(1)\begin{vmatrix} 5 & 1 \\ 0 & -4 \end{vmatrix} = -20$

44. $\begin{vmatrix} 2 & 5 & 1 \\ 2 & 0 & -4 \\ 1 & 0 & 1 \end{vmatrix}$

45. $\begin{vmatrix} 2 & 5 & 0 & 0 \\ 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}$

Solution: Expand along row 4:
$|A| = (-1)(1)\begin{vmatrix} 5 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix}$
Expand the 3×3 determinant along row 1:
$|A| = (-1)(1)(+1)(5)\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} = (-1)(1)(+1)(5)(4) = -20$

46. $\begin{vmatrix} 0 & 2 & 0 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{vmatrix}$

47. $\begin{vmatrix} 2 & 5 & 1 & -1 & 1 \\ 0 & -1 & -4 & 1 & -1 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{vmatrix}$

Solution: $|A| = 18$
Column 4 has the most zeros. Expand along it:
$|A| = (-1)(-1)D_1 + (+1)(1)D_2$ where
$$D_1 = \begin{vmatrix} 0 & -1 & -4 & -1 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 2 & 5 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix}$$
The two 4×4 cross-out determinants D_1, D_2 each have the most zeros in column 4. Expand each along column 4:

$$D_1 = (-1)(-1)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} + (+1)(1)\begin{vmatrix} 0 & -1 & -4 \\ 1 & 2 & 3 \\ 1 & 0 & 3 \end{vmatrix}$$

$D_1 = (-1)(-1)(6) + (+1)(1)(8) = 14$

$$D_2 = (-1)(1)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} + (+1)(1)\begin{vmatrix} 2 & 5 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 3 \end{vmatrix}$$

$D_2 = (-1)(1)(6) + (+1)(1)(10) = 4$

Then

$|A| = (-1)(-1)D_1 + (+1)(1)D_2 = 14 + 4 = 18$

48. $\begin{vmatrix} 2 & 0 & 1 & -1 & 1 \\ 0 & -1 & -4 & 1 & -1 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 \end{vmatrix}$

Minors and Cofactors

Write out and then evaluate the minor and cofactor of each element cited for the matrix $A = \begin{pmatrix} 2 & 5 & y \\ x & -1 & -4 \\ 1 & 2 & z \end{pmatrix}$

49. Row 1 and column 3.

Solution: Let $A = \begin{pmatrix} 2 & 5 & y \\ x & -1 & -4 \\ 1 & 2 & z \end{pmatrix}$. Then **minor**$(A, 1, 3)$ = cross-out determinant of $a_{13} = \begin{vmatrix} x & -1 \\ 1 & 2 \end{vmatrix} = 2x + 1$ and **cof**$(A, 1, 3) = (-1)^{1+3}$ **minor**$(A, 1, 3) = 2x + 1$.

```
# Minors and cofactors: Exercise 49
with(LinearAlgebra):
A:=Matrix([[2,5,y],[x,-1,-4],[1,2,z]]);
Minor(A,1,3);Minor(A,3,2);
# 2x+1, -8-xy
```

50. Row 2 and column 1.

51. Row 3 and column 2.

Solution: Then **minor**$(A, 3, 2)$ = cross-out determinant of $a_{3,2}$ = $\begin{vmatrix} 2 & y \\ x & -4 \end{vmatrix} = -8 - xy$ and **cof**$(A, 3, 2) = (-1)^{3+2}$ **minor**$(A, 3, 2) = 8 + xy$.

52. Row 3 and column 1.

Cofactor Expansion

Use cofactors to evaluate the determinant.

53. $\begin{vmatrix} 2 & 7 & 1 \\ -1 & 0 & -4 \\ 1 & 0 & 3 \end{vmatrix}$

Solution: Expand along column 2, which has the most zeros. Then

$|A| = (-1)(7)\begin{vmatrix} -1 & -4 \\ 1 & 3 \end{vmatrix} = -7$

54. $\begin{vmatrix} 2 & 7 & 7 \\ -1 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$

55. $\begin{vmatrix} 0 & 2 & 7 & 7 \\ 0 & -1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{vmatrix}$

Solution: $|A| = 0$. Expand along column 4. Then

$|A| = (-1)(7)\begin{vmatrix} 0 & -1 & 1 \\ 3 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = (-1)(7)(-1)(3)\begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix} = 0.$

Alternatively, $|A| = 0$ due to equal rows 2 and 4.

56. $\begin{vmatrix} 0 & 2 & 7 & 7 \\ 0 & -1 & y & 0 \\ x & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{vmatrix}$

57. $\begin{vmatrix} 0 & 2 & 7 & 7 & 3 \\ 0 & -1 & 0 & 0 & 1 \\ x & 1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{vmatrix}$

Solution: $|A| = -7x$. Expand along column 4. Then

$|A| = (-1)(7)\,\textbf{minor}(A, 1, 4) = (-1)(7)|A_1|$ where

$A_1 = \begin{pmatrix} 0 & -1 & 0 & 1 \\ x & 1 & 2 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}$

Expand $|A_1|$ along row 1:
$|A_1| = (-1)(-1)\,\textsf{minor}(A_1,1,2) + (-1)(1)\,\textsf{minor}(A_1,1,4)$
$$|A_1| = \begin{vmatrix} x & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} x & 1 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$
$|A_1| = x$ because the second determinant has identical rows 2, 3.
$|A| = (-1)(7)|A_1| = -7x$

58. $\begin{vmatrix} 0 & 2 & 7 & 7 & 3 \\ 0 & -1 & 2 & 0 & 1 \\ x & 1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{vmatrix}$

Adjugate and Inverse Matrix

Find the adjugate of A and the inverse B of A. Check the answers via the formulas $A\,\textbf{adj}(A) = \det(A)I$ and $AB = I$.

59. $A = \begin{pmatrix} 2 & 7 \\ -1 & 0 \end{pmatrix}$

Solution: $\textbf{adj}(A) = \begin{pmatrix} 0 & -7 \\ 1 & 2 \end{pmatrix}$, $|A| = 7$, $B = A^{-1} = \begin{pmatrix} 0 & -1 \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}$

```
# Adjugate and inverse, Exercise 59
A:=Matrix([[2,7],[-1,0]]);
B:=1/A;
Determinant(A); Adjoint(A);
(A . Adjoint(A)) - Determinant(A);# Expect the zero matrix
# Maple auto-inserts Matrix([[1,0],[0,1]])
```

60. $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

61. $A = \begin{pmatrix} 5 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix}$

Solution: $\textbf{adj}(A) = \begin{pmatrix} 0 & -3 & 2 \\ 2 & 14 & -10 \\ 0 & 1 & 0 \end{pmatrix}$, $|A| = 2$,

$B = A^{-1} = \begin{pmatrix} 0 & -3/2 & 1 \\ 1 & 7 & -5 \\ 0 & 1/2 & ,0 \end{pmatrix}$

62. $A = \begin{pmatrix} 5 & 1 & 2 \\ 2 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

63. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 2 \end{pmatrix}$

Solution: $\mathbf{adj}(A) = \begin{pmatrix} -2 & 2 & -2 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & -2 & 2 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$, $|A| = -1$,

$B = A^{-1} \begin{pmatrix} 2 & -2 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$

64. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

Transpose and Inverse

65. Verify that $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ satisfies $A^T = A^{-1}$.

Solution: $A^{-1} = \begin{pmatrix} 1/2\sqrt{2} & -1/2\sqrt{2} \\ 1/2\sqrt{2} & 1/2\sqrt{2} \end{pmatrix}$

```
# Transpose and Inverse, Exercise 65
A:=(1/sqrt(2))*Matrix([[1,1],[-1,1]]);
B:=1/A;
C:=A^+;
B-C;# Expect a zero matrix
```

66. Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\det(A) = 1$ and $A^T = A^{-1}$.

67. Find all 3×3 diagonal matrices A such that $A^T = A^{-1}$.

Solution: Let $A = \mathbf{diag}(a, b, c)$. Then $A^{-1} = \mathbf{diag}(1/a, 1/b, 1/c)$ if and only if $|A| = abc \neq 0$. Because $A^T = A$, then $A^T = A^{-1}$ holds if and only if $\mathbf{diag}(a, b, c) = \mathbf{diag}(1/a, 1/b, 1/c)$ or equivalently $a^2 = b^2 = c^2 = 1$. The matrices are

diag$(1,1,1)$, **diag**$(1,1,-1)$, **diag**$(1,-1,1)$, **diag**$(1,-1,-1)$,
diag$(-1,1,1)$, **diag**$(-1,-1,1)$, **diag**$(-1,1,-1)$, **diag**$(-1,-1,-1)$

68. Find all 3×3 upper triangular matrices A such that $A^T = A^{-1}$.

69. Find all $n \times n$ diagonal matrices A such that $A^T = A^{-1}$.

Solution: See Exercise 67 for $n = 3$. The matrices are **diag**$(a_1, \ldots, a_n)$ for all possible choices of $a_i = \pm 1$, $1 \le i \le n$.

70. Determine the $n \times n$ triangular matrices A such that $\det(A) = 1$ and $A^T = \textbf{adj}(A)$.

Elementary Matrices

Find the determinant of A from the given equation.

71. Let $A = 5E_2E_1$ be 3×3, where E_1 multiplies row 3 of the identity by -7 and E_2 swaps rows 3 and 1 of the identity. Hint: $A = (5I)E_2E_1$.

Solution: $|A| = 875$.

Apply the determinant product rule: $|A| = |5I||E_2||E_1|$. Row operations `mult(3,-7)` and `swap(3,1)` applied to the identity matrix imply $|E_1| = -7$, $|E_2| = -1$. Then $|A| = |5I||E_2||E_1| = 5^3(-7)(-1) = 875$.

72. Let $A = 2E_2E_1$ be 5×5, where E_1 multiplies row 3 of the identity by -2 and E_2 swaps rows 3 and 5 of the identity.

73. Let $A = E_2E_1B$ be 4×4, where E_1 multiplies row 2 of the identity by 3 and E_2 is a combination. Find $|A|$ in terms of $|B|$.

Solution: $|A| = |E_2||E_1||B| = (3)(1)|B| = 3|B|$

74. Let $A = 3E_2E_1B$ be 3×3, where E_1 multiplies row 2 of the identity by 3 and E_2 is a combination. Find $|A|$ in terms of $|B|$.

75. Let $A = 4E_2E_1B$ be 3×3, where E_1 multiplies row 1 of the identity by 2, E_2 is a combination and $|B| = -1$.

Solution: $|A| = |4I||E_2||E_1||B| = 4^3(1)(-1) = -64$

76. Let $A = 2E_3E_2E_1B^3$ be 3×3, where E_1 multiplies row 2 of the identity by -1, E_2 and E_3 are swaps and $|B| = -2$.

Determinants and the Toolkit

Display the toolkit steps for $\mathbf{rref}(A)$. Using only the steps, report:

- The determinant of the elementary matrix E for each step.
- The determinant of A.

77. $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 4 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 4 \end{pmatrix}$

$A_1 = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ `swap(1,3)`, $|E_1| = -1$

$A_2 = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 3 & -3 \end{pmatrix}$ `combo(1,3-2)`, $|E_2| = 1$

$A_3 = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{pmatrix}$ `swap(2,3)`, $|E_3| = -1$

$|A_3| = 6$ Triangular Rule

Result: $E_1 A = A_1$, $E_2 A_1 = A_2$, $E_3 A_2 = A_3$. Summary: $E_3 E_2 E_1 A = A_3$. Then $|E_3||E_2||E_1||A| = |A_3|$. Insert values: $(-1)(1)(-1)|A| = 6$ or $|A| = 6$.

```
# Determinants and the Toolkit, Exercise 77
A:=Matrix([[2,3,1],[0,0,2],[1,0,4]]);
Determinant(A);
```

78. $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

79. $A = \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

Solution: $|A| = -18$

80. $A = \begin{pmatrix} 2 & 3 & 1 & 2 \\ 0 & 3 & 0 & 0 \\ 2 & 6 & 1 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

Determinant Product Rule

Apply the product rule $\det(AB) = \det(A)\det(B)$.

81. Let $\det(A) = 5$ and $\det(B) = -2$. Find $\det(A^2B^3)$.
Solution: $|A^2B^3| = |AABBB| = |A|^2|B|^3 = 5^2(-2)^3 = -200$

82. Let $\det(A) = 4$ and $A(B - 2A) = 0$. Find $\det(B)$.
Solution: Hint: $AB = (2I)A^2$

83. Let $A = E_1E_2E_3$ where E_1, E_2 are elementary swap matrices and E_3 is an elementary combination matrix. Find $\det(A)$.
Solution: $A| = |E_1||E_2||E_3| = (-1)(-1)(1) = 1$

84. Assume $\det(AB + A) = 0$ and $\det(A) \neq 0$. Show that $\det(B + I) = 0$.
Solution: Hint: $AB + A = A(B + I)$

Cramer's 2 × 2 Rule

Assume

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

85. Derive the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} e & b \\ f & d \end{pmatrix}.$$

Solution: The given matrix identity provides equations $ax+by = e$, $cx+dy = f$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} ax+by & 0a+1b \\ cx+dy & 0c+1d \end{pmatrix} = \begin{pmatrix} ax+by & b \\ cx+dy & d \end{pmatrix} = \begin{pmatrix} e & b \\ f & d \end{pmatrix}$$

86. Derive the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} a & e \\ c & f \end{pmatrix}.$$

87. Use the determinant product rule to derive the Cramer's Rule formula

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

Solution: The plan: use Exercise 85 and the determinant product rule.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x & 0 \\ y & 1 \end{vmatrix} = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

88. Derive, using the determinant product rule, the Cramer's Rule formula

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

Cramer's 3×3 Rule

Let A be the coefficient matrix in the equation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

89. Derive the formula

$$A \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}$$

Solution: Definition $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and the given identity provide:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ &a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ &a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{aligned}$$

Let LHS and RHS denote the two sides of the claimed formula. Matrix multiply:

$$\text{LHS} = A \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix}$$

$$\text{LHS} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix}$$

$$\text{LHS} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3, & a_{12} & a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3, & a_{22} & a_{23} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3, & a_{32} & a_{33} \end{pmatrix}$$

$$\text{LHS} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix} \text{ by the three equations above}$$
$$\text{LHS} = \text{RHS} \quad \blacksquare$$

90. Derive the formula

$$A \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}$$

91. Derive, using the determinant product rule, the Cramer's Rule formula

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

Solution: Use Exercise 89 and the determinant product rule:

$$|A| \begin{vmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

Then:

$$|A| x_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$$

92. Use the determinant product rule to derive the Cramer's Rule formula

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

Cayley-Hamilton Theorem

93. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$. Expand $|A - rI|$ to compute the characteristic polynomial of A. Answer: $r^2 - 4r + 5$.

Solution: $|A - rI| = \begin{vmatrix} 1-r & -1 \\ 2 & 3-r \end{vmatrix} = (1-r)(3-r) + 2 = r^2 - 4r + 5$

94. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$. Apply the Cayley-Hamilton theorem to justify the equation

$$A^2 - 4A + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

95. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Expand $|A - rI|$ by Sarrus' Rule to obtain $r^2 - (a+b)r + (ad - bc)$.

Solution: $|A - rI| = \begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+b)r + (ad - bc)$

96. The result of the previous exercise is often written as $(-r)^2 + \textbf{trace}(A)(-r) + |A|$ where $\textbf{trace}(A) = a + d =$ sum of the diagonal elements. Display the details.

97. Let $\lambda^2 - 2\lambda + 1 = 0$ be the characteristic equation of a matrix A. Find a formula for A^2 in terms of A and I.

Solution: Cayley-Hamilton provides equation $A^2 - 2A + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Solve for A^2: $A^2 = 2A - I$.

98. Let A be an $n \times n$ triangular matrix with all diagonal entries zero. Prove that $A^n = 0$.

99. Find all 2×2 matrices A such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, discovered from values of $\textbf{trace}(A)$ and $|A|$.

Solution: Exercise 96 reports the characteristic equation in the form $|A - rI| = (-r)^2 + \textbf{trace}(A)(-r) + |A|$. Cayley-Hamilton provides the identity $(-A)^2 + \textbf{trace}(A)(-A) + |A|I = 0$. If A^2 is the zero matrix, then $\textbf{trace}(A)(-A) + |A|I = 0$. Then $\textbf{trace}(A)A = |A|I$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the requirement becomes: $(a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$ which is the set of equations

$a(a+d) = ad - bc,$
$b(a+d) = 0,$
$c(a+d) = 0,$
$d(a+d) = ad - bc$

The solution:

$c = -a^2/b$, $d = -a$ for all a and $b \neq 0$,
$a = b = c = d = 0$

```
# Cayley-Hamilton Theorem, Exercise 99
 eqs:=a*(a+d) = a*d-b*c,b*(a+d) = 0,
      c*(a+d) = 0,d*(a+d) = a*d-b*c;
 solve({eqs},{a,b,c,d});
```

100. Find four 2×2 matrices A such that $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Applied Definition of Determinant

Miscellany for permutation matrices and the sampled product page 358

$$\begin{aligned} A.P =& (A_1 \cdot P_1)(A_2 \cdot P_2) \cdots (A_n \cdot P_n) \\ =& a_{1\sigma_1} \cdots a_{n\sigma_n}. \end{aligned}$$

101. Compute the sampled product of $\begin{pmatrix} 5 & 3 & 1 \\ 0 & 5 & 7 \\ 1 & 9 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Solution: $A_1.P_1 = (5,3,1) \cdot (1,0,0) = 5$, $A_2 \cdot P_2 = 7$, $A_3 \cdot P_3 = 9$. Then $A.P = (5)(7)(9) = 315$.

102. Compute the sampled product of $\begin{pmatrix} 5 & 3 & 3 \\ 0 & 2 & 7 \\ 1 & 9 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

103. Determine the permutation matrices P required to evaluate $\det(A)$ when A is 2×2.

Solution: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

104. Determine the permutation matrices P required to evaluate $\det(A)$ when A is 4×4.

Three Properties

Reference: Page 359, three properties that define a determinant

105. Assume $n = 3$. Prove that the three properties imply $D = 0$ when two rows are identical.

Solution: Swap the identical rows to obtain determinant F. Then $D = -F$ by the swap property. Since the rows are unchanged, then $F = D$. Then $D = -F = -D$ and finally $D = 0$.

106. Assume $n = 3$. Prove that the three properties imply $D = 0$ when a row is zero.

5.4 Vector Spaces, Independence, Basis

Scalar and Vector General Solution

Given the scalar general solution of $A\vec{x} = \vec{0}$, find the vector general solution

$$\vec{x} = t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots$$

where symbols t_1, t_2, ... denote arbitrary constants and $\vec{u}_1$, $\vec{u}_2$, ... are fixed vectors.

1. $x_1 = 2t_1$, $x_2 = t_1 - t_2$, $x_3 = t_2$

Solution: Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t_1 \\ t_1 - t_2 \\ t_2 \end{pmatrix}$. Compute the partial derivatives of $\vec{x}$ on symbols t_1, t_2:

$$\partial\vec{x}/\partial t_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \partial\vec{x}/\partial t_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Let $\vec{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$

Then the vector general solution is

$$\vec{x} = t_1\vec{u}_1 + t_2\vec{u}_2 = t_1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The vector partial derivatives create vectors $\vec{u}_1$, $\vec{u}_2$, which are called **Strang's Special Solutions**. ■

2. $x_1 = t_1 + 3t_2$, $x_2 = t_1$, $x_3 = 4t_2$, $x_4 = t_2$

3. $x_1 = t_1$, $x_2 = t_2$, $x_3 = 2t_1 + 3t_2$

Solution:

$$\vec{x} = t_1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad ■$$

4. $x_1 = 2t_1 + 3t_2 + t_3$, $x_2 = t_1$, $x_3 = t_2$, $x_4 = t_3$

Vector General Solution

Find the vector general solution $\vec{x}$ of $A\vec{x} = \vec{0}$.

5. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ Shortcut: augmented matrix $\left\langle A|\vec{0}\right\rangle$ not used.

$A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ `combo(1,2,-2)`, found **rref**(A).

Lead variable: x_1. Free variable: x_2

Assign symbols to free variables: $x_2 = t_1$ (invented symbol t_1)

Scalar Equations with isolated lead variables:

$x_1 = -2x_2,$

$0 = 0$

Substitute symbols and list variables in order:

$x_1 = -2t_1,$

$x_2 = t_1$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2t_1 \\ t_1 \end{pmatrix}$. There is only one partial derivative to find:

$\vec{u}_1 = \partial\vec{x}/\partial t_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

The vector general solution is

$\vec{x} = t_1\vec{u}_1 = t_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

There is no known efficient shortcut which finds the vector general solution without also finding the scalar general solution. The paper and pencil solution should be learned by a few examples. An answer check is done by computer as shown below. Computer answers often look different than paper and pencil answers. It can be nontrivial to see that both answers are correct!

```
# Vector General Solution: Exercise 5
with(LinearAlgebra):
A:=Matrix([[1,2],[2,4]]);
X:=LinearSolve(A,<0,0>,free='t');
u1:=map(x->diff(x,t[1]),X);
# u1 = <-2,1>
```

6. $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

7. $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: $\vec{x} = t_1\vec{u}_1 + t_2\vec{u}_2 = t_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. ■

8. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

9. $A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 \end{pmatrix}$

Solution: $\vec{x} = t_1\vec{u}_1 + t_2\vec{u}_2 = t_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. ■

10. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

Dimension

11. Give four examples in $\mathcal{R}^3$ of $S = \mathbf{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ (3 vectors required) which have respectively dimensions $0, 1, 2, 3$.

Solution: Let $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$ be the columns in order of the 3×3 identity matrix. Define
$S_0 = \{\vec{0}, \vec{0}, \vec{0}\}$, $S_1 = \{\vec{0}, \vec{0}, \vec{w}_1\}$, $S_2 = \{\vec{0}, \vec{w}_1, \vec{w}_2\}$, $S_3 = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.
The $\dim(S_i) = i$, $0 \le i \le 3$. ■

12. Give an example in $\mathcal{R}^3$ of 2-dimensional subspaces S_1, S_2 with only the zero vector in common.

13. Let $S = \mathbf{span}(\vec{v}_1, \vec{v}_2)$ in abstract vector space V. Explain why $\dim(S) \le 2$.

Solution: The dimension k of S is the number of vectors in a basis for S. Vectors $\vec{v}_1$, $\vec{v}_2$ already span S because of the equation $S = \mathbf{span}(\vec{v}_1, \vec{v}_2)$. If $\vec{v}_1$, $\vec{v}_2$ are independent then they form a basis for S and $\dim(S) = 2$. If $\vec{v}_1$, $\vec{v}_2$ are dependent and nonzero then one of them is a basis for S and $\dim(S) = 1$. If $\vec{v}_1 = \vec{v}_2 = \vec{0}$ then S is the span of the zero vector and $\dim(S) = 0$. ■

14. Let $S = \mathbf{span}(\vec{v}_1, \ldots, \vec{v}_k)$ in abstract vector space V. Explain why $\dim(S) \le k$.

15. Let S be a subspace of $\mathcal{R}^3$ with basis $\vec{v}_1, \vec{v}_2$. Define $\vec{v}_3$ to be the **cross product** of $\vec{v}_1, \vec{v}_2$. What is $\dim(\mathbf{span}(\vec{v}_2, \vec{v}_3))$?

Solution: $\dim(\mathbf{span}(\vec{v}_2, \vec{v}_3)) = 2$ because it is known that the cross product is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$, hence independent of both vectors. ■

16. Let S_1, S_2 be subspaces of $\mathcal{R}^4$ such that $\dim(S_1) = \dim(S_2) = 2$. Assume S_1, S_2 have only the zero vector in common. Prove or give a counter-example: the span of the union of S_1, S_2 equals $\mathcal{R}^4$.

Independence in Abstract Spaces

17. Assume linear combinations of vectors $\vec{v}_1$, $\vec{v}_2$ are uniquely determined, that is, $a_1\vec{v}_1 + a_2\vec{v}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$ implies $a_1 = b_1$, $a_2 = b_2$. **Prove** this result: If $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$, then $c_1 = c_2 = 0$.

Solution: Let $d_1 = d_2 = 0$. Write the hypothesis $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ in the form $c_1\vec{v}_1 + c_2\vec{v}_2 = d_1\vec{v}_1 + d_2\vec{v}_2$ and apply the uniqueness assumption: $c_1 = d_1 = 0$, $c_2 = d_2 = 0$. ■

18. Assume the zero linear combination of vectors $\vec{v}_1$, $\vec{v}_2$ is uniquely determined, that is, $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ implies $c_1 = c_2 = 0$. **Prove** this result: If $a_1\vec{v}_1 + a_2\vec{v}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$, then $a_1 = b_1$, $a_2 = b_2$.

19. Prove that two **nonzero** vectors $\vec{v}_1$, $\vec{v}_2$ in an abstract vector space V are independent if and only if each of $\vec{v}_1$, $\vec{v}_2$ is not a constant multiple of the other.

Solution: Organize the proof as $A <==> B$ where A is the independence statement and B is the constant multiple statement.

Proof $A => B$: Assume A: the vectors are independent. If B fails then $\vec{v}_1 = c\vec{v}_2$ or $\vec{v}_2 = c\vec{v}_1$ for some constant c. Both possibilities lead to an equation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ with one of c_1, c_2 equal to 1, implying dependence of the vectors, a violation to assumption A. Therefore, $A => B$.

Proof $B => A$: Assume B: neither of $\vec{v}_1$, $\vec{v}_2$ is a constant multiple of the other. To prove: independence of the vectors (conclusion A). Independence test: assume for some constants c_1, c_2 the equation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ holds. We show $c_1 = c_2 = 0$. Let's assume $c_1 = c_2 = 0$ is false. Renumbering allows the assumption $c_1 \neq 0$. Divide: $\vec{v}_1 + (c_2/c_1)\vec{v}_2 = \vec{0}$. Rearrange to equation $\vec{v}_1 = c\vec{v}_2$ where $c = -c_2/c_1$ is a constant. Equation $\vec{v}_1 = c\vec{v}_2$ violates hypothesis B, contradiction. ■

20. Let $\vec{v}_1$ be a vector in an abstract vector space V. Prove that the one-element set $\vec{v}_1$ is independent if and only if $\vec{v}_1$ is not the zero vector.

21. Let V be an abstract vector space and assume $\vec{v}_1, \vec{v}_2$ are independent vectors in V. Define $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{u}_2 = \vec{v}_1 + 2\vec{v}_2$. Prove that $\vec{u}_1$, $\vec{u}_2$ are independent in V.

Advice: Fixed vectors not assumed! Bursting the vector packages is impossible, there are no components.

Solution: The details are to use only the definition of vector space. A common error is to assume that the vectors have components, e.g., the vectors are fixed vectors from some $\mathcal{R}^n$. The error is the assumption that $V = \mathcal{R}^n$, which was never assumed.

The proof is organized as $A \Longrightarrow B$ where A is independence of $\vec{v}_1, \vec{v}_2$ and B is independence of $\vec{u}_1, \vec{u}_2$.

Assume A: Vectors $\vec{v}_1, \vec{v}_2$ are independent in V and $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{u}_2 = \vec{v}_1 + 2\vec{v}_2$.
To prove B: Vectors $\vec{u}_1, \vec{u}_2$ are independent in V.

Independence test: Assume $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$ and prove $c_1 = c_2 = 0$.
Expand the equation $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$ using the definitions of $\vec{u}_1, \vec{u}_2$:

$c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$ hypothesis of the independence test
$c_1(\vec{v}_1 + \vec{v}_2) + c_2(\vec{v}_1 + 2\vec{v}_2) = \vec{0}$ use definitions of $\vec{u}_1, \vec{u}_2$
$(c_1 + c_2)\vec{v}_1 + (c_1 + 2c_2)\vec{v}_2 = \vec{0}$ use the vector space toolkit
$d_1\vec{v}_1 + d_2\vec{v}_2 = \vec{0}$ where $d_1 = c_1 + c_2$, $d_2 = c_1 + 2d_2$

Independence of $\vec{v}_1, \vec{v}_2$ is applied to conclude $d_1 = 0$, $d_2 = 0$, which is the system of equations

$$c_1 + c_2 = 0, \quad c_1 + 2c_2 = 0 \ \text{ or } \ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The 2×2 matrix has an inverse, therefore $c_1 = c_2 = 0$, as was to be proved.

The proof of $A \Longrightarrow B$ is complete. ■

22. Let V be an abstract vector space and assume $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are independent vectors in V. Define $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{u}_2 = \vec{v}_1 + 4\vec{v}_2$, $\vec{u}_3 = \vec{v}_3 - \vec{v}_1$. Prove that $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ are independent in V.

23. Let S be a finite set of independent vectors in an abstract vector space V. Prove that none of the vectors can be the zero vector.

Solution: Let the vectors be listed as $\vec{v}_1, \ldots, \vec{v}_k$. The contrapositive statement $A \Longrightarrow B$ will be proved where A is the statement that one of the vectors is the zero vector and B is the statement that the vectors are dependent.

Assume A. By renumbering if necessary, assume $\vec{v}_1 = \vec{0}$. Then

$$1\vec{v}_1 + \sum_{i=2}^{k} 0\vec{v}_i = \vec{0}$$

By definition, the vectors are dependent. Hence B. ■

24. Let S be a finite set of independent vectors in an abstract vector space V. Prove that no vector in the list can be a linear combination of the other vectors.

The Spaces $\mathcal{R}^n$

25. **(Scalar Multiply)** Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ have components measured in centimeters. Report constants c_1, c_2, c_3 for re-scaled data $c_1\vec{x}$, $c_2\vec{x}$, $c_3\vec{x}$ in units of kilometers, meters and millimeters.

Solution: One meter = 100 cm = 1000 millimeters. One kilometer = 1000 meters.
Then $c_1 = (1/100)/1000$ kilometers, $c_2 = 1/100$ meters, $c_3 = (1/100)(1000) = 1/10$ millimeters

26. **(Matrix Multiply)** Let $\vec{u} = \begin{pmatrix} x_1, x_2, x_3, p_1, p_2, p_3 \end{pmatrix}^T$ have position x-units in kilometers and momentum p-units in kilogram-centimeters per millisecond. Determine a matrix M such that the vector $\vec{y} = M\vec{u}$ has SI units of meters and kilogram-meters per second.

27. Let $\vec{v}_1$, $\vec{v}_2$ be two independent vectors in $\mathcal{R}^n$. Assume $c_1\vec{v}_1 + c_2\vec{v}_2$ lies strictly interior to the parallelogram determined by $\vec{v}_1$, $\vec{v}_2$. Give geometric details explaining why $0 < c_1 < 1$ and $0 < c_2 < 1$.

Solution: If the two vectors are specialized to $\vec{\imath}$ and $\vec{\jmath}$, then the parallelogram is a square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. A vector $c_1\vec{\imath} + c_2\vec{\jmath}$ has tail at $(0,0)$ and head at (c_1, c_2). To be strictly inside the square means the head (c_1, c_2) is strictly inside the square. This happens exactly when the projections c_1, c_2 onto the axes satisfy $0 < c_1 < 1$ and $0 < c_2 < 1$.

A parallelogram maps to the unit square by matrix A chosen by the two requirements $A\vec{v}_1 = \vec{\imath}$, $A\vec{v}_2 = \vec{\jmath}$. The inside of the parallelogram maps to the inside of the unit square (intuitively so, a rigorous proof was not expected). So if $c_1\vec{v}_1 + c_2\vec{v}_2$ is strictly inside the parallelogram then $A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\vec{\imath} + c_2\vec{\jmath}$ is strictly inside the unit square, hence $0 < c_1 < 1$ and $0 < c_2 < 1$.

Why is A invertible? Because $A\left\langle \vec{v}_1 | \vec{v}_2 \right\rangle = I$. The definition of A as the inverse of $\left\langle \vec{v}_1 | \vec{v}_2 \right\rangle$ is possible because the two vectors are independent. ■

28. Prove the 4 scalar multiply toolkit properties for fixed vectors in $\mathcal{R}^3$.

29. Define

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, -\vec{v} = \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix}.$$

Prove the 4 addition toolkit properties for fixed vectors in $\mathcal{R}^3$.

Solution: The four rules are:

Addition	$\vec{X}+\vec{Y}=\vec{Y}+\vec{X}$	commutative
	$\vec{X}+(\vec{Y}+\vec{Z})=(\vec{X}+\vec{Y})+\vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0}+\vec{X}=\vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X}+(-\vec{X})=\vec{0}$	negative

Let $\vec{X}=\begin{pmatrix}-x_1\\-x_2\\-x_3\end{pmatrix}$ and similar notation for $\vec{Y}$ and $\vec{Z}$.

Commutative
LHS $=\vec{X}+\vec{Y}$
LHS $=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$
LHS $=\begin{pmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{pmatrix}$

RHS $=\vec{Y}+\vec{X}$
RHS $=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}+\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$
RHS $=\begin{pmatrix}y_1+x_1\\y_2+x_2\\y_3+x_3\end{pmatrix}$
Then LHS = RHS.

Associative
LHS $=\vec{X}+(\vec{Y}+\vec{Z})$
LHS $=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\left(\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}+\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}\right)$
LHS $=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}y_1+z_1\\y_2+z_2\\y_3+z_3\end{pmatrix}$
LHS $=\begin{pmatrix}x_1+y_1+z_1\\x_2+y_2+z_2\\x_3+y_3+z_3\end{pmatrix}$

RHS $=(\vec{X}+\vec{Y})+\vec{Z}$
RHS $=\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\right)+\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$
RHS $=\begin{pmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{pmatrix}+\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$

$$\text{RHS} = \begin{pmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 \\ x_3 + y_3 + z_3 \end{pmatrix}$$
Then LHS = RHS.

Zero
$\text{LHS} = \vec{0} + \vec{X}$
$$\text{LHS} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$\text{LHS} = \begin{pmatrix} 0 + x_1 \\ 0 + x_2 \\ 0 + x_3 \end{pmatrix}$$
$\text{LHS} = \vec{X}$
$\text{LHS} = \text{RHS}$

Negative
$\text{LHS} = \vec{X} + (-\vec{X})$
$$\text{LHS} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}$$
$$\text{LHS} = \begin{pmatrix} x_1 - x_1 \\ x_2 - x_2 \\ x_3 - x_3 \end{pmatrix}$$
$$\text{LHS} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$\text{LHS} = \vec{0}$
$\text{LHS} = \text{RHS} \quad \blacksquare$

30. Use the 8 property toolkit in $\mathcal{R}^3$ to prove that zero times a vector is the zero vector.

31. Let A be an invertible 3×3 matrix. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be a basis for $\mathcal{R}^3$. Prove that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is a basis for $\mathcal{R}^3$.

Solution: To prove: (1) $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is an independent set. Let $c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3 = \vec{0}$. Multiply my A^{-1} to obtain the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Because $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are given to be independent (they are a basis), then $c_1 = c_2 = c_3 = 0$. This proves $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ are independent.

To prove: (2) Vectors $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ span $\mathcal{R}^3$. Let $\vec{y}$ be any vector in $\mathcal{R}^3$. Constants c_1, c_2, c_3 must be found such that $c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3 = \vec{y}$. Multiply by A^{-1} to obtain the new equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = A^{-1}\vec{y}$. Hypothesis $\mathcal{R}^3 = \textbf{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ implies c_1, c_2, c_3 exist. $\blacksquare$

32. Let A be an invertible 3×3 matrix. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be dependent in $\mathcal{R}^3$. Prove that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is a dependent set in $\mathcal{R}^3$.

Digital Photographs

Let V be the vector space of all 2×3 matrices. A matrix in V is a 6-pixel digital photo, a sub-section of a larger photo.

Let $B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, …, $B_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Each B_j **lights up** one pixel in the 2×3 sub-photo.

33. Prove that B_1, …, B_6 are independent and span V: they are a **basis** for V.

Solution: Because $\sum_{j=1}^{6} c_j B_j = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{pmatrix}$ then $\sum_{j=1}^{6} c_j B_j$ is the zero matrix if and only if c_1 to c_6 are zero. This proves independence and also the span condition $\mathbf{span}(B_1, \ldots, B_6) = V$. ■

34. Let $A = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Assume a black and white image and 0 means black. Describe photo A, from the checkerboard analogy.

Digital RGB Photos

Define red, green and blue monochrome matrices R, G, B by

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 5 & 8 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 5 \end{pmatrix}.$$

35. Define base $x = 16$. Compute $A = R + xG + x^2 B$.

Solution: $A = \begin{pmatrix} 1330 & 0 & 0 \\ 0 & 833 & 1 \\ 261 & 24 & 1281 \end{pmatrix}$

According to the checkerboard analogy, the board has 9 checkers. Number 1330 is an *encoded checker count* at pixel location $(1, 1)$, representing 2 red, 3 green and 5 blue.

```
# Digital RGB Photos, Exercise 35
R:=Matrix([[2,0,0],[0,1,1],[5,8,1]]);
G:=Matrix([[3,0,0],[0,4,0],[0,1,0]]);
B:=Matrix([[5,0,0],[0,3,0],[1,0,5]]);
A:=R+x*G+x^2*B; subs(x=16,A);
# [[1330, 0, 0], [0, 833, 1], [261, 24, 1281]]
```

36. Define base $x = 32$. Compute $A = R + xG + x^2 B$.

Polynomial Spaces

Let V be the vector space of all cubic or less polynomials $p(x) = c_0 + c_1x + c_2x^2 + c_3x^3$.

37. Find a subspace S of V, $\dim(S) = 2$, which contains the vector $1 + x$.

Solution: Let $S = \mathbf{span}(1+x, 1)$. Set S is a subspace by the span theorem. Then $1 + x$ is in S. Because 1, $1 + x$ are independent then $\dim(S) = 2$. ■

38. Let S be the subset of V spanned by x, x^2 and x^3. Prove that S is a subspace of V which does not contain the polynomial $1 + x$.

39. Define set S by the conditions $p(0) = 0, p(1) = 0$. Find a basis for S.

Solution: The conditions on $p(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ are $0 = p(0) = c_0$ and $0 = p(1) = c_1 + c_2 + c_3$. Select initially basis elements x, x^2 which satisfy $p(0) = 0$ and then add $c_1x + c_2x^2 + c_3x^3$ with $c_1 + c_2 + c_3 = 0$ by choosing $c_3 = 2$, $c_1 = c_2 = -1$. Then $S = \mathbf{span}(x, x^2, 2x^3 - x^2 - x)$. Independence is proved by the Wronskian Test:

$$\left|\left.\begin{vmatrix} x & x^2 & 2x^3 - x^2 - x \\ 1 & 2x & 6x^2 - 2x - 1 \\ 0 & 2x & 12x - 2 \end{vmatrix}\right|\right._{x=1} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -2 \end{vmatrix} = 2$$

40. Define set S by the condition $p(0) = \int_0^1 p(x)dx$. Find a basis for S.

The Space $C(E)$

Define $\vec{f}$ to be the vector package with domain $E = \{x \;:\; -2 \le x \le 2\}$ and equation $y = |x|$. Similarly, $\vec{g}$ is defined by equation $y = x$.

41. Show independence of $\vec{f}, \vec{g}$.

Solution: Because f is not differentiable then the Wronskian test does not apply. We'll try to use the sampling test. Select samples $x_1 = 1$ and $x_2 = -1$. Then the sample matrix is

$$S = \begin{pmatrix} f(x_1) & g(x_1) \\ f(x_2) & g(x_2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Because $|S| = -2$, then f and g are independent functions on E, meaning $\vec{f}$ and $\vec{g}$ are independent vectors in $C(E)$. ■

42. Find the dimension of $\mathbf{span}(\vec{f}, \vec{g})$.

43. Let $h(x) = 0$ on $-1 \le x \le 0$, $h(x) = -x$ on $0 \le x \le 1$. Show that $\vec{h}$ is in $C(E)$.

Solution: The issue is continuity of h at $x = 0$. The left and right hand limits at $x = 0$ are both equal to 0, therefore h is continuous a t $x = 0$. ■

44. Let $h(x) = -1$ on $-2 \le x \le 0$, $h(x) = 1$ on $0 \le x \le 2$. Show that $\vec{h}$ is not in $C(E)$.

45. Let $h(x) = 0$ on $-2 \le x \le 0$, $h(x) = -x$ on $0 \le x \le 2$. Show that $\vec{h}$ is in $\mathbf{span}(\vec{f}, \vec{g})$.

Solution: Assume $\vec{h}$ is in $\mathbf{span}(\vec{f}, \vec{g})$. Let $\vec{h} = c_1\vec{f} + c_2\vec{g}$. Then $c_1|x| + c_2x = h(x)$ at $x = -1$ implies equation $c_1 - c_2 = 0$ while at $x = 1$ it implies equation $c_1 + c_2 = 0$. Then $c_1 = c_2 = 0$ and $h(x) = 0$ for all x, a contradiction to the definition of h. ■

46. Let $h(x) = \tan(\pi x/2)$ on $-2 < x < 2$, $h(2) = h(-2) = 0$. Explain why $\vec{h}$ is not in $C(E)$

The Space $C^1(E)$

Define $\vec{f}$ to be the vector package with domain $E = \{x \;:\; -1 \le x \le 1\}$ and equation $y = x|x|$. Similarly, $\vec{g}$ is defined by equation $y = x^2$.

47. Verify that $\vec{f}$ is in $C^1(E)$, but its derivative is not.

Solution: For $y = x|x|$ the derivative is: $y' = 2x$ for $x > 0$, $y' = -2x$ for $x < 0$, $y' = 0$ at $x = 0$. Simplified: $y' = 2|x|$. This function is continuous but not continuously differentiable, therefore $\vec{f}$ is in $C^1(E)$ but $\vec{f'}$ is not in $C^1(E)$. ■

48. Show that $\vec{f}, \vec{g}$ are independent in $C^1(E)$.

The Space $C^k(E)$

49. Compute the first three derivatives of $y(x) = e^{-x^2}$ at $x = 0$.

Solution: Expand as a power series: $y(x) = \sum_{n=0}^{\infty}(-x^2)^n/n!$. Then $y(x) = 1 - x^2 + x^4/2 - \cdots$ which produces the answers $y(0) = 0$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = 0$.

50. Justify that $y(x) = e^{-x^2}$ belongs to $C^k(0, 1)$ for all $k \ge 1$.

51. Prove that the span of a finite list of distinct Euler solution atoms (page 386) is a subspace of $C^k(E)$ for any interval E.

Solution: Euler atoms are in $C^k(E)$. The span of a finite set of vectors is a subspace.

52. Prove that $y(x) = |x|$ is in $C^k(0, 1)$ but not in $C^1(-1, 1)$.

Solution Space

A differential equations solver finds general solution $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$. Use vector space $V = C^4(E)$ where E is the whole real line.

53. Write the solution set S as the span of four vectors in V.

Solution: The technique to discover a basis is to formally differentiate the general solution on the symbols c_1 to c_4. Then basis elements might be 1, x, e^x, e^{-x}. At least S is the span of the four vectors just found. Because distinct Euler solution atoms are independent, then indeed the four vectors are a basis for S.

54. Find a basis for the solution space S of the differential equation. Verify independence using the sampling test or Wronskian test.

55. Find a differential equation $y'' + a_1 y' + a_0 y = 0$ which has solution $y = c_1 + c_2 x$.

Solution: Substitute $y = c_1 + c_2 x$ into $y'' + a_1 y' + a_0 y = 0$ to arrive at $a_1 c_2 + a_0(c_1 + c_2 x) = 0$. Because 1, x are independent then $a_1 c_2 + a_0 c_1 = 0$ and $a_0 c_2 = 0$. The equations are valid for all c_1, c_2 provided $a_0 = a_1 = 0$. The required differential equation is $y'' = 0$.

56. Find a differential equation $y'''' + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0$ which has solution $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Algebraic Independence Test for Two Vectors

Solve for c_1, c_2 in the independence test for two vectors, showing all details.

57. $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution:

$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$

$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$c_1 = c_2 = 0$ because the determinant of coefficients is nonzero.

58. $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Dependence of two vectors

Solve for c_1, c_2 not both zero in the independence test for two vectors, showing all details for dependency of the two vectors.

59. $\vec{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}2\\4\end{pmatrix}$

Solution:

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$

$c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$c_1 + 2c_2 = 0$ or $c_1 = -2t_1$, $c_2 = t_1$ (infinitely many solutions case).
Then for any value of t_1 this dependency relation holds:

$-2t_1\vec{v}_1 + t_1\vec{v}_2 = \vec{0}$

60. $\vec{v}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}-2\\2\\0\end{pmatrix}$

Independence Test for Three Vectors

Solve for the constants c_1, c_2, c_3 in the relation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Report dependent of independent vectors. If dependent, then display a dependency relation.

61. $\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}$

Solution: Exercise 57 has the method, which produces the matrix equation

$$\begin{pmatrix}1 & -1 & 0\\-1 & 2 & 2\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

The coefficient matrix A has rank 2, hence there are infinitely many solutions. Reduce matrix A to $\mathbf{rref}(A) = \begin{pmatrix}1&0&2\\0&1&2\\0&0&0\end{pmatrix}$. Conclude $c_1 = -2t_1$, $c_2 = -2t_1$, $c_3 = t_1$. The dependency relation is

$$-2t_1\begin{pmatrix}1\\-1\\0\end{pmatrix} - 2t_1\begin{pmatrix}-1\\2\\0\end{pmatrix} + t_1\begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

62. $\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$

Independence in an Abstract Vector Space

In vector space V, report independence or a dependency relation for the given vectors.

63. Space $V = C(-\infty,\infty)$, $\vec{v}_1 = 1+x$, $\vec{v}_2 = 2+x$, $\vec{v}_3 = 3+x^2$.

Solution: The independence test:

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$
$c_1(1+x) + c_2(2+x) + c_3(3+x^2) = 0$
$c_1 + 2c_2 + 3c_3 + (c_1+c_2)x + c_3x^2 = 0$
Independence of 1, x, x^2 (distinct Euler solution atoms) implies the equations

$c_1 + 2c_2 + 3c_3 = 0, \quad c_1 + c_2 = 0, \quad c_3 = 0$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then $c_1 = c_2 = c_3 = 0$ because the determinant of coefficients is nonzero. The vectors are independent.

64. Space $V = C(-\infty,\infty)$, $\vec{v}_1 = x^{3/5}$, $\vec{v}_2 = x^2$, $\vec{v}_3 = 2x^2 + 3x^{3/5}$

65. Space V is all 3×3 matrices. Let
$\vec{v}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 2 & 5 \\ 0 & 3 & 5 \end{pmatrix}$.

Solution: Dependent because $2\vec{v}_1 + 3\vec{v}_2 = \vec{v}_3$. The independence test gives rise to 9 equations in 3 unknowns c_1, c_2, c_3. It helps to think of the matrices as column vectors of length 9.

66. Space V is all 2×2 matrices. Let
$\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$,
$\vec{v}_3 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$.

Rank Test

Compute the rank of the augmented matrix to determine independence or dependence of the given vectors.

67. $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The rank is 2, dependent.

68. $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Determinant Test

Evaluate the determinant of the augmented matrix to determine independence or dependence of the given vectors.

69. $\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$. Then $|A| = 0$ because A has a row of zeros. Dependent.

70. $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Sampling Test for Functions

Invent samples to verify independence.

71. $\cosh(x), \sinh(x)$

Solution: Choose samples $x_1 = 0$, $x_2 = 1$. Then the sampling matrix is

$$A = \begin{pmatrix} \cosh(x_1) & \sinh(x_1) \\ \cosh(x_2) & \sinh(x_2) \end{pmatrix} = \begin{pmatrix} \cosh(0) & \sinh(0) \\ \cosh(1) & \sinh(1) \end{pmatrix}$$

Then $|A| = \sinh(1) \neq 0$ which implies independence.

72. $x^{7/3}, x\sin(x)$

73. $1, x, \sin(x)$

Solution: Choose samples $x_1 = 0$, $x_2 = \pi$, $x_3 = \pi/2$. Then the sampling matrix is

$$A = \begin{pmatrix} 1 & x_1 & \sin(x_1) \\ 1 & x_2 & \sin(x_2) \\ 1 & x_3 & \sin(x_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \pi & 0 \\ 1 & \pi/2 & 1 \end{pmatrix}$$

Then $|A| = \pi \neq 0$ which implies independence.

74. $1, \cos^2(x), \sin(x)$

Sampling Test and Dependence

For three functions f_1, f_2, f_3 to be dependent, constants c_1, c_2, c_3 must be found such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0.$$

The trick is that c_1, c_2, c_3 are not all zero and the relation holds **for all** x. The sampling test method can discover the constants, but it is **unable to prove dependence**!

75. Functions $1, x, 1+x$ are dependent. Insert $x = 1, 2, -1$ and solve for c_1, c_2, c_3, to discover a dependency relation.

Solution: The relation is

$c_1 + c_2 x + c_3(1+x) = 0$

Insert samples $x = 0, 1, 2$:

$c_1 + c_2 + 2c_3 = 0 \quad$ for $x = 1$
$c_1 + 2c_2 + 3c_3 = 0 \quad$ for $x = 2$
$c_1 - c_2 = 0 \quad$ for $x = -1$

Then arrange as a system of equations:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The reduced echelon form of is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

giving solution $c_1 = -t_1$, $c_2 = -t_1$, $c_3 = t_1$ and the possible dependency relation

$$-t_1 + (-t_1)x + t_1(1_x) = 0$$

Cancel t_1 to get $-1 + x + (1+x) = 0$, which is true for all x, proving that the three vectors are dependent. ■

The samples $x = 0, 1, 2$ are unsuccessful in this adventure, showing that this discovery method might fail for one set of samples and succeed for a different set of samples.

76. Functions $1, \cos^2(x), \sin^2(x)$ are dependent. Cleverly choose 3 values of x, insert them, then solve for c_1, c_2, c_3, to discover a dependency relation.

Vandermonde Determinant

77. Let $V = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$. Verify by direct computation the formula

$$|V| = x_2 - x_1.$$

Solution: $|V| = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = 1(x_2) - (x_1)(1) = x_2 - x_1.$

78. Let $V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$. Verify by direct computation the formula

$$|V| = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

Wronskian Test for Functions

Apply the Wronskian Test to verify independence.

79. $\cos(x), \sin(x)$.

Solution: Choose $x = 0$, then the Wronskian is $\begin{vmatrix} \cos(0) & \sin(0) \\ -\sin(0) & \cos(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$

80. $\cos(x), \sin(x), \sin(2x)$.

81. $x, x^{5/3}$.

Solution: Choose $x = 1$, then the Wronskian is $\begin{vmatrix} 1 & 1 \\ 1 & 5/3 \end{vmatrix} = 2/3.$

82. $\cosh(x), \sinh(x)$.

Wronskian Test: Theory

83. The functions x^2 and $x|x|$ are continuously differentiable and have zero Wronskian. Verify that they **fail to be dependent** on $-1 < x < 1$.

Solution: Function $y(x) = x|x|$ has derivative $y' = 2|x|$. The Wronskian of x^2 and $x|x|$ is $\begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = x|x| \begin{vmatrix} x & x \\ 2 & 2 \end{vmatrix} = 0$. Independence holds on $-1 < x < 1$ because $c_1x^2 + c_2x|x| = 0$ can be solved for $c_1 = c_2 = 0$ by using the sampling test with samples $x_1 = -1/2$, $x_2 = 1/2$.

84. The Wronskian Test can verify the independence of the powers $1, x, \ldots, x^k$. Show the determinant details.

Extracting a Basis

Given a list of vectors in space $V = \mathcal{R}^4$, extract a largest independent subset.

85. $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix},$
$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

Solution: Let A be the augmented matrix of the vectors:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced row-echelon form of A is

$$\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced form tells us this: the first two vectors are independent, by the rank test, because if A was the augmentation of the first two vectors, then the reduced form would be the first two columns of $\mathbf{rref}(A)$. Similarly, adding columns 3, 5 to A has a reduced form with rank 2, so the added columns cannot be independent of the first two columns. Adding column 4 increases the rank, so column 4 is independent of the first two columns. Adding column 5 does not increase the rank, so column 5 cannot be independent of the preceding columns.

Collecting, a largest independent subset of the vectors is

$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

```
# Extracting a Basis, Exercise 85
 A:=Matrix([[1,-1,0,0,-1],[-1,2,2,-1,1],
           [0,0,0,1,1],[0,0,0,0,0]]);
ReducedRowEchelonForm(A);
# rref=[[1, 0, 2, 0, 0], [0, 1, 2, 0, 1],
#  [0, 0, 0, 1, 1], [0, 0, 0, 0, 0]]
```

86. $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix},$
$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Extracting a Basis

Given a list of vectors in space $V = C(-\infty, \infty)$, extract a largest independent subset.

87. $x, x\cos^2(x), x\sin^2(x), e^x, x + e^x$

Solution: Convert the square terms by trig identities $\cos^2(x) + \sin^2(x) = 1$, $\cos(2x) = 2\cos^2(x) - 1$, $\cos(2x) = 1 - 2\sin^2(x)$. Then the list becomes

$$x, \quad \frac{x}{2} + \frac{x}{2}\cos(2x)), \quad \frac{x}{2} - \frac{x}{2}\cos(2x), \quad e^x, \quad x + e^x$$

The idea is to change the spanning set without changing the span. First fact: Multiplying a spanning vector by a constant $c \neq 0$ does change the span. The replacement set gets rid of the $\frac{1}{2}$ appearing four times:

$$x, \quad x + x\cos(2x)), \quad x - x\cos(2x), \quad e^x, \quad x + e^x$$

Second fact: Vectors $\vec{f}$ and $\vec{g}$ are independent if and only if $\vec{f}$ and $\vec{g} - \vec{f}$ are independent. The proof depends on two vectors being independent if and only if each vector is not a scalar multiple of the other vector. The replacement set gets rid of three occurrences of x:

$$x, \quad x\cos(2x)), \quad -x\cos(2x), \quad e^x, \quad e^x$$

The first fact applies to remove the single minus sign for replacement set

$$x, \quad x\cos(2x)), \quad x\cos(2x), \quad e^x, \quad e^x$$

The first four functions are distinct Euler solution atoms, therefore they are independent. The fifth function is a duplicate. So the first four are independent. In the original set, a largest list of independent spanning vectors is the first four:

$$x, x\cos^2(x), x\sin^2(x), e^x$$

88. $1, 2 + x, \frac{x}{1+x^2}, \frac{x^2}{1+x^2}$

Euler Solution Atom

Identify the Euler solution atoms in the given list. Strictly apply the definition: e^x is an atom but $2e^x$ is not.

89. $1, 2+x, e^{2.15x}, e^{x^2}, \frac{x}{1+x^2}$

Solution: x, e^x

90. $2, x^3, e^{x/\pi}, e^{2x+1}, \ln|1+x|$

Euler Solution Atom Test

Establish independence of set S_1.

Suggestion: First establish an identity $\mathbf{span}(S_1) = \mathbf{span}(S_2)$, where S_2 is an invented list of distinct atoms. The Test implies S_2 is independent. Extract a largest independent subset of S_1, using independence of S_2.

91. Set S_1 is the list $2, 1+x^2, 4+5e^x, \pi e^{2x+\pi}, 10x\cos(x)$.

Solution: Two facts will be used, discussed above in the solution to Exercise 87.

Fact 1: Multiplying a spanning vector by a constant $c \neq 0$ does change the span.
Fact 2: Vectors $\vec{f}$ and $\vec{g}$ are independent if and only if $\vec{f}$ and $\vec{g} - \vec{f}$ are independent.

Using both facts, $\mathbf{span}(S_1) = \mathbf{span}(S_2)$ where

$$S_2 = \{1, x^2, e^x, e^{2x}, x\cos(x)\}$$

Specifically used is exponential identity $\pi e^{2x+\pi} = ce^{2x}$ where $c = \pi e^\pi$. Fact 1 was used to replace S_1 by the set

$$1, 1+x^2, 4+5e^x, e^{2x}, x\cos(x)$$

Fact 2 was then employed to replace the preceding set by

$$1, x^2, 5e^x, e^{2x}, x\cos(x)$$

Fact 1 was used again to replace the above by

$$1, x^2, e^x, e^{2x}, x\cos(x)$$

Conclusion: Set S_2 is a set of distinct Euler solution atoms, therefore it is independent. Then the first five in the original list are independent, so S_1 itself is a largest independent subset.

92. Set S_1 is the list $1+x^2, 1-x^2, 2\cos(3x), \cos(3x)+\sin(3x)$.

Solution: First, third and fourth make a largest independent subset.

5.5 Basis, Dimension and Rank

Basis and Dimension

Compute a basis and the report the dimension of the subspace S.

1. In $\mathcal{R}^3$, S is the solution space of

$$\left|\begin{array}{rrrrrrr} x_1 & & & + & x_3 & = & 0, \\ & & x_2 & + & x_3 & = & 0. \end{array}\right|$$

Solution: Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, the coefficient matrix for $A\vec{x} = \vec{0}$, $\vec{x}$ with components x_1, x_2, x_3. The extra equation $0 = 0$ was appended to create an equivalent 3×3 system.
Matrix A equals $\mathbf{rref}(A)$. The last frame algorithm applies to find general scalar solution $x_1 = -t_1$, $x_2 = -t_1$, $x_3 = t_1$ in terms of invented symbol t_1. The vector general solution is then

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t_1 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

The partial derivative $\partial\vec{x}/\partial t_1$ is a basis, equivalent to setting $t_1 = 1$ in the vector general solution. Then

$$S = \mathbf{span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}, \qquad \dim(S) = 1$$

2. In $\mathcal{R}^4$, S is the solution space of

$$\left|\begin{array}{lrrr} x_1 + 2x_2 + x_3 & & = & 0, \\ & x_4 & = & 0. \end{array}\right|$$

Solution: Follow Exercise 1, A is 4×4 and $\dim(S) = 2$.

3. In $\mathcal{R}^2$, $S = \mathbf{span}(\vec{v}_1, \vec{v}_2)$. Vectors $\vec{v}_1, \vec{v}_2$ are columns of an invertible matrix.

Solution: A matrix is invertible if and only if it is square and the columns are independent. Therefore $\vec{v}_1, \vec{v}_2$ are independent and form a basis for S with $\dim(S) = 2$.

4. Set $S = \mathbf{span}(\vec{v}_1, \vec{v}_2)$, in $\mathcal{R}^4$. The vectors are columns in a 4×4 invertible matrix.

5. Set $S = \mathbf{span}(\sin^2 x, \cos^2 x, 1)$, in the vector space V of continuous functions.

Solution: The first two functions are independent by the sampling test applied with samples $x_1 = 0$ and $x_2 = \pi/2$.

Details: the nonsingular sampling matrix is

$$\begin{pmatrix} \sin^2(x_1) & \cos^2(x_1) \\ \sin^2(x_2) & \cos^2(x_2) \end{pmatrix} = \begin{pmatrix} \sin^2(0) & \cos^2(0) \\ \sin^2(\pi/2) & \cos^2(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The third function satisfies $1 = \cos^2(x) + \sin^2(x)$ for all x, therefore the three functions are dependent. A basis is the first two functions. The dimension of S is two.

6. Set $S = \mathbf{span}(x, x-1, x+2)$, in the vector space V of all polynomials.

7. Set $S = \mathbf{span}(\sin x, \cos x)$, the solution space of $y'' + y = 0$.

Solution: Distinct Euler solution atoms are independent. Therefore S has basis $\sin x, \cos x$ and $\dim(S) = 2$.

8. Set $S = \mathbf{span}\left(e^{2x}, e^{3x}\right)$, the solution space of $y'' - 5y' + 6y = 0$.

Euclidean Spaces

9. Let A be 3×2. Why is it impossible for the columns of A to be a basis for $\mathcal{R}^3$?

Solution: A basis for $\mathcal{R}^3$ has to have 3 independent vectors. To justify this with fewest support theorems, observe that the three columns of the 3×3 identity matrix are a basis for $\mathcal{R}^3$ (the standard basis of $\mathcal{R}^3$). All bases have the same number of elements, so $\mathcal{R}^3$ cannot have a basis of 2 elements.

10. Let A be $m \times n$. What condition on indices m, n implies it is impossible for the columns of A to be a basis for $\mathcal{R}^m$?

11. Find a pairwise orthogonal basis for $\mathcal{R}^3$ which contains $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Solution: A basis for $\mathcal{R}^3$ must have 3 elements. The first will be the given vector. The other two have to be constructed. A geometrical construction idea is to think of the given vector $\vec{v}_1$ as the cross product of two orthogonal vectors $\vec{v}_2$, $\vec{v}_3$. Because the cross product is orthogonal to $\vec{v}_2$, $\vec{v}_3$ then $\vec{v}_1$ is independent of $\vec{v}_2$, $\vec{v}_3$. Already $\vec{v}_2$, $\vec{v}_3$ are independent because they are orthogonal (and nonzero). The construction leads to three orthogonal vectors, known to be independent, and therefore $\mathcal{R}^3 = \mathbf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and the three vectors are a basis for $\mathcal{R}^3$.

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\vec{v}_1 \cdot \vec{v}_2 = a + b - c = 0$ (orthogonality condition)

$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ satisfies $a + b - c = 0$

To find $\vec{v}_3$ requires another solution to $a + b - c = 0$ with additional requirement $\vec{v}_2 \cdot \vec{v}_3 = a + c = 0$.

Choose a, b, c again:

$\vec{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ satisfies $a + b - c = 0$ and $a + c = 0$

The three vectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

We check the conditions $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$ and that all vectors are nonzero. By the Orthogonal basis theorem they form a basis for $\mathcal{R}^3$.

12. Display a basis for $\mathcal{R}^4$ which contains the independent columns of $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

13. Let S be a subspace of $\mathcal{R}^{10}$ of dimension 5. Insert a basis for S into an $m \times n$ augmented matrix A. What are m and n?

Solution: $m = 10 =$ number of component of a vector in $\mathcal{R}^{10}$. $n = 5 =$ number of vectors in a basis for S.

14. Suppose A and B are 3×3 matrices and let $C = AB$. Assume the columns of A are not a basis for $\mathcal{R}^3$. Is there a matrix B so that the columns of C form a basis for $\mathcal{R}^3$?

Solution: No. The determinant product theorem provides $|C| = |A||B|$. Independent columns means the determinant is nonzero. Use these hints to complete the proof.

15. The term **Hyperplane** is used for an equation like $x_4 = 0$, which in $\mathcal{R}^4$ defines a subspace S of dimension 3. Find a basis for S.

Solution: Choose three columns of the 4×4 identity matrix all of which have last component zero.

16. Find a 3-dimensional subspace S of $\mathcal{R}^4$ which has no basis consisting of columns of the identity matrix.

Solution: Define S by an equation $ax_1 + bx_2 + cx_3 + dx_4 = 0$, which makes S have dimension 3. Choose a, b, c, d so that no column of the identity matrix satisfies the equation.

Polynomial Spaces

Symbol V is the vector space of all polynomials $p(x)$. Given subspace S of V, find a basis for S and $\dim(S)$.

17. The subset S of $\mathbf{span}(1, x, x^2)$ is defined by $\frac{dp}{dx}(1) = 0$.

Solution: Let $p(x) = a + bx + cx^2$ and compute the condition $0 = p'(1)$ to be $0 = b + 2cx|_{x=1} = b + 2c$. Then $p(x) = a + bx + cx^2 = a - 2cx + cx^2$ depends only on symbols a, c. Differentiate on symbols a, c to identify a possible basis: $\partial p/\partial a = 1$, $\partial p/\partial c = -2x + x^2$. We must prove that $S = \mathbf{span}\{1, -2x + x^2\}$ and that the two vectors $1, -2x + x^2$ form a basis for S. Any vector in S has to look like $p(x) = a + c(-2x + x^2)$ by the preceding analysis. So the two vectors span S, meaning $S = \mathbf{span}\{1, -2x + x^2\}$. It remains to prove they are independent vectors. Let's appeal to a general vector space result: two vectors are independent if and only if each is not a multiple of the other. Geometry finishes the proof: $y = 1$ is a line of slope 0 while $y = -2x + x^2$ has nonzero slope. $\blacksquare$

18. The subset S of $\mathbf{span}(1, x, x^2, x^3)$ is defined by $p(0) = \frac{dp}{dx}(1) = 0$.

19. The subset S of $\mathbf{span}(1, x, x^2)$ is defined by $\int_0^1 p(x)dx = 0$.

Solution: Let $p(x) = a + bx + cx^2$. Then

$0 = \int_0^1 p(x)dx = ax + bx^2/2 + cx^3/3\Big|_{x=0}^{x=1} = a + b/2 + c/3$

which can be written as the equation $6a + 3b + 2c = 0$. This equation defines a plane in $\mathcal{R}^3$ with vector components a, b, c. The scalar general solution of the equation is $a = -b/2 - c/3$, $b = b$, $c = c$ without inventing symbols, because b, c can be used for the usual symbols t_1, t_2. Independent $\mathcal{R}^3$ solutions are $a = -1, b = 2, c = 0$ and $a = -1, b = 0, c = 3$. These correspond to polynomials $p_1(x) = -1 + 2x$ and $p_2(x) = -1 + 3x^2$. One polynomial is linear, the other is quadratic, so each is not a multiple of the other: they are independent. Both are in S and $S = \mathbf{span}\{p_1, p_2\}$ by the preceding analysis. The basis is p_1, p_2. $\blacksquare$

20. The subset S of $\mathbf{span}(1, x, x^2, x^3)$ is defined by $\int_0^1 xp(x)dx = 0$.

Differential Equations

Find a basis for solution subspace S. Assume the general solution of the 4th order linear differential equation is

$$y(x) = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}.$$

21. Subspace S_1 is defined by $y(0) = \frac{dy}{dx}(0) = 0$.

Solution: Coefficients are determined by the conditions:

$y(0) = 0$ is equivalent to $c_1 = 0$
$y'(0) = 0$ is equivalent to $c_2 + c_3 - c_4 = 0$

There are two linear equations in four unknowns c_1 to c_4. Lead variables are c_1, c_2 and the free variables are c_3, c_4. The scalar general solution is $c_1 = 0$, $c_2 = -t_1 + t_2$, $c_3 = t_1$, $c_4 = t_2$. In $\mathcal{R}^4$ with vector components c_1 to c_4 there are two independent solutions: $c_1 = 0$, $c_2 = -1$, $c_3 = 1$, $c_4 = 0$ and $c_1 = 0$, $c_2 = 0$, $c_3 = 1$, $c_4 = 1$. These solutions correspond to solutions $y_1() = -x + e^x$, $y_2(x) = e^x + e^{-x}$. Geometrically, each of y_1, y_2 is not a scalar multiple of the other: they are independent. Then $S_1 = \mathbf{span}\{y_1, y_2\}$ with basis y_1, y_2.

22. Subspace S_2 is defined by $y(1) = 0$.

23. Subspace S_3 is defined by $y(0) = \int_0^1 y(x)dx$.

Solution: The condition for symbols c_1 to c_4:

$$\begin{aligned}
&y(0) = \int_0^1 y(x)dx \\
&c_1 + c_3 + c_4 = c_1 x + c_2 x^2/2 + c_3 e^x - c_4 e^{-x}\Big|_{x=0}^{x=1} \\
&c_1 + c_3 + c_4 = c_1 + c_2/2 + c_3 e - c_4 e^{-1} - c_3 + c_4 \\
&(-1/2)c_2 + (2 - e)c_3 + c_4 e^{-1} = 0
\end{aligned}$$

This linear equation in variables c_1, c_2, c_3, c_4 is a hyperplane in $\mathcal{R}^4$ of dimension 3. The scalar general solution is $c_1 = t_1$, $c_2 = (4 - 2e)c_3 + 2c_4/e = (4-2e)t_2 + 2t_3/e$, $c_3 = t_2$, $c_4 = t_3$. Three independent solutions are obtained by letting t_1, t_2, t_3 assume the 3 values in each column of the 3×3 identity matrix:

$c_1 = 1$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$ Identity column 1, $t_1 = 1, t_2 = 0, t_3 = 0$
$c_1 = 0$, $c_2 = 4 - 2e$, $c_3 = 1$, $c_4 = 0$ Identity column 2, $t_1 = 0, t_2 = 1, t_3 = 0$
$c_1 = 0$, $c_2 = 2/e$, $c_3 = 0$, $c_4 = 1$ Identity column 3, $t_1 = 0, t_2 = 0, t_3 = 1$

Then correspondingly

$y_1(x) = 1$
$y_2(x) = (4 - 2e)x + e^x$
$y_3(x) = 2x/e + e^{-x}$
$S_3 = \mathbf{span}\{y_1, y_2, y_3\}$ and y_1, y_2, y_3 is the basis.

24. Subspace S_4 is defined by $y(1) = 0, \int_0^1 y(x)dx = 0$.

Largest Subset of Independent Vectors

Find a largest independent subset of the given vectors.

25. The columns of $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$.

Solution: Let A denote the matrix, then

$$\mathbf{rref}(A) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Columns 2 and 3 of A are independent.

26. The columns of $\begin{pmatrix} 3 & 1 & 2 & 0 & 5 \\ 2 & 1 & 1 & 0 & 4 \\ 3 & 2 & 1 & 0 & 7 \\ 1 & 0 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 & 7 \end{pmatrix}$.

27. The polynomials $x, 1+x, 1-x, x^2$.

Solution:

First method: $1 - x = c_1(1) + c_2(1+x)$ for $c_1 = 2$, $c_2 = -1$, therefore the first two are independent and the third depends on the first two. The last one x^2 is not a scalar multiple of either 1 or $1+x$, therefore the largest independent set is $1, 1+x, x^2$.

Second method: The linear map $T: \ a + bx + cx^2 \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is one-to-one and onto from $V = \mathbf{span}\{1, x, x^2\}$ to $\mathcal{R}^3$. Then

$$T(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T(1+x) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad T(1-x) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad T(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{then} \quad \mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Columns 1,2,4 of A are independent. The inverse images of these columns under T are $x, 1+x, x^2$, which is a largest independent set.

28. The continuous functions x, e^x, $x + e^x$, e^{2x}.

Pivot Theorem Method

Extract a largest independent set from the columns of the given matrix A. The answer is a list of independent columns of A, called the pivot columns of A.

29. $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$

Solution: Let A be the matrix. Then $\mathbf{rref}(A)$ is the identity matrix. The pivot theorem applies to conclude all three columns of A are independent.

30. $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

31. $\begin{pmatrix} 0 & 2 & 1 & 0 & 1 \\ 1 & 5 & 2 & 0 & 3 \\ 1 & 3 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 2 & 1 & 0 & 1 \end{pmatrix}$

Solution: Let A be the matrix, then

$$\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Columns 1,2,5 of A are independent by the pivot theorem.

32. $\begin{pmatrix} 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 5 & 2 & 0 & 3 \\ 0 & 1 & 3 & 1 & 0 & 2 \\ 0 & 2 & 4 & 1 & 0 & 3 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 2 & 4 & 1 & 0 & 3 \end{pmatrix}$

Row and Column Rank

Justify by direct computation that $\mathbf{rank}(A) = \mathbf{rank}\left(A^T\right)$, which means that the row rank equals the column rank.

33. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: Already $A = \mathbf{rref}(A)$ with $\mathbf{rank}(A) = 2$. Let $B = A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

Then $\mathbf{rref}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $\mathbf{rank}(B) = 2$.

34. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Nullspace or Kernel

Find a basis for the nullspace of A, which is also called the kernel of A.

35. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: Solve $A\vec{x} = \vec{0}$ for $\vec{x} = t_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ where t_1 is the invented symbol in the last frame algorithm. A basis for the nullspace of A is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

36. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Row Space

Find a basis for the row space of A. There are two possible answers: (1) The nonzero rows of $\mathbf{rref}(A)$, (2) The pivot columns of A^T. Answers (1) and (2) can differ wildly.

37. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution:
(1): Exercise 35 has the same matrix A. Without computation, $\mathbf{rref}(A) = A$ and a basis for the row space is obtained from the nonzero rows of $\mathbf{rref}(A)$ by transposition: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(2): Find the pivot columns of $B = A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ via $\mathbf{rref}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Conclusion: columns 1, 2 of B are independent. A basis is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. The answer happens to duplicate the answer from **(1)**, an unusual event.

38. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Column Space

Find a basis for the column space of A, in terms of the columns of A. Normally, we report the pivot columns of A.

39. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: Without computation, $\mathbf{rref}(A) = A$ and a basis for the column space is obtained from columns 1, 2 of A: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

40. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Dimension Identities

Let A be an $m \times n$ matrix of rank r. Prove the following dimension identities in Theorem 5.46.

41. $\dim(\mathbf{nullspace}(A)) = n - r$

Solution: The rank-nullity theorem in the form **lead count** + **free count** = **variable count** was used to obtain $\mathbf{rank}(A) + \mathbf{nullity}(A) = n$. Because **free count** = $\mathbf{nullity}(A) = \dim(\mathbf{nullspace}(A))$ and **lead count** $=\mathbf{rank}(A) = r$, then the result follows. ■

42. $\dim(\mathbf{colspace}(A)) = r$

43. $\dim(\mathbf{rowspace}(A)) = r$

Solution: Symbol r is the rank of A. The dimension of the row space of A is the number of independent rows of A, which equals the number of nonzero rows of $\mathbf{rref}(A)$. An alterative explanation: the row space of A is the column space of A^T and the dimension of the row space is the number of pivot columns in $\mathbf{rref}(A^T)$. Because $\mathbf{rank}(A) = \mathbf{rank}(A^T)$ then the dimension of the row space of A equals the rank of A, or using notation, $\dim(\mathbf{rowspace}(A)) = r$. ■

44. The dimensions of $\mathbf{nullspace}(A)$ and $\mathbf{colspace}(A)$ add to n.

Orthogonal Complement $S^{\perp}$

Let S be a subspace of vector space $V = \mathcal{R}^n$. Define the **Orthogonal complement** by

(1) $$S^{\perp} = \{\vec{x} \ : \ \vec{x}^T\vec{y} = 0, \ \vec{y} \text{ in } S\}.$$

45. Let $V = \mathcal{R}^3$ and let S be the xy-plane. Compute $S^{\perp}$. Answer: The z-axis.

Solution: Definition: $S^{\perp} = \{\vec{x} \ : \ \vec{x}\cdot\vec{y} = 0, \ \vec{y} \text{ in the } xy\text{-plane}\}$.

Let $\vec{y} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ where a, b are real numbers, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Then $\vec{y}$ is in S and $\vec{x}$ is in V. For $\vec{x}$ to also be in S this requirement is made:

$$0 = \vec{x}\cdot\vec{y} = ax_1 + bx_2$$

The requirement must hold for all a, b. Quickly we decide that $x_1 = x_2 = 0$, leaving x_3 undetermined and $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}$ with x_3 any real number. So $\vec{x}$ is any vector whose head lies on the z-axis and S is the z-axis. ■

46. Prove that $S^{\perp}$ is a subspace, using the **Subspace Criterion**.

47. Prove that the orthogonal complement of $S^{\perp}$ is S. In symbols, $\left(S^{\perp}\right)^{\perp} = S$.

Solution:
Let $W = S^{\perp} = \{\vec{w} : \ \vec{w}\cdot\vec{s} = 0 \text{ for all } s \text{ in } S\}$.
Let $X = W^{\perp} = \{\vec{x} : \ \vec{x}\cdot\vec{w} = 0 \text{ for all } w \text{ in } W\}$.

Orthonormal basis for V: Let $\vec{s}_1, \ldots, \vec{s}_k$ be an orthonormal basis for S. Extend it to an orthonormal basis for V by Gram-Schmidt, adding vectors $\vec{s}_{k+1}, \ldots, \vec{s}_n$. Let $\vec{x} \in X$. By basis expansion, $\vec{x} = \vec{s} + \vec{w}$ where $\vec{s} = \sum_{i=1}^{k} c_i\vec{s}_i$ and $\vec{w} = \sum_{j=k+1}^{n} c_j\vec{s}_j$, the constants defined by $c_m = \vec{x}\cdot\vec{s}_m$, $1 \le m \le n$. By construction, for $k+1 \le j \le n$ each vector $\vec{s}_j$ is orthogonal to every vector in S, meaning $\vec{s}_j$ is in $W = S^{\perp}$. Therefore $\vec{s} \in S$ and $\vec{w} \in W$.

To prove $X \subset S$: Let $\vec{x} \in X$, then for $k+1 \le j \le n$ equation $\vec{x}\cdot\vec{s}_j = 0$ holds (definition of X). Expand $\vec{x} = \sum_{i=1}^{k} c_i\vec{v}_i + \sum_{j=k+1}^{n} c_j\vec{v}_j$. Definition $c_m = \vec{x}\cdot\vec{s}_m$ and $\vec{x}\cdot\vec{s}_j = 0$ implies $c_j = 0$ for $k+1 \le j \le n$. Then

$\vec{x} = \sum_{i=1}^{k} c_i \vec{v}_i$ is in S.

To prove $S \subset X$: Let $\vec{s} \in S$, then $\vec{s} = \sum_{i=1}^{k} c_i \vec{v}_i$ for constants $c_i = \vec{s} \cdot \vec{s}_i$ $(1 \le i \le k)$. We must show $\vec{w} \in W$ implies equation $\vec{s} \cdot \vec{w} = 0$, then $\vec{s} \in X$, as to be proved. Any $\vec{w} \in W$ has expansion $\sum_{j=k+1}^{n} c_j \vec{s}_j$, each vector $\vec{s}_j$ orthogonal to $\vec{s}_1, \ldots, \vec{s}_k$. So $\vec{w}$ is orthogonal to $\vec{s}_1, \ldots, \vec{s}_k$, hence $\vec{w}$ is orthogonal to $\vec{s}$: equation $\vec{s} \cdot \vec{w} = 0$ holds. ■

48. Prove that

$$V = \{\vec{x} + \vec{y} \ : \ \vec{x} \in S, \vec{y} \in S^\perp\}.$$

This relation is called the **Direct Sum** of S and $S^\perp$.

Fundamental Theorem of Linear Algebra

Let A be an $m \times n$ matrix.

49. Write a short proof:

Lemma. Any solution of $A\vec{x} = \vec{0}$ is orthogonal to every row of A.

Solution: Vector $\vec{x}$ is orthogonal to each row of A provided

$$\sum_{j=1}^{n} a_{ij} x_j = 0, \quad 1 \le i \le n \qquad \left(\textbf{row} \quad i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix}^T \right)$$

Let $A\vec{x} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ where $b_i = \sum_{j=1}^{n} a_{ij} x_j$ (**Definition of matrix multiply**).

Because $A\vec{x} = \vec{0}$, matrix equality provides equation $b_i = 0$ and in turn

$$\sum_{j=1}^{n} a_{ij} x_j = b_i = 0, \quad 1 \le i \le n$$

as required to complete the proof.

Alternate Proof: Details in $\mathcal{R}^3$ to get rid of all the summations:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

Then

$$\left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & = & 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & = & 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & = & 0 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{rcl} \textbf{row}(A,1) \cdot \vec{x} & = & 0 \\ \textbf{row}(A,2) \cdot \vec{x} & = & 0 \\ \textbf{row}(A,3) \cdot \vec{x} & = & 0 \end{array} \right.$$

which displays orthogonality of each row of A to vector $\vec{x}$. ■

50. Find the dimension of the kernel and image for both A and A^T. The four answers use symbols $m, n, \textbf{rank}(A)$. The main tool is the rank-nullity theorem.

51. Prove
$\textbf{kernel}(A) = \textbf{Image}\left(A^T\right)^\perp$. Use Exercise 49.

Solution: Let $S = \textbf{kernel}(A)$, the set of solutions to the equation $A\vec{x} = \vec{0}$. Let $W = \textbf{Image}(A^T)$, the set of all linear combinations of columns of A^T, that is, all vectors $\vec{w} = \sum_{i=1}^m c_i\, \textbf{col}(A^T, i)$. We will prove $S = W^\perp$.

Show $S \subset W^\perp$: Exercise 49 shows that $\vec{x} \in S$ is orthogonal to each row of A, which means equation $\vec{x} \cdot \textbf{col}(A^T, i) = 0$ holds, $1 \le i \le m$. Then $\vec{x} \cdot \left(\sum_{i=1}^m c_i\, \textbf{col}(A^T, i)\right) = 0$ or $\vec{x} \cdot \vec{w} = 0$ for all $\vec{w} \in W$. By definition of orthogonal complement, $\vec{x} \in W^\perp$.

Show $W^\perp \subset S$: A vector $\vec{s} \in W^\perp$ is orthogonal to all vectors $\vec{w} \in W$. In particular, $\vec{w} = \textbf{col}(A^T, i)$ is allowed, so $\vec{s}$ satisfies equation $\vec{s} \cdot \textbf{col}(A^T, i) = 0$. Therefore $\vec{s}$ is orthogonal to the rows of A, which by Exercise 49 implies $A\vec{s} = \vec{0}$. Then $\vec{s} \in S$. ■

52. Prove
$\textbf{kernel}\left(A^T\right) = \textbf{Image}\,(A)^\perp$.

Fundamental Subspaces

The kernel and image of both A and A^T are called *The Four Fundamental Subspaces* by Gilbert Strang. Let A denote an $n \times m$ matrix.

53. Prove using Exercise 51:
$\textbf{kernel}(A) = \textbf{rowspace}(A)^\perp$

Solution: Exercise 51 gives $\textbf{kernel}(A) = \textbf{Image}(A^T)^\perp$. We must prove that $\textbf{Image}(A^T) = \textbf{rowspace}(A)$.

The row space of A is the set of all linear combinations of the rows of A, formally the set of all linear combinations of columns of A^T. The set of all linear combinations of the columns of A^T is the image of A^T. ■

54. Establish these four identities.
$\textbf{kernel}(A) = \textbf{Image}\left(A^T\right)^\perp$
$\textbf{kernel}\left(A^T\right) = \textbf{Image}\,(A)^\perp$
$\textbf{Image}\,(A) = \textbf{kernel}(A^T)^\perp$
$\textbf{Image}\left(A^T\right) = \textbf{kernel}(A)^\perp$

Notation. *kernel* is null space, *image* is column space, symbol $\perp$ is orthogonal complement: see equation (4).

Equivalent Bases

Test the given subspaces for equality.

55. $S_1 = \mathbf{span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right),$

$S_2 = \mathbf{span}\left(\begin{pmatrix}3\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$

Solution: Follow the **Equivalent Bases** example in the text.

Let $B = \begin{pmatrix}1&1\\1&1\\0&1\end{pmatrix}$, $C = \begin{pmatrix}3&1\\3&1\\-1&1\end{pmatrix}$.

Let $W = \left\langle B|C \right\rangle = \begin{pmatrix}1&1&3&1\\1&1&3&1\\0&1&-1&1\end{pmatrix}$.

Compute the rank of each to be 2. Then $S_1 = S_2$.

```
# Subspace equality test, Exercise 55
 with(LinearAlgebra):
 B:=<1,1,0|1,1,1>;C:=<3,3,-1|1,1,1>;
 Rank(<B|C>);Rank(B);Rank(C);
# all equal 2 => S1 = S2
```

56. $S_3 = \mathbf{span}\left(\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right),$

$S_4 = \mathbf{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$

57. $S_5 = \mathbf{span}\left(\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\\1\end{pmatrix}\right),$

$S_6 = \mathbf{span}\left(\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right)$

Solution: Follow Exercise 55. Compute the ranks: 3, 2, 2. Then $S_1 \neq S_2$.

```
# Subspace equality test, Exercise 57
 with(LinearAlgebra):
 B:=<1,0,1,1|1,2,1,1>;C:=<1,0,1,1|0,1,0,1>;
 Rank(<B|C>);Rank(B);Rank(C);
# Ranks: 3,2,2 => S1 != S2
```

58. $S_7 = \textbf{span}\left(\begin{pmatrix}2\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\\1\end{pmatrix}\right),$

$S_8 = \textbf{span}\left(\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}3\\3\\2\\2\end{pmatrix}\right)$

Chapter 6

Scalar Linear Differential Equations

Contents

6.1 Linear 2nd Order Constant

General Solution 2nd Order

Solve the constant equation using Theorem 6.1, page 431. Report the general solution using symbols c_1, c_2. Model the solution after Examples 6.1–6.3, page 434.

1. $y'' = 0$
Ans: $y = c_1 + c_2 x$

Solution: Follow Example 6.1 on page 434. Characteristic equation $r^2 = 0$ has a double root $r = 0, 0$. Then $y_1 = e^{0x} = 1$, $y_2 = xe^{0x} = x$ and $y = c_1y_1 + c_2y_2 = c_1 + c_2x$.

2. $3y'' = 0$

3. $y'' + y' = 0$

Solution: Characteristic equation $r^2 + r = 0$ has roots $r = 0$, $r = -1$. Then $y_1 = e^{0x} = 1$, $y_2 = e^{-x}$ and $y = c_1y_1 + c_2y_2 = c_1 + c_2e^{-x}$.

4. $3y'' + y' = 0$

5. $y'' + 3y' + 2y = 0$

Solution: Characteristic equation $r^2 + 3r + 2 = 0$ has roots $r = -2$, $r = -1$. Then $y_1 = e^{-2x}$, $y_2 = e^{-x}$ and $y = c_1y_1 + c_2y_2 = c_1e^{-2x} + c_2e^{-x}$.

6. $y'' - 3y' + 2y = 0$

7. $y'' - y' - 2y = 0$

Solution: Characteristic equation $r^2 - r - 2 = 0$ has roots $r = -1$, $r = 2$. Then $y_1 = e^{-x}$, $y_2 = e^{2x}$ and $y = c_1y_1 + c_2y_2 = c_1e^{-x} + c_2e^{2x}$.

8. $y'' - 2y' - 3y = 0$

9. $y'' + y = 0$

Solution: Follow Example 6.3 page 434. Characteristic equation $r^2 + 1 = 0$ has roots $r = i$, $r = -i$. Then $y_1 = \cos(x)$, $y_2 = \sin(x)$ and $y = c_1y_1 + c_2y_2 = c_1\cos(x) + c_2\sin(x)$.

10. $y'' + 4y = 0$

11. $y'' + 16y = 0$

Solution: $y = c_1\cos(4x) + c_2\sin(4x)$.

12. $y'' + 8y = 0$

13. $y'' + y' + y = 0$

Solution: Use the quadratic formula to find the roots of the characteristic equation $r^2 + r + 1 = 0$. Then $r = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Let $y_1 = e^{-x/2}\cos(\sqrt{3}x/2)$ and $y_2 = e^{-x/2}\sin(\sqrt{3}x/2)$. The general solution is $y = c_1y_1 + c_2y_2 = c_1e^{-x/2}\cos(\sqrt{3}x/2) + c_2e^{-x/2}\sin(\sqrt{3}x/2)$.

14. $y'' + y' + 2y = 0$

15. $y'' + 2y' + y = 0$

Solution: $y = c_1 e^{-x} + c_2 x e^{-x}$.

16. $y'' + 4y' + 4y = 0$

17. $3y'' + y' + y = 0$

Solution: Characteristic equation $3r^2 + r + 1 = 0$ has roots $r = -1/6 \pm (i/6)\sqrt{11}$. Then $y = c_1 e^{-x/6} \cos(\sqrt{11}x/6) + c_2 e^{-x/6} \sin(\sqrt{11}x/6)$.

18. $9y'' + y' + y = 0$

19. $5y'' + 25y' = 0$

Solution: $y = c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x)$.

20. $25y'' + y' = 0$

21. $2y'' + y' - y = 0$

Solution: Characteristic equation $2r^2 + r - 1 = 0$ has roots $r = -1, 1/2$. Then $y = c_1 e^{-x} + c_2 e^{x/2}$.

22. $2y'' - 3y' - 2y = 0$

23. $2y'' + 7y' + 3y = 0$

Solution: $y = c_1 e^{-3x} + c_2 e^{-x/2}$.

24. $4y'' + 8y' + 3y = 0$

25. $6y'' + 7y' + 2y = 0$

Solution: $y = c_1 e^{-x/2} + c_2 e^{-2x/3}$.

26. $6y'' + y' - 2y = 0$

27. $y'' + 4y' + 8y = 0$

Solution: Roots $-2 \pm 2i$. Then $y = c_1 e^{-2x} \cos(2x) + c_2 e^{-2x} \sin(2x)$.

28. $y'' - 2y' + 4y = 0$

29. $y'' + 2y' + 4y = 0$

Solution: Roots $-1 \pm i\sqrt{3}$. Then $y = c_1 e^{-x} \cos(\sqrt{3}\,x) + c_2 e^{-x} \sin(\sqrt{3}\,x)$.

30. $y'' + 4y' + 5y = 0$

31. $4y'' - 4y' + y = 0$
Solution: $y = c_1 e^{x/2} + c_2 x e^{x/2}$.

32. $4y'' + 4y' + y = 0$

33. $9y'' - 6y' + y = 0$
Solution: $y = c_1 e^{x/3} + c_2 x e^{x/3}$.

34. $9y'' + 6y' + y = 0$

35. $4y'' + 12y' + 9y = 0$
Solution: $y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$.

36. $4y'' - 12y' + 9y = 0$

Initial Value Problem 2nd Order

Solve the given problem, modeling the solution after Example 6.4.

37. $6y'' + 7y' + 2y = 0$, $y(0) = 0$, $y'(0) = -1$
Solution: $y = 6e^{-2x/3} - 6e^{-x/2}$. The general solution is $y = c_1 e^{-2x/3} + c_2 e^{-x/2}$, found from roots $-2/3, -1/2$ of $6r^2 + 7r + 2 = 0$. Substitute into equations $y(0) = 0$, $y'(0) = -1$:

$$c_1 + c_2 = 0, \quad -2c_1/3 - c_2/2 = -1$$

Then solve for $c_1 = 6$, $c_2 = -6$ by Cramer's rule.

```
# Exercise 37, answer check
L:=[6,7,2]; A:=0;B:=-1;
solve(L[1]*x^2+L[2]*x+L[3]=0,x);
de:=L[1]*diff(y(x),x,x)+L[2]*diff(y(x),x)+L[3]*y(x)=0;
ic:=y(0)=A,D(y)(0)=B;
dsolve([de,ic],y(x));
# y(x) = 6*exp(-(2/3)*x)-6*exp(-(1/2)*x)
```

38. $2y'' + 7y' + 3y = 0$, $y(0) = 5$, $y'(0) = -5$

39. $y'' - 2y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$
Solution: $y = e^x \cos(\sqrt{3}\, x)$

40. $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 1$

41. $9y'' - 6y' + y = 0$, $y(0) = 3$, $y'(0) = 1$

Solution: $y = 3\,e^{x/3}$

42. $4y'' + 12y' + 9y = 0$, $y(0) = 2$, $y'(0) = 1$

Detecting Euler Solution Atoms

A **Euler solution atom** is defined in Definition 6.1 page 432. Box each list entry that is precisely an atom. Double-box non-atom list entries that are a sum of constants times atoms. Follow Example 6.5 page 436.

43. 1, $e^{x/5}$, -1, $e^{1.1x}$, $2e^x$

Solution: $\boxed{1}$, $\boxed{e^{x/5}}$, -1 (not an atom), $\boxed{e^{1.1x}}$, $\boxed{\boxed{2e^x}}$

44. $-x\cos \pi x$, $x^2 \sin 2x$, x^3, $2x^3$

45. e^{2x}, $e^{-x^2/2}$, $\cos^2 2x$, $\sin 1.57x$

Solution: $\boxed{e^{2x}}$, $e^{-x^2/2}$ not an atom, $\boxed{\boxed{\cos^2 2x}}$ because of a trig double angle identity, $\boxed{\sin 1.57x}$

46. $x^7 e^x \cos 3x$, $x^{10} e^x \sin 4x$

47. $x^7 e^x \cosh 3x$, $x^{10} e^{-x} \sinh 5x$

Solution: Both double-boxed because of definitions $\cosh(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ and $\sinh(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$.

48. $\cosh^2 x$, $x(1+x)$, $x^{1.5}$, $\sqrt{x}e^{-x}$

49. $x^{1/2}e^{x/2}$, $\dfrac{1}{x}e^x$, $e^x(1+x^2)$

Solution: $x^{1/2}e^{x/2}$ not an atom, $\dfrac{1}{x}e^x$ not an atom, $\boxed{\boxed{e^x(1+x^2)}}$

50. $\dfrac{x}{1+x}$, $\dfrac{1}{x}(1+x^2)$, $\ln|x|$

Euler Base Atom

An **Euler base atom** is defined in Definition 6.1 page 432. Find the base atom for each Euler solution atom in the given list.

51. $x\cos \pi x$, x^3, $x^{10}e^{-x}\sin 5x$

Solution: Strip off the power of x: base atoms $= \cos \pi x$, 1, $e^{-x}\sin 5x$.

52. x^6, x^4e^{2x}, $x^2e^{-x/\pi}$, $x^7e^x\cos 1.1x$

Inverse Problems

Find the homogeneous 2nd order differential equation, given the supplied information. Follow Example 6.6.

53. $e^{-x/5}$ and 1 are solutions.
Ans: $5y'' + y' = 0$.

Solution: The roots are obtained from the atoms: $r = -1/5, 0$. Then $(r + 1/5)(r - 0) = 0$ is the characteristic equation: $r^2 + r/5 = 0$. The differential equation is then $y'' + (1/5)y' = 0$. A common error is to report $y'' + (1/5)y = 0$, caused by reading $r/5$ as the constant term (it is not).

54. e^{-x} and 1 are solutions.

55. $e^x + e^{-x}$ and $e^x - e^{-x}$ are solutions.

Solution: Identify atoms e^x and e^{-x}, then roots $r = 1, -1$ to create characteristic polynomial $(r-1)(r+1) = r^2 - 1$. Then the differential equation is $y'' - y = 0$.

56. $e^{2x} + xe^{2x}$ and xe^{2x} are solutions.

57. x and $2 + x$ are solutions.

Solution: Identify atoms $1, x$ and then roots $r = 0, 0$ (double root) to find characteristic equation $r^2 = 0$. The differential equation is $y'' = 0$.

58. $4e^x$ and $3e^{2x}$ are solutions.

59. The characteristic equation is $r^2 + 2r + 1 = 0$.

Solution: $y'' + 2y' + y = 0$

60. The characteristic equation is $4r^2 + 4r + 1 = 0$.

61. The characteristic equation has roots $r = -2, 3$.

Solution: The characteristic polynomial is $(r + 2)(r - 3) = r^2 - r - 6$. The differential equation is $y'' - y' + 6y = 0$.

62. The characteristic equation has roots $r = 2/3, 3/5$.

63. The characteristic equation has roots $r = 0, 0$.

Solution: $y'' = 0$

64. The characteristic equation has roots $r = -4, -4$.

65. The characteristic equation has complex roots $r = 1 \pm 2i$.

Solution: The characteristic polynomial is $(r-1-2i)(r-1+2i) = (r-1)^2+4 = r^2-2r+5$. The differential equation is $y''-2y'+5y=0$.

66. The characteristic equation has complex roots $r = -2 \pm 3i$.

Details of proofs

67. (Theorem 6.1, Background) Expand the relation $Ar^2+Br+C = A(r-r_1)(r-r_2)$ and compare coefficients to obtain the sum and product of roots relations

$$\frac{B}{A} = -(r_1+r_2), \quad \frac{C}{A} = r_1r_2.$$

Solution:

$Ar^2+Br+C = A(r-r_1)(r-r_2)$
$Ar^2+Br+C = A(r^2-r_1r-r_2e+r_1r_2)$

Compare cofficients left and right:

$A = A$
$B == -A(r_1+r_2)$ sum of the roots
$C = Ar_1r_2$

Then:

$B/A = -(r_1+r_2) =$ negative of the sum of the roots
$C/S = r_1r_2 =$ product of the roots

68. (Theorem 6.1, Background)
Let r_1, r_2 be the two roots of $Ar^2+Br+C=0$. The discriminant is $\mathcal{D} = B^2-4AC$. Use the quadratic formula to derive these relations for $\mathcal{D}>0$, $\mathcal{D}=0$, $\mathcal{D}<0$, respectively:

$$r_1 = \frac{-B+\sqrt{\mathcal{D}}}{2A}, r_2 = \frac{-B-\sqrt{\mathcal{D}}}{2A},$$
$$r_1 = r_2 = \frac{\sqrt{\mathcal{D}}}{2A}.$$
$$r_1 = \frac{-B+i\sqrt{-\mathcal{D}}}{2A}, r_2 = \frac{-B-i\sqrt{-\mathcal{D}}}{2A}.$$

69. (Theorem 6.1, Case 1)
Let $y_1 = e^{r_1x}$, $y_2 = e^{r_2x}$. Assume $Ar^2+Br+C = A(r-r_1)(r-r_2)$. Show that y_1, y_2 are solutions of $Ay''+By'+Cy=0$.

Solution:
$$Ay''+By'+Cy\big|_{y=y_1} = A(e^{r_1x})''+B(e^{r_1x})'+Ce^{r_1x}$$

$$\begin{aligned}
&= Ar_1r_1e^{r_1x} + Br_1e^{r_1x} + Ce^{r_1x} \\
&= (Ar_1r_1 + Br_1 + C)e^{r_1x} \\
&= A(r_1 - r_1)(r_1 - r_2)e^{r_1x} \\
&= 0
\end{aligned}$$

Except for indexing the proof is the same for e^{r_2x}. ■

70. (Theorem 6.1, Case 2)
Let $y_1 = e^{r_1x}$, $y_2 = x\,e^{r_1x}$. Assume
$Ar^2 + Br + C{=}A(r - r_1)(r - r_1)$.
Show that y_1, y_2 are solutions of $Ay'' + By' + Cy = 0$.

71. (Theorem 6.1, Case 3)
Let a, b be real, $b > 0$. Let $y_1 = e^{ax}\cos bx$, $y_2 = e^{ax}\sin bx$. Assume factorization
$Ar^2{+}Br{+}C{=}A(r{-}a{-}ib)(r{-}a{+}ib)$
then show that y_1, y_2 are solutions of $Ay'' + By' + Cy = 0$.

Solution:
Let $r_1 = a + ib$, $r_2 = a - ib$, the two complex roots. Let $z = a + ib$. Let $y = e^{zx} = y_1 + iy_2$. Then $Az^2 + Bz + C = 0$ and y is a solution of $Ay'' + By' + Cy = 0$ by these steps:

$Az^2 + Bz + C = 0$ because z is a root of the characteristic equation
$Az^2y + Bzy + Cy = 0$ multiply by y
$Ay'' + By' + Cy = 0$ because $y = e^{zx}$, $y' = ze^{zx} = zy$, $y'' = z^2e^{zx} = z^2y$

Then $y = y_1 + iy_2$ implies:

$$\begin{aligned}
&A(y_1'' + iy_2'') + B(y_1' + iy_2') + C(y_1 + iy_2) = 0 \\
&Ay_1'' + By_1' + Cy_1 + i(Ay_2'' + By_2' + Cy_2) = 0
\end{aligned}$$

The left side is a complex number $X + iY$ equal to zero on the right side, therefore the real and imaginary parts X, Y of the complex number are zero:

$$\begin{aligned}
X &= Ay_1'' + By_1' + Cy_1 = 0 \\
Y &= Ay_2'' + By_2' + Cy_2 = 0
\end{aligned}$$

The result: both y_1 and y_2 are solutions of $Ay'' + By' + Cy = 0$. ■

6.2 Continuous Coefficient Theory

Continuous Coefficients

Determine all intervals J of existence of $y(x)$, according to Picard's theorem.

1. $y'' + y = \ln|x|$

Solution: Let $f(x, y) = -y + \ln|X|$. The domain D is $x \neq 0$ and all y. Both f and $f_y = -1$ are continuous on D. According to Picard's theorem, there is a locally unique solution to each initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ for any y_0 and any $x_0 \neq 0$.

2. $y'' = \ln|x - 1|$

3. $y'' + (1/x)y = 0$

Solution: All y and all $x \neq 0$.

4. $y'' + \frac{1}{1+x}y' + \frac{1}{x}y = 0$

5. $x^2y'' + y = \sin x$

Solution: All y and all $x \neq 0$.

6. $x^2y'' + xy' = 0$

Superposition

Verify that $y = c_1y_1 + c_2y_2$ is a solution.

7. $y'' = 0$, $y_1(x) = 1$, $y_2(x) = x$

Solution: $y'' = c_1y_1'' + c_2y_2'' = c_1(0) + c_2(0) = 0$

8. $y'' = 0$, $y_1(x) = 1 + x$, $y_2(x) = 1 - x$

9. $y''' = 0$, $y_1(x) = x$, $y_2(x) = x^2$

Solution: $y''' = c_1y_1''' + c_2y_2''' = c_1(0) + c_2(0) = 0$

10. $y''' = 0$, $y_1(x) = 1 + x$, $y_2(x) = x + x^2$

Structure

Verify that $y = y_h + y_p$ is a solution.

11. $y'' + y = 2$, $y_h(x) = c_1 \cos x + c_2 \sin x$, $y_p(x) = 2$

Solution: $y'' + y = (y_h + y_p)'' + y_h + y_p = -c_1 \cos x - c_2 \sin x + 0 + c_1 \cos x + c_2 \sin x + 2 = 2$

12. $y'' + 4y = 4$, $y_h(x) = c_1 \cos 2x + c_2 \sin 2x$, $y_p(x) = 1$

13. $y'' + y' = 5$, $y_h(x) = c_1 + c_2 e^{-x}$, $y_p(x) = 5x$

Solution: $y'' + y' = (c_1 + c_2 e^{-x} + 5x)'' + (c_1 + c_2 e^{-x} + 5x)' = 0 + c_2 e^{-x} + 0 + (0 - c_2 e^{-x} + 5) = 5$

14. $y'' + 3y' = 5$, $y_h(x) = c_1 + c_2 e^{-3x}$, $y_p(x) = 5x/3$

15. $y'' + y' = 2x$, $y_h(x) = c_1 + c_2 e^{-x}$, $y_p(x) = x^2 - 2x$

Solution: $y'' + y' = (c_1 + c_2 e^{-x} + x^2 - 2x)'' + (c_1 + c_2 e^{-x} + x^2 - 2x)' = 0 + c_2 e^{-x} + 2 - 0 + (0 - c_2 e^{-x} + 2x - 2 = 2x$

16. $y'' + 2y' = 4x$, $y_h(x) = c_1 + c_2 e^{-2x}$, $y_p(x) = x^2 - x$

Initial Value Problems

Solve for constants c_1, c_2 in the general solution $y_h = c_1 y_1 + c_2 y_2$.

17. $y'' = 0$, $y_1 = 1$, $y_2 = x$, $y(0) = 1$, $y'(0) = 2$

Solution:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 + c_2 x \end{aligned}$$

Translate equations $y(0) = 1$, $y'(0) = 2$:

$c_1 + c_2(0) = 1, \quad c_2 = 2$

Solve: $c_1 = 1$, $c_2 = 2$. Then $y = 1 + 2x$.

18. $y'' = 0$, $y_1 = 1 + x$, $y_2 = 1 - x$, $y(0) = 1$, $y'(0) = 2$

19. $y'' + y = 0$, $y_1 = \cos x$, $y_2 = \sin x$, $y(0) = 1$, $y'(0) = -1$

Solution:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos x + c_2 \sin x \end{aligned}$$

Translate equations $y(0) = 1$, $y'(0) = -1$:

$c_1(1) + c_2(0) = 1, \quad -c_1(0) + c_2(1) = -1$

Solve: $c_1 = 1$, $c_2 = -1$. Then $y = \cos x - \sin x$.

20. $y'' + y = 0$, $y_1 = \sin x$, $y_2 = \cos x$, $y(0) = 1$, $y'(0) = -1$

21. $y'' + 4y = 0$, $y_1 = \cos 2x$, $y_2 = \sin 2x$, $y(0) = 1$, $y'(0) = -1$

Solution:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos 2x + c_2 \sin 2x \end{aligned}$$

Translate equations $y(0) = 1$, $y'(0) = -1$:

$c_1(1) + c_2(0) = 1, \quad -2c_1(0) + 2c_2(1) = -1$

Solve: $c_1 = 1$, $c_2 = -1/2$. Then $y = \cos 2x - (1/2)\sin 2x$.

22. $y'' + 4y = 0$, $y_1 = \sin 2x$, $y_2 = \cos 2x$, $y(0) = 1$, $y'(0) = -1$

23. $y'' + y' = 0$, $y_1 = 1$, $y_2 = e^{-x}$, $y(0) = 1$, $y'(0) = -1$

Solution:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 + c_2 e^{-x} \end{aligned}$$

Translate equations $y(0) = 1$, $y'(0) = -1$:

$c_1(1) + c_2(1) = 1, \quad 0 - c_2(1) = -1$

Solve: $c_1 = 0$, $c_2 = 1$. Then $y = e^{-x}$.

24. $y'' + y' = 0$, $y_1 = 1$, $y_2 = e^{-x}$, $y(0) = 2$, $y'(0) = -3$

25. $y'' + 3y' = 0$, $y_1 = 1$, $y_2 = e^{-3x}$, $y(0) = 1$, $y'(0) = -1$

Solution:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 + c_2 e^{-3x} \end{aligned}$$

Translate equations $y(0) = 1$, $y'(0) = -1$:

$c_1 + c_2(1) = 1, \quad 0 - 3c_2(1) = -1$

Solve: $c_1 = 2/3$, $c_2 = 1/3$. Then $y = 2/3 + (1/3)e^{-3x}$.

26. $y'' + 5y' = 0$, $y_1 = 1$, $y_2 = e^{-5x}$, $y(0) = 1$, $y'(0) = -1$

Recognizing y_h

Extract from the given solution y a particular solution y_p with fewest terms.

27. $y'' + y = x$,
$y = c_1 \cos x + c_2 \sin x + x$

Solution: The Euler solution atoms for the homogeneous equation $y'' + y = 0$ are $\cos x$, $\sin x$. Then $c_1 \cos x + c_2 \sin x$ is a solution of the homogeneous equation, making $y_p = x$ a particular solution.

28. $y'' + y = x$,
$y = \cos x + x$

29. $y'' + y' = x$,
$y = c_1 + c_2 e^{-x} + x^2/2 - x$

Solution: $y_p = x^2/2 - x$

30. $y'' + y' = x$,
$y = e^{-x} - x + 1 + x^2/2$

31. $y'' + 2y' + y = 1 + x$,
$y = (c_1 + c_2 x)e^{-x} + x - 1$
Solution: $y_p = x - 1$

32. $y'' + 2y' + y = 1 + x$,
$y = e^{-x} + x + xe^{-x} - 1$

Reduction of Order

Given solution y_1, find an independent solution y_2 by reduction of order.

33. $y'' + 2y' = 0$, $y_1(x) = 1$
Solution: Let $x_0 = 0$. Let $a = 1$, $b = 2$. Then
$\int_{x_0}^{t}(b/a)dr = \int_0^t 2dr = 2t$
$y_2(x) = y_1(x)\int_{x_0}^{x} \dfrac{e^{-\int_{x_0}^{t}(b/a)dr}}{y_1^2(t)}\,dt = \int_0^x \frac{e^{-2t}}{1^2}\,dt$
$\quad = (1 - e^{-2x})/2$
Answer check:
$y'' + 2y' = (1/2)(1 - e^{-2x})'' + 2(1/2)(1 - e^{-2x})' = (1/2)(4)e^{-2x} - 2e^{-2x} = 0$

34. $y'' + 2y' = 0$, $y_1(x) = e^{-2x}$

35. $2y'' + 3y' + y = 0$, $y_1(x) = e^{-x}$
Solution: Let $x_0 = 0$. Let $a = 2$, $b = 3$. Then
$\int_{x_0}^{t}(b/a)dr = \int_0^t (3/2)dr = 3t/2$
$y_2(x) = y_1(x)\int_{x_0}^{x} \dfrac{e^{-\int_{x_0}^{t}(b/a)dr}}{y_1^2(t)}\,dt = e^{-x}\int_0^x \frac{e^{-3t/2}}{e^{-2t}}\,dt$
$\quad = 2(-e^{-x} + e^{-x/2})$
Answer check:
$$\begin{aligned}
&2y'' + 3y' + y \\
&\quad = 2(-2e^{-x} + 2e^{-x/2})'' + 3(-2e^{-x} + 2e^{-x/2})' + (-2e^{-x} + 2e^{-x/2}) \\
&\quad = -4e^{-x} + e^{-x/2} + 6e^{-x} - 3e^{-x/2} - 2e^{-x} + 2e^{-x/2} \\
&\quad = (-4 + 6 - 2)e^{-x} + (1 - 3 + 2)e^{-x/2} = 0
\end{aligned}$$

36. $2y'' - y' - y = 0$, $y_1(x) = e^x$

Equilibrium Method

Apply the equilibrium method to find y_p, then find the general solution $y = y_h + y_p$.

37. $2y'' = 3$

Solution: $y_h = c_1 + c_2 x$, $y_p = 3x^2/4$.

The equilibrium method applies because the coefficients are constant. Solve $2y'' = 3$ by quadrature, all integration constants zero. Then $y' = 3x/2$, $y = 3x^2/4$.

38. $y'' + 4y' = 5$

39. $y'' + 3y' + 2y = 3$

Solution: $y_h = c_1 e^{-2x} + c_2 e^{-x}$, $y_p = 3/2$.

Factor $r^2 + 3r + 2 = (r+1)(r+2)$, then roots are $r = -1$, $r = -2$ and $y_h = c_1 e^{-2x} + c_2 e^{-x}$. Drop all but the lowest order term in the DE to obtain $2y = 3$, then solve for y (no quadrature required): $y = 3/2$.

40. $y'' - y' - 2y = 2$

41. $y'' + y = 1$

Solution: $y_h = c_1 \cos x + c_2 \sin x$, $y_p = 1$

42. $3y'' + y' + y = 7$

43. $6y'' + 7y' + 2y = 5$

Solution: $y_h = c_1 e^{-2x/3} + c_2 e^{-x/2}$, $y_p = 5/2$

44. $y'' - 2y' + 4y = 8$

45. $4y'' - 4y' + y = 8$

Solution: $y_h = c_1 e^{x/2} + c_2 x\, e^{x/2}$, $y_p = 8$

46. $4y'' - 12y' + 9y = 18$

6.3 Higher Order Linear Constant Equations

Constant Coefficients

Solve for $y(x)$. Proceed as in Examples 6.13–6.20.

1. $3y' - 2y = 0$

Solution: $y = c\,e^{2x/3}$.

The characteristic equation is $3r - 2 = 0$ with root $r = 2/3$. There is one Euler solution atom $e^{2x/3}$. Then y is a linear combination of the atoms.

2. $2y' + 7y = 0$

3. $y'' - y' = 0$

Solution: $y = c_1 + c_2 e^x$.

The characteristic equation is $r^2 - r = 0$ with roots $r = 0$, $r = 1$. The Euler solution atoms are e^{0x} and e^x. Then y is a linear combination of the atoms.

4. $y'' + 2y' = 0$

5. $y'' - y = 0$

Solution: $y = c_1 e^x + c_2 e^{-x}$

6. $y'' - 4y = 0$

7. $y'' + 2y' + y = 0$

Solution: $y = c_1 e^{-x} + c_2 x e^{-x}$

8. $y'' + 4y' + 4y = 0$

9. $y'' + 3y' + 2y = 0$

Solution: $y = c_1 e^{-x} + c_2 e^{-2x}$

10. $y'' - 3y' + 2y = 0$

11. $y'' + y = 0$

Solution: $y = c_1 \cos x + c_2 \sin x$.

The characteristic equation $r^2 + 1 = 0$ has complex roots $\pm i$, with Euler solution atoms $\cos x$, $\sin x$. Then y is a linear combination of the atoms.

12. $y'' + 4y = 0$

13. $y'' + y' + y = 0$

Solution: $y = c_1 e^{-x/2} \cos \sqrt{3}x/2 + c_2 e^{-x/2} \sin \sqrt{3}x/2$

14. $y'' + 2y' + 2y = 0$

15. $y'' = 0$

Solution: $y = c_1 + c_2 x$

16. $y''' = 0$

17. $\frac{d^4y}{dx^4} = 0$

Solution: $y = c_1 + c_2 x + c_2 x^2 + c_4 x^3$.

The characteristic equation $r^4 = 0$ has roots $r = 0, 0, 0, 0$ counted according to multiplicity. The Euler atoms are $1, x, x^2, x^3$ by Euler's multiplicity theorem. Then y is a linear combination of the atoms.

18. $\frac{d^5y}{dx^5} = 0$

19. $y''' + 2y'' = 0$

Solution: $y = c_1 + c_2 x + c_3 e^{-2x}$

The characteristic equation $r^3 + 2r^2 = 0$ has roots $r = 0, 0, -2$ and Euler atoms e^{0x}, xe^{0x}, e^{-2x}. Then y is a linear combination of the atoms.

20. $y''' + 4y' = 0$

21. $\frac{d^4y}{dx^4} + y'' = 0$

Solution: $y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x$.

The characteristic equation $r^4 + r^2 = 0$ has roots $r = 0, 0, i, -i$ and Euler solution atoms $1, x, \cos x, \sin x$. Then y is a linear combination of the atoms.

22. $\frac{d^5y}{dx^5} + y''' = 0$

Detecting Atoms

Decompose each atom into a base atom times a power of x. If the expression fails to be an atom, then explain the failure.

23. $-x$

Solution: Not an atom. Euler solution atoms have coefficient 1.

24. x

25. $x^2 \cos \pi x$

Solution: Base atom = $\cos \pi x$, power = x^2.

26. $x^{3/2} \cos x$

27. $x^{1000} e^{-2x}$

Solution: Base atom = e^{-2x}, power = x^{1000}.

28. $x + x^2$

29. $\dfrac{x}{1+x^2}$

Solution: Not an Euler solution atom. Most fractions fail.

30. $\ln |xe^{2x}|$

31. $\sin x$

Solution: Base atom = $\sin x$, power = x^0.

32. $\sin x - \cos x$

Solution: A linear combination of Euler solution atoms is not an atom.

Higher Order

A homogeneous linear constant-coefficient differential equation can be defined by (1) coefficients, (2) the characteristic equation, (3) roots of the characteristic equation. In each case, solve the differential equation.

33. $y''' + 2y'' + y' = 0$

Solution: $y = c_1 + c_2 e^{-x} + c_3 x e^{-x}$

34. $y''' - 3y'' + 2y' = 0$

35. $y^{(4)} + 4y'' = 0$

Solution: $y = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x$

36. $y^{(4)} + 4y''' + 4y'' = 0$

37. Order 5, $r^2(r-1)^3 = 0$

Solution: $y = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x$

38. Order 5, $(r^3 - r^2)(r^2 + 1) = 0$.

39. Order 6, $r^2(r^2 + 2r + 2)^2 = 0$.

Solution: $y = c_1 + c_2 x + c_3 e^{-x} \cos x + c_4 x e^{-x} \cos x + c_5 e^{-x} \sin x + c_6 x e^{-x} \sin x$

Factor $r^2 + 2r + 2 = (r + 1)^2 + 1$ with roots $-1\pi i$. Then the six roots are $r = 0, 0, -1\pi i, -1\pi i$ and the Euler solution atoms are $1, x$ and $e^{-x} \cos x$, $xe^{-x} \cos x$, $e^{-x} \sin x$, $xe^{-x} \sin x$. Then y is a linear combination of the atoms.

40. Order 6, $(r^2 - r)(r^2 + 4r + 5)^2 = 0$.

41. Order 10, $(r^4 + r^3)(r^2 - 1)^2(r^2 + 1) = 0$.

Solution: Solution y has ten terms as a linear combination of ten atoms $1, x, x^2$, e^x, xe^x, $e^{-x}, xe^{-x}, x^2e^{-x}$. $\cos x, \sin x$.

Factor as $r^3(r + 1)(r - 1)^2(r + 1)^2(r^2 + 1) = 0$ and then collect factors:

$r^3(r - 1)^2(r + 1)^3(r^2 + 1) = 0$

The ten roots are $r = 0, 0, 0$, $r = 1, 1$, $r = -1, -1, -1$, $r = \pm i$. The ten atoms are $1, x, x^2$, e^x, xe^x, $e^{-x}, xe^{-x}, x^2e^{-x}$. $\cos x, \sin x$. Then y is a linear combination of the atoms.

42. Order 10, $(r^3 + r^2)(r - 1)^3(r^2 + 1)^2 = 0$.

43. Order 5, roots $r = 0, 0, 1, 1, 1$.

Solution: $y = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x$

44. Order 5, roots $r = 0, 0, 1, i, -i$.

45. Order 6, roots $r = 0, 0, i, -i, i, -i$.

Solution: $y = c_1 + c_2 x + c_3 \cos x + c_4 x \cos x + c_5 \sin x + c_6 x \sin x$

46. Order 6, roots $r = 0, -1, 1 + i, 1 - i, 2i, -2i$.

47. Order 10, roots $r = 0, 0, 0, 1, 1, -1, -1, -1, i, -i$.

Solution: y is a linear combination of the ten atoms $1, x, x^2$, e^x, xe^x, $e^{-x}, xe^{-x}, x^2e^{-x}$, $\cos x, \sin x$.

48. Order 10, roots $r = 0, 0, 1, 1, 1, -1, i, -i, i, -i$.

Initial Value Problems

Given in each case is a set of independent solutions of the differential equation. Solve for the coefficients c_1, c_2, ... in the general solution, using the given initial conditions.

49. e^x, e^{-x}, $y(0) = 0$, $y'(0) = 1$

Solution: Let $y = c_1e^x + c_2e^{-x}$. Relations $y(0) = 0$, $y'(0) = 1$ translate to $c_1 + c_2 = 0$, $c_1 - c_2 = 1$. Elimination gives $c_1 = 1/2$, $c_2 = -1/2$.

50. xe^x, e^x, $y(0) = 1$, $y'(0) = -1$

51. $\cos x, \sin x$, $y(0) = -1$, $y'(0) = 1$

Solution: Let $y = c_1 \cos x + c_2 \sin x$. Relations $y(0) = 0$, $y'(0) = 1$ translate to $c_1 \cos 0 + c_2 \sin 0 = 0$, $-c_1 \sin 0 + c_2 \cos 0 = 1$. Because $\cos 0 = 1$, $\sin 0 = 0$, then $c_1 = 0$, $c_2 = 1$.

52. $\cos 2x, \sin 2x$, $y(0) = 1$, $y'(0) = 0$

53. $e^x, \cos x, \sin x$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0$

Solution: Let $y = c_1 \cos x + c_2 \sin x + c_3e^x$. Relations $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0$ translate to $c_1 + c_3 = -1$, $c_2 + c_3 = 1$, $-c_1 + c_3 = 0$. Add the first and third equation to get $c_3 = -1/2$. Then $c_1 = -1/2$, $c_2 = 3/2$, $c_3 = -1/2$.

```
# Exercise 53 answer check
u:=x->-1/2*cos(x)+3/2*sin(x)-1/2*exp(x);
u(0);D(u)(0);D(D(u))(0);
#  -1, 1, 0
```

54. $1, \cos x, \sin x$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0$

55. $e^x, xe^x, \cos x, \sin x$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 0$

Solution: Let $y = c_1 \cos x + c_2 \sin x + c_3e^x + c_4xe^x$. Relations $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 0$ translate to $c_1 + c_3 = -1$, $c_2 + c_3 + c_4 = 1$, $-c_1 + c_3 + 2c_4 = 0$, $-c_2 + c_3 + 3c_4 = 0$. Use computer assist to find $c_1 = 1/2$, $c_2 = 3/2$, $c_3 = -3/2$, $c_4 = 1$.

```
# Exercise 55, solve and answer check
sys:=[c1+c3=-1, c2+c3+c4=1, -c1+c3+2*c4=0, -c2+c3+3*c4=0];
solve(sys,{c1,c2,c3,c_});
# {c1 = 1/2, c2 = 3/2, c3 = -3/2, c4 = 1}
u:=x->1/2*cos(x)+3/2*sin(x)-3/2*exp(x)+x*exp(x);
u(0);D(u)(0);D(D(u))(0);D(D(D(u)))(0);
#  -1, 1, 0, 0
```

56. $1, x, \cos x, \sin x$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$, $y'''(0) = 0$

57. $1, x, x^2, x^3, x^4$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 1$, $y'''(0) = 3$, $y^{(4)}(0) = 0$

Solution: Let $y = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4$. Relations $y(0) = 1$, $y'(0) = 2$, $y''(0) = 1$, $y'''(0) = 3$, $y^{(4)}(0) = 0$ translate to $c_1 = 1$, $c_2 = 2$, $2c_3 = 1$, $6c_4 = 3$, $24c_5 = 0$. Then $y = 1 + 2x + x^2/2 + x^3/2$ and $c_1 = 1$, $c_2 = 2$, $c_3 = 1/2$, $c_4 = 1/2$. $c_5 = 0$.

58. $e^x, xe^x, x^2e^x, 1, x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$, $y^{(4)}(0) = 0$

Inverse Problem

Find a linear constant-coefficient homogeneous differential equation from the given information. Follow Example 6.21.

59. The characteristic equation is $(r + 1)^3(r^2 + 4) = 0$.

Solution: Expand to $r^5 + 3r^4 + 7r^3 + 13r^2 + 12r + 4 = 0$. Then the DE is $y^{(5)} + 3y^{(4)} + 7y''' + 13y'' + 12y' + 4y = 0$

60. The general solution is a linear combination of the Euler solution atoms $e^x, e^{2x}, e^{3x}, \cos x, \sin x$.

Solution: The atoms $e^x, e^{2x}, e^{3x}, \cos x, \sin x$ correspond to roots $1, 2, 3, \pm i$. The characteristic polynomial is then $(r-1)(r-2)(r-3)(r^2+1) = r^5 - 6r^4 + 12r^3 - 12r^2 + 11r - 6$. Then the DE is $y^{(5)} - 6y^{(4)} + 12y''' - 12y'' + 11y' - 6y = 0$.

61. The roots of the characteristic polynomial are $0, 0, 2 + 3i, 2 - 3i$.

Solution: The roots imply characteristic polynomial $(r-0)(r-0)((r-2)^2 + 9) = r^4 - 4r^3 + 13r^2$. Then the DE is $y^{(4)} - 4y''' + 13y'' = 0$.

62. The equation has order 4. Known solutions are $e^x + 4\sin 2x$, xe^x.

63. The equation has order 10. Known solutions are $\sin 2x$, x^7e^x.

Solution: Derivatives of solutions are also solutions which amasses a longer list of ten atoms $\sin 2x, \cos 2x$, e^x, xe^x, x^2e^x, x^3e^x, x^4e^x, x^5e^x, x^6e^x, x^7e^x. Then the characteristic polynomial is $(r^2 + 4)(r - 1)^8 = r^10 - 8r^9 + 32r^8 - 88r^7 + 182r^6 - 280r^5 + 308r^4 - 232r^3 + 113r^2 - 32r + 4$. Then the DE is $y^{(10)} - 8y^{(9)} + 32y^{(8)} - 88y^{(7)} + 182y^{(6)} - 280y^{(5)} + 308y^{(4)} - 232y''' + 113y'' - 32y' + 4y = 0$.

64. The equation is $my'' + cy' + ky = 0$ with $m = 1$ and c, k positive. A solution is $y(x) = e^{-x/5}\cos(2x - \theta)$ for some angle θ.

Independence of Euler Atoms

65. Apply the independence test page 378 to atoms 1 and x: form equation $0 = c_1 + c_2x$, then solve for $c_1 = 0$, $c_2 = 0$. This proves Euler atoms $1, x$ are independent.

Solution: Equation $0 = c_1 + c_2x$ holds for all x. Set $x = 0$ to conclude $c_1 = 0$. Then $0 + c_2x = 0$ for all x. Set $x = 1$ to conclude $c_2 = 0$. By the independence test, 1 and x are independent on $-\infty < x < \infty$.

66. Show that Euler atoms $1, x, x^2$ are independent using the independence test page 378 ,

67. A Taylor series is zero if and only if its coefficients are zero. Use this result to give a complete proof that the list $1, \ldots, x^k$ is independent. Hint: a polynomial is a Taylor series.

Solution: Let $y(x) = \sum_{n=0}^{k} c_n x^n$. Apply the independence test: let $y(x) = 0$ for all x and solve for c_0 to c_k. By the theory of Taylor series, $y(x) = 0$ means all Taylor coefficients are zero, because $c_n = y^{(n)}(0)/n!$. Therefore, c_0 to c_k are all zero, proving the powers 1 to x_k are independent on $-\infty < x < \infty$.

68. Show that Euler atoms e^x, xe^x, x^2e^x are independent using the independence test page 378 .

Solution: Hint: reduce the problem to Exercise 67 by canceling e^x from the independence test equation.

Wronskian Test

Establish independence of the given lists of functions by using the Wronskian test page 385 :

Functions $f_1, f_2, \ldots, f_n$ are independent if $W(x_0) \neq 0$ for some x_0, where $W(x)$ is the $n \times n$ determinant

$$\begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ & \vdots & \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

69. $1, x, e^x$

Solution: Because $W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$, then the Wronskian test applies and $1, x, x^2$ are independent on $-\infty < x < \infty$.

70. $1, x, x^2, e^x$

71. $\cos x, \sin x, e^x$

Solution: Let $W(x) = \begin{vmatrix} \cos x & \sin x & e^x \\ -\sin x & \cos x & e^x \\ -\cos x & -\sin x & e^x \end{vmatrix}$. Then $W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 2$. The Wronskian test applies. proving the three functions are independent on $-\infty < x < \infty$.

72. $\cos x, \sin x, \sin 2x$

Kümmer's Lemma

73. Compute the characteristic polynomials $p(r)$ and $q(r)$ for

$$\begin{aligned} &y'' + 3y' + 2y = 0 \text{ and} \\ &z'' + z' = 0. \end{aligned}$$

Verify the equations are related by $y = e^{-x}z$ and $p(r-1) = q(r)$.

Solution: The characteristic polynomials are $p(r) = r^2+3r+2 = (r+1)(r+2)$ and $q(r) = r^2 + r$. Then $p(u-1) = (u-1+1)(u-1+2) = u(u+1) = u^2 + u = q(u)$. Compute:

$$\begin{aligned} &y' = \tfrac{d}{dx}(e^{-x}\,z) = -e^{-x}z + e^{-x}z' \\ &y'' = e^{-x}z - 2e^{-x}z' + e^{-x}z'' \\ &y'' + 3y' + 2y = e^{-x}(z - 2z' + z'') - 3e^{-x}z + 3e^{-x}z' + 2e^{-x}z \\ &\quad = e^{-x}(z - 2z' + z'' - 3z + 3z' + 2z) \\ &\quad = e^{-x}(z'' + z') \end{aligned}$$

Then $y'' + 3y' + 2y = 0$ if and only if $z'' + z' = 0$.

74. Compute the characteristic polynomials $p(r)$ and $q(r)$ for

$$\begin{aligned} &ay'' + by' + cy = 0 \text{ and} \\ &az'' + (2ar_0 + b)z' + \\ &\quad (ar_0^2 + br_0 + c)z = 0. \end{aligned}$$

Verify the equations are related by $y = e^{r_0x}z$ and $p(r + r_0) = q(r)$.

6.4 Variation of Parameters

Independence: Constant Equation

Find solutions y_1, y_2 of the given homogeneous differential equation using Theorem 6.1 page 431. Then apply the Wronskian test page 464 to prove independence, following Example 6.22.

1. $y'' - y = 0$

Solution: Characteristic equation $r^2 - 4 = 0$ has roots $r = 2, -2$. Euler solution atoms are e^{2x}, e^{-2x}. The Wronskian of the two atoms is $W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$. The Wronskian test applies: the two atoms are independent.

2. $y'' - 4y = 0$

3. $y'' + y = 0$

Solution: $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$.

4. $y'' + 4y = 0$

5. $4y'' = 0$

Solution: Atoms are 1, x and $W = 1$.

6. $y'' = 0$

7. $4y'' + y' = 0$

Solution: Atoms are 1, $e^{-x/4}$. Then $W = \begin{vmatrix} 1 & e^{-x/4} \\ 0 & -e^{-x/4}/4 \end{vmatrix} = \frac{-1}{4}e^{-x/4}$.

8. $y'' + y' = 0$

9. $y'' + y' + y = 0$

Solution: The roots of $r^2 + r + 1 = 0$ are $-1/2 \pm i\sqrt{3}/2$. Atoms are $e^{-x/2}\cos\sqrt{3}x/2$, $e^{-x/2}\sin\sqrt{3}x/2$. Let $W = \begin{vmatrix} e^a x \cos bx & e^a x \sin bx \\ (e^a x \cos bx)' & (e^a x \sin bx)' \end{vmatrix}$. Then $W = be^{2ax}$ because $\cos^2(bx) + \sin^2(bx) = 1$. Substitute $a = -1/2$, $b = \sqrt{3}/2$. Then $W = \dfrac{\sqrt{3}}{2}e^{-x}$. The Wronskian test applies to prove the two atoms are independent.

10. $y'' - y' + y = 0$

11. $y'' + 8y' + 2y = 0$

Solution: The roots of $r^2 + 8r + 2 = 0$ are $-4 \pm \sqrt{14}$. Let $W = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = (b-a)(e^{ax+bx}$. Then $W \neq 0$ if $a \neq b$. Substitute $a, b = -4 \pm \sqrt{14}$ and apply the Wronskian test to prove the atoms are independent.

12. $y'' + 16y' + 4y = 0$

Independence for Euler's Equation

Change variables, $x = e^t$, $u(t) = y(x)$ in $Ax^2y''(x) + Bxy'(x) + Cy(x) = 0$ to obtain a constant-coefficient equation $A\left(\frac{d^2u}{dt^2} - \frac{du}{dt}\right) + B\frac{du}{dt} + Au = 0$. Solve for $u(t)$ and then substitute $t = \ln|x|$ to obtain $y(x)$. Find two solutions y_1, y_2 which are independent by the Wronskian test page 464.

13. $x^2y'' + y = 0$

Solution: The transformed equation is $u'' - u' + u = 0$ where $' = d/dt$, $t = \ln|x|$, $u(t) = y(x)$. The roots and atoms are $1/2 \pm i\sqrt{3}/2$, $e^{x/2}\cos\sqrt{3}x/2$, $e^{x/2}\sin\sqrt{3}x/2$. The calculation of Exercise 9 proves independence of the atoms.

14. $x^2y'' + 4y = 0$

15. $x^2y'' + 2xy' + y = 0$

Solution: The transformed equation is $u'' - u' + 2u' + u = 0$ where $' = d/dt$, $t = \ln|x|$, $u(t) = y(x)$. The roots and atoms are $-1/2 \pm i\sqrt{3}/2$, $e^{-x/2}\cos\sqrt{3}x/2$, $e^{-x/2}\sin\sqrt{3}x/2$. The calculation of Exercise 9 proves independence of the atoms.

16. $x^2y'' + 8xy' + 4y = 0$

Wronskian

Compute the Wronskian, up a constant multiple, without solving the differential equation: Example 6.23 page 466.

17. $y'' + y' - xy = 0$

Solution: Abel's identity $W(x) = W(x_0)e^{-\int_{x_0}^{x} \frac{b(t)}{a(t)}dt}$ applies to $a(x)y'' + b(x)y' + c(x)y = 0$. Translate $y'' + y' - xy = 0$ to $a = b = 1$, $c = -1$. Choose $x_0 = 0$. Then

$$W(x) = W(0)e^{-\int_0^x dt} = W(0)e^{-x}$$

The Wronskian $W(x) = e^{-x}$ up to a constant multiple.

18. $y'' - y' + xy = 0$

19. $2y'' + y' + \sin(x)y = 0$

Solution: $W(x) = e^{-x/2}$ up to a constant.

20. $4y'' - y' + \cos(x)y = 0$

21. $x^2y'' + xy' - y = 0$

Solution: The integral in Abel's identity is $\int_1^x (-t/t^2)dt = -\ln|x| + \ln|1|$. Abel's identity is $W(x) = W(1)e^{-\ln|x|} = c/x$ for some constant c.

22. $x^2y'' - 2xy' + y = 0$

Variation of Parameters

Find the general solution $y_h + y_p$ by applying a variation of parameters formula: Example 6.24 page 466.

23. $y'' = x^2$

Solution: Because $y_h = c_1 + c_2x$, let $y_1 = 1$ and $y_2 = x$. Compute $W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$. Follow Example 6.24 page 466:

$$\begin{aligned} y_p &= -y_1(x)\int y_2(x)x^2dx + y_2(x)\int y_1(x)x^2dx \\ &= -\int x^3dx + x\int x^2dx \\ &= -x^4/4 + x^4/3 + c_3 + c_4x \\ &= x^4/12 \quad \text{by taking integration constants } c_3 = c_4 = 0. \end{aligned}$$

Then $y = y_h + y_p = c_1 + c_2x + x^4/12$.

Answer check: $y'' = (c_1 + c_2x + x^4/12)'' = (c_2 + 4x^3/12)' = x^2$.

24. $y'' = x^3$

25. $y'' + y = \sin x$

Solution: Let $y_1 = \cos x$, $y_2 = \sin x$, which are Euler atoms for $r^2 + 1 = 0$. Then $W = 1$ and

$$\begin{aligned} y_p &= -y_1(x)\int y_2(x)\sin x dx + y_2(x)\int y_1(x)\sin x dx \\ &= -\cos x\int \sin x \sin x dx + \sin x\int \cos x \sin x dx \\ &= -\cos(x)(-\tfrac{1}{2}\cos(x)\sin(x) + \tfrac{1}{2}x) + \tfrac{1}{2}\sin^3(x) + c_3\cos x + c_4\sin x \\ &= \tfrac{1}{2}\sin x - \tfrac{1}{2}x\cos x \quad \text{by taking } c_3 = c_4 = 0. \end{aligned}$$

Then $y_h = c_1\cos x + c_2\sin x$, $y_p = \frac{1}{2}\sin x - \frac{1}{2}x\cos x$

Answer check: $y'' + y = (\frac{1}{2}\sin x - \frac{1}{2}x\cos x)'' + (\frac{1}{2}\sin x - \frac{1}{2}x\cos x)$

$$\begin{aligned} &= -\tfrac{1}{2}\sin x + \sin x + \tfrac{1}{2}x\cos x + (\tfrac{1}{2}\sin x - \tfrac{1}{2}x\cos x) \\ &= \sin x. \end{aligned}$$

```
# Exercise 25, Variation of Parameters
 Y1:=cos(x);Y2:=sin(x);
 YP:=-Y1*int(Y2*sin(x),x) + Y2*int(Y1*sin(x),x);
 simplify(YP);
 # (1/2)*sin(x)-(1/2)*cos(x)*x
 dsolve(D(D(y))(x) + y(x)=sin(x),y(x));# double-check answer
```

26. $y'' + y = \cos x$

27. $y'' + y' = e^x$
Solution: $y_1 = 1$, $y_2 = e^{-x}$, $W = -e^{-x}$, $y_p = \frac{1}{2}e^x$.

28. $y'' + y' = -e^x$

29. $y'' + 2y' + y = e^{-x}$
Solution: $y_1 = e^{-x}$, $y_2 = xe^{-x}$, $W = -e^{-2x}$, $y_p = \frac{1}{2}x^2e^{-x}$.

30. $y'' - 2y' + y = e^x$

6.5 Undetermined Coefficients

Polynomial Solutions

Determine a polynomial solution y_p for the given differential equation.

1. $y'' = x$

Solution: Quadrature equation, $y_p = x^3/6$.

2. $y'' = x - 1$

3. $y'' = x^2 - x$

Solution: Quadrature equation, $y_p = x^4/12 - x^3/6$.

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

Solution: Trial solution $y = d_1 + d_2 x$ or guess by experience $y_p = -x$.

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

Solution: Trial solution $y = d_1 x + d_2 x^2$. Calculate $2d_2 - d_1 - 2d_2 x = x$ and then linear algebraic equations $-d_1 + 2d_2 = 0$, $-2d_2 = 1$. Solution $d_2 = -1/2$ and $d_1 = -1$ gives particular solution $y_p = -x - x^2/2$.

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

Solution: No roots of characteristic equation $r^2 - r + 1 = 0$ match root=0 of RHS atom 1. Then $y = d_1$ is the trial solution. Substitution gives $d_1 = 1$ and $y_p = 1$. A shortcut is provided by theorems: cancel all higher order terms and deduce $y = 1$.

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

Solution: No shortcut available. Trial solution $y = d_1 + d_2 x$ obtained from the RHS has roots $0, 0$, which do not conflict with the roots $\pm i$ of the characteristic equation $r^2+1 = 0$. The first trial solution is also the corrected trial solution. Substitute to obtain $d_1 + d_2 x = 1 - x$, then $y_p = 1 - x$. The shortcut of canceling higher derivatives does not apply, however the answer provided is correct. Guessing the answer and then checking the answer is always a valid technique.

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

Solution: No shortcut available. Trial solution $y = d_1 + d_2 x + d_3 x^2$ obtained from the RHS has roots $0, 0, 0$, which do not conflict with the roots ± 1 of the characteristic equation $r^2 - 1 = 0$. The first trial solution is also the corrected trial solution. Substitute to obtain $2d_3 - d_1 - d_2 x - d_3 x^2 = x^2$, then match coefficients left and right. The linear algebraic equations are $d_3 = -1$, $-d_1 + 2d_3 = 0$, $d_2 = 0$. Then $y_p = -2 - x^2$. The shortcut of canceling higher derivatives does not apply and the answer from this false method is incorrect.

14. $y'' - y = x^3$

Polynomial-Exponential Solutions

Determine a solution y_p for the given differential equation.

15. $y'' + y = e^x$

Solution: $y_p = \frac{1}{2} e^x$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

Solution: $y_p = d_1 e^{2x}$, $d_1 = \frac{1}{4}$.

18. $y'' = e^{-2x}$

19. $y'' - y = (x + 1)e^{2x}$

Solution: $y_p = d_1 e^{2x} + d_2 x e^{2x}$, $d_1 = -\frac{1}{9}$, $d_2 = \frac{1}{3}$.

```
# Exercise 19
de:=diff(y(x),x,x) - y(x) = (x+1)*exp(2*x);
dsolve(de,y(x));
```

20. $y'' - y = (x - 1)e^{-2x}$

21. $y'' - y' = (x + 3)e^{2x}$

Solution: $y_p = d_1 e^{2x} + d_2 x e^{2x}$, $d_1 = \frac{3}{4}$, $d_2 = \frac{1}{2}$.

22. $y'' - y' = (x - 2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2 + 3)e^{3x}$

Solution: $y_p = d_1 e^{3x} + d_2 x e^{3x} + d_3 x^2 e^{3x}$, $d_1 = \frac{13}{4}$, $d_2 = -\frac{3}{2}$, $d_3 = \frac{1}{2}$.

24. $y'' - 3y' + 2y = (x^2 - 2)e^{-3x}$

Sine and Cosine Solutions

Determine a solution y_p for the given differential equation.

25. $y'' = \sin(x)$

Solution: $y_p = d_1 \cos(x) + d_2 \sin(x)$, $d_1 = 0$, $d_2 = -1$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

Solution: $y_p = d_1 x \cos(x) + d_2 x \sin(x)$, $d_1 = -\frac{1}{2}$, $d_2 = 0$

28. $y'' + y = \cos(x)$

29. $y'' = (x + 1)\sin(x)$

Solution: $y_p = d_1 \cos(x) + d_2 \sin(x) + d_3 x \cos(x) + d_4 x \sin(x)$, $d_1 = -2$, $d_2 = -1$, $d_3 = 0$, $d_4 = -1$.

30. $y'' = (x + 1)\cos(x)$

31. $y'' - y = (x + 1)e^x \sin(2x)$

Solution: $y_p = d_1 e^x \cos(2x) + d_2 e^x \sin(2x) + d_3 x e^x \cos(x) + d_4 x e^x \sin(x)$, $d_1 = -\frac{3}{16}$, $d_2 = 0$, $d_3 = -\frac{1}{8}$, $d_4 = -\frac{1}{8}$.

32. $y'' - y = (x + 1)e^x \cos(2x)$

33. $y'' - y' - y = e^x \sin(2x)$

Solution: $y_p = d_1 e^x \cos(2x) + d_2 e^x \sin(2x)$, $d_1 = 2$, $d_2 = -\frac{5}{29}$.

34. $y'' - y' - y = (x^2 + x)e^x \cos(2x)$

Undetermined Coefficients Algorithm

Determine a solution y_p for the given differential equation.

35. $y'' = x + \sin(x)$

Solution: $y_p = (d_1 + d_2 x)x^2 + d_3 \cos(x) + d_4 \sin(x) = \frac{1}{6}x^3 - \sin(x)$. Superposition implies $y_p = u + v$ where $u'' = x$ and $v'' = \sin(x)$. Guess answers $u = x^3/6$, $v = -\sin(x)$ and check.

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

Solution: $y_p = d_1 + d_2 x + x(d_3\cos(x) + d_4\sin(x)) = x - \frac{1}{2}x\cos(x)$. Superposition implies $y_p = u + v$ where $u'' + u = x$ and $v'' + v = \sin(x)$. Guess answers $u = x$, $v = -\frac{1}{2}x\cos(x)$ and check.

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

Solution: $y_p = x(d_1\cos(x) + d_2\sin(x)) = -\frac{1}{2}x\cos(x) + \frac{1}{2}x\sin(x)$

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

Solution: $y_p = (d_1 + d_2 x)x^2 + d_3 e^x + d_4 xe^x + d_5\cos(x) + d_6\sin(x) = \frac{1}{6}x^3 - e^x + xe^x - \sin(x)$.

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

Solution: Write the RHS $= \frac{1}{2}e^x - \frac{1}{2}e^{-x} + \frac{1}{2} + \frac{1}{2}\cos(2x)$. Then $y_p = x(d_1 e^x + d_2 e^{-x}) + d_3 + d_4\cos(2x) + d_5\sin(2x) = \frac{1}{4}xe^x + \frac{1}{4}xe^{-x} - \frac{1}{2} - \frac{1}{10}\cos(2x)$.

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2 e^x$

Solution: $y_p = d_1 e^x + d_2 xe^x + d_3 x^2 e^x = 16e^x - 6xe^x + x^2 e^x$.

46. $y'' + y' - y = xe^x\sin(2x)$

Roots and Related Atoms

Euler atoms A and B are said to be **related** if and only if the derivative lists A, A', ... and B, B', ... share a common Euler atom.

47. Find the roots, listed according to multiplicity, for the atoms 1, x, x^2, e^{-x}, $\cos 2x$, $\sin 3x$, $x\cos\pi x$, $e^{-x}\sin 3x$.

Solution: Roots $= 0, 0, 0, -1, \pm 2i, \pm 3i, \pi i, \pi i, -1 \pm 3i$.

48. Find the roots, listed according to multiplicity, for the atoms 1, x^3, e^{2x}, $\cos x/2$, $\sin 4x$, $x^2\cos x$, $e^{3x}\sin 2x$.

49. Let $A = xe^{-2x}$ and $B = x^2e^{-2x}$. Verify that A and B are related.

Solution: Distinct atoms in derivatives of $A = e^{-2x}$, xe^{-2x}. Distinct atoms in derivatives of $B = e^{-2x}$, xe^{-2x}, x^2e^{-2x}. The lists have two atoms in common.

50. Let $A = xe^{-2x}$ and $B = x^2e^{2x}$. Verify that A and B are not related.

51. Prove that atoms A and B are related if and only if their base atoms have the same roots.

Solution: An atom can be written as $x^n e^{ax}\cos(bx)$ or $x^n e^{ax}\sin(bx)$ where $n \ge 0$ is an integer, $a =$ real number, $b \ge 0$ is a real number.

If A ad B are related then their derivative lists have an atom in common, say $x^n e^{ax}\cos(bx)$. The base atom strips off the power of x: base atom $=$ $e^{ax}\cos(bx)$. Then $e^{ax}\cos(bx)$ is an atom in common with the two derivative lists. So both A and B have base atom $e^{ax}\cos(bx)$ with roots $a \pm bi$.

If A and B have the same base atom, say $e^{ax}\cos(bx)$, then this atom appears in both derivative lists. Therefore A and B are related. $\blacksquare$

52. Prove that atoms A and B are related if and only if they are in the same **group**. See page 474 for the definition of a group of atoms.

Modify a Trial Solution

Apply Rule II to modify the given Rule I trial solution into the shortest trial solution.

53. The characteristic equation has factors r^3, $(r^3 + 2r^2 + 2)$, $(r-1)^2$, $(r+1)$, $(r^2+4)^3$ and the Rule I trial solution is constructed from atoms 1, x, e^x, xe^x, e^{-x}, $\cos 2x$, $\sin 2x$, $\cos x$, $\sin x$.

Solution: The shortest trial solution is a linear combination of atoms x^3, x^4, x^2e^x, x^3e^x, xe^{-x}, $x^3\cos 2x$, $x^3\sin 2x$, $\cos x$, $\sin x$.

54. The characteristic equation has factors r^2, $(r^3 + 3r^2 + 2)$, $(r+1)$, $(r^2+4)^3$ and the Rule I trial solution is constructed from atoms 1, x, e^x, xe^x, e^{-x}, $\cos 2x$, $\sin 2x$.

Annihilators and Laplace Theory

Laplace theory can construct the annihilator of $f(t)$. The example $y'' + 4y = t + 2t^3$ is used to discuss the techniques. Formulas to be justified: $p(s) = \mathcal{L}(f)/\mathcal{L}(y)$ and $q(s) = \mathbf{denom}(\mathcal{L}(f(t)))$.

55. **(Transfer Function)** Find the characteristic polynomial $q(r)$ for the homogeneous equation $y'' + 4y = 0$. The transfer function for $y'' + 4y = f(t)$ is $\mathcal{L}(y)/\mathcal{L}(f)$, which equals $1/q(s)$.

Solution: $q(r) = r^2 + 4$; the transfer function can be formally obtained by solving with Laplace's method for the special solution with zero initial data and input Dirac impulse function. The details are in the solution of Exercise 56.

56. (Laplace of $y_p(t)$)
The Laplace of $y(t)$ for problem
$y'' + 4y = f(t)$, $y(0) = y'(0) = 0$ must equal the Laplace of $f(t)$ times the transfer function. Justify and explain what it has to do with finding y_p.

Solution: Laplace details: $(s^2 + 4)\mathcal{L}(y) = \mathcal{L}(f)$. Then y is a particular solution y_p, found by Laplace methods.

57. (Annihilator of $f(t)$)
Let $g(t) = t + 2t^3$. Verify that $\mathcal{L}(g(t)) = \dfrac{s^2 + 12}{s^4}$, which is a proper fraction with denominator s^4. Then explain why one annihilator of $g(t)$ has characteristic polynomial r^4. The result means that $y = g(t) = t + 2t^3$ is a solution of $y'''' = 0$.

Solution: Laplace tables: $\mathcal{L}(g) = \mathcal{L}(t) + \mathcal{L}(2t^3) = 1/s^2 + 2(6/s^4) = (s^2 + 12)/s^4$.

A differential equation $H(y) = 0$ with one solution being $y = g(t)$ is called an annihilator of g.

Solution $y(t) = t + 2t^3$ of $H(y) = 0$ has initial data $y(0) = 0$, $y'(0) = 1$. Formal Laplace methods applied to find $\mathcal{L}(y)$ from $H(y) = 0$ would collect symbols $y(0)$ and $y'(0)$ on the left side, with $\mathcal{L}(0) = 0$ on the right side. We always collect $q(s)\mathcal{L}(y)$ on the left and move the lower order terms to the right side. Then divide. Therefore, $\mathcal{L}(y) =$ polynomial in s divided by $q(s)$. Look at fraction $\mathcal{L}(g) = (s^2 + 12)/s^4$. It has lower order terms in the numerator. So $q(s) = s^4$ could be the characteristic polynomial.

Check: $q(r) = r^4$ would imply $H(y) = 0$ is $y'''' = 0$. Test $H(g) = 0$. It works.

58. (Laplace Theory finds y_p)
Show that the problem $y'' + 4y = t + 2t^3$, $y(0) = y'(0) = 0$ has Laplace transform

$$\mathcal{L}(y) = \frac{s^2 + 12}{(s^2 + 4)s^4}.$$

Explain why $y(t)$ must be a solution of the constant-coefficient homogeneous differential equation having characteristic polynomial $w(r) = (r^2 + 4)r^4$.

Annihilator Method Justified

The method of annihilators can be justified by successive differentiation of a non-homogeneous differential equation, then forming a linear combination of the resulting formulas. It is carried out here, for exposition efficiency, for the non-homogeneous equation $y'' + 4y = x + 2x^3$. The right side is $f(x) = x + 2x^3$ and the homogeneous equation is $y'' + 4y = 0$.

59. (Homogeneous equation)
Verify that $y'' + 4y = 0$ has characteristic polynomial $q(r) = r^2 + 4$.

Solution: Euler's substitution $y = e^{rx}$ gives $r^2 + 4 = 0$ as characteristic equation.

60. (Annihilator)
Verify that $y^{(4)} = 0$ is an annihilator for $f(x) = x + 2x^3$, with characteristic polynomial $q(r) = r^4$.

61. (Composite Equation)
Differentiate four times across the equation $y'' + 4y = f(x)$ to obtain $y^{(6)} + 4y^{(4)} = f^{(4)}(x)$. Argue that $f^{(4)}(x) = 0$ because $y^{(4)} = 0$ is an annihilator of $f(x)$. This proves that y_p is a solution of higher order equation $y^{(6)} + 4y^{(4)} = 0$. Then argue that $w(r) = r^4(r^2 + 4)$ is the characteristic polynomial of the equation $y^{(6)} + 4y^{(4)} = 0$.

Solution: Details are short proofs or calculations. Because y_p is a solution of $y'' + 4y = f(x)$ then it is legal to differentiate the equation repeatedly to obtain homogeneous higher order equation $y^{(6)} + 4y^{(4)} = f^{(4)}(x) = 0$.

62. (General Solution)
Solve the homogeneous composite equation $y^{(6)} + 4y^{(4)} = 0$ using its characteristic polynomial $w(r) = r^4(r^2 + 4)$.

Solution: $y = d_1 \cos 2x + d_2 \sin 2x + d_3 + d_4 x + d_5 x^2 + d_6 x^3$

63. (Extraneous Atoms)
Argue that the general solution from the previous exercise contains two terms constructed from atoms derived from roots of the polynomial $q(r) = r^2 + 4$. Remove these terms to obtain the shortest expression for y_p and explain why it works.

Solution: Remove $d_1 \cos 2x + d_2 \sin 2x$. Then $y_p = d_3 + d_4 x + d_5 x^2 + d_6 x^3$. The argument: $y = y_1 + y_2$ where $y_1 = d_1 \cos 2x + d_2 \sin 2x$ and $y_2 = d_3 + d_4 x + d_5 x^2 + d_6 x^3$. Because $y'' + 4y = 0$ has general solution $y_h = c_1 \cos 2x + c_2 \sin 2x$ then y_1 equals y_h with specialized coefficients $c_1 = d_1$, $c_2 = d_2$. Therefore, $y = y_1 + y_2$ has the form $y_h + y_p$ and we remove y_1 to obtain the shortest particular solution y_2, announced as particular solution y_p.

64. (Particular Solution)
Report the form of the shortest particular solution of $y'' + 4y = f(x)$, according to the previous exercise.

Solution: $y_p = d_3 + d_4 x + d_5 x^2 + d_6 x^3$

6.6 Undamped Mechanical Vibrations

Simple Harmonic Motion

Determine the model equation $mx''(t) + kx(t) = 0$, the natural frequency $\omega = \sqrt{k/m}$, the period $2\pi/\omega$ and the solution $x(t)$ for the following spring–mass systems.

1. A mass of 4 Kg attached to a spring of Hooke's constant 20 Newtons per meter starts from equilibrium plus 0.05 meters with velocity 0.

Solution: Initial data: $x(0) = 0.05$, $x'(0) = 0$. Parameters: $m = 4$, $k = 20$. The model is $4x'' + 20x = 0$ with solution $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, $\omega^2 = k/m = 5$. Initial data $x(0) = 0.05$, $x'(0) = 0$ provides a system of linear algebraic equations for unknowns c_1, c_2:

$$c_1 \cos(0) + c_2 \sin(0) = 0.05, \quad -\omega c_1 \sin(0) + \omega c_2 \cos(0) = 0$$

Solve for $c_1 = 0.05$, $c_2 = 0$. Then $x(t) = 0.05 \cos(\sqrt{5}t)$.

2. A mass of 2 Kg attached to a spring of Hooke's constant 20 Newtons per meter starts from equilibrium plus 0.07 meters with velocity 0.

3. A mass of 2 Kg is attached to a spring that elongates 20 centimeters due to a force of 10 Newtons. The motion starts at equilibrium with velocity -5 meters per second.

Solution: Initial data: $x(0) = 0$, $x'(0) = -5$. Parameters: $m = 2$ and $k =$ force/elongation $= 10/(20/100) = 50$. The model is $2x'' + 50x = 0$ with solution $x(t) = c_1 \sin(5t) + c_2 \cos(5t)$, $c_1 = 0$, $c_2 = -1$.

```
# Exercise 3 Answer Check
DE:=m*diff(x(t),t,t) + k*x(t)=0;
de:=subs(m=2,k=50,DE);
dsolve(de,x(t));
dsolve([de,x(0)=0,D(x)(0)=-5],x(t));
```

4. A mass of 4 Kg is attached to a spring that elongates 20 centimeters due to a force of 12 Newtons. The motion starts at equilibrium with velocity -8 meters per second.

5. A mass of 3 Kg is attached to a coil spring that compresses 2 centimeters when 1 Kg rests on the top coil. The motion starts at equilibrium plus 3 centimeters with velocity 0.

Solution: Newton's law: force = mass $\times$ acceleration $= 3g$. Hooke's law: $3g =$ force $= k$(elongation) $= k(2/100)$. Then $k = 3g/(2/100) = 150g$. Units are MKS: $g = 9.8$. Model: $3x'' + 150gx = 0$. Answer: $x(t) = -\dfrac{1}{14\sqrt{10}} \sin\left(7\sqrt{10}\,t\right)$

6. A mass of 4 Kg is attached to a coil spring that compresses 2 centimeters when 2 Kg rests on the top coil. The motion starts at equilibrium plus 4 centimeters with velocity 0.

7. A mass of 5 Kg is attached to a coil spring that compresses 1.5 centimeters when 1 Kg rests on the top coil. The motion starts at equilibrium plus 3 centimeters with velocity −5 meters per second.

Solution: Newton's law: force = mass × acceleration = $5g$. Hooke's law: $5g$ = force = k(elongation) = $k(1.5/100)$. Then $k = 5g/(1.5/100) = 1000g/3$. Units are MKS: $g = 9.8$. Model: $15x'' + 1000gx = 0$. Answer: $x(t) = -\dfrac{1}{28\sqrt{30}}\sin(\omega t) + \dfrac{3}{100}\cos(\omega t)$, $\omega = \frac{14}{3}\sqrt{30}$.

8. A mass of 4 Kg is attached to a coil spring that compresses 2.2 centimeters when 2 Kg rests on the top coil. The motion starts at equilibrium plus 4 centimeters with velocity −8 meters per second.

9. A mass of 5 Kg is attached to a spring that elongates 25 centimeters due to a force of 10 Newtons. The motion starts at equilibrium with velocity 6 meters per second.

Solution: Model: $5x'' + 20x = 0$, $x(0) = 0$, $x'(0) = 6$. Answer: $x(t) = 3\sin(2t)$

10. A mass of 5 Kg is attached to a spring that elongates 30 centimeters due to a force of 15 Newtons. The motion starts at equilibrium with velocity 4 meters per second.

Phase–amplitude Form

Solve the given differential equation and report the general solution. Solve for the constants c_1, c_2. Report the solution in phase–amplitude form

$$x(t) = A\cos(\omega t - \alpha)$$

with $A > 0$ and $0 \le \alpha < 2\pi$.

11. $x'' + 4x = 0$,
$x(0) = 1$, $x'(0) = -1$

Solution: General solution: $x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$, $\omega = 2$. Constants: $c_1 = 1$, $c_2 = -1/2$. Amplitude: $A = \sqrt{(c_1^2 + c_2^2)} = \frac{1}{2}\sqrt{5}$. Phase: $\alpha = \arctan(c_2/c_1) = \arctan(-1/2)$. Because $\alpha < 0$ then replace it by $\alpha + 2\pi = -\arctan(1/2) + 2\pi = 5.819537699$ radians. Then $x(t) = A\cos(\omega t - \alpha)$ with $A > 0$ and $0 \le \alpha < 2\pi$.

```
# Exercise 11, Answer Check
DE:=m*diff(x(t),t,t) + k*x(t)=0;
de:=subs(m=1,k=4,DE);dsolve(de,x(t));
q:=dsolve([de,x(0)=1,D(x)(0)=-1],x(t));
# Convert to phase-amplitude form
findAlphaAmplitude:=proc(c1,c2)
local A,alpha;
A:=sqrt(c1^2+c2^2);alpha:=arctan(c2,c1);
if evalf(alpha)>=0 then RETURN([A,alpha,alpha]);
else RETURN([A,alpha+2*Pi,alpha]); fi;
end proc:
c1:=1:c2:=-1/2:
p:=findAlphaAmplitude(c1,c2);
A:=p[1]:ALPHA:=p[2]:
printf("A=%a, ALPHA=%a, %a\n",A,ALPHA,evalf(ALPHA));
printf("c1=%a,c2=%a,tan(alpha)=%a\n",c1,c2,tan(ALPHA));
simplify(expand(A*cos(u-ALPHA)));
```

12. $x'' + 4x = 0$,
$x(0) = 1$, $x'(0) = 1$

13. $x'' + 16x = 0$,
$x(0) = 2$, $x'(0) = -1$

Solution: Solution: $x\left(t\right) = \frac{1}{2}\,\sin\left(2\,t\right) + \cos\left(2\,t\right)$. Values: $\alpha = \arctan(1/2) = 0.4636476090$, $c_1 = 1$, $c_2 = 1/2$, $\tan(\alpha) = 1/2$, $A = \frac{1}{2}\sqrt{5}$. Then $x(t) = A\cos(\omega t - \alpha)$ with $A > 0$ and $0 \le \alpha < 2\pi$.

14. $x'' + 16x = 0$,
$x(0) = -2$, $x'(0) = -1$

15. $5x'' + 11x = 0$,
$x(0) = -4$, $x'(0) = 1$

Solution: Solution: $x\left(t\right) = \frac{1}{11}\,\sqrt{55}\sin\left(\frac{1}{5}\,\sqrt{55}t\right) - 4\,\cos\left(\frac{1}{5}\,\sqrt{55}t\right)$. Values: $\alpha = -\arctan(\sqrt{55}/44) + \pi = 2.974612139$, $c_1 = -4$, $c_2 = \sqrt{55}/11$, $\tan(\alpha) = -\sqrt{55}/11$, $A = \sqrt{1991}/11$. Then $x(t) = A\cos(\omega t - \alpha)$ with $A > 0$ and $0 \le \alpha < 2\pi$.

16. $5x'' + 11x = 0$,
$x(0) = -4$, $x'(0) = -1$

17. $x'' + x = 0$,
$x(0) = 1$, $x'(0) = -2$

Solution: Solution: $x\left(t\right) = 2\,\sin\left(t\right) - \cos\left(t\right)$. Values: $\alpha = -\arctan(2) + 2\pi = 5.176036590$, $c_1 = 1$, $c_2 = -2$, $\tan(\alpha) = -2$, $A = \sqrt{5}$. Then $x(t) = A\cos(\omega t - \alpha)$ with $A > 0$ and $0 \le \alpha < 2\pi$.

18. $x'' + x = 0$,
$x(0) = -1$, $x'(0) = 2$

19. $x'' + 36x = 0$,
$x(0) = 1$, $x'(0) = -4$

Solution: Solution: $x(t) = -\frac{2}{3}\sin(6t) + \cos(6t)$. Values: $\alpha = -\arctan(2/3) + 2\pi = 5.695182704$, $c_1 = 1$, $c_2 = -2/3$, $\tan(\alpha) = -2/3$, $A = \sqrt{13}/3$. Then $x(t) = A\cos(\omega t - \alpha)$ with $A > 0$ and $0 \le \alpha < 2\pi$.

20. $x'' + 64x = 0$,
$x(0) = -1$, $x'(0) = 4$

Pendulum

The formula

$$\frac{P_1}{P_2} = \frac{R_1}{R_2}\sqrt{\frac{L_1}{L_2}}$$

is valid for the periods P_1, P_2 of two pendulums of lengths L_1, L_2 located at distances R_1, R_2 from the center of the earth. The formula implies that a pendulum can be used to find the radius of the earth at a location. It is also useful for designing a pendulum clock adjustment screw.

21. Derive the formula, using $\omega = \sqrt{g/L}$, period $P = 2\pi/\omega$ and the gravitational relation $g = GM/R^2$.

Solution:
Pendulum 1: $\omega_1 = \sqrt{g/L_1}$, period $P_1 = 2\pi/\omega_1$.

Pendulum 2: $\omega_2 = \sqrt{g/L_2}$, period $P_2 = 2\pi/\omega_2$.

Divide:

$$\frac{P_1}{P_2} = \frac{\omega_2}{\omega_1}, \quad \text{now use } \omega = \sqrt{g/L} \text{ and } g = GM/R^2:$$

$$= \frac{\sqrt{GM/R_2^2/L_2}}{\sqrt{GM/R_1^2/L_1}}$$

$$= \frac{\sqrt{L_1}}{\sqrt{L_2}}\frac{\sqrt{1/R_2^2}}{\sqrt{1/R_1^2}}$$

$$= \sqrt{\frac{L_1}{L_2}}\frac{1/R_2}{1/R_1}$$

$$= \frac{R_1}{R_2}\sqrt{\frac{L_1}{L_2}}$$

22. A pendulum clock taken on a voyage loses 2 minutes a day compared to its exact timing at home. Determine the altitude change at the destination.

23. A pendulum clock with adjustable length L loses 3 minutes per day when $L = 30$ inches. What length L adjusts the clock to perfect time?

Solution: Answer: $L = 2.4896157952$ feet $= 29.87538954$ inches. Details: The time lost is $3/60$ hours in one day and $L_1 = 30.0/12$ feet is the current length of the pendulum. We seek $L = L_2$ so that 0 hours are lost. Let $P_1 = 24 + 3/60$ hours, $P_2 = 24$ hours. The radii are $R_1 = R_2 =$ radius of the earth. Then: $P_1/P_2 = 1\sqrt{L_1/L_2}$ or $L_1 = L_2(P_1/P_2)^2$. Solve for $L_2 = L_1(P_2/P_1)^2 = (30.0/12)(24/(24 + 3/60))^2 = 29.88$ inches.

24. A pendulum clock with adjustable length L loses 4 minutes per day when $L = 30$ inches. What fineness length F is required for a 1/4–turn of the adjustment screw, in order to have 1/4–turns of the screw set the clock to perfect time plus or minus one second per day?

Torsional Pendulum

Solve for $\theta_0(t)$.

25. $\theta_0''(t) + \theta_0(t) = 0$

Solution: Answer: $\theta_0(t) = c_1 \cos t + c_2 \sin t$

26. $\theta_0''(t) + 4\theta_0(t) = 0$

27. $\theta_0''(t) + 16\theta_0(t) = 0$

Solution: Answer: $\theta_0(t) = c_1 \cos 4t + c_2 \sin 4t$

28. $\theta_0''(t) + 36\theta_0(t) = 0$

Shockless Auto

Find the period and frequency of oscillation of the car on four springs. Use model $mx''(t) + kx(t) = 0$.

29. Assume the car plus occupants has mass 1650 Kg. Let each coil spring have Hooke's constant $k = 20000$ Newtons per meter.

Solution: Follow the shockless auto example. Model: $mx''(t) + kx(t) = 0$, $x(t) = A\cos(\omega t - \alpha)$, $m = 1650/4$, $k = 20000$, $\omega^2 = k/m = 1600/33$. The period is $2\pi/\omega = 2\pi/\sqrt{1600/33} = 2\pi\sqrt{33}/40 = 0.9023537906$, frequency $=$ $1/\text{period} = 1.108212777$.

30. Assume the car plus occupants has mass 1850 Kg. Let each coil spring have Hooke's constant $k = 20000$ Newtons per meter.

31. Assume the car plus occupants has mass 1350 Kg. Let each coil spring have Hooke's constant $k = 18000$ Newtons per meter.

Solution: Model: $mx''(t)+kx(t) = 0$, $x(t) = A\cos(\omega t-\alpha)$, $m = 1350/4$, $k = 18000$, $\omega^2 = k/m = 40/3$. The period is $2\pi/\omega = 2\pi/\sqrt{40/3} = \pi/\sqrt{10/3} = 1.7207211636$, frequency $= 1/\text{period} = 0.5811516831$.

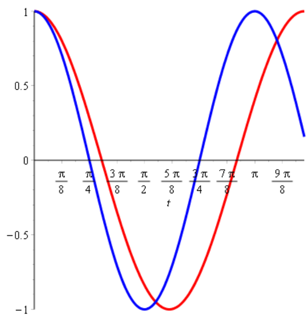
32. Assume the car plus occupants has mass 1350 Kg. Let each coil spring have Hooke's constant $k = 16000$ Newtons per meter.

Rolling Wheel on a Spring

Solve the rolling wheel model $mx''(t) + \frac{2}{3}kx(t) = 0$ and also the frictionless model $mx''(t) + kx(t) = 0$, each with the given initial conditions. Graph the two solutions $x_1(t), x_2(t)$ on one set of axes.

33. $m = 1$, $k = 4$,
$x(0) = 1$, $x'(0) = 0$

Solution: The two equations are harmonic oscillators with general solutions $x_1(t) = c_1\cos\omega t + c_2\sin\omega t$, $\omega^2 = 2k/(3m) = 8/3$, and $x_2(t) = c_3\cos 2t + c_4\sin 2t$. Evaluate from initial data constants $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $c_4 = 0$. Plot $\cos\omega t$, $\cos 2t$ on one set of axes.

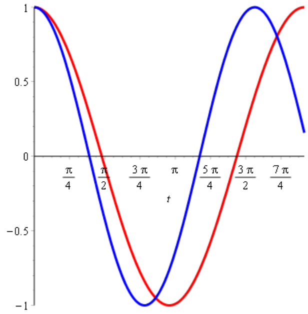



```
# Exercise 33, Rolling wheel graphics
omega:=sqrt(8/3):F:=[cos(omega*t),cos(2*t)];
plot(F,t=0..2*Pi/omega,color=[red,blue],thickness=3);
```

34. $m = 5$, $k = 18$,
$x(0) = 1$, $x'(0) = 0$

35. $m = 11$, $k = 18$,
$x(0) = 0$, $x'(0) = 1$

Solution: The two equations are harmonic oscillators with general solutions $x_1(t) = c_1\cos\omega_1 t + c_2\sin\omega_1 t$, $\omega_1^2 = 2k/(3m) = 36/33 = 12/11$, and $x_2(t) = c_3\cos\omega_2 t + c_4\sin\omega_2 t$, $\omega_2^2 = k/m = 18/11$. Evaluate from initial data constants $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $c_4 = 0$. Plot $\cos\omega_1 t$, $\cos\omega_2 t$ on one set of axes.




```
# Exercise 35, Rolling wheel graphics
omomega1:=sqrt(12/11):omega2:=sqrt(18/11):
F:=[cos(omega1*t),cos(omega2*t)];
plot(F,t=0..2*Pi/omega1,color=[red,blue],thickness=3);
```

36. $m = 7$, $k = 18$,
$x(0) = 0$, $x'(0) = 1$

6.7 Forced and Damped Vibrations

Forced Undamped Vibration

Solve the given equation.

1. $x'' + 100x = 20\cos(5t)$

Solution: Answer: $x(t) = x_h(t) + x_p(t)$, $x_h(t) = c_1\cos 10t + c_2\sin 10t$, $x_p(t) = \dfrac{75}{20}\cos 5t$.

The method of undetermined coefficients applies. Trial solution: $x_p(t) = d_1\cos 5t$ because of the trig shortcut for 2-termed second order differential equations (the expected $d_2\sin 5t$ upon substitution gives $d_2 = 0$). Substitute x_p and find the linear equation(s): $(100-25)d_1 = 20$. Then $x_p(t) = \dfrac{75}{20}\cos 5t$.

2. $x'' + 16x = 100\cos(10t)$

3. $x'' + \omega_0^2 x = 100\cos(\omega t)$, when the internal frequency ω_0 is twice the external frequency ω.

Solution: Answer: $x(t) = x_h(t) + x_p(t)$, $x_h(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t$, $x_p(t) = \dfrac{\omega_0^2 - \omega^2}{200}\cos\omega t$. Details follow Exercise 1.

4. $x'' + \omega_0^2 x = 5\cos(\omega t)$, when the internal frequency ω_0 is half the external frequency ω.

Black Box in the Trunk

5. Construct an example $x'' + \omega_0^2 x = F_1\cos(\omega t)$ with a solution $x(t)$ having beats every two seconds.

Solution: Two beats correspond to two consecutive extrema (max-min or min-max) in the slow;y varying envelope curve. For $x(t) = 2\sin(4t)\sin(40t)$ the slowly varying envelope curves are $\pm 2\sin 4t$. Two consecutive extrema for this example occur in one period, which is $2\pi/4$. We replace 4 by a larger number A so that $2\pi/A = 1$: choose $A = 2\pi$. The example would have $x(t) = 2\sin(2\pi t)\sin(40t)$ as a solution. Needed was $2\pi < 40$ to keep the rapidly varying curve $x = \sin 40t$.

It remains to find the differential equation.

Let's use the equations in the textbook, subsection **Black Box in the Trunk**. Required: $\omega_0 > \omega$. Equations: $\frac{1}{2}(\omega_0 - \omega) = 2\pi$, $\frac{1}{2}(\omega_0 + \omega) = 40$. Then $\omega_0 = 40 + 2\pi$ and $\omega = 40 - 2\pi$.

The differential equation is found by differentiation of the solution $x(t) = \cos(\omega_0 t) - \cos(\omega t)$:

$$\begin{aligned}
x''(t) + (\omega_0)^2\, x(t) &= (\omega^2 - \omega_0^2)\cos(\omega t), \\
x''(t) + (40 + 2\pi)^2\, x(t) &= (\omega - \omega_0)(\omega_0 + \omega)\, \cos(\omega t) \\
&= (-4\pi)(80)\, \cos(\omega t) \\
&= -320\pi\, \cos((40 - 2\pi)t)
\end{aligned}$$

```
# Exercise 5, Answer check, black box in the trunk
omega0:=40+2*Pi:omega:=40-2*Pi;
LHS:=diff(x(t),t,t)+(40+2*Pi)^2 * x(t);
RHS:=-320*Pi*cos((40-2*Pi)*t);
p:=subs(x(t)=cos(omega0*t)-cos(omega*t),LHS);
simplify(p-RHS);#expect zero)
# Check college algebra:
# 2*sin(2*Pi*t)*cos(40*t)=cos(omega0*t)-cos(omega*t)
x3:=expand(cos(a-b)-cos(a+b));subs(a=2*Pi*t,b=40*t,x3);
x4:=cos(a-b)-cos(a+b);subs(a=2*Pi*t,b=40*t,x4);
```

6. A solution $x(t)$ of $x'' + 25x = 100\cos(\omega t)$ has beats every two seconds. Find ω.

Rotating Drum

Solve the given equation.

7. $x'' + 100x = 500\,\omega^2\cos(\omega t)$, $\omega \ne 10$.

Solution: Answers: $x = x_h + x_p$, $x_h = c_1]cos 10t + c_2\sin 10t$, $x_p = \dfrac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t) = x_p = \dfrac{F_0}{199 - \omega^2}\cos(\omega t)$ where $F_0 = 500\omega^2$. Equation (1) in the textbook was used with $m = 1$.

8. $x'' + \omega_0^2 x = 5\,\omega^2\cos(\omega t)$, $\omega \ne \omega_0$.

Harmonic Oscillations

Express the general solution as a sum of two harmonic oscillations of different frequencies, each oscillation written in phase-amplitude form.

9. $x'' + 9x = \sin 4t$

Solution: Answer: $x = x_h + x_p$, $x_h = c_1\cos 3t + c_2\sin 3t = A\cos(3t - \alpha)$, $x_p = d_1\sin 4t = \dfrac{1}{9 - 4^2}\sin 4t$ by the method of undetermined coefficients. The challenge: write $\sin 4t = \cos(4t - \pi/2)$ using trig identity $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$. Then $x_p = -\frac{1}{7}\cos(4t - \pi/2)$ and $x = x_h + x_p = A\cos(3t - \alpha) - \frac{1}{7}\cos(4t - \pi/2)$ is the sum of two harmonic oscillations of different frequencies, each harmonic term in phase-amplitude form.

10. $x'' + 100x = \sin 5t$

11. $x'' + 4x = \cos 4t$

Solution: Answer: $x(t) = A\cos(2t - \alpha) - \dfrac{1}{12}\cos(4t)$

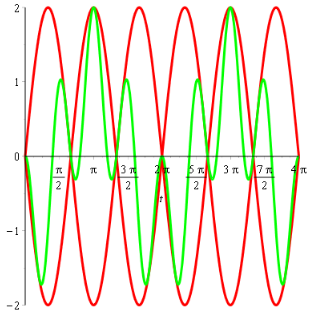
12. $x'' + 4x = \sin t$

Beats: Convert and Graph

Write each linear combination as $x(t) = C \sin at \sin bt$. Then graph the slowly-varying envelope curves and the curve $x(t)$.

13. $x(t) = \cos 4t - \cos t$

Solution: Let $\omega_0 = 4$, $\omega = 1$. Use the textbook formulas from subsection **Black Box in the Trunk** to write $x(t) = 2\sin((\omega - \omega_0)t/2)\sin((\omega 0 + \omega)t/2) = -2\sin(3t/2)\sin(5t/2)$.

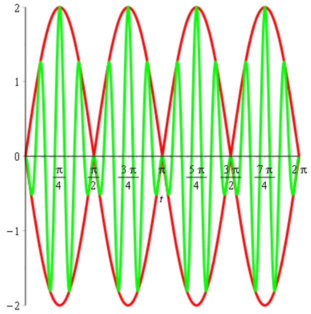


```
# Exercise 13, Graph envelope curves, Beats
x1:=2*sin(3*t/2);x2:=cos(4*t)-cos(t);
plot([x1,-x1,x2],t=0..4*Pi,color=[red,red,green],thickness=3);
```

14. $x(t) = \cos 10t - \cos t$

15. $x(t) = \cos 16t - \cos 12t$

Solution: Let $\omega_0 = 16$, $\omega = 12$. Then $x(t) = 2\sin(-4t/2)\sin(28t/2)$.



16. $x(t) = \cos 25t - \cos 23t$

Beats: Solve, find Envelopes

Solve each differential equation with $x(0) = x'(0) = 0$ and determine the slowly-varying envelope curves.

17. $x'' + x = 99 \cos 10t.$

Solution: Answer: $x(t) = \cos(t) - \cos(10t) = 2\sin(9t/2)\sin(11t/2)$, envelope curves $\pm \sin(9t/2)$.

Details: use superposition $x = x_h + x_p$. The homogeneous solutionis $x_h = c_1 \cos t + c_2 \sin t$. The undetermined coefficients method finds $x_p = -\cos 10t$. Use the initial data and the general solution $x = x_h + x_p$ to find linear equations for c_1, c_2:

$$c_1(1) + c_2(0) - \cos(0) = 0, \quad -c_1(0) + c_2(1) + 10(0) = 0$$

Solve for $c_1 = 1$, $c_2 = 0$. Then $x = \cos t - \cos 10t$. Use the textbook formulas from subsection **Black Box in the Trunk** to write $x = -2\sin(9t/2)\sin(11t/2)$. The envelope curves use the sine factor with smaller natural frequency.

18. $x'' + 4x = 252 \cos 10t.$

19. $x'' + x = 143 \cos 12t.$

Solution: Answer: $x(t) = \cos(t) - \cos(12t) = 2\sin(11t/2)\sin(13t/2)$, envelope curves $\pm \sin(11t/2)$.

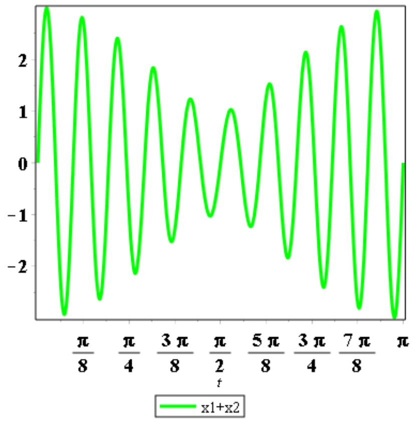
20. $x'' + 256x = 252 \cos 2t.$

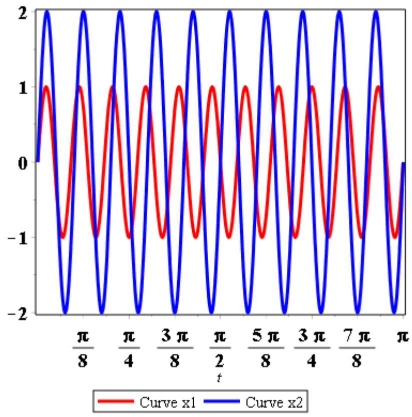
Waves and Superposition

Graph the individual waves x_1, x_2 and then the superposition $x = x_1 + x_2$. Report the apparent period of the superimposed waves.

21. $x_1(t) = \sin 22t$, $x_2(t) = 2\sin 20t$

Solution: The periods of the two waves: $2\pi/20$ and $2\pi/22$. The waves share a common period T provided $T = 2n\pi/20 = 2m\pi/22$ for some positive integers n, m. The requirement on n, m: $n/10 = m/11$ or $11n = 10m$. Find the least period by trying $n = 1, 2, 3, \ldots$ until $11n = 10m$ can be satisfied for some m. This happens at $n = 10$ and $m = 11$. The least period common to both waves is $T = n(2\pi/20) = \pi$.

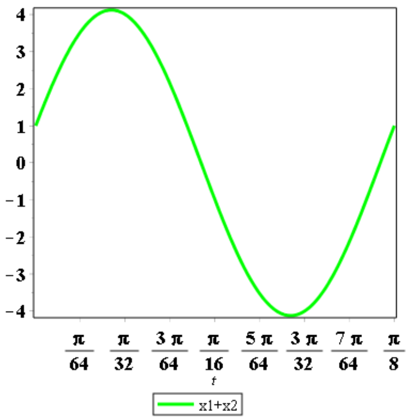


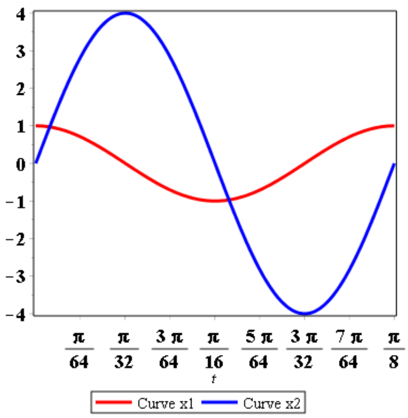



```
# Exercise 21, Waves and superposition
isolve(11*n=10*m,a);
# {m = 11 a, n = 10 a}, a=integer=1=least solution
x1:=sin (22*t); x2:=2*sin( 20*t);
plot(x1+x2,t=0..Pi,color=[green],thickness=3);
plot({x1,x2},t=0..Pi,color=[red,blue],thickness=3);
```

22. $x_1(t) = \cos 16t$, $x_2(t) = 4\cos 20t$

23. $x_1(t) = \cos 16t$, $x_2(t) = 4\sin 16t$
Solution: Fundamental period $= T = 2\pi/16$.






```
# Exercise 23, Waves and superposition
x1:=cos (16*t); x2:=4*sin( 16*t);
opts:=thickness=3,font=[courier,14,bold];
plot(x1+x2,t=0..Pi/8,color=[green],opts);
plot([x1,x2],t=0..Pi/8,color=[red,blue],opts);
```

24. $x_1(t) = \cos 25t$, $x_2(t) = 4\cos 27t$

Periodicity

25. Let $x_1(t) = \cos 25t$, $x_2(t) = 4\cos 27t$. Their sum has period $T = m\frac{2\pi}{25} = n\frac{2\pi}{27}$ for some integers m, n. Find all m, n and the least period T.

Solution: Solve $27m = 25n$ for positive integers n, m: $m = 25a$, $n = 27a$, $a = 1, 2, 3, \ldots$. The smallest period is for $a = 1$. Then $T = m\frac{2\pi}{25} = (25a)\frac{2\pi}{25}$, therefore $T = 2\pi$. Maple code to solve the equation $27m = 25n$ appears above in Exercise 21.

26. Let $x_1(t) = \cos\omega_1 t$, $x_2(t) = \cos\omega_2 t$. Find a condition on ω_1, ω_2 which implies that the sum $x_1 + x_2$ is periodic.

27. Let $x(t) = \cos(t) - \cos(\sqrt{2}t)$. Explain without proof, from a graphic, why $x(t)$ is not periodic.

Solution: The graphic displayed on a large interval does not show repeating extrema. So it cannot be periodic.

A proof can be done by expanding the relation $x(t+T) = x(t)$:

$$
\begin{aligned}
&x(t+T) = x(t)\\
&\cos(t+T) - \cos(\sqrt{2}t + \sqrt{2}T) = \cos(t) - \cos(\sqrt{2}t)\\
&\cos(t)\cos(T) - \sin(t)\sin(T) - \cos(\sqrt{2}T)\cos(\sqrt{2}T) +\\
&\quad \sin(\sqrt{2}t)\sin(\sqrt{2}T) = \cos(t) - \cos(\sqrt{2}t)
\end{aligned}
$$

Transform to a system of four nonlinear equations by matching coefficients of Euler solution atoms $\cos(t)$, $\sin(t)$, $\cos(\sqrt{2}t)$, $\sin(\sqrt{2}t)$ on each side (independence of atoms used here):

$$
\begin{cases}
\cos(T) &= 1\\
-\sin(T) &= 0\\
-\cos(\sqrt{2}T) &= 1\\
\sin(\sqrt{2}T) &= 0
\end{cases}
$$

The first two equations imply $T = 2n\pi$ for integers $n = 0, 1, 2, \ldots$. The last two equations violate $T = 2n\pi$. There is no solution T: function $x(t)$ fails to be periodic. ■

28. Let $x(t) = \cos(5t) + \cos(5\sqrt{2}t)$. Is $x(t)$ is periodic? Explain without proof.

Rotating Drum

Let $x(t)$ and $x_p(t)$ be defined as in Example 4, page 509. Replace Hooke's constant $k = 10$ by $k = 1$, all other constants unchanged.

29. Re-compute the amplitude $A(t)$ of solution $x_p(t)$. Find the decimal value for the maximum of $|A(t)|$.

Solution: Answer: $x_p = -\dfrac{275\,\pi^2\cos(20\,\pi\,t)}{4120\,\pi^2 - 2}$, amplitude $A(t) = \dfrac{275\,\pi^2}{4120\,\pi^2 - 2}$.
The maximum of $|A(t)|$ is $|A(0)| \approx 0.067$.

30. Find $x(t)$ when $x(0) = x'(0) = 0$. It is known that $x(t)$ fails to be periodic. Let $t_1 = 0, \ldots, t_{29}$ be the consecutive extrema on $0 \le t \le 1.4$. Verify graphically or by computation that $|x(t_{i+1}) - x(t_i)| \approx 0.133$ for $i = 1, \ldots, 28$.

Solution: Answer:

$$
x(t) = \frac{275\,\pi^2}{4120\,\pi^2 - 2}\cos\left(\frac{2\sqrt{515}t}{103}\right) - 275\,\frac{\pi^2\cos(20\,\pi\,t)}{4120\,\pi^2 - 2}
$$

Function $x(t)$ is not periodic: identity $x(t+T) = x(t)$ fails for all $T > 0$. A function that fails to be periodic is called **Aperiodic**. Functions like $x(t)$ are called **Quasiperiodic**.

```
# Exercise 30, Rotating drum, quasiperiodic x(t)
de:=m*diff(x(t),t,t)+k*x(t)=R*M*omega^2*cos(omega*t);
DE:=subs(k=1,M = 0.275, m =5.15, R = 1.25, omega = 20*Pi,de);
p:=dsolve([DE,x(0)=0,D(x)(0)=0],x(t));
X:=unapply(evalf(rhs(p)),t);
plot(X(t),t=0..1.4);
seq(abs(X(0.1*j+0.05)-X(0.1*j)),j=0..7);
```

Musical Instruments

Melodious tones are superpositions of harmonics $\sin(n\omega t)$, with $n =$ an integer, $\omega =$ fundamental frequency.

In 1885 Alexander J. Ellis introduced a measurement unit **Cent** by the equation **one cent** $= 2^{\frac{1}{12}} \approx 1.0005777895$. On most pianos, the frequency ratio between two adjacent keys equals 100 **cents**, called an **equally tempered semitone**. Two piano keys of frequencies 480 Hz and 960 Hz span 1200 **cents** and have tones $\sin(\omega t)$ and $\sin(2\omega t)$ with $\omega = 480$. A span of 1200 **cents** between two piano key frequencies is called an **Octave**.

31. **(Equal Temperament)** Find the 12 frequencies of equal temperament for octave 480 Hz to 960 Hz. The first two frequencies are 480, 508.5422851.

Solution: The frequencies are

```
480.0, 508.5422851, 538.7817830, 570.8194152,
604.7621040, 640.7231299, 678.8225098, 719.1873970,
761.9525050, 807.2605589, 855.2627693, 906.1193400,
960.0
```

```
# Exercise 31, Equal temperament
seq(480.0*2.0^(n/12),n=0..12);
```

32. **(Flute or Noise)** Equation $x(t) = \sin 220\pi t + 2\sin 330\pi t$ could represent a tone from a flute or just a dissonant, unpleasing sound. Discuss the impossibility of answering the question with a simple yes or no.

33. **(Guitar)** Air inside a guitar vibrates a little like air in a bottle when you blow across the top. Consider a flask of volume $V = 1$ liter, neck length $L = 5$ cm and neck cross-section $S = 3$ cm^2. The vibration has model $x'' + f^2 x = 0$ with $f = c\sqrt{\frac{S}{VL}}$, where $c = 343$ m/s is the speed of sound in air. Compute $\frac{f}{2\pi}$ and $\lambda = \frac{2\pi c}{f}$, the frequency and wavelength. The answers are about 130 Hz and $\lambda = 2.6$ meters, a low sound.

Solution: Answers: $\omega := 343.2\sqrt{6}$, $F = 133.7959711$, $\lambda = 2.565099660$

```
# # Exercise 33, Guitar
S:=3*(1/100)^2; # 1cm=m/100
L:=5*(1/100);
V:=1/1000; # 1000 liters = 1 cubic meter
c:=343.2;
omega:=c*sqrt(S/(V*L));
F:=omega/(2*Pi); # Frequency in Hz
lambda:=evalf(c/F); # wavelength
```

34. **(Helmholtz Resonance)** Repeat the previous exercise calculations, using a flask with neck diameter 2.0 cm and neck length 3 cm. The tone should be lower, about 100 Hz, and the wavelength λ should be longer.

Seismoscope

35. Verify that x_p given in (14) and x_p^* given by (15), page 519, have the same initial conditions when $u(0) = u'(0) = 0$, that is, the ground does not move at $t = 0$. Conclude that $x_p = x_p^*$ in this situation.

Solution: Given $u(0) = u'(0) = 0$ then $x_p^*(0) = 0$ by (15). Differentiate (15) to obtain $\frac{d}{dt}x_p * (0) = -u'(0) + K(0)u(0) + \int_0^0 K_t(0 - x)u(x)dx = 0$.

Equation (14) implies $x_p(0) = x_p'(0) = 0$ similarly. Then both x_p and x_p^* have the same initial data. Apply Picard-Lindelöf to show they are identical solutions. ■

36. A **release test** begins by starting a vibration with $u = 0$. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. Explain how to find β, Ω_0 in the equation $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$, using Exercises 69 and 70, *infra*.

Free Damped Motion

Classify the homogeneous equation $mx'' + cx' + kx = 0$ as **over-damped**, **critically damped** or **under-damped**. Then solve the equation for the general solution $x(t)$.

37. $m = 1, c = 2, k = 1$

Solution: Answer: Critically-damped, $x = c_1 e^{-t} + c_2 t e^{-t}$

Characteristic equation: $r^2 + 2r + 1 = 0$ with roots $r = 1, 1$ and atoms e^{-t}, te^{-t}. The discriminant of $ar^2 + br + c$ is zero: $D = b^2 - 4ac = 4 - 4 = 0$. Critical damping corresponds to the double root case.

38. $m = 1, c = 4, k = 4$

39. $m = 1, c = 2, k = 3$

Solution: Answer: under-damped, $x = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$ where $a\pi ib$ are the two complex roots of the characteristic equation $r^2 + 2r + 3 = 0$. The roots are $-1 \pm \sqrt{2}i$. The discriminant of $ar^2 + br + c$ is $D = b^2 - 4ac = 4 - 12 < 0$, which implies complex roots and therefore oscillation. The only oscillatory case is for complex roots and then the classification is underdamped.

40. $m = 1, c = 5, k = 6$

41. $m = 1, c = 2, k = 5$

Solution: Roots of $r^2 + 2r + 5 = 0$ are complex: $-1 \pm 2i$. Classification: under-damped. Solution: $x = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$.

42. $m = 1, c = 12, k = 37$

43. $m = 6, c = 17, k = 7$

Solution: Roots of $r^2 + 2r + 5 = 0$ are real distinct: $-1/2, -7/3$. Classification: over-damped. Solution: $x = c_1 e^{-t/2} + c_2 e^{-7t/3}$.

44. $m = 10, c = 31, k = 15$

45. $m = 25, c = 30, k = 9$

Solution: Roots of $25r^2 + 30r + 9 = 0$ are real repeated: $-3/5, -3/5$. Classification: critically-damped. Solution: $x = c_1 e^{-3t/5} + c_2 t e^{-3t/5}$.

46. $m = 9, c = 30, k = 25$

47. $m = 9, c = 24, k = 41$

Solution: Roots of $9r^2 + 24r + 41 = 0$ are complex: $-4/3 \pm 5i/3$. Classification: under-damped. Solution: $x = c_1 e^{-4t/3}\cos 5t/3 + c_2 e^{-4t/3}\sin 5t/3$.

48. $m = 4, c = 12, k = 34$

Cafe and Pet Door

Classify as a cafe door model and/or a pet door model. Solve the equation for the general solution and identify as oscillatory or non-oscillatory.

49. $x'' + x' = 0$

Solution: Cafe door. The pet door always has a nonzero $x(t)$-term. Non-oscillatory because the classification is over-damped. Discriminant of $ar^2 + br + c = D = b^2 - 4ac = 1 - 0 > 0$ and two distinct real roots $0, 1$.

50. $x'' + 2x' + x = 0$

Solution: Can be either a pet door or a cafe door. Non-oscillatory because the discriminant $= 0$, the critically-damped case with two equal real roots $-1, -1$.

51. $x'' + 2x' + 5x = 0$

Solution: The roots are $-1 \pm \sqrt{5}i$, so the equation is under-damped oscillatory. Cafe door or pet door.

52. $x'' + x' + 3x = 0$

53. $9x'' + 24x' + 41x = 0$

Solution: The roots are $-4/3 \pm \sqrt{5}i/3$, so the equation is under-damped oscillatory. Cafe door or pet door.

54. $6x'' + 17x' = 0$

55. $9x'' + 24x' = 0$

Solution: Cafe door. The pet door always has a nonzero $x(t)$-term. Non-oscillatory because the classification is over-damped, discriminant $= 24^2 - 0 > 0$ and two distinct real roots $0, -8/3$.

56. $6x'' + 17x' + 7x = 0$

Classification

Classify $mx'' + cx' + kx = 0$ as **over-damped**, **critically damped** or **under-damped** without solving the differential equation.

57. $m = 5, c = 12, k = 34$

Solution: It is enough to compute the discriminant of $ar^2 + br + c$: $b^2 - 4ac = -536 < 0$. Under-damped.

58. $m = 7, c = 12, k = 19$

59. $m = 5, c = 10, k = 3$

Solution: Under-damped.

60. $m = 7, c = 12, k = 3$

61. $m = 9, c = 30, k = 25$

Solution: Under-damped.

62. $m = 25, c = 80, k = 64$

Critically Damped

The equation $mx'' + cx' + kx = 0$ is critically damped when $c^2 - 4mk = 0$. Establish the following results for $c > 0$.

63. The mass undergoes no oscillations, because

$$x(t) = (c_1 + c_2 t)e^{-\frac{ct}{2m}}.$$

Solution: The roots of $mr^2 + cr + k = 0$ are $-c/2, -c/2$, the critically-damped case. Then the solution is a linear combination of Euler atoms $e^{-ct/2}$, $te^{-ct/2}$. There are no trig terms in the solution: non-oscillatory.

64. The mass passes through $x = 0$ at most once.

Over-Damped

Equation $mx'' + cx' + kx = 0$ is defined to be over-damped when $c^2 - 4mk > 0$. Establish the following results for $c > 0$.

65. The mass undergoes no oscillations, because if r_1, r_2 are the roots of $mr^2 + cr + c = 0$, then

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Solution: Oscillation is caused by trig terms. There are none so no oscillation.

66. The mass passes through equilibrium position $x = 0$ at most once.

Under-Damped

Equation $mx'' + cx' + kx = 0$ is defined to be under-damped when $c^2 - 4mk < 0$. Establish the following results.

67. The mass undergoes infinitely many oscillations. If $c = 0$, then the oscillations are harmonic.

Solution: The roots are complex $a \pi bi$ with $a = -c/2m < 0$ and $b = \sqrt{4mk - c^2} > 0$. The solutions are linear combinations of Euler atoms $e^{at}\cos bt$, $e^{at}\sin bt$, which is always oscillatory. If $c = 0$ then $a = 0$ and the Euler atoms are $\cos bt$, $\sin bt$. Then $x(t)$ is a pure harmonic.

68. The solution $x(t)$ can be factored as an exponential function $e^{-\frac{ct}{2m}}$ times a harmonic oscillation. In symbols:

$$x(t) = e^{-\frac{ct}{2m}}\left(A\cos(\omega t - \alpha)\right).$$

Experimental Methods

Assume model $mx'' + cx' + kx = 0$ is oscillatory. The results apply to find non-negative constants m, c, k from one experimentally known solution $x(t)$. Provide details.

69. Let $x(t)$ have consecutive maxima at $t = t_1$ and $t = t_2 > t_1$. Then $t_2 - t_1 = T = \frac{2\pi}{\omega}$ = pseudo period of $x(t)$.

Solution: The solution is $x(t) = A(t)\cos(\omega t - \alpha)$ in phase-amplitude form. As in the work on envelope curves and the pseudo-period, the amplitude is $A(t) = Ce^{at}$ and ω is the natural frequency with $a \pm \omega i$ the two complex roots of characteristic equation $mr^2 + cr + k = 0$. Maxima of $x(t)$ occur when $x'(t) = 0$, which is the equation $0 = x'(t) = aCe^{at}\cos(\omega t - \alpha) - Ce^{at}\omega\sin(\omega t - \alpha)$. This equation simplifies to $\tan(\omega t - \alpha) = \frac{a}{\omega} = -\frac{c}{2m\omega}$. Consecutive maxima at t_1, t_2 then satisfy $\frac{a}{\omega} = \tan(\omega t_1 - \alpha) = \tan(\omega t_2 - \alpha)$. For $t_2 > t_1$ the possible solutions are $\omega t_2 - \alpha = \omega t_1 + n\pi$, $n = 1, 2, 3, \ldots$, because the tangent has period π. Suppose amplitude $A(t_1)$ is positive, equivalent to coefficient $C > 0$. Then $x(t_1) > 0$ implies $\cos(\theta) > 0$ for $\theta = \omega t_1 - \alpha$. At a later maximum $t = t_2 > t_1$ the values of $x(t_2)$ and $\cos(\theta + n\pi)$ must be positive. For $n = 1$: $\cos(\theta + \pi) = cos(\theta)\cos(\pi) = -\cos(\theta)$, so the cosine is not positive. Then $n = 1$ is not a solution. Let's try $n = 2$: it works because the sine and cosine are 2π-periodic. So the correct answer is $n = 2$. Choosing $n = 2$ gives equation $\omega t_1 - \alpha + 2\pi = \omega t_2 - \alpha$. Solve: $t_2 - t_1 = \frac{2\pi}{\omega}$, which is the pseudo period.

70. Let (t_1, x_1) and (t_2, x_2) be two consecutive maximum points of the graph of a solution $x(t) = Ce^{-ct/(2m)}\cos(\omega t - \alpha)$ of $mx'' + cx' + kx = 0$. Let $a \pm \omega i$ be the two complex roots of $mr^2 + cr + k = 0$ where $a = -c/(2m)$ and $\omega = \frac{1}{2m}\sqrt{4mk - c^2}$. Then

$$\ln\frac{x_1}{x_2} = \frac{c\pi}{m\omega},$$

Solution: The equations: $A(t) = Ce^{at}$, $x_1 = A(t_1)\cos(\omega t_1 - \alpha) = A(t_1)(1) = Ce^{at_1}$, $x_2 = A(t_2)\cos(\omega t_2 - \alpha) = A(t_2)(1) = Ce^{at_2}$. Then Exercise 69 gives

$$\frac{x_1}{x_2} = e^{a(t_1 - t_2)} = e^{(-1)\frac{c}{2m}(-1)\frac{2\pi}{\omega}}$$

Take logs:

$$\ln|\frac{x_1}{x_2}| = \frac{c}{2m}\frac{2\pi}{\omega} = \frac{c\pi}{m\omega}$$

71. **(Bike Trailer)** Assume *fps* units. A trailer equipped with one spring and one shock has mass $m = 100$ in the model $mx'' + cx' + kx = 0$. Find c

and k from this experimental data: two consecutive maxima of $x(t)$ are $(0.35, 10/12)$ and $(1.15, 8/12)$.
Hint: Use exercises 69 and 70.

72. **(Auto)** Assume *fps* units. An auto weighing 2.4 tons is equipped with four identical springs and shocks. Each spring-shock module has damped oscillations satisfying $mx'' + cx' + kx = 0$. Find m. Then find c and k from this experimental data: two consecutive maxima of $x(t)$ are $(0.3, 3/12)$ and $(0.7, 2/12)$.
Hint: Use exercises 69 and 70.

Solution: Answer: $m = 41.113$, $c = 83.34943495$, $\omega = 5\pi$, $k = 10860.46092$.

Let $t_1 = 0.3$, $x_1 = 3/12$, $t_2 = 0.7$, $x_2 = 2/12$. Let $m = 164.452/4 = 41.113$ slugs, for 2.4 tons divided among the four springs equally. The model is under-damped from the data. The pseudo period is $2/pi/\omega = t_2 - t_1 = 0.4$ by the result of Exercise 69. Then $\omega = 5\pi$. The logarithmic decrement $= c\pi/(m\omega) = \ln|x_1/x_2|$ by Exercise 70. Then $c\pi/(m\omega) = \ln(x_1/x_2) = \ln(3/2)$. Because $m = 41.113$ and $\omega = 5\pi$ then $c = \frac{1}{\pi}(m\omega\ln(3/2)) = 83.34943495$. To find k use $\omega = \frac{1}{2m}\sqrt{4mk - c^2}$ and solve for $k = 10860.46092$.

Structure of Solutions

Establish these results for the damped spring-mass system $mx'' + cx' + kx = 0$. Assume $m > 0$, $c > 0$, $k > 0$.

73. **(Monotonic Factor)** Let the equation be critically damped or over-damped. Prove that

$$x(t) = e^{-pt} f(t)$$

where $p \geq 0$ and $f(t)$ is monotonic (f' one-signed).

Solution: Case 1: over-damped. Then the roots of the characteristic equation are two distinct real roots $r_1 > r_2$. The roots satisfy $(r - r_1)(r - r_2) = 0$ which upon expansion gives $r^2 - (r_1 + r_2)r + r_1 r_2 = 0$. Because $-r_1 - r_2 = c/m$ and $r_1 r_2 = k/m$ then both roots are negative. The Euler atoms give general solution $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_2 t} f(t)$ where $f(t) = c_1 e^{r_1 t - r_2 t} + c_2$. The derivative $f'(t)$ is either zero ($c_1 = 0$) or else never vanishes. Then $f'(t)$ is one-signed: $f(t)$ is monotonic.

Case 2: critically-damped, Then the characteristic equation has a double root $r_1 = r_2$ and the general solution is $x(t) = e^{r_1 t}(c_1 + c_2 t)$. Root r_1 is negative because $-r_1 - r_1 = c$, following Case 1. Let $f(t) = c_1 + c_2 t$. Then $f'(t)$ is zero or never vanishes: $f(t)$ is monotonic.

74. **(Harmonic Factor)** Let the equation be under-damped. Prove that

$$x(t) = e^{-at} f(t)$$

where $a > 0$ and $f(t) = c_1 \cos \omega t + c_2 \sin \omega t = A \cos(\omega t - \alpha)$ is a harmonic oscillation.

75. **(Limit Zero and Transients)** A term appearing in a solution is called **transient** if it has limit zero at $t = \infty$. Prove that positive damping $c > 0$ implies that the homogeneous solution satisfies $\lim_{t\to\infty} x(t) = 0$.

Solution: The decompositions of $x(t)$ in the last few exercises show that $x(t)$ equals an exponential factor e^{rt} with $r < 0$ multiplied by a function $f(t)$. There are three cases:

(1) $f(t) = c_1 e^{-at} + c_2$ with $a > 0$, a monotone function with $\lim_{t=\infty} f(t) = c_2$,
(2) $f(t) = c_1 + c_2 t$, a linear function,
(3) $f(t) = c_1 \cos(bt) + c_2 \sin(bt)$, a harmonic function.

In cases (1), (2), (3) the exponential factor e^{rt} dominates at $t = \infty$ with $\lim_{t=\infty} x(t) = 0$.

76. **(Steady-State)** An **observable** or **steady-state** is expression obtained from a solution by excluding all terms with limit zero at $t = \infty$. The **Transient** is the expression excluded to obtain the steady state. Assume $mx'' + cx' + kx = 25 \cos 2t$ has a solution

$$x(t) = 2te^{-t} - \cos 2t + \sin 2t.$$

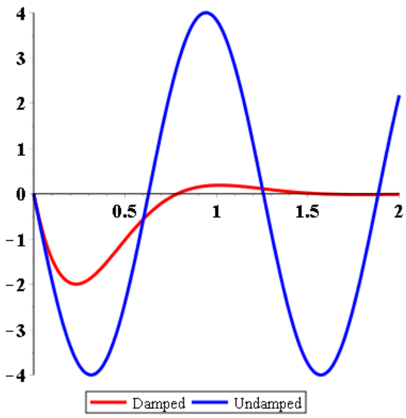
Find the transient and steady-state terms.

Damping Effects

Construct a figure on $0 \le t \le 2$ with two curves, to illustrate the effect of removing the dashpot. Curve 1 is the solution of $mx''+cx'+kx = 0$, $x(0) = x_0$, $x'(0) = v_0$. Curve 2 is the solution of $my'' + ky = 0$, $y(0) = x_0$, $y'(0) = v_0$.

77. $m = 2, c = 12, k = 50,$
$x_0 = 0, v_0 = -20$

Solution:

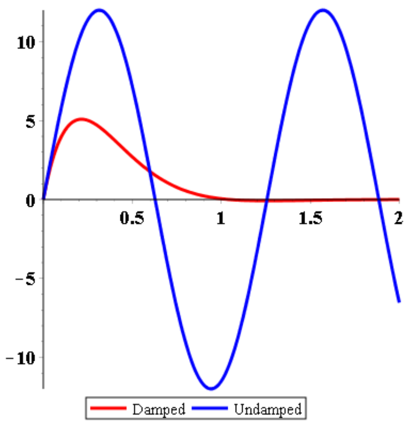



```
# Exercise 77, Damping effects
de:=m*diff(x(t),t,t)+c*diff(x(t),t)+k*x(t);
ic:=x(0)=0,D(x)(0)=-20;
de1:=subs(m=2,c=12,k=50,de);
p:=dsolve({de1,ic},x(t));X:=unapply(rhs(p),t);
de2:=subs(m=2,c=0,k=50,de);
q:=dsolve({de2,ic},x(t));Y:=unapply(rhs(q),t);
opts:=font=[courier,14,bold],thickness=3,color=[red,blue];
plot([X,Y],0..2,opts);
```

78. $m = 1, c = 6, k = 25,$
$x_0 = 0, v_0 = 20$

79. $m = 1, c = 8, k = 25,$
$x_0 = 0, v_0 = 60$

Solution:



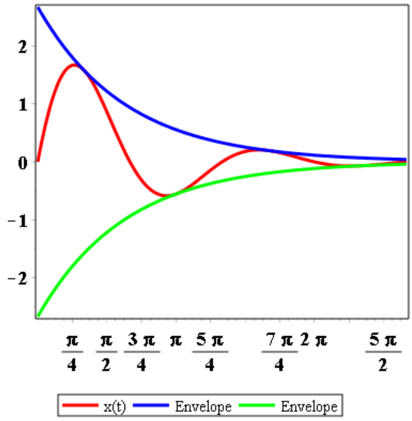
80. $m = 1, c = 4, k = 20,$
$x_0 = 0, v_0 = 4$

Envelope and Pseudo-period

Plot on one graphic the envelope curves and the solution $x(t)$, over two pseudo-periods. Use initial conditions $x(0) = 0$, $x'(0) = 4$.

81. $x'' + 2x' + 5x = 0$

Solution: Answer: $x(t) = \frac{8}{3}\,\mathrm{e}^{-t/2}\sin(3t/2)$, envelope curves $y(t) = \pm\frac{8}{3}\,\mathrm{e}^{-t/2}$

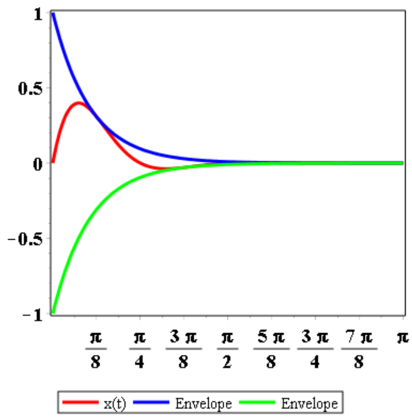



```
# Exercise 81, Envelope and pseudo-period
de:=m*diff(x(t),t,t)+c*diff(x(t),t)+k*x(t);
ic:=x(0)=0,D(x)(0)=4;
de1:=subs(m=2,c=2,k=5,de);
p:=dsolve({de1,ic},x(t));X:=unapply(rhs(p),t);
Y:=t->(8/3)*exp(-(1/2)*t);Z:=t->(-8/3)*exp(-(1/2)*t);
T:=4*Pi/3;
opts:=font=[courier,14,bold],thickness=3,color=[red,blue];
plot([X,Y,Z],0..2*T,opts);
```

82. $x'' + 2x' + 26x = 0$

83. $2x'' + 12x' + 50x = 0$

Solution: Answer: $x(t) = \mathrm{e}^{-3t}\sin(4t)$, envelope curves $y(t) = \pm\mathrm{e}^{-3t}$



84. $4x'' + 8x' + 20x = 0$

6.8 Resonance

Beats

Each equation satisfies the beats relation $\omega \neq \omega_0$. Find the general solution. See Example 6.53, page 538.

1. $x'' + 100x = 10\sin 9t$

Solution: Answer: $x = x_h + x_p$, $x_h = c_1\cos 10t + c_2\sin 10t$, $x_p = \frac{5}{19}\sin(9t)$

2. $x'' + 100x = 5\sin 9t$

3. $x'' + 25x = 5\sin 4t$

Solution: Answer: $x = x_h + x_p$, $x_h = c_1\cos 5t + c_2\sin 5t$, $x_p = \frac{5}{9}\sin(4t)$

4. $x'' + 25x = 5\cos 4t$

Pure Resonance

Each equation satisfies the pure resonance relation $\omega = \omega_0$. Find the general solution. See Example 6.53, page 538.

5. $x'' + 4x = 10\sin 2t$

Solution: Answer: $x = x_h + x_p$, $x_h = c_1\cos 2t + c_2\sin 2t$, $x_p = -\frac{5}{2}\,t\cos(2t)$

6. $x'' + 4x = 5\sin 2t$

7. $x'' + 16x = 5\sin 4t$

Solution: Answer: $x = x_h + x_p$, $x_h = c_1\cos 4t + c_2\sin 4t$, $x_p = -\frac{5}{8}\,t\cos(4t)$

8. $x'' + 16x = 10\sin 4t$

Practical Resonance

For each model, find the **tuned practical resonance** frequency Ω and the **resonant amplitude** C:

$$\begin{aligned}&\Omega = \sqrt{k/m - c^2/(2m^2)},\\ &C = F_0/\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}\end{aligned}$$

9. $x'' + 2x' + 17x = 100\cos(4t)$

Solution: Answer: $m = 1$, $c = 2$, $k = 17$, $F_0 = 100$, $\omega = 4$, $C = \frac{20}{13}\sqrt{65}$, $\Omega = \sqrt{60}/2$, $x_p = \frac{20}{13}\cos 4t$, $x_h = c_1e^{-t}\cos 4t + c_2e^{-t}\sin 4t$

```
# Exercise 9, practical resonance
F:=t->100*cos(4*t);
de:=m*diff(x(t),t,t)+c*diff(x(t),t)+k*x(t)=F(t);
de1:=subs(m=1,c=2,k=17,de);
p:=dsolve(de1,x(t));X:=unapply(rhs(p),t);
C:=F(0)/sqrt( (k-m*omega^2)^2 + (c*omega)^2 );
Omega:=sqrt(k/m - c^2/(2*m^2));
subs(m=1,c=2,k=17,Omega);
subs(m=1,c=2,k=17,omega=4,C);
```

10. $x'' + 2x' + 10x = 100\cos(4t)$

11. $x'' + 4x' + 5x = 10\cos(2t)$

Solution: Answer: $m = 1$, $c = 4$, $k = 5$, $F_0 = 100$, $\omega = 2$, $C = \frac{2\sqrt{65}}{13}$, $\Omega = -3$ which means no practical resonant frequency exists, $x_p = \frac{2}{13}\cos 2t + \frac{16}{13}\sin 2t$, $x_h = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$

12. $x'' + 2x' + 6x = 10\cos(2t)$

Transient Solution

Identify from superposition $x = x_h + x_p$ a shortest particular solution, given one particular solution.

13. $x'' + 2x' + 10x = 26\cos(3t)$,
$x = 100e^{-t}\cos(3t) + 3\cos(2t) + 2\sin(2t)$

Solution: $x_p = 3\cos(2t) + 2\sin(2t)$

14. $x'' + 4x' + 13x = 920\cos(3t)$,
$x = 5\,e^{-2t}\cos(3t) + 23\cos(3t) + 69\sin(3t)$

15. $x'' + 2x' + 2x = 2\cos(t)$,
$x = 3\,e^{-t}\sin(t) + 5\,e^{-t}\cos(t) + \cos(t) + 2\sin(t)$

Solution: $x_p = \cos(t) + 2\sin(t)$

16. $x'' + 2x' + 17x = 65\cos(4t)$,
$x = -2\,e^{-t}\sin(4t) + 7\,e^{-t}\cos(4t) + \cos(4t) + 8\sin(4t)$

Steady-State Periodic Solution

Consider the model $mx'' + cx' + kx = F_0\cos(\omega t)$ of external frequency ω. Compute the unique steady-state solution $A\cos(\omega t) + B\sin(\omega t)$ and its amplitude $C(\omega) = \sqrt{A^2 + B^2}$. Graph the ratio $100C(\omega)/C(\Omega)$ on $0 < \omega < \infty$, where Ω is the tuned practical resonance frequency.

17. $x'' + 2x' + 17x = 100\cos(4t)$

Solution: $x_{ss} = \dfrac{20\,\cos(4\,t)}{13} + \dfrac{160\,\sin(4\,t)}{13}$, because the steady-state consists of terms left over in the general solution after the transient terms have been removed.

```
# Exercise 17, Steady-state periodic solution
F:=t->100*cos(4*t);
de:=m*diff(x(t),t,t)+c*diff(x(t),t)+k*x(t)=F(t);
de1:=subs(m=1,c=2,k=17,de);
p:=dsolve(de1,x(t));X:=unapply(rhs(p),t);
```

18. $x'' + 2x' + 10x = 100\cos(4t)$

19. $x'' + 4x' + 5x = 10\cos(2t)$

Solution: $x_{ss} = \dfrac{16\,\sin(2\,t)}{13} + 2/13\,\cos(2\,t)$

20. $x'' + 2x' + 6x = 10\cos(2t)$

21. $x'' + 4x' + 5x = 5\cos(2t)$

Solution: $x_{ss} = \dfrac{8\,\sin(2\,t)}{13} + 1/13\,\cos(2\,t)$

22. $x'' + 2x' + 5x = 5\cos(1.5t)$

Phase-Amplitude

Solve for a particular solution in the form $x(t) = C\cos(\omega t - \alpha)$.

23. $x'' + 6x' + 13x = 174\sin(5t)$

Solution: Answer: $x_p = \sqrt{(}29)\cos(\arctan(5/2) - \pi)$.

First solve for the general solution

$$x(t) = c_1 e^{-3\,t}\cos(2\,t) + c_2 e^{-3\,t}\sin(2\,t) - 2\,\sin(5\,t) - 5\,\cos(5\,t)$$

Extract the steady-state solution $x_p = -2\,\sin(5\,t) - 5\,\cos(5\,t)$ and convert to phase-amplitude form as follows:

$$\begin{aligned} x_p &= -5\,\cos(5\,t) - 2\,\sin(5\,t) \\ &= \sqrt{c_1^2 + c_2^2}\cos(5t - \alpha) \text{ where } c_1 = -5,\ c_2 = -2 \\ &= \sqrt{29}\cos(5t - \alpha) \text{ where } \tan(\alpha) = c_1/c_2 = 5/2 \end{aligned}$$

Then $\alpha = \arctan(5/2) + \pi$ because α is the angle in quadrant 3 formed by point $(-5, -2)$. Check: expand $\sqrt{29}\cos(u - \arctan(5/2) - \pi)$ using the trig sum identities to get $-2\cos u - 5\sin u$.

24. $x'' + 8x' + 25x = 100\cos(t) + 260\sin(t)$

Chapter 7

Topics in Linear Differential Equations

Contents

7.1 Higher Order Homogeneous

Higher Order Factored

Solve the higher order equation with the given characteristic equation. Display the roots according to multiplicity and list the corresponding solution atoms.

1. $(r-1)(r+2)(r-3)^2 = 0$

Solution: Roots: $1, -2, 3, 3$. Atoms: e^x, e^{-2x}, e^{3x}, xe^{3x}.

```
eq:=(r-1)*(r+2)*(r-3)^2;
solve(eq=0,r);
```

2. $(r-1)^2(r+2)(r+3) = 0$

3. $(r-1)^3(r+2)^2r^4 = 0$

Solution: Roots: $1, 1, 1, -2, -2, 0, 0, 0, 0$. Atoms: e^x, xe^x, x^2e^x e^{-2x}, xe^{-2x}, 1, x, x^2, x^3

4. $(r-1)^2(r+2)^3r^5 = 0$

5. $r^2(r-1)^2(r^2+4r+6) = 0$

Solution: Roots: $0, 0, 1, 1, -2 \pm \sqrt{2}i$. Atoms: 1, x, e^x, xe^x, $e^{-2x}\cos\sqrt{2}x$, $e^{-2x}\sin\sqrt{2}x$

6. $r^3(r-1)(r^2+4r+6)^2 = 0$

7. $(r-1)(r+2)(r^2+1)^2 = 0$

Solution: Roots: $1, -2, \pm i$. Atoms: e^x, e^{-2x}, $\cos x$, $\sin x$

8. $(r-1)^2(r+2)(r^2+1) = 0$

9. $(r-1)^3(r+2)^2(r^2+4) = 0$

Solution: Roots: $1, 1, 1, -2, -2, \pm 2i$. Atoms: e^x, xe^x, x^2e^x, e^{-2x}, xe^{-2x}, $\cos 2x$, $\sin 2x$

10. $(r-1)^4(r+2)(r^2+4)^2 = 0$

Higher Order Unfactored

Completely factor the given characteristic equation, then the roots according to multiplicity and the solution atoms.

11. $(r-1)(r^2-1)^2(r^2+1)^3 = 0$

Solution: $(r-1)^3(r+1)^2(r^2+1)^3 = 0$, roots $1, 1, 1-1, -1, i, i, i, -i, -i, -i$, atoms e^x, xe^x, x^2e^x, e^{-x}, xe^{-x}, $\cos x$, $x\cos x$, $x^2\cos x$, $\sin x$, $x\sin x$, $x^2\sin x$

12. $(r+1)^2(r^2-1)^2(r^2+1)^2 = 0$

13. $(r+2)^2(r^2-4)^2(r^2+16)^2 = 0$

Solution: $(r+2)^4(r-2)^2(r^2+16)^3 = 0$, roots $-2, -2, -2, -2, 2, 2, , 4i, 4i, 4i, -4i, -4i, -4i$, atoms e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}, e^{2x}, xe^{2x}, $\cos 4x$, $x\cos 4x$, $x^2\cos 4x$, $\sin 4x$, $x\sin 4x$, $x^2\sin 4x$

14. $(r+2)^3(r^2-4)^4(r^2+5)^2 = 0$

15. $(r^3-1)^2(r-1)^2(r^2-1)=0$

Solution: $(r-1)^5(r^2+r+1)^2(r+1)=0$, roots $1,1,1,1,1,-1/2+\sqrt{3}i/2,-1/2+\sqrt{3}i/2,-1/2-\sqrt{3}i/2,-1/2-\sqrt{3}i/2,$, -1, atoms 1, x, x^2, x^3, x^4, $e^{-x/2}\cos\sqrt{3}x/2$, $xe^{-x/2}\cos\sqrt{3}x/2$, $e^{-x/2}\sin\sqrt{3}x/2$, $xe^{-x/2}\sin\sqrt{3}x/2$, e^{-x}

16. $(r^3-8)^2(r-2)^2(r^2-4)=0$

17. $(r^2-4)^3(r^4-16)^2=0$

Solution: $(r-2)^3(r+2)^3(r^2-4)^2(r^2+4)^2=0$ or $(r-2)^5(r+2)^5(r^2+4)^2=0$, roots $2,2,2,-2,-2,-2,2i,2i,-2i,-2i$, atoms e^{2x} multiplied by $1,x,x^2,x^3,x^4$, e^{-2x} multiplied by $1,x,x^2,x^3,x^4$, $\cos 2x$, $x\cos 2x$, $\sin 2x$, $x\sin 2x$

18. $(r^2+8)(r^4-64)^2=0$

19. $(r^2-r+1)(r^3+1)^2=0$

Solution: $(r^2-r+1)^3(r+1)^2=0$, roots $-1,-1,1/2+\sqrt{3}i/2,1/2+\sqrt{3}i/2,1/2+\sqrt{3}i/2,1/2-\sqrt{3}i/2,1/2-\sqrt{3}i/2,1/2-\sqrt{3}i/2$, atoms e^{-x}, xe^{-x}, $e^{x/2}\cos\sqrt{3}x/2$, $xe^{x/2}\cos\sqrt{3}x/2$, $x^2e^{x/2}\cos\sqrt{3}x/2$, $e^{x/2}\sin\sqrt{3}x/2$, $xe^{x/2}\sin\sqrt{3}x/2$, $x^2e^{x/2}\sin\sqrt{3}x/2$

20. $(r^2+r+1)^2(r^3-1)=0$

Higher Order Equations

The exercises study properties of Euler atoms and nth order linear differential equations.

21. (Euler's Theorem)
Explain why the derivatives of atom x^3e^x satisfy a higher order equation with characteristic equation $(r-1)^4=0$.

Solution: Euler's theorem says that x^3e^x is a solution if and only if $r=1$ is a root of the characteristic of multiplicity 4. Therefore, $(r-1)^4$ is a factor of the characteristic equation. The simplest such equation comes from expanding $(r-1)^4$ and then recovering the differential equation from the powers of r: $r^n\mapsto(d/dx)^ny(x)$.

22. (Euler's Theorem)
Explain why the derivatives of atom $x^3\sin x$ satisfy a higher order equation with characteristic equation $(r^2+1)^4=0$.

23. (Kümmer's Change of Variable)
Consider a fourth order equation with characteristic equation $(r+a)^4 = 0$ and general solution y. Define $y = ue^{-ax}$. Find the differential equation for u and solve it. Then solve the original differential equation.

Solution: Let $v = e^{bx}$, $b = -a$. Then $v' = bv$. The product rule for derivatives gives

$$(d/dx)(uv) = u'v + buv = (u' + bu)v$$

Expand $(D+a)y = (D+a)(ue^{bx}) = u'v + buv + auv = u'v$. Replace u by u' and repeat the expansion: $(D+a)^2 y = (D+a)(u'v) = u''v$. Conclusion: $(D+a)^4 u = u''''v$. Because y is a solution of $(D+a)^4 y = 0$ then $0 = u''''v$. Cancel $v = e^{-ax}$ to reach the differential equation for u: $u'''' = 0$.

The solution of $u'''' = 0$ is $u = \sum_{i=1}^{3} c_i x^{i-1}$. Then $y = \left(\sum_{i=1}^{3} c_i x^{i-1}\right) e^{-ax}$.

24. (Kümmer's Change of Variable)
A polynomial $u = c_0 + c_1 x + c_2 x^2$ satisfies $u''' = 0$. Define $y = ue^{ax}$. Prove that y satisfies a third order equation and determine its characteristic equation.

25. (Ziebur's Derivative Lemma)
Let y be a solution of a higher order constant-coefficient linear equation. Prove that the derivatives of y satisfy the same differential equation.

Solution: The proof consists of differentiation of the differential equation, n times to find a new differential equation for $y^{(n)}$. ■

26. (Ziebur's Lemma: atoms)
Let $y = x^3 e^x$ be a solution of a higher order constant-coefficient linear equation. Prove that Euler atoms e^x, xe^x, $x^2 e^x$ are solutions of the same differential equation.

27. (Ziebur's Atom Lemma)
Let y be an Euler atom solution of a higher order constant-coefficient linear equation. Prove that the Euler atoms extracted from the expressions $y, y', y'', \ldots$ are solutions of the same differential equation.

Solution: The ideas are in Exercise 22 and Exercise 26. The proof of Exercise 27 is inductive, motivated by Ziebur's Derivative Lemma, Exercise 25.

Induction Hypothesis:
Let y be an Euler atom solution decomposed as $y = x^n z$, where z is a base atom. If z has associated complex root $a + ib$ ($b = 0$ if real) then for some constants c_i, d_i

$$y' = \sum_{i=1}^{n} x_{i-1}(c_i \cos bx + d_i \sin bx)\vec{\mathbf{e}}^{ax}$$

The identity for y' says that y' is a linear combination of Euler solution atoms w. Each atom w is a solution by Euler's theorem.

Induction $n = 0$:
Let $y = x^0 z$ where z is a base atom. Then $z = e^{at}\cos bt$ or $z = e^{at}\sin bt$, with $b = 0$ allowed, the latter excluding $z = 0$ from the cases. Differentiate $y = z = e^{at}\cos bt$ and $y = z = e^{at}\sin bt$, proving directly from Euler's theorem that $y' = z' =$ linear combination of solution atoms of the differential equation.

Induction $n \geq 1$:
Assume the induction hypothesis is true for all powers less than or equal to $n-1$. Differentiate $y = x^n z$ to obtain $nx^{n-1}z + x^n z'$. The induction hypothesis applies to $nx^{n-1}z$: it equals some $\sum_{i=1}^{n-1} x_{i-1}(c_i \cos bx + d_i \sin bx)\vec{\mathbf{e}}^{ax}$ and the atoms w involved are solutions of the differential equation. Euler's theorem implies z' is a linear combination of at most two base atoms which are solutions of the same differential equation. Multiply by x^n to produce $x^n z'$ as a linear combination of Euler atoms w. Each atom w is known by Euler's theorem and the hypothesis on y to be a solution. Then y' is a sum of linear combinations of Euler solution atoms. ■

28. (Differential Operators)
Let y be a solution of a differential equation with characteristic equation $(r-1)^3(r+2)^6(r^2+4)^5 = 0$. Explain why y''' is a solution of a differential equation with characteristic equation $(r-1)^3(r+2)^6(r^2+4)^5 r^3 = 0$.

29. (Higher Order Algorithm)
Let atom $x^2\cos x$ appear in the general solution of a linear higher order equation. Find the pair of complex conjugate roots that constructed this atom, and the multiplicity k. Report the $2k$ atoms which must also appear in the general solution.

Solution: The root pair for the base atom $\cos x$ is $\pm i$. The base atoms for this pair are $\cos x$, $\sin x$. The multiplicity according to Euler's theorem is $k = 3$, always one higher than the highest power of x in the atom. The $2k = 6$ atoms according to Euler's theorem: $\cos x$, $x\cos x$, $x^2\cos x$, $\sin x$, $x\sin x$, $x^2\sin x$.

30. (Higher Order Algorithm)
Let Euler atom $xe^x\cos 2x$ appear in the general solution of a linear higher order equation. Find the pair of complex conjugate roots that constructed this atom and estimate the multiplicity k. Report the $2k$ atoms which are expected to appear in the general solution.

31. (Higher Order Algorithm)
Let a higher order equation have characteristic equation $(r-9)^3(r-5)^2(r^2+4)^5 = 0$. Explain precisely using existence-uniqueness theorems why the general solution is a sum of constants times Euler atoms.

Solution: The number of independent solutions is the order n of the differential equation, which is the degree of the characteristic equation: $n = 15$. It suffices by existence-uniqueness theory to find a basis of n independent solutions, because then the general solution is a linear combination of these solutions. Euler's theorem provides the n solutions from the roots listed according to multiplicity: $3, 3, 3$, $5, 5$, and $\pm 2i$ repeated 5 times. The atoms: e^{9x}, xe^{9x}, x^2e^{9x}, e^{5x}, xe^{5x}, $\cos x$, $x\cos x$, $x^2\cos x$, $x^3\cos x$, $x^4\cos x$, $\sin x$, $x\sin x$, $x^2\sin x$, $x^3\sin x$, $x^4\sin x$. The atoms are independent by a theorem: *A finite list of distinct Euler atoms is independent.*

32. (Higher Order Algorithm)
Explain why any higher order linear homogeneous constant-coefficient differential equation has general solution a sum of constants times Euler atoms.

7.2 Differential Operators

Operator Arithmetic

Compute the operator and solve the corresponding differential equation.

1. $D(D+1)+D$

Solution: Answers: D^2+2D, $y''+2y'=0$ has solution $y=c_1+c_2e^{-2x}$.

2. $D(D+1)+D(D+2)$

3. $D(D+1)^2$

Solution: Answers: D^3+2D^3+D, $y'''+2y''+y'=0$ has solution $y=c_1+c_2e^{-x}+c_3xe{-}x$.

4. $D(D^2+1)^2$

5. $D^2(D^2+4)^2$

Solution: Answers: $D^6+8D^4+16D^2$, solution $y=c_1+c_2x+c_3\cos 2x+c_4\sin 2x+c_5x\cos 2x+c_6x\sin 2x$.

6. $(D-1)((D-1)^2+1)^2$

Operator Properties.

7. **(Operator Composition)** Multiply $P=D^2+D$ and $Q=2D+3$ to get $R=2D^3+5D^2+3D$. Then compute $P(Qy)$ and $Q(Py)$ for $y(x)$ 3-times differentiable, and show both equal Ry.

Solution: Multiply: $PQ=(D^2+D)(2D+3)=2D^3+5D^2+3D=R$.

Compute $P(Qy)$: Let $u=Qy=(2D+3)(y)=2y'+3y$. Then $P(u)=(D^2+D)(u)=u''+u'=(2y'+3y)''+(2y'+3y)'=2y'''+3y''+2y''+3y'=2y'''+5y''+3y'$.

Compute $Q(Py)$: Let $u=Py=(D^2+D)(y)=y''+y'$. Then$Q(Py)=Q(u)=(2D+3)(u)=2u'+3u=2(y''+y')'+3(y''+y')=2y'''+2y''+3y''+3y'=2y'''+5y''+3y'$

Because $R(y)=(2D^3+5D^2+3D)(y)=2y'''+5y''+3y'$, then $P(Qy)=Q(Py)=Ry$.

8. **(Kernels)**
The operators $(D-1)^2(D+2)$ and $(D-1)(D+2)^2$ share common factors. Find the Euler solution atoms shared by the corresponding differential equations.

9. (Operator Multiply)
Let differential equation $(D^2 + 2D + 1)y = 0$ be formally differentiated four times. Find its operator and solve the equation. What does this have to do with operator multiply?

Solution: After four differentiations: $y^{(6)} + 2y^{(5)} + y^{(4)} = 0$. The operator is $D^6 + 2D^5 + D^4$, which can be factored according to theorems in a number of ways, one of which is $D^4(D^2 + 2D + 1)$. The meaning: to differentiate the differential equation four times, multiply the operator equation by D^4.

10. (Non-homogeneous Equation) The differential equation $(D^5 + 4D^3)y = 0$ can be viewed as $(D^2 + 4)u = 0$ and $u = D^3 y$. On the other hand, y is a linear combination of the atoms generated from the characteristic equation $r^3(r^2 + 4) = 0$. Use these facts to find a particular solution of the non-homogeneous equation $y''' = 3\cos 2x$.

Kümmer's Change of Variable

Kümmer's change of variable $y = ue^{ax}$ changes a y-differential equation into a u-differential equation. It can be used as a basis for solving homogeneous nth order linear constant coefficient differential equations.

11. Supply details: $y = ue^{ax}$ changes $y'' = 0$ into $u'' + 2au' + a^2 u = 0$.

Solution: Differentiate $y = ue^{ax}$: $y' = u'e^{ax} + aue^{ax} = (u' + au)e^{ax}$, then differentiate again to get $y'' = (u'' + 2au' + a^2 u)e^{ax}$.

12. Supply details: $y = ue^{ax}$ changes $(D^2 + 4D)y = 0$ into $((D + a)^2 + 4(D + a))u = 0$.

13. Supply details: $y = ue^{ax}$ changes the differential equation $D^n y = 0$ into $(D + a)^n u = 0$.

Solution: The details in Exercise 11 give $D(we^{ax}) = (w' + aw)e^{ax} = ((D + a)(w))e^{ax}$. Then $y' = ((D + a)(u))e^{ax}$, $y'' = D(we^{ax}) = ((D + a)(w))e^{ax}$ where $w = (D + a)(u)$. Repeat or use induction to give $D^n y = ((D + a)^n(u))e^{ax}$.

14. Kümmer's substitution $y = ue^{ax}$ changes the differential equation $(D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = 0$ into $(F^n + a_{n-1}F^{n-1} + \cdots + a_0)u = 0$, where $F = D + a$. Write the proof.

7.3 Higher Order Non-Homogeneous

Variation of Parameters

Solve the higher order equation given by its characteristic equation and right side $f(x)$. Display the Cauchy kernel $\mathcal{K}(x)$ and a particular solution $y_p(x)$ with fewest terms. Use a computer algebra system to evaluate integrals, if possible.

1. $(r-1)(r+2)(r-3)^2 = 0$,
$f(x) = e^x$

Solution: Answers: $y = \frac{1}{12}\, x\mathrm{e}^x + c_1\, \mathrm{e}^x + c_2\, \mathrm{e}^{-2\,x} + c_3\, \mathrm{e}^{3\,x} + c_4\, x\mathrm{e}^{3\,x}$,
$y_p = \frac{1}{12}\, x\mathrm{e}^x$, $k(x) = \frac{1}{12}\, \mathrm{e}^x - \frac{1}{75}\mathrm{e}^{-2\,x} - \frac{7}{100}\mathrm{e}^{3\,x} + \frac{1}{10}\, x\mathrm{e}^{3\,x}$

```
# Exercise 1,Variation of Parameters
F:=x->exp(x);
sol:=c1*exp(x)+c2*exp(-2*x)+c3*exp(3*x)+c4*x*exp(3*x);
eqs:=[subs(x=0,sol)=0,
subs(x=0,diff(sol,x))=0,
subs(x=0,diff(sol,x,x))=0,
subs(x=0,diff(sol,x,x,x))=1];
solve(eqs,[c1,c2,c3,c4]);
kk:=unapply(subs(c1 = 1/12, c2 = -1/75, c3 = -7/100,
 c4 = 1/10,sol),x);
int(kk(x-t)*F(t),t=0..x);# Var of parameters formula
# (1/900*(45*exp(5*x)*x-54*exp(5*x)+75*x*exp(3*x)+
 50*exp(3*x)+4))*exp(-2*x)
# Answer check
expand((r-1)*(r+2)*(r-3)^2);
# r^4-5*r^3+r^2+21*r-18
de:=diff(u(x),x,x,x,x)-5*diff(u(x),x,x,x)+diff(u(x),x,x)+
 21*diff(u(x),x)-18*u(x)=F(x);
dsolve(de,u(x));
```

2. $(r-1)^2(r+2)(r+3) = 0$,
$f(x) = e^x$

3. $(r-1)^3(r+2)^2r^4 = 0$,
$f(x) = x + e^{-2x}$

Solution: Answers: $k\,(x) = -\dfrac{\mathrm{e}^{-2\,x}}{144} - \dfrac{x\mathrm{e}^{-2\,x}}{432} + \dfrac{13\,\mathrm{e}^x}{9} - \dfrac{14\,x\mathrm{e}^x}{27} +$
$\dfrac{1}{18}\,\mathrm{e}^x x^2 - \dfrac{23}{16} - \dfrac{15\,x}{16} - \dfrac{1}{4}\,x^2 - \dfrac{1}{24}\,x^3$,
$y_h = c_1\,\mathrm{e}^{-2\,x} + c_2\,x\mathrm{e}^{-2\,x} + c_3\mathrm{e}^x + c_4\,x\,\mathrm{e}^x + c_5\,x^2\,\mathrm{e}^x +$
$c_6 + c_7\,x + c_8\,x^2 + c_9\,x^3$,
$y_p = \dfrac{x^3\mathrm{e}^x}{54} - \dfrac{2}{9}\,\mathrm{e}^x x^2 + \dfrac{10\,x\mathrm{e}^x}{9} - \dfrac{20\,\mathrm{e}^x}{9}$

4. $(r-1)^2(r+2)^3r^5 = 0$,
$f(x) = x + e^{-2x}$

5. $r^2(r-1)^2(r^2+4r+6) = 0$,
$f(x) = x + e^{x}$

Solution: $k(x) = 2/9 + x/6 - \frac{28\,\mathrm{e}^{x}}{121} + \frac{1}{11}x\mathrm{e}^{x} + \frac{10\,\mathrm{e}^{-2\,x}\cos\left(\sqrt{2}x\right)}{1089} - \frac{17\sqrt{2}\mathrm{e}^{-2\,x}\sin\left(\sqrt{2}x\right)}{4356}$,

$y_p(t) = \frac{1}{36}x^3 + \frac{4\,x\mathrm{e}^{x}}{33} + \frac{1}{9}x^2 - \frac{545\,\mathrm{e}^{x}}{1089} + \frac{11\,x}{27} + \frac{313}{648} + \frac{263\,\mathrm{e}^{-2x}\sin\left(\sqrt{2}x\right)\sqrt{2}}{78408} + \frac{139\,\cos\left(\sqrt{2}x\right)}{39204\,\left(\mathrm{e}^{x}\right)^2} + \frac{1}{72}\,\mathrm{e}^{-2x}$

```
# Exercise 5,Variation of Parameters
F:=x->x+exp(x);
solve(r^2*(r-1)^2*(r^2+4*r+6)=0,r);
 # 0, 0, 1, 1, -2+I*sqrt(2), -2-I*sqrt(2)
sol:=c1+c2*x+c3*exp(x)+c4*x*exp(x)+
 c5*exp(-2*x)*cos(sqrt(2)*x)+c6*exp(-2*x)*sin(sqrt(2)*x);
eqs:=[subs(x=0,sol)=0,subs(x=0,diff(sol,x))=0,
 subs(x=0,diff(sol,x$2))=0,subs(x=0,diff(sol,x$3))=0,
 subs(x=0,diff(sol,x$4))=0,subs(x=0,diff(sol,x$5))=1];
p:=solve(eqs,[c1,c2,c3,c4,c5,c6]);
q:=convert(p[1],set,nested=true);
kk:=unapply(subs(q,sol),x);
int(kk(x-t)*F(t),t=0..x);# Var of parameters formula
```

6. $r^3(r-1)(r^2+4r+6)^2 = 0$,
$f(x) = x^2 + e^{x}$

7. $(r-1)(r+2)(r^2+1)^2 = 0$,
$f(x) = \cos x + e^{-2x}$

Solution: $k(x) = -\frac{1}{75}e^{-2x} + \frac{1}{12}e^{x} - \frac{7}{100}\cos(x) - \frac{13}{50}\sin(x) + \frac{3}{20}x\cos(x) - \frac{1}{20}x\sin(x)$,

$y_p(x) = -\frac{67\,\left(\cos\left(x\right)\right)^3}{400} - \frac{67\,\left(\sin\left(x\right)\right)^2\cos\left(x\right)}{400} + \frac{3\,\cos\left(x\right)x^2}{80} - \frac{\sin\left(x\right)x^2}{80} + \frac{19\,x\cos\left(x\right)}{400} - \frac{33\,x\sin\left(x\right)}{400} + \frac{237\,\cos\left(x\right)}{2000} - \frac{289\,\sin\left(x\right)}{2000} + \frac{5\,\mathrm{e}^{x}}{72} + \frac{\left(\cos\left(x\right)\right)^2}{500\,\left(\mathrm{e}^{x}\right)^2} + \frac{\left(\sin\left(x\right)\right)^2}{500\,\left(\mathrm{e}^{x}\right)^2} - \frac{x}{75\,\left(\mathrm{e}^{x}\right)^2} - \frac{101}{4500\,\left(\mathrm{e}^{x}\right)^2}$

8. $(r-1)^2(r+2)(r^2+1)=0$,
$f(x)=\sin x+e^{-2x}$

9. $(r-1)^3(r+2)^2(r^2+4)=0$,
$f(x)=\cos 2x+e^{x}$

Solution: $k(x) = \frac{e^{-2x}}{144} - \frac{xe^{-2x}}{216} + \frac{14\,e^{x}}{1125} - \frac{16\,xe^{x}}{675} + \frac{e^{x}x^2}{90} - \frac{11\,\cos(2\,x)}{2000} + \frac{\sin(2\,x)}{1000}$,

$$y_p(x) = \frac{(\sin(x))^2(\cos(x))^4}{500} + \frac{(\cos(x))^6}{500} - \frac{7\,e^{x}(\sin(x))^2(\cos(x))^2}{2500} - \frac{7\,e^{x}(\cos(x))^4}{2500} - \frac{(\sin(x))^2(\cos(x))^2}{1000} - \frac{3\,(\cos(x))^4}{1000} + \frac{x^3e^{x}}{270} + \frac{\sin(x)\cos(x)\,x}{1000} + \frac{7\,e^{x}(\cos(x))^2}{2500} - \frac{11\,(\cos(x))^2\,x}{2000} - \frac{13\,e^{x}x^2}{1350} + \frac{93\,\sin(x)\cos(x)}{20000} + \frac{71\,(\cos(x))^2}{20000} + \frac{7\,xe^{x}}{675} - \frac{887\,e^{x}}{151875} + \frac{11\,x}{4000} - \frac{51}{40000} + \frac{7\,x}{2592\,(e^{x})^2} + \frac{71}{15552\,(e^{x})^2}$$

10. $(r-1)^4(r+2)(r^2+4)^2=0$,
$f(x)=\sin 2x+e^{x}$

Undetermined Coefficient Method

A higher order equation is given by its characteristic equation and right side $f(x)$. Display (a) a trial solution, (b) a system of equations for the undetermined coefficients, and (c) a particular solution $y_p(x)$ with fewest terms. Use a computer algebra system to solve for undetermined coefficients, if possible.

11. $(r-1)(r+2)(r-3)^2=0$,
$f(x)=e^{x}$

Solution: (a) Trial solution $y=c_1xe^{x}$
The char equation roots: $-2,1,3,3$
The roots for $f(x)$: 1

(b) System
The differential equation is $(D-1)(D+2)(D-3)^2y=e^{x}$.
Substitute the trial solution into the DE to get $12c_1e^{x}=e^{x}$. Then only one equation occurs by matching coefficients of atoms:
$c_1=1/12$

(c) Particular solution: $y_p=\frac{1}{12}xe^{x}$

```
F:=x->exp(x);
solve((r-1)*(r+2)*(r-3)^2=0,r);
 #  -2, 1, 3, 3
sol:=c1*exp(-2*x)+c2*exp(x)+c3*exp(3*x) + c4*x*exp(3*x);
L:=PolynomialTools[CoefficientList]((r-1)*(r+2)*(r-3)^2,r)
 # [-18, 21, 1, -5, 1]
n:=numelems(L);
expand((r-1)*(r+2)*(r-3)^2);
de:=L[1]*y(x):for i from 1 to n-1 do
de:= de + diff(y(x),x$i)*L[i+1];
od:
trial:=c1*x*exp(x);
eqs:=subs(y(x)=trial,de=F(x)):simplify(eqs);
 #   12 c1 exp(x) = exp(x)
solve(eqs,c1);
 # c1 = 1/12
 # Answer check:
 dsolve(de = F(x),y(x))
 # yp = (1/12)*x*exp(x) by setting y_h=0
```

12. $(r-1)^2(r+2)(r+3)=0$,
$f(x)=e^x$

13. $(r-1)^3(r+2)^2r^4=0$,
$f(x)=x+e^{-2x}$

Solution: Roots of the char equation: $-2,-2,1,1,1,0,0,0,0$
Roots for f: $0,0,-2$
Trial solution: $y=c_1x^4+c_2x^5+c_3x^2e^{-2x}$
Differential equation:

$$-4y'''' + 8y^{(5)} - y^{(6)} - 5y^{(7)} + y^{(8)} + y^{(9)} = f(x)$$

Equations:

$$\begin{cases} -96\,c_1 - 864\,c_3 + 960\,c_2 &= 1 \\ -480\,c_2 + 1728\,c_3 &= -1 \\ -3456\,c_3 &= 4 \end{cases}$$

Solution to the equations: $c_1=-1/48, c_2=-1/480, c_3=-1/864$
Particular solution:
$y_p = -\dfrac{x^4}{48} - \dfrac{x^5}{480} - \dfrac{x^2\mathrm{e}^{-2\,x}}{864}$

```
# Exercise 13, Undetermined coefficients
F:=x->x+exp(-2*x);charpoly:=(r-1)^3*(r+2)^2*r^4;
solve(charpoly=0,r); # -2, -2, 1, 1, 1, 0, 0, 0, 0
sol:=c1*exp(-2*x)+c2*x*exp(-2*x) + c3*exp(x)+c4*x*exp(x)
 + c5*x^2*exp(x) + c6 + c7*x + c8*x^2+c9*x^3;
L:=PolynomialTools[CoefficientList](charpoly,r)
n:=numelems(L);expand(charpoly);
de:=L[1]*y(x):for i from 1 to n-1 do
de:= de + diff(y(x),x$i)*L[i+1];
od:
# roots F: 0,0,-2
trial:=c1*x^4 + c2*x^5+c3*x^2*exp(-2*x);
eqs:=subs(y(x)=trial,de=F(x)):simplify(eqs);
eq1:=simplify(subs(x=0,eqs));
eq2:=simplify(subs(x=0,diff(eqs,x)));
eq3:=simplify(subs(x=0,diff(eqs,x$2)));
solve([eq1,eq2,eq3],[c1,c2,c3]);
 # c1 = -1/48, c2 = -1/480, c3 = -1/864
# Answer check:
dsolve(de = F(x),y(x))
 # yp = -x^5/480-x^4/48+2* exp(-2*x) -x^2*exp(-2*x)/864
# Extra terms from y_h removed
```

14. $(r-1)^2(r+2)^3r^5 = 0$,
$f(x) = x + e^{-2x}$

15. $r^2(r-1)^2(r^2+4r+6) = 0$,
$f(x) = x + e^x$

Solution: Roots of char equation: $0, 0, 1, 1, -2 \pm i\sqrt{2}$
Roots for f: $0, 0, 1$
Trial solution: $y = c_1x^2 + c_2x^3 + c_3x^2e^x$
Differential equation:
$6D^2y - 8D^3y - D^4y + 2D^5y + D^6y = f(x)$
Particular solution:
$$y_p = \frac{1}{36}\,x^3 + \frac{2}{9}\,x^2 + \frac{1}{22}\,\mathrm{e}^x x^2$$

16. $r^3(r-1)(r^2+4r+6)^2 = 0$,
$f(x) = x^2 + e^x$

17. $(r-1)(r+2)(r^2+1)^2 = 0$,
$f(x) = \cos x + e^{-2x}$

Solution: Roots of char equation: $-2, 1, \pm i, \pm i$
Roots for f: $\pm i, -2$

Trial solution: $y = c_1x^2\cos(x) + c_2x^2\sin(x) + c_3xe^{-2x}$
Differential equation:
$-2y + Dy - 3D^2y + 2D^3y + D^5y + D^6y = f(x)$
Particular solution:

$$y_p = 3\,x^2\frac{\cos(x)}{80} - x^2\frac{\sin(x)}{80} - \frac{x\mathrm{e}^{-2\,x}}{75}$$

18. $(r-1)^2(r+2)(r^2+1) = 0,$
$f(x) = \sin x + e^{-2x}$

19. $(r-1)^3(r+2)^2(r^2+4) = 0,$
$f(x) = \cos 2x + e^x$

Solution: Roots of char equation: $-2, -2, 1, 1, 1, \pm 2i$
Roots for f: $\pm 2i, 1$
Trial solution: $y = c_1x\cos(2x) + c_2x\sin(2x) + c_3x^3e^x$
Differential equation:
$-16y + 32Dy - 8D^2y - 12D^3y + 3D^4y - D^5y + D^6y + D^7y = f(x)$
-16, 32, -8, -12, 3, -1, 1, 1 Particular solution:

$$y_p = \frac{11\,x\,\cos(2\,x)}{4000} + \frac{x\sin(2\,x)}{2000} + \frac{x^3e^x}{270}$$

20. $(r-1)^4(r+2)(r^2+4)^2 = 0,$
$f(x) = \sin 2x + e^x$

7.4 Cauchy-Euler Equation

Cauchy-Euler Equation

Find solutions y_1, y_2 of the given homogeneous differential equation which are independent by the Wronskian test, page 464.

1. $x^2y'' + y = 0$

Solution: $y_1(x) = \sqrt{x}\sin\left(1/2\sqrt{3}\ln(x)\right)$, $y_2(x) = \sqrt{x}\cos\left(1/2\sqrt{3}\ln(x)\right)$

2. $x^2y'' + 4y = 0$

3. $x^2y'' + 2xy' + y = 0$

Solution: $y_1(x) = \dfrac{\sin\left(1/2\sqrt{3}\ln(x)\right)}{\sqrt{x}}$, $y_2(x) = \dfrac{\cos\left(1/2\sqrt{3}\ln(x)\right)}{\sqrt{x}}$

4. $x^2y'' + 8xy' + 4y = 0$

Variation of Parameters

Find a solution y_p using a variation of parameters formula.

5. $x^2y'' = x$

6. $x^3y'' = e^x$

Solution: $y_p(x) = x\ln(x) - x$

Because $y_h = c_1 + c_2x$ then variation of parameters in Cauchy kernel form gives

$$y_p = \int_1^x k(x-t)f(t)dt/t^2 = \int_1^x k(x-t)(1/t)dt$$

Compute $k(x) = x$. Then integrate:

$$y_p = \int_1^x k(x-t)(1/t)dt = \int_1^x \left(\frac{x}{t} - 1\right)dt = x\ln|x| - x + c$$

for some constant c. Choose $c = 0$.

7. $y'' + 9y = \sec 3x$

Solution: $y_p = (1/3)x\sin(3x) + (1/9)\cos(3x)\ln|\cos(3x)|$

The Cauchy kernel for $y'' + 9y = 0$ is $k(x) = \sin(3x)/3$. To keep the integration result simple, feed the computer algebra system integrator this t-expression for the integrand $k(x-t)\sec(3t)$ in the variation of parameters formula (x held fixed):

$$(1/3)(\sin(3x) - \cos(3x)\sin(3t)/\cos(3t))$$

8. $y'' + 9y = \csc 3x$

7.5 Variation of Parameters Revisited

Cauchy Kernel

Find the Cauchy kernel $\mathcal{K}(x,t)$ for the given homogeneous differential equation.

1. $y'' - y = 0$

Solution: Solve the equation with initial data $x(0) = 0$, $x'(0) = 1$. Then $k(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x} = \sinh(x)$.

2. $y'' - 4y = 0$

3. $y'' + y = 0$

Solution: $k(x) = \sin x$

4. $y'' + 4y = 0$

5. $4y'' + y' = 0$

Solution: $k(x) = c_1 + c_2 e^{-x/4} = 4 - 4e^{-x/4}$

6. $y'' + y' = 0$

7. $y'' + y' + y = 0$

Solution: $k(x) = \frac{2}{3}\sqrt{3}\, e^{-x/2}\sin(\sqrt{3}x/2)$

8. $y'' - y' + y = 0$

Variation of Parameters

Find the general solution $y_h + y_p$ by applying a variation of parameters formula.

9. $y'' = x^2$

Solution: $y_h = c_1 + c_2 x$, $y_p = x^4/12$

10. $y'' = x^3$

11. $y'' + y = \sin x$

Solution: $y_h = c_1 \cos x + c_2 \sin x$, $k(x) = \sin x$, $y_p = \int_0^x k(x-t)\sin(t)dt = \frac{1}{2}\sin(x) - \frac{1}{2}x\cos(x)$. A shortened $y_p = -\frac{1}{2}x\cos(x)$.

12. $y'' + y = \cos x$

13. $y'' + y' = \ln|x|$

Solution: $y_h = c_1 + c_2 x$, $k(x) = x$, $y_p = \int_1^x k(x-t)\ln(t)dt = x - \frac{1}{4} + \frac{1}{2}x^2\ln(x) - \frac{3}{4}x^2$. A shortened $y_p = \frac{1}{2}x^2\ln(x) - \frac{3}{4}x^2$.

14. $y'' + y' = -\ln|x|$

15. $y'' + 2y' + y = e^{-x}$

Solution: $y_h = c_1 e^{-x} + c_2 x e^{-x}$, $k(x) = xe^{-x}$,
$y_p = \int_0^x k(x-t)e^{-t}dt = \frac{1}{2}x^2 e^{-x}$.

16. $y'' - 2y' + y = e^x$

7.6 Undetermined Coefficients Library

Polynomial Solutions

Determine a polynomial solution y_p for the given differential equation. Apply Theorem 7.8, page 581, and model the solution after Examples 7.5, 7.6, 7.7 and 7.8.

1. $y'' = x$

Solution: The example needs no special method: use quadrature. Answer: $y_p = x^3/6$.

2. $y'' = x - 1$

3. $y'' = x^2 - x$

Solution: The example needs no special method: use quadrature. Answer: $y_p = x^4/12 - x^2/6$.

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

Solution: Equilibrium method is an easy shortcut: drop term y'' and solve $-y' = 1$ by quadrature. Then $y_p = -x$.

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

Solution: The polynomial method applies. Differentiate the DE until the RHS becomes constant:

$$\begin{cases} y'' - y' & = & x \\ y''' - y'' & = & 1 \end{cases}$$

Use the equilibrium method on the last equation: drop y''' and solve $-y'' = 1$ by quadrature. Then $y = p(x) = -x^2/2 + d_0 + d_1 x$ is a polynomial trial solution. Insert the trial solution into the DE:

$$(-x^2/2 + d_0 + d_1 x)'' - (-x^2/2 + d_0 + d_1 x)' = x$$

$$(-1) - (-x + d_1) = x$$

Match Euler atom coefficients left and right to find the equation(s) for d_0, d_1: $-1 - d_1 = 0$. Then $d_1 = -1$ and d_0 is a free variable. Let $c_0 = 0$. Then $y_p = p(x) = -x^2/2 - x$.

Check the answer:

$$(-x^2/2 - x)'' - (-x^2/2 - x)' = (-1) - (-x - 1) = x$$

The method agrees with case **One Root** $r = 0$ of characteristic equation $r^2 - r = 0$.

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

Solution: Equilibrium method: replace terms y'' and y' by zero, then solve $0+0+y = 1$ to find $y_p = 1$. The answer can be checked without pencil and paper.

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

Solution: The polynomial method applies. The trial solution arises from $y''' + y' = -1$, solved by replacing y''' by zero, then apply quadrature to $0+y' = -1$ get $y = p(x) = d_0 - x$. Substitute back into the DE: $(d_0 - x)'' + (d_0 - x) = 1 - x$. Then $d_0 = 1$ and $y = p(x) = 1 - x$. The answer can be checked without pencil and paper.

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

Solution: The polynomial method applies. Reduced equation: $0 - y'' = 2$. Trial solution: $y = p(x) = x^2 + d_1 x + d_0$. Euler atom equation for d_0, d_1:

$$(x^2 + d_1 x + d_0)'' + (x^2 + d_1 x + d_0) = x^2$$

$$(2) + (x^2 + d_1 x + d_0) = x^2$$

Then $d_0 = -2$, $d_1 = 0$ and $y_p = p(x) = x^2 - 2$
Answer check:

$$(x^2 - 2)'' + (x^2 - 2) = (2) + x^2 - 2 = x^2$$

14. $y'' - y = x^3$

Polynomial-Exponential Solutions

Determine a solution y_p for the given differential equation. Apply Theorem 7.9, page 581, and model the solution after Example 7.9.

15. $y'' + y = e^x$

Solution: The polynomial times exponential method applies. Kummer's transformation $y = e^x Y$ is used to obtain the new equation $(D+1)^2 Y + Y = 1$, which expands to $Y'' + 2Y' + 2Y = 1$. The latter is solved by the equilibrium shortcut: replace terms Y'' and Y' by zero, then solve the reduced equation: $2Y = 1$. Then $Y = 1/1$ and $y = e^x Y = \frac{1}{2}e^x$.
Answer check:

$$(\frac{1}{2}e^x)'' + (\frac{1}{2}e^x) = (\frac{1}{2} + \frac{1}{2})e^x = e^x$$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

Solution: Solved by quadrature: $y_p = \frac{1}{4}e^{2x}$.

18. $y'' = e^{-2x}$

19. $y'' - y = (x+1)e^{2x}$

Solution: The polynomial times exponential method applies: Replace D by $D+2$ in the DE and cancel e^{2x} on the RHS:

$$(D+2)^2 Y - Y = x + 1, \quad \text{or} \quad Y'' + 4Y' + 3Y = x + 1$$

The solution will be $y_p = e^{2x}Y$, which is Kummer's transformation.
The trial solution: solve $0 + 0 + 3Y' = 1$, which is obtained by one differentiation of the equation $Y'' + 4Y' + 3Y = x + 1$ then replace Y'' and Y' by zero. The trial solution: $Y = p(x) = x/3 + d_0$. Substitute Y into the original equation $Y'' + 4Y' + 3Y = x + 1$ to determine d_0:

$$(x/3 + d_0)'' + 4(x/3 + d_0)' + 3(x/3 + d_0) = x + 1$$

$$(0) + 4(1/3) + 3(x/3 + d_0) = x + 1$$

Then $4/3 + 3d_0 = 1$ and $d_0 = -1/9$, giving $Y = x/3 - 1/9$.
Answer check:
$Y'' + 4Y' + 3Y = (x/3 - 1/9)'' + 4(x/3 - 1/9)' + 3(x/3 - 1/9) = 0 + 4/3 + x - 1/3 = x + 1$.
Final answer: $y = e^{2x}Y = (x/3 - 1/9)e^{2x}$.

20. $y'' - y = (x-1)e^{-2x}$

21. $y'' - y' = (x+3)e^{2x}$

Solution: The polynomial times exponent method applies. Replace D by $D+2$ in the DE and cancel e^{2x} on the RHS:

$$(D+2)^2Y - (D+2)Y = x+3, \quad \text{or} \quad Y'' + 3Y' + 2Y = x+3$$

The solution will be $y_p = e^{2x}Y$, which is Kummer's transformation.
The trial solution: solve $0 + 0 + 2Y' = 1$, which is obtained by one differentiation of the equation $Y'' + 3Y' + 2Y = x + 3$ then replace Y''' and Y'' by zero. The trial solution is $Y = x/2 + d_0$. Substitute Y into the original equation $Y'' + 3Y' + 2Y = x + 3$ to determine d_0:

$$(x/2 + d_0)'' + 3(x/2 + d_0)' + 2(x/2 + d_0) = x + 3$$

$$(0) + 3(1/2) + 2(x/2 + d_0) = x + 3$$

Then $3/2 + 2d_0 = 3$ and $d_0 = 3/4$, giving $Y = x/2 + 3/4$.
Answer check:
$Y'' + 3Y' + 2Y = (x/2+3/4)'' + 3(x/2+3/4)' + 2(x/2+3/4) = (0) + (3/2) + (x + 3/2) = x + 3$.
Final answer: $y = e^{2x}Y = (x/2 + 3/4)e^{2x}$.

22. $y'' - y' = (x-2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2+3)e^{3x}$

Solution: The polynomial times exponent method applies. Final answer: $y = \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{13}{4}\right)e^{3x}$.

24. $y'' - 3y' + 2y = (x^2-2)e^{-3x}$

Sine and Cosine Solutions

Determine a solution y_p for the given differential equation. Apply Theorem 7.10, page 581, and model the solution after Examples 7.10 and 7.11.

25. $y'' = \sin(x)$

Solution: The polynomial times exponential times sine method applies. The root: $z = 0 + i$. The reduced equation: $(D+z)^2Y = 1$. Expand: $(D^2 + 2zD + z^2 = D^2 + 2iD - 1$. Solve $Y'' + 2iY' - Y = 1$ by the equilibrium method: $Y = -1$. Then $y = e^{0x}\,\mathcal{I}\text{m}(e^{ix}Y) = -\sin x$.
Answer check:
$y'' = (-\sin x)'' = \sin x$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

Solution: The polynomial times exponential times sine method applies. The root: $z = 0 + i$. The reduced equation: $(D+z)^2Y + Y = 1$. Expand: $(D^2 + 2zD + z^2 + 1 = D^2 + 2iD$. Solve $Y'' + 2iY' = 1$ by the equilibrium method: $2iY = x$ ($d_0 = 0$ to simplify). Then $y = e^{0x}\,\mathcal{I}\mathrm{m}(e^{ix}\,Y) = \mathcal{I}\mathrm{m}(e^{ix}\,x/(2i)) = -\frac{1}{2}\,x\cos x$.
Answer check:
$y'' = \frac{1}{2}(-x\cos x)'' + \frac{1}{2}(-x/\cos x) = \frac{1}{2}(-\cos x + x\sin x)' + \frac{1}{2}(-x/\cos x) = \frac{1}{2}(\sin x + \sin x + x\cos x) + \frac{1}{2}(-x/\cos x) = \sin x$.

28. $y'' + y = \cos(x)$

Solution: $y_p = \frac{1}{2}x\sin x$. See also Exercise 39.

29. $y'' = (x+1)\sin(x)$

Solution: The polynomial times exponential times sine method applies.
$y = -2\cos(x) - x\sin(x) - \sin(x)$

30. $y'' = (x+1)\cos(x)$

31. $y'' - y = (x+1)e^x\sin(2x)$

Solution: The polynomial times exponential times sine method applies.
The root: $z = 1+2i$. The reduced equation: $(D+z)^2Y - Y = x+1$. Expand: $(D+z)^2 - 1 = D^2 + 2D + (4i)D - 4 + 4i$. Solve $D^2+2D+(4i)D-4+4i = x+1$ by the equilibrium method: $Y''' + 2Y'' + (4i)Y'' - 4Y' + 4iY' = 1$ reduces to $(-4+4i)Y' = 1$ and then $Y = x/(4i-4) + d_0$. Find d_0 by substitution into $Y'' + 2Y' + (4i)Y' + (-4+4i)Y = x+1$. Then $y = e^x\,\mathcal{I}\mathrm{m}(e^{2ix}\,Y) = e^x\,\mathcal{I}\mathrm{m}(e^{2ix}\,(x/(4i-4) + d_0))$.
Final answer: $y = \dfrac{1}{16}(-2x-3)e^x\,\cos(2x) - \dfrac{1}{8}x\,e^x\sin(2x)$
Answer check by `maple` `dsolve`.

```
# Exercise 31
F:=x->(x+1)*exp(x)*sin(2*x);
de:=m*diff(y(x),x,x)+c*diff(y(x),x)+k*y(x)=F(x);
de1:=subs(m=1,c=0,k=-1,de);
p:=dsolve(de1,y(x));
```

32. $y'' - y = (x+1)e^x\cos(2x)$

33. $y'' - y' - y = (x^2+x)e^x\sin(2x)$

Solution: The polynomial times exponential times sine method applies.
The root: $z = 1+2i$. The reduced equation: $(D+z)^2Y - (D+z)Y - Y = x^2 + x$.

Final answer:

$$y = \frac{\left(-1682\,x^2 - 7714\,x - 956\right) e^x \cos\left(2\,x\right)}{24389} - \frac{5\,e^x \sin\left(2\,x\right)}{29}\left(x^2 + \frac{27\,x}{145} - \frac{943}{841}\right)$$

34. $y'' - y' - y = (x^2 + x)e^x \cos(2x)$

Undetermined Coefficients Algorithm

Determine a solution y_p for the given differential equation. Apply the polynomial algorithm, page 576, and model the solution after Example 7.12.

35. $y'' = x + \sin(x)$

Solution: Break the problem into two equations:

$$y_1'' = x, \quad y_2'' = \sin(x)$$

Solve each problem by quadrature, dropping homogeneous terms. Then $y_p = y_1 + y_2 = x^3/6 - \sin(x)$.

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

Solution: Break the problem into two equations:

$$y_1'' + y_1 = x, \quad y_2'' + y_2 = \sin(x)$$

Solve for y_1 by the equilibrium method for polynomials: $y_1 = x$ (or guess the answer). Solve for y_2 by the polynomial exponential sine method: $y_2 = -\frac{1}{2}x\cos x$ by Exercise 27. Then $y_p = y_1 + y_2 = x - \frac{1}{2}x\cos x$.

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

Solution: Break the problem into two equations:

$$y_1'' + y_1 = \sin x, \quad y_2'' + y_2 = \cos(x)$$

Solve for y_1 by the polynomial exponential sine method: $y_1 = -\frac{1}{2}x\cos x$ by Exercise 27. Then $y_p = y_1 + y_2 = x - \frac{1}{2}x\cos x$.
Solve for y_2 by the polynomial exponential cosine method: $y_2 = \frac{1}{2}x\sin x$ by Exercise 28. Then $y_p = y_1 + y_2 = -\frac{1}{2}x\cos x + \frac{1}{2}x\sin x$.

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

Solution: Break the problem into three equations:

$$y_1'' = x, \quad y_2'' = xe^x,' quady_3'' = \sin(x)$$

All three can be solved by quadrature.
Final answer: $y_p = y_1 + y_2 + y_3 = (1/6) * x^3 + x * exp(x) - 2 * exp(x) - sin(x)$

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

Solution: Use identities $\sinh u = \frac{1}{2}e^u - \frac{1}{2}e^{-u}$, $\cos 2\theta = \cos(\theta + \theta) = cos^2\theta - \sin^2\theta$ and $\cos^2\theta + \sin^2\theta = 1$ to write the RHS of the DE as $f(x) = \sinh(x) + \cos^2(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x} + \frac{1}{2}(\cos 2x + 1)$. Then split into four differentia equations:

$$y_1'' - y_1 = \frac{1}{2}e^x, \quad y_2'' - y_2 = -\frac{1}{2}e^{-x}, \quad y_3'' - y_3 = \frac{1}{2}\cos 2x, \quad y_4'' - y_4 = \frac{1}{2}$$

Apply classical undetermined coefficients to find y_1, y_2 and y_3. Guess $y_4 = -\frac{1}{2}$. Then
$y_p = y_1 + y_2 + y_3 + y_4 = \frac{1}{40}(10x + 5)e^{-x} - \frac{1}{10}\cos(2x) - \frac{1}{2} + \frac{1}{40}(10x - 5)e^x$
Answer check: Use `maple dsolve`

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2e^x + xe^x\cos(2x)$

Solution: Break into two equations:

$$y_1'' + y_1' - y_1 = x^2e^x, \quad y_2'' + y_2' - y_2 = xe^x\cos(2x)$$

Alternatively, apply Kummer's transformation $y = ze^x$ to reduce the problem to polynomial type and polynomial cosine type.
The steps to solution are challenging. Final answer:
$y_p = -(1/45)e^x(3x\cos(2x) - 6x\sin(2x) - 45x^2 - 5\cos(2x) + 270x - 720)$

46. $y'' + y' - y = x^2e^{-x} + xe^x\sin(2x)$

Additional Proofs

The exercises below fill in details in the text. The hints are in the proofs in the textbook. No solutions will be given for the odd exercises.

47. **(Theorem 7.8)**
Supply the missing details in the proof of Theorem 7.8 for case 1. In particular, give the details for back-substitution.

48. (Theorem 7.8)
Supply the details in the proof of Theorem 7.8 for case 2. In particular, give the details for back-substitution and explain fully why it is possible to select $y_0 = 0$.

49. (Theorem 7.8)
Supply the details in the proof of Theorem 7.8 for case 3. In particular, explain why back-substitution leaves y_0 and y_1 undetermined, and why it is possible to select $y_0 = y_1 = 0$.

50. (Superposition)
Let Ly denote $ay'' + by' + cy$. Show that solutions of $Lu = f(x)$ and $Lv = g(x)$ add to give $y = u + v$ as a solution of $Ly = f(x) + g(x)$.

51. (Easily Solved Equations)
Let Ly denote $ay'' + by' + cy$. Let $Ly_k = f_k(x)$ for $k = 1, \ldots, n$ and define $y = y_1 + \cdots + y_n$, $f = f_1 + \cdots + f_n$. Show that $Ly = f(x)$.

Chapter 8

Laplace Transform

Contents

8.1 Laplace Method Introduction

Laplace method

Solve the given initial value problem using Laplace's method.

1. $y' = -2$, $y(0) = 0$.

Solution: Answer: $y(t) = -2t$.

$\mathcal{L}(y') = \mathcal{L}(-2)$	Apply $\mathcal{L}$ across the DE
$s\mathcal{L}(y) - y(0) = -2/s$	Laplace derivative rule, forward Laplace table
$\mathcal{L}(y) =)y(0) - 2)/s$	Isolate $\mathcal{L}(y)$ left
$\mathcal{L}(y) = -2/s = \mathcal{L}(-2t)$	Use $y(0) = 0$ and the backward Laplace table
$y = -2t$	Lerch's cancellation law

```
# Exercise 1, Laplace method
with(inttrans):
de:=diff(y(t),t)=-2;
p:=laplace(de,t,s); # Apply L across DE
q:=solve(p,laplace(y(t), t, s)); # Isolate L(y)
y(t)= subs(y(0)=0,invlaplace(q,s,t)); # Solve for y
 # y(t)=-2t
```

2. $y' = 1$, $y(0) = 0$.

3. $y' = -t$, $y(0) = 0$.
Solution: $y(t) = -(1/2)t^2$

4. $y' = t$, $y(0) = 0$.

5. $y' = 1 - t$, $y(0) = 0$.
Solution: $y(t) = t - (1/2)t^2$

6. $y' = 1 + t$, $y(0) = 0$.

7. $y' = 3 - 2t$, $y(0) = 0$.
Solution: $y(t) = -t^2 + 3t$

8. $y' = 3 + 2t$, $y(0) = 0$.

9. $y'' = -2$, $y(0) = y'(0) = 0$.
Solution: $y = -t^2$

$$
\begin{array}{ll}
\mathcal{L}(y'') = \mathcal{L}(-2) & \text{apply } \mathcal{L} \text{ across the DE} \\
s^2\mathcal{L}(y) - y'(0) - y(0)s = -2/s & \text{Derivative theorem, forward Laplace table} \\
\mathcal{L}(y) = \dfrac{y'(0) + y(0)s - 2/s}{s^2} & \text{Isolate } \mathcal{L}(y) \text{ left} \\
\mathcal{L}(y) = -2/s^3 & \text{Insert } y(0) = 0 \text{ and } y'(0) = 0 \\
\mathcal{L}(y) = \mathcal{L}(-t^2) & \text{backward Laplace table} \\
y = -t^2 & \text{Lerch's cancellation law}
\end{array}
$$

```
# Exercise 9, Laplace method
with(inttrans):
de:=diff(y(t),t,t)=-2;
p:=laplace(de,t,s);
q:=solve(p,laplace(y(t), t, s));
y(t)= subs(y(0)=0,D(y)(0)=0,invlaplace(q,s,t));
```

10. $y'' = 1$, $y(0) = y'(0) = 0$.

11. $y'' = 1 - t$, $y(0) = y'(0) = 0$.
Solution: $y(t) = (1/2)t^2 - (1/6)t^3$

12. $y'' = 1 + t$, $y(0) = y'(0) = 0$.

13. $y'' = 3 - 2t$, $y(0) = y'(0) = 0$.
Solution: $y(t) = (3/2)t^2 - (1/3)t^3$

14. $y'' = 3 + 2t$, $y(0) = y'(0) = 0$.

Exponential order

Show that $f(t)$ is of exponential order, by finding a constant $\alpha \geq 0$ in each case such that $\lim_{t\to\infty} \dfrac{f(t)}{e^{\alpha t}} = 0$.

15. $f(t) = 1 + t$

Solution: Let $\alpha > 0$, e.g., $\alpha = 1$. Then $\lim_{t\to\infty} \dfrac{f(t)}{e^{\alpha t}} = 0$.

16. $f(t) = e^t \sin(t)$

17. $f(t) = \sum_{n=0}^{N} c_n t^n$, for any choice of the constants $c_0, \ldots, c_N$.

Solution: Let $\alpha > 0$, e.g., $\alpha = 1$. Then $\lim_{t\to\infty} \dfrac{f(t)}{e^{\alpha t}} = 0$. The limit is zero because an exponential $e^{\alpha t}$ grows faster than any power x^k. The latter is proved in calculus using L'Hôpital's Rule.

18. $f(t) = \sum_{n=1}^{N} c_n \sin(nt)$, for any choice of the constants $c_1, \ldots, c_N$.

Existence of transforms

Let $f(t) = te^{t^2} \sin(e^{t^2})$. Establish these results.

19. The function $f(t)$ is not of exponential order.

Solution: Let α be any real number. Then $f(t)/e^{\alpha t} = te^{t^2-\alpha t}\sin(e^{t^2})$. Define sequence $\{t_n\}$ by the equation $e^{t_n^2} = (4n+1)\pi/2$. Then $\sin(e^{t_n^2}) = \sin(2n\pi + \pi/2) = 1$ by periodicity of the sine function. Fraction $f(t_n)/e^{\alpha t_n}$ then equals $t_n e^{t_n^2 - \alpha t_n}$, which has limit infinity as $n \to \infty$ (t^2 grows faster than αt for any fixed α). Therefore, $f(t)/e^{\alpha t}$ cannot have limit zero at infinity for any value of α. $\blacksquare$

20. The Laplace integral of $f(t)$, $\int_0^\infty f(t)e^{-st}dt$, converges for all $s > 0$.

Jump Magnitude

For f piecewise continuous, define the **jump** at t by

$$J(t) = \lim_{h\to 0+} f(t+h) - \lim_{h\to 0+} f(t-h).$$

Compute $J(t)$ for the following f.

21. $f(t) = 1$ for $t \ge 0$, else $f(t) = 0$

Solution: The left limit at $t = 0$ is one, the right limit at $t = 0$ is zero. Then $J(0) = 1$ and by continuity elsewhere$J(t) = 0$.

22. $f(t) = 1$ for $t \ge 1/2$, else $f(t) = 0$

23. $f(t) = t/|t|$ for $t \ne 0$, $f(0) = 0$

Solution: $J(0) = 1 - (-1) = 2$ and elsewhere $J(t) = 0$.

24. $f(t) = \sin t/|\sin t|$ for $t \ne n\pi$, $f(n\pi) = (-1)^n$

Taylor series

The series relation $\mathcal{L}(\sum_{n=0}^{\infty} c_n t^n) = \sum_{n=0}^{\infty} c_n \mathcal{L}(t^n)$ often holds, in which case the result $\mathcal{L}(t^n) = n! s^{-1-n}$ can be employed to find a series representation of the Laplace transform. Use this idea on the following to find a series formula for $\mathcal{L}(f(t))$.

25. $f(t) = e^{2t} = \sum_{n=0}^{\infty} (2t)^n/n!$

Solution: $\mathcal{L}(f(t)) = \sum_{n=0}^{\infty} \dfrac{2^n}{n!}\mathcal{L}(t^n) = \sum_{n=0}^{\infty} \dfrac{2^n}{n!}\dfrac{n!}{s^{n+1}} = \sum_{n=0}^{\infty} \dfrac{2^n}{s^{n+1}}$

26. $f(t) = e^{-t} = \sum_{n=0}^{\infty} (-t)^n/n!$

Transfer of Radiance

The differential equation $\frac{d}{dr}N + \alpha N = N^*$ models laser beam radiance (absorption and scattering out of the beam) in a medium like water, where r is the distance from the source.

27. Solve $\frac{d}{dr}N + 2N = 1, N(0) = 20$ by Laplace's method.
Ans: $N(r) = \frac{1}{2} + \frac{39}{2}\,\mathrm{e}^{-2\,r}$.
Hint: Obtain $\mathcal{L}(N(t)) = \frac{1+20\,s}{s(s+2)} = \frac{1}{2s} + \frac{39}{2(s+2)}$ using $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ from the Forward Table page 601.

Solution: Let $y = N(t)$ and use $y' + 2y = 1$, $y(0) = 20$ with Laplace's method as follows.

$$\begin{aligned}
&\mathcal{L}(y' + 2y) = \mathcal{L}(1) && \text{Apply } \mathcal{L} \text{ across the DE.}\\
&s\mathcal{L}(y) - y(0) + 2\mathcal{L}(y) = \frac{1}{s} && \text{Derivative rule, forward table.}\\
&\mathcal{L}(y) = \frac{20 + \frac{1}{s}}{s+2} && \text{Isolate } \mathcal{L}(y) \text{ left, use } y(0) = 20.\\
&\mathcal{L}(y) = \frac{a}{s+2} + \frac{b}{s} && \text{Partial fractions, } a, b \text{ found later.}\\
&\mathcal{L}(y) = \mathcal{L}(ae^{-2t} + b) && \text{Backward table.}\\
&y = ae^{-2t} + b && \text{Lerch's cancellation law.}\\
&y = ae^{-2t} + b && \text{Found } a =
\end{aligned}$$

28. Solve $\frac{d}{dr}N + 2N = 1 - e^{-r}, N(0) = 25$ by any method.
Ans: $N(r) = \frac{1}{2} - \mathrm{e}^{-r} + \frac{51}{2}\,\mathrm{e}^{-2\,r}$.
Hint: A particular solution is $N_p = \frac{1}{2} - e^{-r}$. Superposition applies. See also Example 8.11 page 609.

Piecewise-Defined Functions

29. Define a piecewise continuous function $f(t)$ on $[-1, 1]$ that agrees with $\frac{\sin(t)}{|t|}$ except at $t = 0$. Suggestion: use Taylor expansion $\sin(t) = t - t^3/6 + \cdots$ to define continuous functions f_1, f_2 on $-\infty < t < \infty$.

Solution: Let $f_1 = -f_2$ and $f_2(t) = 1 - t^2/6 + t^4/5! - \cdots = \frac{1}{t}\sum_{n=0}^{\infty}(-1)^n t^{2n+1}/(2n+1)! = \sum_{n=0}^{\infty}(-1)^n t^{2n}/(2n+1)!$. Power series are infinitely differentiable, therefore continuous. Define

$$f(t) = \begin{cases} f_1(t) & t < 0, \\ 1 & t = 0, \\ f_2(t) & t > 0. \end{cases}$$

Then $f(t) = \dfrac{\sin t}{|t|}$ except at $t = 0$ where the fraction is undefined.

30. Explain in detail why $1/t$ is not piecewise continuous on $[-1, 1]$. ■

31. Find $\mathcal{L}(f(t))$, given
$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ 0 & \text{otherwise.} \end{cases}$

Solution: A basic solution:
$\mathcal{L}(f(t)) = \int_0^{\infty} f(t)e^{-st}dt = \int_1^2 (1)e^{-st}dt = \dfrac{e^{-s}}{s} - \dfrac{e^{-2s}}{s}$.

A second solution:

Define $\mathbf{pulse}(t,a,b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise.} \end{cases}$ as in Exercise 32, *infra*. Then $f(t) = \mathbf{pulse}(t,1,2)$. Because $\mathbf{pulse}(t,a,b) = \mathbf{u}(t-a) - \mathbf{u}(t-b)$ and $\mathcal{L}(\mathbf{u}(t-c)) = \dfrac{e^{-cs}}{s}$ then $\mathcal{L}(f(t)) = \mathcal{L}(\mathbf{u}(t-1)) - \mathcal{L}(\mathbf{u}(t-2)) = \dfrac{e^{-s}}{s} - \dfrac{e^{-2s}}{s}$.

32. Find $\mathcal{L}(\mathbf{pulse}(t,a,b))$, given
$\mathbf{pulse}(t,a,b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise.} \end{cases}$

33. Define

$$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ 2 & 3 \le t < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Find the weights c_1, c_2 such that
$f(t) = c_1\,\mathbf{pulse}(t,1,2) +$
$c_2\,\mathbf{pulse}(t,3,4)$.

Solution: $c_1 = 1$, $c_2 = 2$

$$\begin{aligned} f(t) &= \begin{cases} 1 & 1 \le t < 2, \\ 2 & 3 \le t < 4, \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1 & 1 \le t < 2, \\ 0 & \text{otherwise.} \end{cases} + 2 \begin{cases} 1 & 3 \le t < 4, \\ 0 & \text{otherwise.} \end{cases} \\ &= \mathbf{pulse}(t,1,2) + 2\,\mathbf{pulse}(t,3,4) \end{aligned}$$

34. Let
$f(t) = \cos(t)\,\mathbf{pulse}(t,0,\pi) +$
$(\sin(t)-1)\,\mathbf{pulse}(t,\pi,2\pi)$
Write f as a piecewise-defined function and graph it.

Piecewise Continuous Definition

Let $g(t)$ be zero for $t < 0$ and have on $t \ge 0$ at most finitely many points of discontinuity, at which finite right and left hand limits exist.

This definition is an alternative way to define *piecewise continuous*, crafted for Laplace theory.

35. Let t_1, t_2 be consecutive points of discontinuity of g. Define a function $g_1(t)$ continuous on $-\infty < t < \infty$ such that $g(t) = g_1(t)$ on $t_1 \le t \le t_2$.

The whole real line is the required domain of g_1, which must be defined using g itself and right and left hand limit values of g.

Solution: The plan: define $g_1 = g$ on $t_1 \le t \le t_2$ with the endpoint definition taken to mean the appropriate left or right limit at the point. Then extend

g_1 to the whole real line as a constant on $t < t_1$ and also on $t > t_2$.

$$g_1(t) = \begin{cases} g(t_1 + 0) & t \le t_1, \\ g(t) & t_1 < t < t_2, \\ g(t_2 - 0) & t \ge t_2. \end{cases}$$

Then g_1 is continuous except possibly at $t = t_1$, $t = t_2$. Compute the left and right limits at these two points:

$(g_1(t_1 - 0) = g(t_1 + 0) = g_1(t_1 + 0)$
$(g_1(t_2 - 0) = g(t_2 - 0) = g_1(t_2 + 0)$
Because right and left limits match at $t = t_1, t_2$ then g_1 is continuous. ■

36. Let t_1, t_2, t_3 be consecutive points of discontinuity of g. Invent functions $g_1(t)$, $g_2(t)$ continuous on $-\infty < t < \infty$ such that $g(t) = g_1(t)$ on $t_1 \le t \le t_2$ and $g(t) = g_2(t)$ on $t_2 \le t \le t_3$.

37. Define g_1, g_2 as in Exercise 36 above. Compute the **jump** at $t = t_2$, $J(t_2) = g(t_2 + 0) - g(t_2 - 0)$, in terms of g_1, g_2.

Solution:

$g(t_2 + 0) = g_2(t_2 + 0) = g_2(t_2)$
$g(t_2 - 0) = g_1(t_2 - 0) = g_1(t_2)$
Then: $J(t_2) = g_2(t_2) - g_1(t_2)$

38. Using the preceding steps, prove that g is piecewise continuous according to the definition given in the text.

8.2 Laplace Integral Table

Laplace Transform Forward Table

Using the basic Laplace table and linearity properties of the transform, compute $\mathcal{L}(f(t))$. Do not use the direct Laplace transform!

1. $\mathcal{L}(2t)$

Solution: $\mathcal{L}(2t) = 2\mathcal{L}(t) = 2\,\dfrac{1}{s^2}$

2. $\mathcal{L}(4t)$

3. $\mathcal{L}(1 + 2t + t^2)$

Solution: $\mathcal{L}(1 + 2t + t^2) = \mathcal{L}(1) + \mathcal{L}(2t) + \mathcal{L}(t^2) = \dfrac{1}{s} + 2\,\dfrac{1}{s^2} + \dfrac{2}{s^3}$

4. $\mathcal{L}(t^2 - 3t + 10)$

5. $\mathcal{L}(\sin 2t)$

Solution: $\mathcal{L}(\sin 2t) = \left.\dfrac{b}{s^2 + b^2}\right|_{b=2} = \dfrac{2}{s^2 + 4}$

6. $\mathcal{L}(\cos 2t)$

7. $\mathcal{L}(e^{2t})$

Solution: $\mathcal{L}(e^{2t}) = \left.\dfrac{1}{s - a}\right|_{a=2} = \dfrac{1}{s - 2}$

8. $\mathcal{L}(e^{-2t})$

9. $\mathcal{L}(t + \sin 2t)$

Solution: $\mathcal{L}(t + \sin 2t) = \mathcal{L}(t) + \mathcal{L}(\sin 2t) = \dfrac{1}{s^2} + \dfrac{2}{s^2 + 4}$

10. $\mathcal{L}(t - \cos 2t)$

11. $\mathcal{L}(t + e^{2t})$

Solution: $\mathcal{L}(t + e^{2t}) = \mathcal{L}(t)\mathcal{L}(+e^{2t}) = \dfrac{1}{s^2} + \dfrac{1}{s - 2}$

12. $\mathcal{L}(t - 3e^{-2t})$

13. $\mathcal{L}((t+1)^2)$

Solution: $\mathcal{L}((t+1)^2) = \mathcal{L}(t^2+2t+1) = \mathcal{L}(t^2)+\mathcal{L}(2t)+\mathcal{L}(1) = \dfrac{2}{s^3}+\dfrac{2}{s^2}+\dfrac{1}{s}$

14. $\mathcal{L}((t+2)^2)$

15. $\mathcal{L}(t(t+1))$

Solution: $\mathcal{L}(t(t+1)) = \mathcal{L}(t^2+t) = \mathcal{L}(t^2)+\mathcal{L}(t) = \dfrac{2}{s^3}+\dfrac{1}{s^2}$

16. $\mathcal{L}((t+1)(t+2))$

17. $\mathcal{L}(\sum_{n=0}^{10} t^n/n!)$

Solution:

$$\begin{aligned}\mathcal{L}(\textstyle\sum_{n=0}^{10} t^n/n!) &= \textstyle\sum_{n=0}^{10} \mathcal{L}(t^n/n!)\\ &= \textstyle\sum_{n=0}^{10} \dfrac{n}{s^{n+1}(n!)}\\ &= \textstyle\sum_{n=0}^{10} \dfrac{1}{s^{n+1}((n-1)!)}\end{aligned}$$

18. $\mathcal{L}(\sum_{n=0}^{10} t^{n+1}/n!)$

19. $\mathcal{L}(\sum_{n=1}^{10} \sin nt)$

Solution:

$\mathcal{L}(\sum_{n=1}^{10} \sin nt) = \sum_{n=1}^{10} \mathcal{L}(\sin nt) = \sum_{n=1}^{10} \dfrac{n}{s^2+n^2}$

20. $\mathcal{L}(\sum_{n=0}^{10} \cos nt)$

Laplace Backward Table

Solve the given equation for the function $f(t)$. Use the basic table and linearity properties of the Laplace transform.

21. $\mathcal{L}(f(t)) = s^{-2}$

Solution: $\mathcal{L}(f(t)) = s^{-2} = \mathcal{L}(t)$ by the backward table, then $f(t) = t$ by Lerch's cancellation law.

22. $\mathcal{L}(f(t)) = 4s^{-2}$

23. $\mathcal{L}(f(t)) = 1/s + 2/s^2 + 3/s^3$

Solution: $\mathcal{L}(f(t)) = 1/s+2/s^2+3/s^3 = \mathcal{L}(1)+2\mathcal{L}(t)+\frac{3}{2}\dfrac{2}{s^3} = \mathcal{L}(1+2t+\frac{3}{2}t^2)$ by the backward table, then $f(t) = 1+2t+\frac{3}{2}t^2$ by Lerch's cancellation law.

```
# Exercise 23, Laplace backward table
with(inttrans):
invlaplace(1/s+2/s^2+3/s^3,s,t);
```

24. $\mathcal{L}(f(t)) = 1/s^3 + 1/s$

25. $\mathcal{L}(f(t)) = 2/(s^2 + 4)$

Solution: $\mathcal{L}(f(t)) = 2/(s^2 + 4) = \left.\dfrac{b}{s^2 + b^2}\right|_{b=2} = \mathcal{L}(\sin(bt)) = \mathcal{L}(\sin(2t))$ by the backward table, then $f(t) = \sin(2t)$ by Lerch's cancellation law.)

26. $\mathcal{L}(f(t)) = s/(s^2 + 4)$

27. $\mathcal{L}(f(t)) = 1/(s - 3)$

Solution: $\mathcal{L}(f(t)) = 1/(s - 3) = \left.\dfrac{1}{s - a}\right|_{a=3} = \mathcal{L}(e^{3t})$ by the backward table, then $f(t) = e^{3t}$ by Lerch's cancellation law.

28. $\mathcal{L}(f(t)) = 1/(s + 3)$

29. $\mathcal{L}(f(t)) = 1/s + s/(s^2 + 4)$

Solution: $\mathcal{L}(f(t)) = 1/s+s/(s^2+4) = \mathcal{L}(1)+\mathcal{L}(\cos(bt))|_{b=2} = \mathcal{L}(1+\cos(2t))$ by the backward table, then $f(t) = 1 + \cos(2t)$ by Lerch's cancellation law.

30. $\mathcal{L}(f(t)) = 2/s - 2/(s^2 + 4)$

31. $\mathcal{L}(f(t)) = 1/s + 1/(s - 3)$

Solution: $\mathcal{L}(f(t)) = 1/s + 1/(s - 3) = \dfrac{1}{s} + \left.\dfrac{1}{s - a}\right|_{a=3} = \mathcal{L}(1) + \mathcal{L}(e^{3t}) = \mathcal{L}(1+e^{3t})$ by the backward table, then $f(t) = 1+e^{3t}$ by Lerch's cancellation law.

32. $\mathcal{L}(f(t)) = 1/s - 3/(s - 2)$

33. $\mathcal{L}(f(t)) = (2 + s)^2/s^3$

Solution: $\mathcal{L}(f(t)) = (2+s)^2/s^3 = \dfrac{4 + 4s + s^2}{s^3} = 2\dfrac{2}{s^3} + 4\dfrac{1}{s^2} + \dfrac{1}{s} = 2\mathcal{L}(t^2) + 4\mathcal{L}(t)+\mathcal{L}(1) = \mathcal{L}(2t^2+4t+1)$ by the backward table, then $f(t) = 2t^2+4t+1$ by Lerch's cancellation law.

34. $\mathcal{L}(f(t)) = (s + 1)/s^2$

35. $\mathcal{L}(f(t)) = s(1/s^2 + 2/s^3)$

Solution: $\mathcal{L}(f(t)) = s(1/s^2+2/s^3) = \dfrac{1}{s} + \dfrac{2}{s^2} = \mathcal{L}(1) + 2\mathcal{L}(t) = \mathcal{L}(1+2t)$ by the backward table, then $f(t) = 1 + 2t$ by Lerch's cancellation law.

36. $\mathcal{L}(f(t)) = (s+1)(s-1)/s^3$

37. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{1+n}$

Solution: $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{1+n} = \sum_{n=0}^{10} \mathcal{L}(t^n) = \mathcal{L}\left(\sum_{n=0}^{10} t^n\right)$ by the backward table, then $f(t) = \sum_{n=0}^{10} t^n$ by Lerch's cancellation law.

38. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{2+n}$

39. $\mathcal{L}(f(t)) = \sum_{n=1}^{10} \dfrac{n}{s^2+n^2}$

Solution: $\mathcal{L}(f(t)) = \sum_{n=0}^{10} \dfrac{s}{s^2+n^2} = \sum_{n=0}^{10} \mathcal{L}(\sin(nt)) = \mathcal{L}\left(\sum_{n=0}^{10} \sin(nt)\right)$ by the backward table, then $f(t) = \sum_{n=0}^{10} \sin(nt)$ by Lerch's cancellation law.

40. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} \dfrac{s}{s^2+n^2}$

Laplace Table Extension

Compute the indicated Laplace integral using the extended Laplace table, page 602.

41. $\mathcal{L}(u(t-2) + 2u(t))$

Solution: $\mathcal{L}(u(t-2) + 2u(t)) = \mathcal{L}(u(t-2)) + 2\mathcal{L}(u(t)) = \left.\dfrac{e^{-as}}{s}\right|_{a=2} + 2\left.\dfrac{e^{-as}}{s}\right|_{a=0} = \dfrac{e^{-2s}}{s} + \dfrac{2}{s}$

42. $\mathcal{L}(u(t-3) + 4u(t))$

43. $\mathcal{L}(u(t-\pi)(u(t) + u(t-1)))$

Solution: $\mathcal{L}(u(t-\pi)(u(t)+u(t-1))) = \mathcal{L}(u(t-\pi)u(t) + u(t-\pi)u(t-1)) = \mathcal{L}(u(t-\pi) + u(t-\pi)) = 2\mathcal{L}(u(t-\pi))) = 2\left.\dfrac{e^{-as}}{s}\right|_{a=\pi} = \dfrac{2e^{-\pi s}}{s}$

```
# Exercise 43, Laplace table extension
with(inttrans):u:=Heaviside:
laplace( u(t-Pi)*(u(t)+u(t-1) ),t,s);
```

44. $\mathcal{L}(u(t-2\pi)+3u(t-1)u(t-2))$

45. $\mathcal{L}(\delta(t-2))$

Solution: $\mathcal{L}(\delta(t-2)) = e^{-as}|_{a=2} = e^{-2s}$

```
# Exercise 45, Laplace table extension
with(inttrans):
laplace( Dirac(t-2),t,s );
```

46. $\mathcal{L}(5\delta(t-\pi))$

47. $\mathcal{L}(\delta(t-1)+2\delta(t-2))$

Solution: $\mathcal{L}(\delta(t-1)+2\delta(t-2)) = \mathcal{L}(\delta(t-1))+2\mathcal{L}(\delta(t-2)) = e^{-as}|_{a=1} + 2\,e^{-as}|_{a=2} = e^{-s}+2e^{-2s}$

48. $\mathcal{L}(\delta(t-2)(5+u(t-1)))$

49. $\mathcal{L}(\textbf{floor}(3t))$

Solution: $\mathcal{L}(\textbf{floor}(3t)) = \left.\dfrac{e^{-as}}{s(1-e^{-as})}\right|_{a=1/3} = \dfrac{e^{-s/3}}{s(1-e^{-s/3})} = \dfrac{1}{s(e^{s/3}-1)}$

```
# Exercise 49, Laplace table extension
with(inttrans):
laplace( floor(3*t),t,s );
```

50. $\mathcal{L}(\textbf{floor}(2t))$

51. $\mathcal{L}(5\,\textbf{sqw}(3t))$

Solution: $\mathcal{L}(5\,\textbf{sqw}(3t)) = \mathcal{L}(5\,\textbf{sqw}(t/a))|_{a=1/3} = 5\left.\dfrac{\tanh(as/2)}{s}\right|_{a=1/3} = 5\dfrac{\tanh(s/6)}{s}$

`maple` does not have a laplace table entry for the square wave in 2022.

52. $\mathcal{L}(3\,\textbf{sqw}(t/4))$

53. $\mathcal{L}(4\,\textbf{trw}(2t))$

Solution: $\mathcal{L}(4\,\textbf{trw}(2t)) = 8\mathcal{L}(\frac{1}{2}\,\textbf{trw}(t/(1/2))) = 8\;\mathcal{L}(a\,\textbf{trw}(t/a))|_{a=1/2} = 8\dfrac{\tanh(s/4)}{s^2}$

54. $\mathcal{L}(5\,\textbf{trw}(t/2))$

55. $\mathcal{L}(t + t^{-3/2} + t^{-1/2})$

Solution:

$$\begin{aligned}
\mathcal{L}(t + t^{-3/2} + t^{-1/2}) &= \mathcal{L}(t) + \mathcal{L}(t^\alpha)\big|_{\alpha=-3/2} + \mathcal{L}(t^\alpha)\big|_{\alpha=-1/2} \\
&= \mathcal{L}(t) + \left.\frac{\Gamma(1+\alpha)}{s^{1+\alpha}}\right|_{\alpha=-3/2} + \left.\frac{\Gamma(1+\alpha)}{s^{1+\alpha}}\right|_{\alpha=-1/2} \\
&= \frac{1}{s^2} + \frac{\Gamma(-1/2)}{s^{-1/2}} + \frac{\Gamma(1/2)}{s^{1/2}} \\
&= \frac{1}{s^2} + \frac{\Gamma(-1/2)}{s^{-1/2}} + \frac{\sqrt{\pi}}{\sqrt{s}}
\end{aligned}$$

56. $\mathcal{L}(t^3 + t^{-3/2} + 2t^{-1/2})$

Inverse Laplace, Extended Table

Solve the given equation for $f(t)$, using the extended Laplace integral table.

57. $\mathcal{L}(f(t)) = e^{-s}/s$

Solution: $\mathcal{L}(f(t)) = e^{-s}/s = \left.\frac{e^{-as}}{s}\right|_{a=1} = \mathcal{L}(\delta(t-a))\big|_{a=1} = \mathcal{L}(\delta(t-1))$ by the extended Laplace table.
Then $f(t) = \delta(t-1)$ by Lerch's cancellation law.

58. $\mathcal{L}(f(t)) = 5e^{-2s}/s$

59. $\mathcal{L}(f(t)) = e^{-2s}$

Solution: $\mathcal{L}(f(t)) = e^{-2s} = e^{-as}\big|_{a=2} = \mathcal{L}(\delta(t-a))\big|_{a=2} = \mathcal{L}(\delta(t-2))$ by the extended Laplace table.
Then $f(t) = \delta(t-2)$ by Lerch's cancellation law.

60. $\mathcal{L}(f(t)) = 5e^{-3s}$

61. $\mathcal{L}(f(t)) = \dfrac{e^{-s/3}}{s(1-e^{-s/3})}$

Solution:

$\mathcal{L}(f(t)) = \dfrac{e^{-s/3}}{s(1-e^{-s/3})} = \left.\dfrac{e^{-as}}{s(1-e^{-as})}\right|_{a=1/3} = \mathcal{L}(\mathbf{floor}(t/a)\big|_{a=1/3} = \mathcal{L}(\mathbf{floor}(3t))$ by the extended Laplace table.
Then $f(t) = \mathbf{floor}(3t)$ by Lerch's cancellation law.

62. $\mathcal{L}(f(t)) = \dfrac{e{-}2s}{s(1-e^{-2s})}$

63. $\mathcal{L}(f(t)) = \dfrac{4\tanh(s)}{s}$

Solution:
$\mathcal{L}(f(t)) = \dfrac{4\tanh(s)}{s} = 4\left.\dfrac{\tanh(as/2)}{s}\right|_{a=2} = 4\,\mathcal{L}(\mathbf{sqw}(t/a))\big|_{a=2} = \mathcal{L}(4\,\mathbf{sqw}(t/2))$ by the extended Laplace table.
Then $f(t) = 4\,\mathbf{sqw}(t/2)$ by Lerch's cancellation law.

64. $\mathcal{L}(f(t)) = \dfrac{5\tanh(3s)}{2s}$

65. $\mathcal{L}(f(t)) = \dfrac{4\tanh(s)}{3s^2}$

Solution: $f(t) = 4a\,\mathbf{trw}(t/a)$ where $a/2 = 1$ by the extended Laplace table.
Final answer: $f(t) = 8\,\mathbf{trw}(t/2)$.

66. $\mathcal{L}(f(t)) = \dfrac{5\tanh(2s)}{11s^2}$

67. $\mathcal{L}(f(t)) = \dfrac{1}{\sqrt{s}}$

Solution: $f(t) = \dfrac{1}{\sqrt{\pi s}}$ by the extended Laplace table entry $\mathcal{L}(t^{-1/2}) = \sqrt{\pi/s}$.

68. $\mathcal{L}(f(t)) = \dfrac{1}{\sqrt{s^3}}$

8.3 Laplace Transform Rules

First Order Linear DE

Display the Laplace method details which verify the supplied answer.

The first two exercises use forward and backward Laplace tables plus the first shifting theorems. The others require a calculus background in partial fractions.

1. $x' + x = e^{-t}$, $x(0) = 1$;
$x(t) = (1+t)e^{-t}$.

Solution:

Transform and isolate $\mathcal{L}(x)$:

$\mathcal{L}(x' + x) = \mathcal{L}(e^{-t})$	Apply $\mathcal{L}$ across the DE.
$s\mathcal{L}(x) - x(0) + \mathcal{L}(x) = \dfrac{1}{s+1}$	Derivative rule, forward table.
$\mathcal{L}(x) = \dfrac{x(0) + \dfrac{1}{s+1}}{s+1}$	Isolate $\mathcal{L}(x)$ left.
$\mathcal{L}(x) = \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2}$	Use $x(0) = 1$ and expand in partial fractions.
$\mathcal{L}(x) = \mathcal{L}(e^{-t}) + \mathcal{L}(t)\big\rvert_{s\to s+1}$	Backward Laplace table.
$\mathcal{L}(x) = \mathcal{L}(e^{-t}) + \mathcal{L}(te^{-t})$	Shift theorem.
$x(t) = e^{-t} + te^{-t}$	Linearity, Lerch's theorem.

```
f:=proc(de1) local q;
q:=subs(laplace(x(t),t,s)=F,laplace(de1,t,s));
collect(q,F);# Collect on F=Laplace of x(t)
end proc:
#
fx:=proc(de,label)global qx,qxx;
qx:=f(de); qxx:=solve(qx,F);# Isolate F=laplace(x(t))
printf("%a: %a\n%a\nF=%a\n",label,convert(de,D),qx,qxx);
dsolve([de,ic],x(t));
end proc:
#
# Exercise 1, First Order Linear DE
ic:=x(0)=1;de1:=diff(x(t),t)+x(t)=exp(-t):
fx(de1,DE1);
# DE1: D(x)(t)+x(t) = exp(-t)
# (s+1)*F-x(0) = 1/(s+1)
# F=(x(0)*s+x(0)+1)/(s+1)^2
# x(t) = (t + 1) exp(-t)
```

2. $x' + 2x = -e^{-2t}$, $x(0) = 1$;
$x(t) = (1-t)e^{-2t}$.

3. $x' + x = 1$, $x(0) = 1$; $x(t) = 1$.

Solution: $\mathcal{L}(x(t)) = (x(0)s+1)/(s(s+1)) = \dfrac{1}{s} = \mathcal{L}(1)$, then $x(t) = 1$ by Lerch's theorem.

4. $x' + 4x = 4$, $x(0) = 1$; $x(t) = 1$.

5. $x' + x = t$, $x(0) = -1$; $x(t) = t - 1$.

Solution: $\mathcal{L}(x(t)) = -(s-1)/s^2 = \dfrac{-1}{s} + \dfrac{1}{s^2} = \mathcal{L}(-1+t)$. Then $x(t) = -1+t$ by Lerch's theorem.

6. $x' + x = t$, $x(0) = 1$;
$x(t) = t - 1 + 2e^{-t}$.

Second Order Linear DE

Display the Laplace method details which verify the supplied answer.
The first 4 exercises require only forward and backward Laplace tables and the first shifting theorems. The others require methods in partial fractions beyond a calculus background.

7. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 1$; $x(t) = \cos t + \sin t$.

Solution:
Transform and isolate $\mathcal{L}(x)$:

$\mathcal{L}(x'' + x) = 0$	Apply $\mathcal{L}$ across the DE.
$s\mathcal{L}(x') - x'(0) + \mathcal{L}(x) = 0$	Derivative rule on x'.
$s(s\mathcal{L}(x) - x(0)) - x'(0) + \mathcal{L}(x) = 0$	Derivative rule on x.
$\mathcal{L}(x) = \dfrac{x(0) + \dfrac{1}{s+1}}{s+1}$	Isolate $\mathcal{L}(x)$ left.
$\mathcal{L}(x) = (x(0)s + x'(0))/(s^2+1)$	Expand and simplify.
$\mathcal{L}(x) = s/(s^2+1) + 1/(s^2+1)$	Use $x(0) = x'(0) = 1$ and expand in partial fractions.
$\mathcal{L}(x) = \mathcal{L}(\cos t) + \mathcal{L}(\sin t)$	Backward Laplace table.
$x(t) = \cos t + \sin t$	Linearity, Lerch's theorem.

```
# Exercise 7, Second Order Linear DE
de7:=diff(x(t),t,t)+x(t)=0;ic:=x(0)=1,D(x)(0)=1;
fx(de7,DE7);# See Exercise 1 for the code
# F=(x(0)*s+D(x)(0))/(s^2+1)
# x(t) = sin(t)+cos(t)
```

8. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 2$; $x(t) = \cos t + 2\sin t$.

9. $x'' + 2x' + x = 0$, $x(0) = 0$, $x'(0) = 1$; $x(t) = te^{-t}$.

Solution: $\mathcal{L}(x(t)) = (x(0)s + x'(0) + 2x(0))/(s^2 + 2s + 1) = 1/(s+1)^2 = \mathcal{L}(te^{-t})$ by the backward table and the first shifting theorem. Then $x(t) = te^{-t}$ by Lerch's theorem.

10. $x'' + 2x' + x = 0$, $x(0) = 1$, $x'(0) = -1$; $x(t) = e^{-t}$.

11. $x'' + 3x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$; $x(t) = e^{-t}$.

Solution: $\mathcal{L}(x(t)) = (x(0)s + x'(0) + 3x(0))/(s^2 + 3s + 2) = (s - 1 + 3)/((s+1)(s+2)) = \dfrac{1}{s+1} = \mathcal{L}(e^{-t})$ by the backward table. Then $x(t) = e^{-t}$ by Lerch's theorem.

12. $x'' + 3x' + 2x = 0$, $x(0) = 1$, $x'(0) = -2$; $x(t) = e^{-2t}$.

13. $x'' + 3x' = 0$, $x(0) = 5$, $x'(0) = 0$; $x(t) = 5$.

Solution: $\mathcal{L}(x(t)) = (x(0)s + D(x)(0) + 3x(0))/(s(s+3)) = (5s+15)/(s(s+3)) = 5/s = \mathcal{L}(5)$. Then $x(t) = 5$ by Lerch's theorem.

14. $x'' + 3x' = 0$, $x(0) = 1$, $x'(0) = -3$; $x(t) = e^{-3t}$.

15. $x'' + x = 1$, $x(0) = 1$, $x'(0) = 0$; $x(t) = 1$.

Solution: $\mathcal{L}(x(t)) = (x(0)s^2 + x'(0)s + 1)/(s(s^2+1)) = (s^2+1)/(s(s^2+1)) = 1/s = \mathcal{L}(1)$. Then $x(t) = 1$ by Lerch's theorem.

16. $x'' = 2$, $x(0) = 0$, $x'(0) = 0$; $x(t) = t^2$.

Forward Integral Rule

The rule is $\mathcal{L}\left(\int_0^t g(r)dr\right) = \frac{1}{s}\mathcal{L}(g(t))$

17. Relate this rule to the convolution rule with $f(t) = 1$.

Solution: The integral $\int_0^t g(r)dr$ is the convolution of 1 and g. Therefore, $\mathcal{L}(\int_0^t g(r)dr) = \mathcal{L}(1)\mathcal{L}(g) = \frac{1}{s}\mathcal{L}(g)$ by the convolution rule.

18. Compute $\mathcal{L}\left(\int_0^t \sin(r)dr\right)$.

19. Compute $\mathcal{L}\left(\int_0^t (r+1)^3\, dr\right)$.

Solution: Answer: $(s^3 + 3 * s^2 + 6 * s + 6)/s^5$.

Details: Let $g(t) = (t+1)^3$. Then

$$\begin{aligned}
&\mathcal{L}(\textstyle\int_0^t g(r)dr) = \mathcal{L}(1)\mathcal{L}(g) = \dfrac{1}{s}\mathcal{L}(g) \\
&= \dfrac{1}{s}\mathcal{L}(t^3 + 3t^2 + 3t + 1) \\
&= \dfrac{1}{s}\left(\dfrac{6}{s^4} + 3\dfrac{2}{s^3} + 3\dfrac{1}{s^2} + \dfrac{1}{s}\right) \\
&= \dfrac{6}{s^5} + \dfrac{6}{s^4} + \dfrac{3}{s^3} + \dfrac{1}{s^2}
\end{aligned}$$

20. Compute $\mathcal{L}\left(\int_0^t \textsf{sqw}(r)dr\right)$, where **sqw** is the square wave of period 2. Use the Extended Laplace Table.

Backward Integral Rule

Apply rule $\frac{1}{s}\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t g(r)dr\right)$
and Lerch's theorem to solve for $f(t)$.

21. $\mathcal{L}(f(t)) = \frac{1}{s(s^2+1)}$

Solution: $\mathcal{L}(f(t)) = \frac{1}{s}\frac{1}{s^2+1} = \frac{1}{s}\mathcal{L}(\sin t) = \mathcal{L}\left(\int_0^t \sin(r)dr\right) = \mathcal{L}(-\cos(t)+1)$. Then $f(t) = 1 - \cos(t)$ by Lerch's theorem.

22. $\mathcal{L}(f(t)) = \frac{1}{s}\frac{s+1}{s^2+1}$

23. $\mathcal{L}(f(t)) = \frac{1}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$

Solution: $\mathcal{L}(f(t)) = \frac{1}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) = \frac{1}{s}\mathcal{L}\left(e^{-t} - e^{-2t}\right)$ by the backward table. Then $\mathcal{L}(f(t)) = \mathcal{L}\left(\int_0^t (e^{-r} - e^{-2r})dr\right) = \mathcal{L}\left(1 - e^{-t} - 1/2 + e^{-2t}/2\right)$. By Lerch's theorem, $f(t) = 1/2 - e^{-t} + e^{-2t}/2$.

24. $\mathcal{L}(f(t)) = \frac{1}{s}\frac{e^{-s}}{s}$
Hint: $\mathcal{L}(u(t-a)) = \frac{1}{s}e^{-as}$.

The s–Integral Rule

Identity $\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty \mathcal{L}(f(t))\,ds$
requires piecewise continuous $f(t)$ of exponential order with $\lim_{t\to 0+}\frac{f(t)}{t} = L$.

25. Prove the identity.

Solution: A statement of the known theorem appears in Joel Schiff's textbook *The Laplace Transform: Theory and Applications*, Springer New York (1999), page 33, ISBN 0-0387-98698-7. The proof found there has missing details.

Let $g(t) = f(t)/t$ with definition $g(0) = L$ and $g(t) = 0$ for $t < 0$.
Lemma 1. Function $g(t)$ is piecewise continuous and of exponential order.

Lemma 2. Function $F(x) = \int_0^\infty f(t)e^{-xt}dt$ is continuous, of exponential order and $\int_s^\infty F(x)dx$ exists for $s \geq 0$.

Details:
Proofs of the lemmas require details but the details will not be supplied.

$\mathcal{L}(tg(t)) = \mathcal{L}(f(t)$ definition $g(t) = f(t)/t$

$-\frac{d}{ds}\mathcal{L}(g) = \mathcal{L}(f)$ Derivative theorem

$-\int_s^\infty \frac{d}{ds}\mathcal{L}(g)ds = \int_s^\infty \mathcal{L}(f)ds$ integrate, valid by Lemma 2

$-0 + \mathcal{L}(g) = \int_s^\infty \mathcal{L}(f)ds$ because $\lim_{s\to\infty}\mathcal{L}(g) = 0$

$\mathcal{L}(f(t)/t) = \int_s^\infty \mathcal{L}(f)ds$ because $g(t) = f(t)/t$

■

26. Compute $\mathcal{L}\left(\frac{\sin(t)}{t}\right)$.

Forward First Shifting Rule

Apply $\mathcal{L}(f(t)e^{at}) = \mathcal{L}(f(t))|_{s\to s-a}$ to find the Laplace transform.

27. $\mathcal{L}(te^t)$

Solution: Let $f(t) = t$. Then

$$\begin{aligned}
\mathcal{L}(te^t) &= \mathcal{L}(f(t)e^{at})\Big|_{a=1} \\
&= \mathcal{L}(f)|_{s\to s-a}\Big|_{a=1} \\
&= \mathcal{L}(t)|_{s\to s-a}\Big|_{a=1} \\
&= \left.\left.\frac{1}{s^2}\right|_{s\to s-a}\right|_{a=1} \\
&= \left.\frac{1}{(s-a)^2}\right|_{a=1} \\
&= \frac{1}{(s-1)^2}
\end{aligned}$$

28. $\mathcal{L}(te^t + e^{2t})$

29. $\mathcal{L}(\sin(t)e^t)$

Solution: $\dfrac{1}{(s-1)^2+1}$

30. $\mathcal{L}(\sin(2t)e^{2t} + \cos(t)e^t)$

31. $\mathcal{L}(t\cosh(2t))$ using identity $\cosh(w) = \frac{1}{2}e^{w} + \frac{1}{2}e^{-w}$.

Solution: $\dfrac{s^2+4}{(s^2-4)^2}$

32. $\mathcal{L}((t+1)^3\, e^t)$

Backward First Shifting Rule

Apply $\mathcal{L}(f(t))|_{s\to s-a} = \mathcal{L}(f(t)e^{at})$ and Lerch's theorem to solve for $f(t)$.

33. Explain for $\mathcal{L}(t^2)\big|_{s\to s-4}$ the rule
Erase a shift $|_{s\to s-a}$ *by inserting* e^{at} *inside the scope of* $\mathcal{L}$.

Solution: Rule $\mathcal{L}(f(t))|_{s\to s-a} = \mathcal{L}(f(t)e^{at})$ on the left has a shift while on the right there is no shift. The effect of the shift on the left is to multiply $f(t)$ by an exponential. Then:

$\mathcal{L}(t^2)\big|_{s\to s-4} = \mathcal{L}(g(t))$ where $g(t)$ is t^2 multiplied by e^{4t}:

$$\mathcal{L}(t^2)\big|_{s\to s-4} = \mathcal{L}(t^2e^{4t})$$

34. $\mathcal{L}(f(t)) = \left.\frac{s}{s^2+1}\right|_{s\to s-1}$

35. $\mathcal{L}(f(t)) = \frac{s-1}{(s-1)^2+4}$

Solution: $\mathcal{L}(f(t)) = \frac{s-1}{(s-1)^2+4} = \left.\frac{s}{s^2+4}\right|_{s\to s-1} = \mathcal{L}(\cos(2t))|_{s\to s-1} = \mathcal{L}(\cos(2t)\, e^t)$. Then $f(t)) = \cos(2t)\, e^t$ by Lerch's theorem.

36. $\mathcal{L}(f(t)) = \frac{8}{(s+1)^2+4}$

37. $\mathcal{L}(f(t)) = \frac{s+1}{s^2+2s+5}$

Solution: $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2+2s+5} = \dfrac{s+1}{(s+1)^2+4} = \left.\dfrac{s}{s^2+4}\right|_{s\to s+1} = \mathcal{L}(\cos(2t))|_{s\to s+1} = \mathcal{L}(\cos(2t)e^{-t})$. Then $f(t)) = \cos(2t)\, e^{-t}$ by Lerch's theorem.

38. $\mathcal{L}(f(t)) = \frac{4}{s^2+8s+17}$

39. $\mathcal{L}(f(t)) = \frac{2}{(s+1)^2}$

Solution: $\mathcal{L}(f(t)) = \dfrac{2}{(s+1)^2} = \left.\dfrac{2}{s^2}\right|_{s\to s+1} = \mathcal{L}(2t)|_{s\to s+1} = \mathcal{L}(2te^{-t})$. Then $f(t)) = 2te^{-t}$ by Lerch's theorem.

40. $\mathcal{L}(f(t)) = \frac{1}{(s+2)^{101}}$

Forward s-Differentiation

Apply $\mathcal{L}((-t)f(t)) = \frac{d}{ds}\mathcal{L}(f(t))$ to find the Laplace transform.

41. Explain for $\mathcal{L}((-t)\cos(t))$ the rule
Multiplying by $(-t)$ differentiates the Laplace transform..

Solution: One explanation:
$\mathcal{L}((-t)\cos(t)) = \frac{d}{ds}\mathcal{L}(\cos t)$ effectively differentiates on s the expression $\mathcal{L}(\cos t) = \dfrac{s}{s^2+1}$ to obtain the final answer $\frac{d}{ds}\dfrac{s}{s^2+1} = \dfrac{-(s^2-1)}{(s^2+1)^2}$.

Another explanation:

$$\begin{aligned}
\mathcal{L}((-t)\cos(t)) &= \int_0^\infty \cos(t)(-t)e^{-st}dt \\
&= \int_0^\infty \cos(t)\tfrac{d}{ds}\left(e^{-st}\right)dt \\
&= \tfrac{d}{ds}\int_0^\infty \cos(t)e^{-st}dt \\
&= \tfrac{d}{ds}\mathcal{L}(\cos(t))
\end{aligned}$$

42. $\mathcal{L}((-t)\sin(2t))$

43. $\mathcal{L}((-t)\sinh(2t))$, using identity
$\sinh(w) = \frac{1}{2}e^{w} - \frac{1}{2}e^{-w}$.

Solution: $\mathcal{L}((-t)\sinh(2t)) = \frac{1}{2}\mathcal{L}(e^{2t}) - \frac{1}{2}\mathcal{L}(e^{-2t}) = \frac{1}{2}(1/(s-2) - 1/(s+2))$.
Further simplifications would give $\dfrac{-4s}{(s^2-4)^2}$.

44. $\mathcal{L}(te^{t}\sin(2t) + te^{2t}\cos(t))$

Backward s-Differentiation

Apply $\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}((-t)f(t))$ and Lerch's theorem to solve for $f(t)$.

45. Explain for $\frac{d}{ds}\mathcal{L}(\cos(t))$ the rule
Erase $\frac{d}{ds}$ by inserting factor $(-t)$ inside the scope of $\mathcal{L}$.

Solution:

$$\begin{aligned}
\tfrac{d}{ds}\mathcal{L}(\cos(t)) &= \tfrac{d}{ds}\int_0^\infty \cos(t)e^{-st}dt \\
&= \int_0^\infty \cos(t)\tfrac{d}{ds}(e^{-st})dt \\
&= \int_0^\infty \cos(t)(-t)e^{-st}dt \\
&= \mathcal{L}((-t)\cos(t))
\end{aligned}$$

46. $\mathcal{L}(f(t)) = \frac{d}{ds}\frac{s}{s^2+4}$

47. $\mathcal{L}(f(t)) = \frac{d^2}{ds^2}\frac{1}{(s+1)^5}$

Solution:

$$\begin{aligned}\mathcal{L}(f(t)) &= \frac{d^2}{ds^2}\frac{1}{(s+1)^5} \\ &= \frac{d^2}{ds^2}\,\mathcal{L}(t^4/24)\Big|_{s\to s+1} \\ &= \mathcal{L}((-t)(-t)t^4/24)\Big|_{s\to s+1} && \text{one } (-t) \text{ for each } d/ds \\ &= \mathcal{L}((-t)(-t)t^4e^{-t}/24) && \text{first shifting theorem}\end{aligned}$$

Then $f(t) = t^6e^{-t}/24$ by Lerch's theorem.

48. $\mathcal{L}(f(t)) = \frac{d^3}{ds^3}\frac{s+1}{s^2+2s+5}$

Unit Step and Pulse

Define

$$\textbf{pulse}(t,a,b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{else,}\end{cases}$$

which is a tool for encoding and decoding piecewise-defined functions.

49. Prove the identity
$\textbf{pulse}(t,a,b)=u(t-a)-u(t-b)$,
where u is the **unit step**.

Solution:

$$\textbf{pulse}(t,a,b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{else,}\end{cases}$$

$$u(t-a) = \begin{cases} 1 & t \ge a \\ 0 & \text{else,}\end{cases}$$

$$u(t-b) = \begin{cases} 1 & t \ge b \\ 0 & \text{else,}\end{cases}$$

$$\begin{aligned}u(t-a)-u(t-b) &= \begin{cases} 1 & t \ge a \\ 0 & \text{else,}\end{cases} - \begin{cases} 1 & t \ge b \\ 0 & \text{else,}\end{cases} \\ &= \begin{cases} 1-1 & t \ge b, \\ 1 & a \le t < b, \\ 0 & \text{else,}\end{cases} \\ &= \textbf{pulse}(t,a,b)\end{aligned}$$

50. Prove the Laplace formula
$\mathcal{L}(\textbf{pulse}(t,a,b))=\frac{e^{-at}-e^{-bt}}{s}$

51. Verify that $f(t)$ defined by
$\begin{cases} 2 & 1 \le t < 2, \\ 0 & \text{else}\end{cases} + \begin{cases} 3 & 3 \le t < 4, \\ 0 & \text{else}\end{cases}$

encodes to representation
$2\,\textbf{pulse}(t,1,2)+3\,\textbf{pulse}(t,3,4)$.

Solution: Let

$$\text{LHS} = \begin{cases} 2 & 1 \le t < 2, \\ 0 & \text{else} \end{cases} + \begin{cases} 3 & 3 \le t < 4, \\ 0 & \text{else} \end{cases}$$

$$\text{LHS} = 2\,\textbf{pulse}(t,1,2) + 3\,\textbf{pulse}(3,4)$$

To use Laplace calculations in a computer algebra system it is necessary to write a function for $\textbf{pulse}(t,s,b)$ or to rewrite in unit step form:

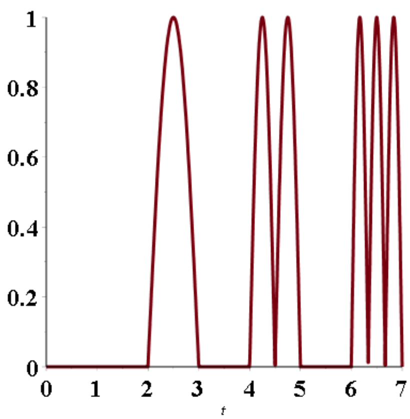
$$= 2(u(t-1) - u(t-2)) + 3(u(t-3) - u(t-4))$$

52. Decode $f(t)$ into a piecewise–defined function and graph it by hand, no computer, given $f(t)$ is
$e^t\,\textbf{pulse}(t,1,3)+e^{-t}\,\textbf{pulse}(t,4,6)$

53. Decode $f(t)$ into a piecewise–defined function and graph it, no computer, given $f(t)$ is the sum
$\sum_{n=1}^{3} |\sin(n\pi t)|\,\textbf{pulse}(t,2n,2n{+}1)$

Solution: Let

$f(t) = |\sin(\pi t)|\,\textbf{pulse}(t,2,3) + |\sin(2\pi t)|\,\textbf{pulse}(t,4,5) + |\sin(3\pi t)|\,\textbf{pulse}(t,6,7)$. Pulses are sine graphs: 1/2 period, 1 period, 1.5 periods. The graph is all in quadrant I.




```
# Exercise 53, Answer check
step:=t->piecewise(t >=0,1,0);
pulse:=(t,a,b)->step(t-a)-step(t-b);
f:=t->abs(sin(Pi*t))*pulse(t,2,3) +
 abs(sin(2*Pi*t))*pulse(t,4,5) +abs(sin(3*Pi*t))*pulse(t,6,7);
 plot(f(t),t=0..7,font=[courier,18,bold],thickness=3)
```

54. Encode as a combination of pulses

$$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ -2 & 3 \le t < 4, \\ 1 & 5 \le t < 6, \\ 0 & \text{else}, \end{cases}$$

showing all encoding details. Ans: $f(t) = \mathbf{pulse}(t,1,2) - 2\,\mathbf{pulse}(t,3,4)$ $+\,\mathbf{pulse}(t,5,6)$.

Alternate Second Shifting Rule

$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(w)|_{w=t+a}\right)$. No Laplace here. The focus is on function notation and finding $g(t+a) = g(w)|_{w=t+a}$, which means *substitute* $w = t + a$ *into the* $g(w)$*–formula.*

55. Let $g(t) = te^{-t}$. Verify identity $g(w)|_{w=t+2} = e^{-2}(te^{-t} + 2e^{-t})$.

Solution:

$$\begin{aligned} g(w)|_{w=t+2} &= we^{-w}|_{w=t+2} \\ &= (t+2)e^{-t-2} \\ &= e^{-2}(t+2)e^{-t} \\ &= e^{-2}\,(te^{-t} + 2e^{-t}). \end{aligned}$$

56. Let $g(t) = t^3$. Verify identity $g(w)|_{w=t+2} = 8 + 12t + 6t^2 + t^3$.

57. Typical polynomial $g(w) = 1 + 2w^2 + 3w^4$ upon substitution $w = t + a$ requires expansions for $(t+a)^2$ and $(t+a)^4$. Pascal's Triangle can be useful. Find the answer for $g(t+a) = g(w)|_{w=t+a}$.

Solution:

$$\begin{aligned} g(t+a) &= g(w)|_{w=t+a} \\ &= 1 + 2w^2 + 3w^4\big|_{w=t+a} \\ &= 1 + 2(t+a)^2 + 3(t+a)^4 \\ &= 1 + 2(t^2 + 2at + a^2) + 3(a^4 + 4a^3t + 6a^2t^2 + 4at^3 + t^4) \\ &= a^4 + 4\,a^3t + 6\,a^2t^2 + 4\,at^3 + t^4 + 2\,a^2 + 4\,ta + 2\,t^2 + 1 \end{aligned}$$

58. Polynomial $1+2w^2+3w^4$ upon substitution $w = t-b$ is a Taylor polynomial expansion

$f(t) = \sum_{n=0}^{4} \frac{f^{(n)}(b)}{n!}\,(t-b)^n$.

Find the Maclaurin expansion

$f(t) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!}\,t^n$.

Forward Second Shifting Rule

$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}(g(t+a))$

Find $\mathcal{L}(f(t))$, where u is the unit step.

59. $f(t) = u(t - \pi)$

Solution:

$f(t) = \mathcal{L}(u(t - \pi)) = e^{-as}\mathcal{L}(u(t))|\, a = \pi = e^{-as}\frac{1}{s}\Big|\, a = \pi = e^{-\pi s}\frac{1}{s}$

60. $f(t) = e^t\, u(t - 1)$

61. $f(t) = t^3 u(t - \pi)$

Solution:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(t^3 u(t - \pi) \\
&= e^{-as}\mathcal{L}((t + a)^3 u(t))\big|_{a=\pi} \\
&= e^{-as}\mathcal{L}(t^3 + 3at^2 + 3a^2 t + a^3)\big|_{a=\pi} \\
&= e^{-as}\left(\frac{6}{s^4} + \frac{6a}{s^3} + \frac{3a^2}{s^2} + \frac{a^3}{s}\right)\bigg|_{a=\pi} \\
&= e^{-\pi s}\left(\frac{6}{s^4} + \frac{6\pi}{s^3} + \frac{3\pi^2}{s^2} + \frac{\pi^3}{s}\right)
\end{aligned}$$

62. $f(t) = e^t\,$ **pulse**$(t, 1, 2)$, where
pulse(t, a, b)=$u(t - a)$–$u(t - b)$.

63. $f(t) = te^t u(t - 2)$

Solution: $\mathcal{L}(f) = \dfrac{(-1 + 2s)e^{-2s+2}}{(s - 1)^2}$

64. $f(t) = t\sin(t)u(t - \pi)$

Backward Second Shifting Rule

$e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t - a)u(t - a))$

Find $f(t)$ using the rule and Lerch's theorem, giving a piecewise–defined display and a unit step or pulse formula.

65. $\mathcal{L}(f(t)) = \frac{1}{s}e^{-3s}$

Ans: $f(t)$=$u(t - 3)$=$\begin{cases} 1 & t \geq 3, \\ 0 & \text{else,} \end{cases}$

Solution:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \frac{1}{s}e^{-3s} \\
&= e^{-3s}\mathcal{L}(u(t)) \\
&= \mathcal{L}(u(t - a)u(t - a))\big|_{a=3} \\
&= \mathcal{L}(u(t - a))\big|_{a=3}
\end{aligned}$$

$$= \mathcal{L}(u(t-3))$$

Then

$$f(t) = u(t-3) = \begin{cases} 0 & t < 3 \\ 1 & 3 \le t \end{cases}$$

66. $\mathcal{L}(f(t)) = \dfrac{1}{s^2} e^{3-3s}$

67. $\mathcal{L}(f(t)) = \dfrac{4}{s^2 + 8s + 17} e^{-2s}$

Solution: $f(t) = 4u(t-3)e^{-4t+12}\sin(t-3)$ or

$$f(t) = 4\,\mathrm{e}^{-4\,t+12}\sin(t-3)\begin{cases} 1 & 0 \le t-3 \\ 0 & \text{otherwise} \end{cases}$$

68. $\mathcal{L}(f(t)) = \dfrac{4+s}{s^2 + 8s + 17} e^{-3s}$

69. $\mathcal{L}(f(t)) = \left(\dfrac{1}{s^2} + \dfrac{2}{s^3}\right) e^{-2s}$

Solution: $f(t) = (t-2)(t-1)\,u(t-2)$ or

$$f(t) = \begin{cases} 0 & t < 2 \\ (t-2)(t-1) & 2 \le t \end{cases}$$

70. $\mathcal{L}(f(t)) = \dfrac{1}{(s-4)^2}\, e^{-2s}$

Trigonometric Formulas

Supply the details in Example 8.21.

71. $\mathcal{L}(t \sin at) = \dfrac{2as}{(s^2 + a^2)^2}$

Solution:

$$\begin{aligned} \mathcal{L}(t\sin at) &= -\mathcal{L}((-t)\sin(at)) \\ &= -\tfrac{d}{ds}\mathcal{L}(\sin(at)) \\ &= -\tfrac{d}{ds}\frac{a}{s^2+a^2} \\ &= \frac{2as}{(s^2+a^2)^2} \qquad \text{calculus quotient rule } (1/u)' = -u'/u^2 \end{aligned}$$

72. $\mathcal{L}(t^2 \sin at) = \dfrac{6s^2 a - a^3}{(s^2+a^2)^3}$

Exponential Formulas

Supply the details in Example 8.22.

73. $\mathcal{L}(e^{at} \sin bt) = \dfrac{b}{(s-a)^2+b^2}$

Solution:

$$\begin{aligned}\mathcal{L}(e^{at}\sin bt) &= \mathcal{L}(\sin(bt))\big|_{s\to s-a}\\ &= \left.\frac{b}{s^2+b^2}\right|_{s\to s-a}\\ &= \frac{b}{(s-a)^2+b^2}\end{aligned}$$

74. $\mathcal{L}(te^{at} \sin bt) = \dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$

Hyperbolic Functions

Supply the details in Example 8.23.

75. $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

Solution:

$$\begin{aligned}\mathcal{L}(\sinh at) &= \mathcal{L}\left(\tfrac{1}{2}e^{at} - \tfrac{1}{2}e^{-at}\right) \qquad \text{because } \sinh u = (e^u - e^{-u})/2\\ &= \tfrac{1}{2}\mathcal{L}(e^{at}) - \tfrac{1}{2}\mathcal{L}(e^{-at})\\ &= \frac{1}{2(s-a)} - \frac{1}{2(s+a)}\\ &= \frac{4a}{4(s-1)(s+a)}\\ &= \frac{a}{s^2-a^2}\end{aligned}$$

76. $\mathcal{L}(t \cosh at) = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

Waves

Use Laplace ideas from Examples 8.24 and 8.25. Each $f(t)$ can be expressed as a **pulse train**, which is an expression $\sum_{n=1}^{\infty} f_n(t)\,\mathbf{pulse}(t, a_i, b_i)$ to which the second shifting theorem applies.

77. Find $\mathcal{L}(f(t))$ for the square wave
$f(t) = \sum_{n=0}^{\infty}(-1)^n\,\mathbf{pulse}(t,n,n+1)$

Solution: First, establish $\mathcal{L}(\mathbf{pulse}(t,a,b)) = \mathcal{L}(u(t-a)) - \mathcal{L}(u(t-b)) = e^{-as}/s - e^{-bs}/s$. Then

$$\begin{aligned}
\mathcal{L}(f(t)) &= \textstyle\sum_{n=0}^{\infty}(-1)^n\mathcal{L}(\mathbf{pulse}(t,n,n+1)) \\
&= \textstyle\sum_{n=0}^{\infty}(-1)^n\left(e^{-ns}/s - e^{-ns-s}/s\right) \\
&= \textstyle\sum_{n=0}^{\infty}(-1)^n e^{-ns}\left(\dfrac{1-e^{-s}}{s}\right) \\
&= \dfrac{1-e^{-s}}{s}\textstyle\sum_{n=0}^{\infty}(-1)^n e^{-ns} \\
&= \dfrac{1-e^{-s}}{s}\textstyle\sum_{n=0}^{\infty} r^n\big|_{r=-e^{-s}} \\
&= \dfrac{1-e^{-s}}{s}\left(\left.\dfrac{1}{1-r}\right|_{r=-e^{-s}}\right) && \text{by geometric series} \\
&= \dfrac{1-e^{-s}}{s}\,\dfrac{1}{1+e^{-s}} \\
&= \dfrac{1-e^{-s}}{s}\,\dfrac{e^s}{e^s+1} && \text{multiply by } \dfrac{e^s}{s^s} \\
&= \dfrac{e^s-1}{s(e^s+1)} \\
&= \dfrac{e^{s/2}-e^{-s/2}}{s(e^{s/2}+e^{-s/2})} && \text{multiply by } \dfrac{e^{-s/2}}{s^{-s/2}} \\
&= \dfrac{1}{s}\,\dfrac{\sinh(s/2)}{s\,\cosh(s/2)} \\
&= \dfrac{1}{s}\,\tanh(s/2)
\end{aligned}$$

The answer agrees with the Extended Laplace Table: $\mathcal{L}(\textsf{sqw}(t/a)) = \frac{1}{s}\tanh(as/2)$ when $a=1$.

78. Define pulse train
$f(t) = \sum_{n=0}^{\infty} f_n(t)\,\mathbf{pulse}(t,n,n+1)$,
$f_{2n}(t) = t-2n$, $f_{2n+1}(t) = 2-t+2n$. Show that $f(t+2) = f(t)$ and
$f(t) = \begin{cases} t & 0 \le t < 1, \\ 2-t & 1 \le t \le 2. \end{cases}$

79. Find $\mathcal{L}(f(t))$ for
$f(t) = \begin{cases} |\sin(2t)| & 0 \le t \le \pi, \\ 0 & \pi \le t \le 2\pi, \end{cases}$

and $f(t + r\pi) = f(t)$.

Solution: Answer: $\mathcal{L}(f) = \dfrac{2\,(e^{s\pi} - 1)\,e^{-s\pi}}{(s^2+4)(1 - e^{-2\pi s}}$

The Rule for P-periodic functions will be applied:

$$\mathcal{L}(f) = \frac{\displaystyle\int_0^P f(t)e^{-st}dt}{1 - e^{-Ps}} \qquad \text{where} \qquad P = 2\pi$$

$$\begin{aligned}\textstyle\int_0^P f(t)e^{-st}dt &= \textstyle\int_0^\pi \sin(2t)e^{-st}dt \\ &= \frac{2\,(e^{s\pi} - 1)\,e^{-s\pi}}{s^2+4}\end{aligned}$$

```
# Exercise 79, P-Periodic Function Rule
int(sin(2*t)*exp(-s*t),t=0..Pi);
# (2*(exp(s*Pi)-1))*exp(-s*Pi)/(s^2+4)
```

80. Find $\mathcal{L}(f(t))$ for
$f(t) = \begin{cases} 1 & 0 \le t \le \pi, \\ |\sin(t)| & \pi \le t \le 2\pi, \end{cases}$
and $f(t + 2\pi) = f(t)$.

81. Given $f(t) = \frac{1}{2}(|\sin t| + \sin t)$, called the **Half–wave rectification** of the sine wave, derive $\mathcal{L}(f(t))=\frac{1}{(s^2+1)(1-e^{-\pi s})}$

Solution: Answer: Following the method in Exercise 79, `maple` gives:

$$\mathcal{L}(f) = \frac{1 + e^{-\pi s}}{(s^2+1)(1 - e^{-2\pi s})}$$

The problem reduces to rewriting the answer in reduced form:

$$\begin{aligned}\mathcal{L}(f) &= \frac{1}{s^2+1}\frac{1 + e^{-\pi s}}{1 - e^{-2\pi s}}. \\ &= \frac{1}{s^2+1}\frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(1 + e^{-\pi s})} \qquad \text{factor by } a^2 - b^2 = (a-b)(a+b) \\ &= \frac{1}{s^2+1}\frac{1}{1 - e^{-\pi s}}\end{aligned}$$

82. Solve for 2–periodic function $f(t)$:
$\mathcal{L}(f(t)) = \dfrac{1}{s}\tanh\left(\dfrac{s}{2}\right)$.
Use the Extended Laplace Integral Table.

8.4 Heaviside's Method

Partial Fraction Mistakes

1. How many real constants appear in the partial fraction expansion of the fraction $\dfrac{s+1}{s^2(s+2)(s+3)^2}$?

Solution: The numerator does not divide the denominator. The degree of the denominator is 5. The number of constants is 5.

2. How many real constants appear in the partial fraction expansion of $\dfrac{s+1}{s^2(s^2+4)(s^2+2s+5)^2}$?

3. Guido expanded $\dfrac{s+1}{s(s+2)(s+3)^2}$
to get $\dfrac{a}{s}+\dfrac{b}{s+2}+\dfrac{c}{(s+3)^2}$.
What is the mistake?

Solution: The numerator does not divide the denominator. The degree of the denominator is 4. The number of constants is 4. Guido omitted fraction $\dfrac{d}{s+3}$.

4. Helena made this expansion:
$\dfrac{s+1}{s(s+2)}=\dfrac{a}{s}+\dfrac{b}{s+2}+\dfrac{c}{s+3}$
The expansion is correct! Explain how you know that $c = 0$ without computing anything.
This example explains why fractions on the right have denominators dividing the denominator on the left.

5. Marco made an expansion:
$\dfrac{s+1}{s(s^2+4)}=\dfrac{a}{s}+\dfrac{b}{s+2}+\dfrac{c}{s-2}$
Explain why it is a mistake.

This example explains why sanity checks have more than one item to check.

Solution: Marco incorrectly factored s^2+4 as though the roots are $\pi 2$. The roots are complex: $\pm 2i$. The correct term is $\dfrac{cs+d}{s^2+4}$ or if using complex numbers then instead $\dfrac{C}{s-2i}+\dfrac{D}{s+2i}$ with C, D complex.

6. Violeta made an expansion
$$\frac{s+2}{s(s-2)(s+2)}=\frac{a}{s}+\frac{b}{s-2}+\frac{c}{s+2}$$
Explain why $c=0$ without computing anything.

This example explains why common factors of numerator and denominator should be removed.

7. Find the mistake in expansion
$$\frac{(s+2)^2}{s(s-2)}=\frac{a}{s}+\frac{b}{s-2}$$
This example explains why the degree of the numerator and denominator are checkpoints.

Solution: The degree of the numerator is not less than the degree of the denominator. Long division should be applied followed by partial fraction theory:

$$\frac{(s+2)^2}{s(s-2)}=1+\frac{6s+4}{s(s-2)}=1-\frac{2}{s}+\frac{8}{(s-2}$$

8. Is there a mistake here?
$$\frac{(s+2)^2}{s^2(s-2)}=\frac{a}{s}+\frac{b}{s^2}+\frac{c}{s-2}$$

Solution: No mistake.

Sampling Method

Apply the sampling method (a *failsafe method*) to verify the given equation.

9. $\dfrac{s}{s^2-1}=\dfrac{1/2}{s-1}+\dfrac{1/2}{s+1}$

Solution:

Seek a, b in the equation $\dfrac{s}{(s-1)(s+1)}=\dfrac{a}{s-1}+\dfrac{b}{s+1}$

$s=a(s+1)+b(s-1)$ Clear fractions.

Then substitute roots of the denominator.

$$\begin{cases} 1 &= a(2)+b(0) & \text{substitute } s=1 \\ -1 &= a(0)+b(-2) & \text{substitute } s=-1 \end{cases}$$

Solution of the system of equations: $a=1/2$, $b=1/2$

Then

$$\frac{s}{(s-1)(s+1)}=\frac{a}{s-1}+\frac{b}{s+1}=\frac{1/2}{s-1}+\frac{1/2}{s+1}$$

10. $\dfrac{s}{s^4-1}=\dfrac{1/4}{s-1}+\dfrac{1/4}{s+1}+\dfrac{-s/2}{s^2+1}$

Method of Atoms

Apply the method of atoms to verify the given equation.

11. $\dfrac{2s}{s^2-1}=\dfrac{1}{s-1}+\dfrac{1}{s+1}$

Solution:

Seek a, b in the equation $\dfrac{2s}{(s-1)(s+1)}=\dfrac{a}{s-1}+\dfrac{b}{s+1}$

$2s = a(s+1)+b(s-1)$ Clear fractions.

$2s = (a+b)s+(a-b)1$ collect coefficients on atoms $s, 1$.

$$\begin{cases} 2 = a+b & \text{match coefficient of atom} = s \\ 0 = a-b & \text{match coefficient of atom} = 1 \end{cases}$$

Solution of the system of equations: $a=1$, $b=1$

Then

$$\frac{2s}{(s-1)(s+1)}=\frac{a}{s-1}+\frac{b}{s+1}=\frac{1}{s-1}+\frac{1}{s+1}$$

12. $\dfrac{4s}{s^4-1}=\dfrac{1}{s-1}+\dfrac{1}{s+1}+\dfrac{-2s}{s^2+1}$

Heaviside's 1890 Shortcut

Apply Heaviside's shortcut to verify the given equation.

13. $\dfrac{2s}{s^2-4}=\dfrac{1}{s-2}+\dfrac{1}{s+2}$

Solution: Solve for a, b in the equation

$$\frac{2s}{(s-2)(s+2)}=\frac{a}{s-2}+\frac{b}{s+2}$$

To find a, let $H=s-2$ be the denominator of the fraction $\dfrac{a}{s-2}$. Mentally multiply the equation by H to get

$$\frac{2s}{(s+2)(H \text{ removed})}=\frac{a}{H \text{ removed}}+\frac{bH}{s+2}$$

Set $H=0$ and solve for $s=2$. Substitute $s=2$ and $H=0$ into the mentally multiplied equation. This step removes all symbols from the equation except for symbol a.

$$\frac{2(2)}{((2)+2)(H \text{ was removed})}=\frac{a}{(H \text{ was removed})}+\frac{b(0)}{(2)+2}$$

Simplify: $\dfrac{2(2)}{4}=a$

Then $1=a$.

The process repeats using $H = s+2$ to find symbol b:

$$\frac{2s}{(s-2)(H \text{ covered up})} = \frac{a(H)}{s-2} + \frac{b}{(H \text{ covered up})}$$

Then set $s = -2$ and $H = 0$:

$$\frac{2(-2)}{(-2-2)(H \text{ remvoved})} = \frac{a(0)}{-2-2} + \frac{b}{(H \text{ removed})}$$

Simplify: $1 = b$.

Conclusion:

$$\frac{2s}{(s-2)(s+2)} = \frac{a}{s-2} + \frac{b}{s+2} = \frac{1}{s-2} + \frac{1}{s+2}$$

14. $\dfrac{s+4}{s^3+4s} = \dfrac{1}{s} + \dfrac{-s+1}{s^2+4}$

Residues and Poles

Compute the residue for the given pole.

15. Residue at $s = 2$ for $\dfrac{2s}{s^2-4}$.

Solution: The process is the same as Heaviside's Coverup method but with steps eliminated. The residue calculation imagines the coverup method in progress with $H = s-2$ so that $s = 2$ results from $H = 0$. The given fraction $\dfrac{2s}{s^2-4}$ is imagined as the LHS of the equation before mentally multiplying by H. Suppose constant a in fraction $\dfrac{a}{H}$ is being determined by the coverup method. Then multiplying by H and setting $H = 0$ and $s = 2$ would result in

$$\left.\frac{2sH}{s^2-4}\right|_{s=2, H=0} = a$$

This is the essence of the residue formula: multiply by H then set $s = 2$ (which also means $H = 0$). Because H cancels in the fraction then it is only required to set $s = 2$:

$$\left.\frac{2s(s+2)}{(s-2)(s+2)}\right|_{s=2} = a$$

$$\left.\frac{2s}{(s+2)}\right|_{s=2} = a \qquad H = s-2 \text{ cancelled}$$

Then: $1 = a$ is the residue.

16. Residue at $s = 0$ for $\dfrac{s+4}{s^3+16s}$.

Scalar Differential Equations

The **transfer function** of $x'' + x = f(t)$ is $H(s) = \frac{1}{s^2+1}$. A common definition is $H(s) = \mathcal{L}(f(t))$ divided by $\mathcal{L}(x(t))$, assuming $x(0) = x'(0) = 0$.

17. Verify for $x'' + x = e^{-t}$ with $x(0) = 0$, $x'(0) = 0$ that $\mathcal{L}(x)=\frac{1}{s+1}\,\frac{1}{s^2+1}$. Then compute $H(s)$.

Solution: Step 1:

$$\begin{aligned}
&\mathcal{L}(x'' + x) = \mathcal{L}(e^{-t}) && \text{Apply } \mathcal{L} \text{ across the DE.}\\
&s\mathcal{L}(x') - x'(0) + \mathcal{L}(x) = \frac{1}{s+1} && \text{Derivative rule on } x'.\\
&s(s\mathcal{L}(x) - x(0)) - x'(0) + \mathcal{L}(x) = \frac{1}{s+1} && \text{Derivative rule on } x.\\
&(s^2+1)\mathcal{L}(x) = \frac{1}{s+1} && \text{Use } x(0) = x'(0) = 0 \text{ and collect on } \mathcal{L}(x).\\
&\mathcal{L}(x) = \frac{1}{(s^2+1)(s+1)} && \text{Use } x(0) = x'(0) = 0 \text{ Isolate } \mathcal{L}(x) \text{ left.}
\end{aligned}$$

Step 2:

$$\begin{aligned}
H(s) &= \text{transfer function} = \mathcal{L}(f)/\mathcal{L}(x)\\
&= \mathcal{L}(e^{-t})/\mathcal{L}(x)\\
&= \frac{1/(s+1)}{1/((s^2+1)(s+1))}\\
&= \frac{1(s^2+1)(s+1)}{(s+1)}\\
&= s^2+1
\end{aligned}$$

The most often-used shortcut: $H(s)$ is the characteristic polynomial of the homogeneous DE with the variable changed to s: $r^2 + 1$ with $r \to s$ is $H(s) = s^2 + 1$.

18. Explain the transfer function equation $H(s) = \frac{1}{\text{characteristic equation}}$.

19. Solve $\mathcal{L}(x(t))=\frac{1}{s+1}\,\frac{1}{s^2+1}$ by Heaviside cover–up for output $x(t) = \frac{1}{2}(e^{-t} - \cos t + \sin t)$.

Solution: The roots of $s^2+1 = 0$ are complex: $s = \pm i$. The plan is to expand $\mathcal{L}(x(t))$ in partial fractions and then use the backward Laplace table:

$$\mathcal{L}(x(t)) = \frac{1}{s+1}\,\frac{1}{s^2+1} = \frac{a}{s+1} + \frac{bs+c}{s^2+1}, \qquad \text{Real } a, b, c.$$

The coverup method applies because the roots are distinct: $-1, i, -i$.

Let $H = s - i$ for root $s = i$. Multiply by H mentally and cancel H where possible, then set $H = 0$ which implies symbol s is replaced by i:

$$\frac{1}{s+1}\,\frac{s-i}{(s-i)(s+i)} = \frac{a(s-i)}{(s+i)(s-i)} + \frac{(bs+c)(s-i)}{(s-i)(s+i)}$$

$$\left.\frac{1}{s+1}\,\frac{1}{s+i}\right|_{s=i,H=0} = 0 + \left.\frac{bs+c}{s+i}\right|_{s=i}$$

$$\frac{1}{s+1}\,\frac{1}{s+i} = \left.\frac{bs+c}{s+i}\right|_{s=i} \qquad \boxed{1}$$

$$\frac{1}{i+1}\,\frac{1}{i+i} = \frac{bi+c}{i+i}$$

$$\frac{1}{i+1} = bi + c$$

$$1 = (bi+c)(i+1) = -b + ci + bi + c$$

Then $1 = -b + c$ and $0 = c + b$ by matching real and imaginary parts of the complex number on each side of the equation. Solve to get $b = -1/2$, $c = 1/2$. The usual Heaviside coverup method quickly finds $a = 1$. Lerch's theorem and the backward Laplace table then imply

$$x(t) = ae^{-t} + b\cos(t) + c\sin(t) = e^{-t} - \tfrac{1}{2}\cos(t) + \tfrac{1}{2}\sin(t)$$

Shortcut.
The substitution of $s = i$ found both $c = -1/2$ and $c = 1/2$ in one step. It turns out that the answer can be found by manipulation of s instead of substitution of $s = i$ at stage $\boxed{1}$ above. Using just symbol s gives:

$$1 = (bs+c)(s+1)$$

$$1 = bs^2 + (b+c)s + c$$

$$1 = (b+c)s + (c-b) \qquad \text{because } s^2 = i^2 = -1.$$

It is correct to analyze the equation as a linear equation in s and match coefficients: $1 = c - b$, $0 = b + c$. This is due to independence of complex numbers i and 1, imagined as 2-vectors in the plane.The result is a method to find b, c *without using complex arithmetic.*

20. Given $x'' + x = te^{-t}$, $x(0) = x'(0) = 0$, show all steps to find $\mathcal{L}(x(t)) = \frac{1}{(s+1)^2}\frac{1}{s^2+1}$.

First Order System

Using Example 8.29 as a guide, solve the system for $x_1(t)$ by Laplace's method.

21. $\begin{cases} x_1' = x_2, \\ x_2' = 4x_1 + 12e^{-t}, \\ x_1(0) = x_2(0) = 0. \end{cases}$

Ans: $x_1(t) = e^{2t} + 3e^{-2t} - 4e^{-t}$.

Solution:
Step 1.
Apply $\mathcal{L}$ across each of the two differential equations. Use the Derivative Theorem and the Forward Table to reduce the equations to a system in $\mathcal{L}(x_1$, $\mathcal{L}(x_2)$ with coefficients in variable s.

$$\begin{cases} s\mathcal{L}(x_1) - x_1(0) &= \mathcal{L}(x_2), \\ s\mathcal{L}(x_2) - x_2(0) &= 4\mathcal{L}(x_1) + 12/(s+1) \end{cases}$$

Step 2.
Insert the initial data $x_1(0) = x_2(0) = 0$ to obtain the nonhomogeneous linear algebraic system for variables $X_1 = \mathcal{L}(x_1)$, $X_2 = \mathcal{L}(x_2)$:

$$\begin{cases} sX_1 - X_2 &= 0, \\ -4X_1 + sX_2 &= 12/(s+1) \end{cases}$$

Solve by linear algebra:

$$X_1 = \frac{12}{(s-2)(s+2)(s+1)}, \quad X_2 = \frac{12s}{(s-2)(s+2)(s+1)}$$

Step 3.
Solve the preceding equations for x_1 and x_2. For instance,

$$\begin{aligned} \mathcal{L}(x_1) = X_1 &= \frac{12}{(s-2)(s+2)(s+1)} \\ &= \frac{a}{s-2} + \frac{b}{s+2} + \frac{c}{s+1} \end{aligned}$$

Then $a = 1$, $b = 3$, $c = -4$ by Heaviside coverup and the backward table. Final answer:

$\mathcal{L}(x_1) = \mathcal{L}(ae^{2t} + be^{-2t} + ce^{-t})$

$x_1 = e^{2t} + 3e^{-2t} - 4e^{-t}$ by Lerch's theorem.

Details for $x_2 = 2e^{2t} - 6e^{-2t} + 4e^{-t}$ are similar.

22. $\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' = 4x_1 - 4x_2 + x_3 + 10e^{-t}, \\ x_1(0) = x_2(0) = x_3(0) = 0. \end{cases}$
Ans: $x_1(t) = e^t - e^{-t} - \sin(2t)$.

Second Order System

Using Example 8.29 as a guide, compute $x(t), y(t)$.

23. $\mathcal{L}(x(t)) = \frac{3s^2+2}{(s-1)(s^2+4)}$,
$\mathcal{L}(y(t)) = \frac{10}{(s-1)(s^2+4)}$.

Ans: x=$2\cos(2t)$+$\sin(2t)$+e^t,
y=$-2\cos(2t)$−$\sin(2t)$+$2e^t$

Solution: The methods are in Exercise 19. To solve for $x(t)$:

$$\frac{3s^2+2}{(s-1)(s^2+4)} = \frac{a}{s-1} + \frac{bs+c}{s^2+4}$$

Then $a = 1$ by Heaviside coverup. To find b, c multiply by $H = s^2 + 4$ and then set $H = 0$:

$\dfrac{3s^2+2}{s-1} = bs + c$ subject to $H = s^2 + 4 = 0$

$3s^2 + 2 = (bs + c)(s - 1)$ cross-multiply

$3s^2 + 2 = bs^2 + cs - bs - c$

$3(-4) + 2 = b(-4) + cs - bs - c$ because $H = 0$ implies $s^2 = -4$ for both roots of $H = 0$

$-10 = -4b - c$, $0 = c - b$ match coefficients of 1 and s

Solve: $b - 2$, $c = 2$. Then

$$\begin{aligned}\mathcal{L}(x(t)) &= \frac{3s^2+2}{(s-1)(s^2+4)}\\ &= \frac{1}{s-1} + \frac{2s+2}{s^2+4}\\ &= \mathcal{L}(e^t) + \mathcal{L}(2\cos 2t + \sin 2t)\end{aligned}$$

Lerch's theorem gives $x(t) = e^t + 2\cos 2t + \sin 2t$. The details for $y(t)$ are similar.

```
# Exercise 23, Answer check
convert((3*s^2+2)/((s-1)*(s^2+4)),parfrac);
# 1/(s-1)+(2*s+2)/(s^2+4)
```

24. $\mathcal{L}(x(t))$=$\frac{2s^2+4}{(s+1)(s^2+1)}$,
$\mathcal{L}(y(t))$=$\frac{2}{(s+1)(s^2+1)}$.
Ans: x=$-\cos(t)$+$\sin(t)$+$3e^{-t}$,
y=$-\cos(t)$+$\sin(t)$+e^{-t}.

8.5 Heaviside Step and Dirac Impulse

Unit Step and Heaviside

1. The unit step $\mathbf{u}(t)$ is defined on the whole real line. Is it piecewise continuous on the whole line?

Solution: Yes.

2. Is there a continuous function on the real line that agrees with the Heaviside function except at $t = 0$?

3. The piecewise continuous function $\mathbf{pulse}(t, a, b)$ is defined everywhere. Redefine $\mathbf{pulse}(t, a, b)$ using $H(t)$ instead of $\mathbf{u}(t)$.

Solution: Replace in the $\mathbf{pulse}(t, a, b)$ definition symbol $\mathbf{u}(t)$ by symbol $H(t)$, the Heaviside function. There is a difference at $t = a$ and $t = b$, because $H(0)$ is undefined. The piecewise definition after the replacement:

$$\mathbf{pulse}(t,a,b) = \begin{cases} 0 & t < a \\ \textit{undefined} & t = a \\ 1 & a < t < b \\ \textit{undefined} & t = b \\ 0 & t > b \end{cases}$$

4. Write $f(t) = \textbf{floor}(t)\,\mathbf{u}(t)$ as a sum of terms, each of which has the form $g(t)\,\mathbf{pulse}(t, a, b)$.

Solution: An infinite series is required.

Dirac Impulse

5. Verify $\int_{-\infty}^{\infty} \frac{\mathbf{pulse}(t,a,b)}{b-a}\,dt = 1$.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathbf{pulse}(t,a,b)}{b-a}\,dt &= \int_{-\infty}^{\infty} \frac{1}{b-a}\left(\begin{cases} 1 & a \le t < b \\ 0 & \text{otherwise} \end{cases}\right) dt \\ &= \int_a^b \frac{1}{b-a} dt \qquad \text{zero integrand outside } a \le t \le b \\ &= 1 \end{aligned}$$

6. Verify by direct integration that $f(t) = 10\,\mathbf{pulse}(t, -0.001, 0.001)$ represents a simple impulse of 10 at $t = 0$ of duration 0.002. Graph it without using technology.

7. Find $\mathcal{L}(\delta(t-1)+\delta(t-2))$.

Solution: $\mathcal{L}(\delta(t-1)+\delta(t-2)) = e^{-s}+e^{-2s}$

8. Find $\mathcal{L}(10\,\delta(t-1)-5\,\delta(t-2))$.

9. Solve for $f(t)$ in terms of δ:
$\mathcal{L}(f(t)) = 10e^{-s}$

Solution: There is no piecewise continuous function $f(t)$ of exponential order satisfying the equation $\mathcal{L}(f(t)) = 10e^{-s}$. Nevertheless, most experts would write $\mathcal{L}(f(t)) = 10e^{-s} = \mathcal{L}(10\delta(t-1))$, then write $f(t) = 10\delta(t-1)$ which represents an impulse of 10 at $t = 1$ (a hammer hit). It is technically incorrect to claim that $f(t)$ is a function. It is not, but it is approximated by the function $f_\epsilon(t) = 10\frac{1}{2\epsilon}\,\textbf{pulse}(t, 1-\epsilon, 1+\epsilon)$ as $\epsilon \to 0$. What allows the formal result is the equation $\lim_{\epsilon\to 0}\mathcal{L}(f_\epsilon) = e^{-s} = \mathcal{L}(\delta(t-1))$. The formal calculation looks like we used Lerch's theorem. But Lerch's theorem does not apply to equations involving δ.

10. Solve for $f(t)$ in terms of δ:
$\mathcal{L}(f(t)) = 10e^{-s} + \frac{s}{s^2+1}\,e^{-2s}$

11. Find $\mathcal{L}\left(\sum_{n=1}^{10}(1+n)\delta(t-n)\right)$.

Solution:
$\mathcal{L}\left(\sum_{n=1}^{10}(1+n)\delta(t-n)\right) = \sum_{n=1}^{10}(1+n)\mathcal{L}(\delta(t-n)) = \sum_{n=1}^{10}(1+n)e^{-ns}$

12. A sequence of camshaft impulses happening periodically in a finite time interval have transform $\mathcal{L}(f(t)) = \sum_{i=1}^{N} e^{-c_i\,s}$. Find the idealized impulse train f.

Riemann–Stieltjes Integral

Evaluate the integrals either directly from the definition or else by using Theorem 8.15.

13. $\int_0^2 d\,\mathbf{u}(t-1)$

Solution: Theorem 8.15 part (2) does not apply directly because the limits of integration do not match.

$\int_0^2 d\,\mathbf{u}(t-1) = \int_{-\infty}^{\infty}\textbf{pulse}(t,-1,3)d\,\mathbf{u}(t-1)$ because $d\,\mathbf{u}(t-1) = 0$ outside a small interval containing $t = 1$.

$\int_0^2 d\,\mathbf{u}(t-1) = \lim_{t\to 1+}\textbf{pulse}(t,-1,3)$ by Theorem 8.15 part (2)

$\int_0^2 d\,\mathbf{u}(t-1) = 1$

14. $\int_0^{\infty} d\,\mathbf{u}(t-2)$

15. $\int_0^2 \tanh(t^2+1)\,d\,\mathbf{u}(t-1)$

Solution:

$$\begin{aligned}\int_0^2 \tanh(t^2+1)\,d\,\mathbf{u}(t-1) &= \int_{-\infty}^{\infty} \mathbf{pulse}(t,-1,3)\tanh(t^2+1)\,d\,\mathbf{u}(t-1)\\ &= \lim_{t\to 1+} \mathbf{pulse}(t,-1,3)\tanh(t^2+1)\\ &= \tanh(2) \text{ by Theorem 8.15 part (2).}\end{aligned}$$

16. $\int_0^{\infty} \frac{t}{1+t^2}\,d\,\mathbf{u}(t-2)$

8.6 Modeling

Oscillatory and Non–oscillatory

Assume $x'' + px' + qx = 0$ with p, q nonnegative.

Parameter p is imagined as a set screw adjustment on a screen door dashpot, larger p meaning more damping effect.

Parameter q is the Hooke's constant for the spring restoring force.

1. Let $q = 100$, $p = 99$. Verify that the equation is over–damped in two ways:
(1) Graph $x(t)$;
(2) Justify that $r^2 + pr + q = 0$ has real negative roots.

Solution: The graph can be made by hand for

$$x(t) = c_1\, e^{\left(-99+\sqrt{9401}\right)t/2} + c_2\, e^{-\left(99+\sqrt{9401}\right)t/2}$$

It looks like $x = e^{-t}$ or $x = -e^{-t}$ as $t \to \infty$ no matter the nonzero values of c_1, c_2.

The characteristic equation $r^2 + pr + q = 0$ has roots -1.02062294, -97.97937706, both negative.

```
# Exercise 1, Over-damped
de:=diff(x(t),t,t)+p*diff(x(t),t)+q*x(t)=0;
eq:=r^2+p*r+q=0;
p:=99;q:=100;
X:=dsolve(de,x(t));
# x(t) = _C1*exp((1/2*(-99+sqrt(9401)))*t)+
#        _C2*exp(-(1/2*(99+sqrt(9401)))*t)
R:=solve(eq,r);evalf([R]);
# [-1.02062294, -97.97937706]
XX:=subs(_C1=1,_C2=1,rhs(X));
plot(XX,t=0..5);
```

2. Let $q = 100$. The case which is called *critically–damped* happens at exactly one value $p = p^*$ between 0 and 99. Compute p^* numerically. Graph $x(t)$ using $q = 100$, $p = p^*$, $x(0) = 0$, $x'(0) = 1$.

3. Let $q = 100$. Verify that $p = 0$ produces the **harmonic oscillator** $x'' + \omega^2 x = 0$, $\omega = 10$.

Small set screw changes from $p = 0$ to $p > 0$ are still oscillatory. Under–damped means weak dashpot reaction.

Solution: Equation $x'' + px' + qx = 0$ becomes $x'' + 100x = 0$ or $x'' + 10^2 x = 0$. Then $\omega = 10$ and the equation is the harmonic oscillator.

4. Let $q = 100$, $p = 2$. Justify oscillatory under–damped from the graph of $x(t)$ and also by solving $r^2 + pr + q = 0$.

Simplistic Dirac Impulse

Define $g(t) = 7\,e^{-153800\,t}\ \mathbf{u}(t)$ and
$f(t,a) = \frac{1}{a}\,(u(t) - u(t-a))$, $a > 0$.
The impulse of force h is $\int_{-\infty}^{\infty} h(t)\,dt$.

5. Compute the impulse for $f(t,a)$.
Ans: 1.

Solution: Let $h(t) = f(t,a)$ be the force. The impulse of h is

$$\begin{aligned}\int_{-\infty}^{\infty} h(t)\,dt &= \int_{-\infty}^{\infty} \tfrac{1}{a}\,\mathbf{pulse}(t,0,a)dt \\ &= \int_{0}^{a} \tfrac{1}{a}\,\mathbf{pulse}(t,0,a)dt \\ &= \int_{0}^{a} \tfrac{1}{a}\,(1)dt \\ &= 1\end{aligned}$$

6. Plot $f(t,a)$ for $a = 0.1,\ 0.001,\ 0.0001$.

7. Calculate the impulse for $g(t)$.
Ans: About 46 times 10^{-6}.

Solution:

Given: $g(t) = 7\,e^{-153800\,t}\ \mathbf{u}(t)$, compute $\int_{-\infty}^{\infty} g(t)\,dt$.

$$\begin{aligned}\int_{-\infty}^{\infty} g(t)\,dt &= \int_{-\infty}^{\infty} 7\,e^{-153800\,t}\ \mathbf{u}(t)\,dt \\ &= 7\int_{0}^{\infty} e^{-153800\,t}\,(1)\,dt \\ &= \left.\frac{7e^{-153800\,t}}{-153800}\right|_{t=0}^{\infty} \\ &= 0 - \frac{7}{-153800} \\ &== 0.00004551365410\end{aligned}$$

8. Try to find an **RC** discharge circuit with 10 volt *emf* and output $g(t)$.
Circuit response $g(t)$ simulates Dirac impulsive force $\frac{45.5}{1000000}\delta(t)$.

Parameters: Over–Damped

Find $a, b, \omega = \sqrt{ab}, \zeta = \frac{a+b}{2\omega}$ given the plot and two dots on the graph.

9. Step input Figure 9, dots
$(1, 0.1998)$, $(4, 0.4819)$.
Ans: $a = 1.0000$, $b = 1.9997$, $\omega = 1.4141$, $\zeta = 1.0607$.

Solution: The exercise follows the solution to the example following Figure 9 page 660. The work should be carried out with a computer.

```
# Exercise 9, Parameters: Over-Damped
F:=(t,a,b)->1/(a*b)+exp(-a*t)/(a^2-a*b)+exp(-b*t)/(b^2-a*b);
t1:=1;x1:=0.1998;t2:=4;x2:=0.4819;
ans:=fsolve(eval({F(t1,a,b)=x1,F(t2,a,b)=x2}),{a,b});
 # ans := {a = 1.000041775, b = 1.999717461}
omega:=eval(sqrt(a*b),ans);
 # omega := 1.414143203
zeta:=eval((a+b)/(2*omega),ans);
 # zeta := 1.060627817
```

10. Impulse input Figure 10, dots $(0.5, 0.1193)$, $(2, 0.0585)$.
Ans: $a = 0.9991$, $b = 2.0021$, $\omega = 1.4143$, $\zeta = 1.0610$.

Parameters: Under–Damped

Find $a, b, \omega = \sqrt{a^2 + b^2}, \zeta = \frac{a}{\omega}$ given the plot and two dots on the graph.

11. Zero input like Figure 11, but consecutive maxima at $(2.5107, 0.0257)$, $(4.6051, 0.0032)$.
Ans: Approximately $a = 1$, $b = 3$.

Solution: Follow the solution to the Example after Figure 11 page 663.

```
# Exercise 11, Parameters: Under-Damped
t1:=2.5107; x1:=0.0257; t2:=4.6051;x2:=0.0032;y0:=0;
 a=ln((x1-y0)/(x2-y0))/(t2-t1); # a = 0.9947193382
 b=2*Pi/(t2-t1);# b = 2.999992986
```

12. Step input like Figure 13, but steady–state $y_0 = 1/26$ and consecutive maxima at $(0.6283, 0.0205)$, $(1.8850, 0.0058)$.
Ans: Approximately $a = 1$, $b = 5$.

Chapter 9

Eigenanalysis

Contents

9.1 Matrix Eigenanalysis

Eigenanalysis

Classify as true or false. If false, then explain.

1. The purpose of eigenanalysis is to discover a new coordinate system.

Solution: True.

2. Eigenanalysis can discover an opportunistic change of coordinates.

3. A matrix can have eigenvalue 0.

Solution: True: the zero matrix.

4. Eigenvalues are scale factors, imagined to be measurement units.

5. Eigenvectors are directions.

Solution: True. A physical example is a football or ellipsoid. The eigenvectors are the three semiaxis directions.

6. For each eigenvalue of a matrix A, there always exists at least one eigenpair.

7. If A^{-1} has eigenvalue λ, then A has eigenvalue $1/\lambda$.

Solution: True. If A^{-1} exists then identity $A^{-1} = \mathbf{adj}(A)/|A|$ prevents $|A| = 0$. Matrix A^{-1} cannot have eigenvalue zero, due to characteristic equation $|A^{-1} - \lambda I| = 0$ being impossible for $\lambda = 0$ (use product identity $|A||A^{-1}| = |AA^{-1}| = |I| = 1$). If $A^{-1}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ then $\vec{\mathbf{x}} = \lambda A\vec{\mathbf{x}}$. Because $\lambda \neq 0$, then division is possible and $(1/\lambda, \vec{\mathbf{x}})$ is an eigenpair of A.

8. Eigenvectors cannot be $\vec{\mathbf{0}}$.

9. The transpose of A has the same eigenvalues as A.

Solution: True. Eigenvalues of A are found from algebraic equation $|A - \lambda I| = 0$, called the characteristic equation of A. Then $|A^T - \lambda I| = |(A - \lambda I)^T| = |A - \lambda I| = 0$ by the determinant property $|B| = |B^T|$. Therefore A and A^T have the same eigenvalues.

10. Eigenpairs $(\lambda, \vec{\mathbf{v}})$ of A satisfy the equation $(A - \lambda I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Eigenpairs of a Diagonal Matrix

Find eigenpairs of A without computation. Use Theorem 9.7.

11. $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Solution: $\lambda = 2, 3$

12. $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

13. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution: $\lambda = 2, 3, 1$

14. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

15. $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

Solution: $\lambda = 7, 2, -6$

16. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Fourier Replacement

Let symbols c_1, c_2 represent arbitrary constants. Let 2×2 matrix A have Fourier replacement equation

$$A\left(c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\2\end{pmatrix}\right) = 2c_1\begin{pmatrix}1\\1\end{pmatrix}-5c_2\begin{pmatrix}1\\2\end{pmatrix}$$

17. Display the eigenpairs of A.

Solution: $\left(2, \begin{pmatrix}1\\1\end{pmatrix}\right), \left(-5, \begin{pmatrix}1\\2\end{pmatrix}\right)$

18. Display the replacement equation if the eigenvalues $2, -5$ are replaced by $1, 0$.

19. Display the eigenpair packages P, D such that $AP = PD$.

Solution: By Exercise 17, the eigenpairs of A are $\left(2, \begin{pmatrix}1\\1\end{pmatrix}\right), \left(-5, \begin{pmatrix}1\\2\end{pmatrix}\right)$. Then

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad D = \textbf{diag}(2, -5)$$

20. Find A.

Eigenanalysis Facts

Mark as true or false, then explain your answer.

21. If matrix A has all eigenvalues zero, then A is the zero matrix.

Solution: False. A triangular matrix with zeros on the diagonal has all eigenvalues zero.

22. If 2×2 matrix A has all eigenvalues zero, then Fourier's replacement equation is

$$A\left(c_1\vec{\mathbf{v}}_1+c_2\vec{\mathbf{v}}_2\right) = \vec{\mathbf{0}}.$$

23. There are infinitely many 2×2 matrices A with complex eigenvalues $1 + i, 1 - i$.

Solution: True. Let $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Then B has eigenvalues $1 + i, 1 - i$. Let $P = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for arbitrary real x. Let $A(x) = P^{-1}BP$. Then $A(x)$ and B have the same eigenvalues and there are infinitely many distinct matrices $A(x)$.

24. A real 2×2 matrix A with eigenvalues $2+3i, 2-3i$ cannot have a real eigenvector.

25. A real 2×2 matrix A with real eigenvalues has only real eigenvectors.

Solution: False. If $\lambda\vec{\mathbf{v}} = A\vec{\mathbf{v}}$ and $\vec{\mathbf{v}}$ is real then $\vec{\mathbf{w}} = z\vec{\mathbf{v}}$ is complex for purely complex z and $\lambda\vec{\mathbf{w}} = z(\lambda\vec{\mathbf{v}}) = zA\vec{\mathbf{v}} = A\vec{\mathbf{w}}$. Then $(\lambda, \vec{\mathbf{w}})$ is an eigenpair with complex eigenvector.

What is known:
The eigenpairs $(\lambda, \vec{\mathbf{v}})$ of A in the case of a real eigenvalue λ can always be selected so that $\vec{\mathbf{v}}$ is real. This is because $A - \lambda I$ is a real matrix and Gaussian elimination finds a real vector $\vec{\mathbf{v}}$ solution to the homogeneous system $(A - \lambda I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

26. A real 2×2 matrix A with complex eigenvalues has only complex eigenvectors.

Eigenpair Packages and equation $AP = PD$

27. Suppose A has eigenpair packages. Explain why there are so many different answers for P, D.

Solution: The packages contain eigenpairs which can be listed in many different orders, resulting in different P and D. Further, while eigenvalues are determined from $|A - \lambda I| = 0$, the eigenvectors are not unique: even for 2×2 matrices an eigenvector is either determined up to a constant multiple or else **kernel**$(A - \lambda I)$ is two dimensional leaving infinitely many choices for two independent eigenvectors. For instance, the eigenvectors of the zero matrix can be any two independent vectors in $\mathcal{R}^2$.

28. Suppose $AP = PD$ and $AQ = QD$ hold (same diagonal matrix D). Does $P = Q$?

29. Find one choice of P and D for $A = 2 \times 2$ diagonal matrix.

Solution: Let $A = \mathbf{diag}(a, b)$. The eigenpairs can be $\left(a, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \left(b, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$.
Then $P = I$ and $D = A$.

30. Given $A = 3 \times 3$ zero matrix, find one choice of P and D with column one of P equal to $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Matrix Eigenanalysis Method

31. The eigenvalues of $\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ satisfy a quadratic equation. Find the equation and solve for the eigenvalues.

Solution: The equation is $|A - \lambda I| = 0$ which is $\begin{vmatrix} 1-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0.$ The equation can be written directly as $(-\lambda)^2 + \mathbf{trace}(A)(-\lambda) + |A| = 0$ or $\lambda^2 - 5\lambda + 1 = 0$. The roots are found by the quadratic formula: $5/2 \pm \sqrt{21}/2$.

32. Find the eigenvalues of $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

33. Find all eigenpairs of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

Solution: The eigenvalues are the diagonal elements of A: $1, 2, 3$.

Eigenvectors are found from solving the equation $(A - I)\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$, $(A - 2I)\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$, $(A - 3I)\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$. Each of the three homogeneous systems is solved by finding the general solution by swap, combo, multiply. Take as the eigenvector in each case $\partial\vec{\mathbf{v}}/\partial t_1$ where t_1 is the free variable. We know in advance that each eigenvalue has at least one eigenvector. Distinct eigenvalues implies the dimension of the solution space, which equals the number of free variables, is in each case exactly one.

To find an eigenpair $(1, \vec{\mathbf{v}})$, solve $A_1\vec{\mathbf{v}} = \vec{\mathbf{0}}$ where

$$A_1 = A - (1)I = \begin{pmatrix} 1-(1) & 2 & 0 \\ 0 & 2-(1) & 2 \\ 0 & 0 & 3-(1) \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Then $\mathbf{rref}(A_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The general solution is $\vec{\mathbf{v}} = t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Let

$$\vec{\mathbf{v}}_1 = \partial\vec{\mathbf{v}}/\partial t_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and then the eigenpair $(1, \vec{\mathbf{v}}_1)$ is

$$\left(1, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

The other two eigenpairs are found similarly:

$$\left(2, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right)$$

```
# Exercise 33
A:=Matrix([[1,2,0],[0,2,2],[0,0,3]]);
CharacteristicPolynomial(A,r);
 #  r^3  - 6 r^2  + 11 r - 6
Eigenvalues(A);Eigenvectors(A);
A1:=A-(1)*IdentityMatrix(3);
ReducedRowEchelonForm(A1);
LinearSolve(A1,ZeroVector(3));
```

34. A triangular $n \times n$ matrix with distinct diagonal entries has n eigenpairs. Provide a detailed proof for the case $n = 3$.

35. Find all eigenpairs of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

Solution: There is only one eigenpair, all eigenvalues $= 1$:

$$\left(1, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

```
# Exercise 35
A:=Matrix([[1,2,0],[0,2,2],[0,0,3]]);
CharacteristicPolynomial(A,r);
 #  r^3  - 6 r^2  + 11 r - 6
Eigenvalues(A);Eigenvectors(A);
A1:=A-(1)*IdentityMatrix(3);
ReducedRowEchelonForm(A1);
LinearSolve(A1,ZeroVector(3));
```

36. A triangular $n \times n$ matrix may not have n eigenpairs. Provide a series of examples for dimensions $n = 2, 3, 4, 5$.

37. Prove that equations $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ and $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have exactly the same solutions $\vec{\mathbf{x}}$.

Solution:

Proof:
Part I. Let $\vec{\mathbf{x}}$ solve $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$. Then

$(A - \lambda I)\vec{\mathbf{x}} = A\vec{\mathbf{x}} - \lambda I\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}} - \lambda\vec{\mathbf{x}} = \vec{\mathbf{0}}$

Part II. Let $\vec{\mathbf{x}}$ solve $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Then

$A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}} + (A\vec{\mathbf{x}} - \lambda\vec{\mathbf{x}}) = \lambda\vec{\mathbf{x}} + \vec{\mathbf{0}} = \lambda\vec{\mathbf{x}}$

Combine Part I and Part II: the equations have the same solutions. ■

38. Cite basic linear algebra theorems to prove that $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a nonzero solution $\vec{\mathbf{x}}$ if and only if λ is a root of the characteristic equation $|A - \lambda I| = 0$.

Basis of Eigenvectors

The problem $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ has a standard general solution $\vec{x}$ with invented symbols $t_1, t_2, t_3, \ldots$. **Strang's special solutions** are defined to be the vector partial derivatives of $\vec{x}$ with respect to the invented symbols.

39. Why are Strang's special solutions independent?

Solution:
First solution: We can cite a theorem which says they are independent: Theorem 5.22 page 370 .

Second solution: The plan is prove that a linear combination of the special solutions equal to the zero vector has all weights zero.
The special solutions are given as vector partial derivatives on the free variables $t_1, \ldots, t_k$ where k is the rank of the matrix. A linear combination of the special solutions with weights $c_1, \ldots, c_k$ is the same as the vector general solution with substitutions $t_1 = c_1, \ldots, t_k = c_k$. Setting this linear combination equal to the zero vector is the same as setting the vector general solution equal to the zero vector, except for notation. Then the corresponding scalar general solution is also zero. But the free variables $t_1, \ldots, t_k$ then appear in the zero scalar general solution in k scalar equations $0 = t_1, \ldots, 0 = t_k$. The other scalar equations in the general solution not of this form are ignored for this analysis. This means all free variables $t_1, \ldots, t_k$ are zero, which also means all weights $c_1, \ldots, c_k$ are zero. The special solutions are proved independent. ■

40. Prove that linear combinations of Strang's special solutions provide all possible solutions of $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

Independence of Eigenvectors

Eigenvectors of matrix A for eigenvalue λ are the nonzero solutions of $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

41. Invent a 2×2 example A with eigenpairs $\left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, $\left(2, \begin{pmatrix} 5 \\ 5 \end{pmatrix}\right)$. Then explain why an eigenvector for eigenvalue λ is never unique.

Solution:
Explanation: Only the first eigenpair is used to construct the example because $\begin{pmatrix} 5 \\ 5 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $c = 5$. A constant multiple of an eigenvector is also an eigenvector, therefore an eigenvector for eigenvalue λ is never unique. Eigenvectors can be unique up to a constant multiple. If an eigenvalue has

algebraic multiplicity greater than 1, then uniqueness up to a constant multiple fails.

An example: let $D = \textbf{diag}(1, 2)$, $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $A = PDP^{-1}$. Then A has two distinct eigenvalues $1, 2$ and two independent eigenvectors. One eigenpair is $\left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ and another is is $\left(2, 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.

42. Explain: *For a given eigenvalue λ, there are infinitely many eigenvectors.*

43. Explain: *Each solution $\vec{\mathbf{x}}$ of $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ is a linear combination of Strang's special solutions for $B = A - \lambda I$.*

Solution:
Solution 1. Apply Theorem 5.22 page 370.

Solution 2. Equation $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ has the same solutions as equation $(A - \lambda I)\,\vec{\mathbf{x}}$, or $B\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Then $\vec{\mathbf{x}}$ is a linear combination of Strang's special solutions by elimination methods for solving homogeneous linear algebraic equations of the form $B\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

44. Let P be an invertible 3×3 matrix. Construct a matrix A which has eigenvectors equal to the columns of P and corresponding eigenvalues $-1, 0, 0$.

Eigenspaces

Let $\mathcal{B}(\lambda)$ denote some basis of eigenvectors for the eigenpair equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. The **eigenspace** for λ is the subspace $\mathbf{span}(\mathcal{B}(\lambda))$.

45. Explain: The eigenspace of λ does not depend on the choice of basis.

Solution: An eigenspace E is a subspace of vector space $\mathcal{R}^n$. It is a vector space itself using the toolkit of $\mathcal{R}^n$. Then $E = \mathbf{span}\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\}$ for every basis $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k$ of E. The vectors in any basis of E satisfy equation $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

46. Every nonzero vector in eigenspace $\mathbf{span}(\mathcal{B}(\lambda))$ is an eigenvector of A for eigenvalue λ. Provide details of proof.

47. Justify that $\mathbf{span}(\mathcal{B}(\lambda))$ is a vector subspace of $\mathcal{R}^n$, one possible basis being Strang's special solutions for matrix $B = A - \lambda I$.

Solution: Apply to matrix B the Kernel Theorem 5.2 page 300.

48. Find a 4×4 matrix A with only one eigenvalue $\lambda = 1$ such that eigenspace $\mathcal{B}(\lambda)$ (defined above) has dimension two.

Independence of Unions of Eigenvectors

Denote by $\mathcal{B}(\lambda)$ some basis for the eigenpair equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$.

49. Define U_1 to be the union of lists $\mathcal{B}(\lambda_1)$, $\mathcal{B}(\lambda_2)$ and define U_2 to be the union of lists $\mathcal{B}(\lambda_3)$, $\mathcal{B}(\lambda_4)$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is a list of distinct eigenvalues of A. Prove that subspaces $V_1 = \mathbf{span}(U_1)$ and $V_2 = \mathbf{span}(U_2)$ intersect in only the zero vector.

Solution:
Let $\vec{\mathbf{x}} \in V_1 \cap V_2$. We will prove $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

Vector $\vec{\mathbf{x}}$ is a linear combination $\sum_{i=1}^{k} c_i\vec{\mathbf{v}}_i$ of basis vectors $\vec{\mathbf{v}}_i$ from U_1. Also vector $\vec{\mathbf{x}}$ is a linear combination $\sum_{j=1}^{\ell} d_j\vec{\mathbf{w}}_j$ of basis vectors $\vec{\mathbf{w}}_j$ from U_2. The proof will be completed by showing that all weights are zero: $c_i = d_j = 0$.

Theorem 9.5 page 673 about unions of eigenvectors tell us that the list $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{w}}_1, \ldots, \vec{\mathbf{w}}_\ell$ is independent. Then independence and the equation

$\vec{\mathbf{x}} - \vec{\mathbf{x}} = \sum_{i=1}^{k} c_i\vec{\mathbf{v}}_i - \sum_{j=1}^{\ell} d_j\vec{\mathbf{w}}_j = \vec{\mathbf{0}}$

results in all weights zero: $c_i = d_j = 0$. Conclusion: zero is the only vector in the intersection of V_1 and V_2. ■

50. Complete the details of the induction proof of Theorem 9.5, using the textbook details for $k = 3$.

51. Let U^* be a subset of the list U of independent vectors in Theorem 9.5. Explain why U^* is an independent set.

Solution: Subsets of independent sets are independent: Theorem 5.24 page 378.

52. Let B_i be a subset of the list of independent vectors in $\mathcal{B}(\lambda_i)$, $i = 1, \ldots, p$. Explain why the union U^* of $B_1, \ldots, B_p$ is an independent set.

Diagonalization Theory

53. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

(a) Find Strang's special solutions for each eigenvalue.
(b) Compare to Theorem 9.7 on diagonal matrices.

Solution: (a) The eigenvalues are the diagonal elements. The eigenpairs are $(2, \vec{\mathbf{v}}_1), (2, \vec{\mathbf{v}}_2), (2, \vec{\mathbf{v}}_3)$ where $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_3$ are the columns of the 3×3 identity matrix, in order left to right. The method: subtract $\lambda = 2$ from the diagonal of A then row-reduce to $\mathbf{rref}(A-2I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $x_1 = t_1 =$ free variable, $x_2 = x_3 = 0$. The eigenvector is $\partial_{t_1}\vec{\mathbf{x}}$ = column 1 of I. Similar for the other two eigenvectors.

(b) Same result as the theorem.

54. Let $\vec{\mathbf{v}}_!, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ be independent vectors in $\mathcal{R}^3$. Explain why $(0, \vec{\mathbf{v}}_1)$, $(0, \vec{\mathbf{v}}_2)$, $(0, \vec{\mathbf{v}}_3)$ is a complete set of eigenpairs for the 3×3 zero matrix. Does this contradict Theorem 9.7?

55. Write a proof of Theorem 9.7 for $n = 3$.

Solution: The eigenvalues are the diagonal elements of A, which are symbols a, b, c. Details: expand $|A - \lambda I| = (\lambda - a)(\lambda - b)(\lambda - c)$ and solve for λ.

(1) Subtract $\lambda = a$ from the diagonal of A and reduce to row echelon form $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Convert the matrix problem $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$ to scalar form $0 = 0$, $x_2 = 0$, $x_3 = 0$. Then $x_1 = t_1 =$ free variable, $x_2 = x_3 = 0$. The eigenvector is $\partial_{t_1}\vec{\mathbf{x}}$ = column 1 of I.
(2) Repeat (1) for $\lambda = b$, result $\vec{\mathbf{v}}_2$ = column 2 of I.
(2) Repeat (1) for $\lambda = c$, result $\vec{\mathbf{v}}_2$ = column 3 of I.

56. State Theorem 9.7 for $n \times n$ diagonal matrices and outline a proof.

Non-diagonalizable Matrices

Verify that the matrix is not diagonalizable by using the equation $AP = PD$.

57. $A = \begin{pmatrix} 5 & 2 \\ 0 & 5 \end{pmatrix}$

Solution: Eigenvalues are on the diagonal of A. Then $D = \mathbf{diag}(5, 5)$. Use $AP = PD$ to reach a contradiction. Compute $PD = 5P$. Then $AP = 5P$. Multiply right by P^{-1}, assumed to exist. Then $APP^{-1} = 5PP^{-1}$ simplifies to $\begin{pmatrix} 5 & 2 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, which is false. Contradiction reached. ■

58. $A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$

Distinct Eigenvalues

Find the eigenvalues.

59. $A = \begin{pmatrix} 2 & 6 \\ 5 & 3 \end{pmatrix}$ Ans: $8, -3$

Solution: Characteristic equation: $|A - \lambda I| = \lambda^2 - 5\lambda - 24$ with roots $8, -3$.

60. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ Ans: $0, 5$

61. $A = \begin{pmatrix} 2 & 6 & 2 \\ 9 & 3 & 9 \\ 1 & 3 & 1 \end{pmatrix}$ Ans: $0, 12, -6$

Solution: $|A - \lambda I| = \lambda^3 - 6\,\lambda^2 - 72\,\lambda$ with roots $0, 12, -6$ found by factoring.

62. $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 3 \end{pmatrix}$ Ans: $0, 1, 3$

63. $A = \begin{pmatrix} 7 & 12 & 6 \\ 2 & 2 & 2 \\ -7 & -12 & -6 \end{pmatrix}$ Ans: $0, 1, 2$

Solution: $|A - \lambda I| = \lambda^3 - 3\,\lambda^2 + 2\,\lambda$ with roots $0, 1, 2$ found by factoring.

64. $A = \begin{pmatrix} 2 & 2 & -6 \\ -3 & -4 & 3 \\ -3 & -4 & -1 \end{pmatrix}$ Ans: $0, 1, 4$

Computing 2×2 Eigenpairs

65. Verify eigenpairs: $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$,
$\left(-1, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$, $\left(5, \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}\right)$

Solution: The plan: check the answer, do not compute eigenvalues or eigenvectors.

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, $\vec{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\lambda_1 = -1$. Then:

$$A\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_1 \vec{\mathbf{v}}_1 = (-1) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then $A\vec{\mathbf{v}}_1 = \lambda_1 \vec{\mathbf{v}}_1$, verifying the first eigenpair.
The second eigenpair is done similarly.

```
# Exercise 65, Answer check
A:=<1,2|4,3>^+;Eigenvectors(A);
 #[5, -1]), v1=[1/2, 1], v2=[-1, 1]
```

66. Verify eigenpairs: $\begin{pmatrix} 1 & 6 \\ 2 & -3 \end{pmatrix}$,
$\left(-5, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} 3 \\ -1 \end{pmatrix}\right)$

Solution:

```
# Exercise 66, Answer check
A:=<1,6|2,-3>^+;Eigenvectors(A);
 # [3, -5], v1=[1/2, 1], v2=[-1, 1]
```

67. Verify eigenpairs: $\begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix}$,
$\left(7, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, $\left(-3, \begin{pmatrix} -3 \\ 2 \end{pmatrix}\right)$

Solution:

```
# Exercise 67, Answer check
A:=<1,2|4,3>^+;Eigenvectors(A);
 # [7, -3], v1=[1, 1], v2=[-3/2, 1]
```

68. Verify eigenpairs: $\begin{pmatrix} 7 & 4 \\ -1 & 3 \end{pmatrix}$,
$\left(5, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$, only one eigenpair

Solution:

```
# Exercise 68, Answer check
A:=<7,4|-1,3>^+;Eigenvectors(A);
 # [5, 5], v1=[1/2, 1], v2=[0, 0] invalid
```

Computing 2×2 Complex Eigenpairs

69. Verify eigenpairs: $\begin{pmatrix} -2 & -6 \\ 3 & 4 \end{pmatrix}$,
$\left(1+3i, \begin{pmatrix} -1+i \\ 1 \end{pmatrix}\right)$, $\left(1-3i, \begin{pmatrix} -1-i \\ 1 \end{pmatrix}\right)$

Solution: The first eigenpair is checked like in Exercise 65.

Let $A = \begin{pmatrix} -2 & -6 \\ 3 & 4 \end{pmatrix}$, $\vec{\mathbf{v}}_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$, $\lambda_1 = 1+3i$. Then:

$$A\vec{\mathbf{v}}_1 = \begin{pmatrix} -2 & -6 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2-2i-6 \\ -3+3i+4 \end{pmatrix} = \begin{pmatrix} -4-2i \\ 1+3i \end{pmatrix}$$

$$\lambda_1\vec{\mathbf{v}}_1 = (1+3i)\begin{pmatrix} -1+i \\ 1 \end{pmatrix} = \begin{pmatrix} -4-2i \\ 1+3i \end{pmatrix}$$

Then $A\vec{\mathbf{v}}_1 = \lambda_1\vec{\mathbf{v}}_1$, verifying the first eigenpair.

The second eigenpair is checked by replacing i by $-i$ throughout the first eigenpair.

```
# Exercise 69, Answer check
A:=<-2,-6|3,4>^+;Eigenvectors(A);
 # [1+3*I, 1-3*I], v1=[-1+I, 1], v2=[-1-I, 1]
lambda:=1+3*I;v:=<-1+I,1>;A.v - lambda*v;# zero expected
```

70. Verify eigenpairs: $\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$,
$\left(2+3i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$, $\left(2-3i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$

Solution:

```
# Exercise 70, Answer check
A:=<2,-3|3,2>^+;Eigenvectors(A);
 # [2+3*I, 2-3*I]
 # v1=[I, 1], v2=[-I, 1]
lambda:=2+3*I;v:=<I, 1>;
simplify(A.v - lambda*v);# zero expected
```

71. Let a, b be real with $b \neq 0$. Assume $n \times n$ real matrix A has eigenpair $(a+ib, \vec{\mathbf{v}})$. Replace i by $-i$ throughout expression $\vec{\mathbf{v}}$ to obtain vector $\vec{\mathbf{w}}$. Prove that $(a-ib, \vec{\mathbf{w}})$ is an eigenpair.

Solution:

Proof:

To be verified is equation $A\vec{\mathbf{w}} = \lambda\vec{\mathbf{w}}$ where $\lambda = a - ib$. Notation: an overline on a symbol denotes complex conjugation, which is replacing i by $-i$.

$$\begin{aligned} A\vec{\mathbf{w}} &= A\overline{\vec{\mathbf{v}}} \\ &= \overline{A}\overline{\vec{\mathbf{v}}} && \text{because } A \text{ is a real matrix} \\ &= \overline{A\vec{\mathbf{v}}} && \text{by college algebra rule } \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} \\ &= \overline{\lambda\vec{\mathbf{v}}} && \text{by } A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} \\ &= (a-ib)\vec{\mathbf{w}} && \blacksquare \end{aligned}$$

72. Explain: Eigenpairs of a 2×2 real matrix A with complex eigenvalues are computed with just one row-reduction sequence.

Computing 3×3 Eigenpairs

73. Show algorithm steps to compute eigenpairs of $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Answers: $\left(1, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

Solution: There are only two eigenpairs, not three. The eigenvalues $\lambda = 1, 1, 3$ are found from the characteristic equation $|A - \lambda I| = 0$, which is the cubic equation $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = (3 - \lambda)(\lambda^2 - 2\lambda + 1) = 0$.

> Cofactor expansion of the determinant produces a pre-factored equation. Make this your default method of attack on paper. Computer algebra systems like `maple` will display the expanded polynomial and then extra steps are required to factor it or to find the roots.

Steps for $\lambda = 1$:

Create matrix $B = A - \lambda I = A - (1)I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Row-reduce B to $\mathbf{rref}(B)$ and find Strang's Special Solutions:

$$B = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \texttt{mult(3,1/2)}$$

$$B_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \texttt{combo(1,2,1)}$$

$$B_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \texttt{swap(2,3)}$$

Then $B_3 = \mathbf{rref}(B)$ and the scalar equations for $B\vec{\mathbf{x}} = \vec{\mathbf{0}}$ are $x_1 + x_2 = 0$, $x_3 = 0$. The lead variables are x_1, x_3 and $x_2 =$ free variable. Let $x_2 = t_1 =$ invented symbol. Due to only one free variable, there will be only one eigenvector. The scalar solution is $x_1 = -t_1$, $x_2 = t_1$, $x_3 = 0$ and Strang's solution is

$$\vec{\mathbf{v}}_1 = \partial_{t_1}\vec{\mathbf{x}} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

There is no eigenvector $\vec{\mathbf{v}}_2$.

Steps for $\lambda = 3$:
The eigenvector for $\lambda = 3$ is found similarly by creating matrix $B = A - \lambda I = A - (3)I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Row-reduce B to $\mathbf{rref}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The scalar equations for $B\vec{\mathbf{x}} = \vec{\mathbf{0}}$ are $x_1 = 0$, $x_2 = 0$. The lead variables are x_1, x_2 and $x_3 =$ free variable. Let $x_3 = t_1 =$ invented symbol. Due to only one free variable, there will be only one eigenvector. The scalar solution is $x_1 = 0$, $x_2 = 0$, $x_3 = t_1$ and Strang's solution is

$$\vec{\mathbf{v}}_3 = \partial_{t_1}\vec{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

```
# Exercise 73, Compute 3x3 eigenpairs
A:=<2,1,0|-1,0,0|0,0,3>^+;p:=Eigenvectors(A);
 # lambda=  [1, 1, 3]
 # v1=[-1, 1, 0], v2=[0, 0, 0], v3=[0, 0, 1]
CharacteristicPolynomial(A,'lambda');
 # lambda^3-5*lambda^2+7*lambda-3
Z:=<0,0,0>;# Solve (A-I)x=Z, (A-3I)x=Z
ReducedRowEchelonForm(A-1);LinearSolve(A-1,Z);
ReducedRowEchelonForm(A-3);LinearSolve(A-3,Z);
```

74. Show algorithm steps to compute eigenpairs of $A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 4 & -4 & -1 \end{pmatrix}$.

Answers:
$\left(1, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right)$, $\left(-1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$,
$\left(-1, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

75. Suppose A is row-reduced to a triangular form B. Are the eigenvalues of B also the eigenvalues of A? Give a proof or a counter-example.

Solution: Counterexample: Choose an invertible matrix A with at least two complex eigenvalues.Then $\mathbf{rref}(A) = I$, which has all eigenvalues equal to one. ■

```
# Exercise 75, Counterexample
A:=<1 , 2 , 4|-2 , 1 , 0|0 , 0 , -1>^+;
with(LinearAlgebra):
Eigenvalues(A);
 # lambda = -1. 1 + 2i, 1-2i
B:=ReducedRowEchelonForm(A);
Eigenvalues(B);
 # lambda = 1,1,1
```

76. Suppose $A - \lambda I$ is row-reduced to a triangular form B. Explain: The eigenvalues of A are usually unrelated to the roots λ of $|B| = 0$.

Decomposition $A = PDP^{-1}$

Compute the eigenpairs. If diagonalizable, then display D, P and Fourier's replacement equation.

77. $A = \begin{pmatrix} 7 & 4 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Ans: only 2 eigenpairs

Solution: The expected solution is by hand. It terminated when $\lambda = 5$ produced $\mathbf{rank}(A-5I) = 2$, meaning only one free variable.

```
# Exercise 77, A=P.D.(1/P) failed
A:=<7,4,0|-1,3,0|0,0,3>^+; Eigenvectors(A);
 # lambda = 5,5,3
 # v1=[-2,1,0], v2=[0,0,0] (not an eigenvector), v3=[0,0,1]
 Rank(A-5);
 # rank = number lead vars = 2, nullity = number free vars = 1
```

.

78. $A = \begin{pmatrix} 1 & 6 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Ans: $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \begin{pmatrix} 3 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Fourier equation: $AP\vec{\mathbf{c}} = PD\vec{\mathbf{c}}$.

Diagonalization

Report **diagonalizable** or not and explain why.

79. $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}$

Ans: diagonalizable

Solution: Cofactor expansion of $|A - \lambda I|$ along the last row produces the factored form $((1-\lambda)^2 - 4)(3-\lambda)(-3-\lambda)$. Then $\lambda = -1, -3, 3, 3$. To decide diagonalizability we only need to find the nullity of $A - 3I$, by row reduction.

$$A - 3I = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{the RREF after several row operations.}$$

Then $\mathbf{rank}(A-3I) = 2$, $\mathbf{nullity}(A-3I) = 4 - \mathbf{rank}(A-3I) = 2$ and matrix A is diagonalizable.

```
# Exercise 79, Diagonalization
A:=<1,2,0,0|2,1,0,0|0,0,3,1|0,0,0,-3>^+;
Eigenvectors(A);
RowDimension(A)-Rank(A-3);
```

80. $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
Ans: not diagonalizable

Non-diagonalizable Matrices

81. Verify $A = \begin{pmatrix} 1 & 2 \\ -8 & 9 \end{pmatrix}$ is not diagonalizable.

Solution: The eigenvalues: $\lambda = 5, 5$. To decide compute the nullity of $A-5I$ by row operations.

$A - 5I = \begin{pmatrix} -4 & 2 \\ -8 & 5 \end{pmatrix}$ subtract 5 along the diagonal of A

$\mathbf{rref}(A - 5I) = \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$ after some row operations

Then $\mathbf{rank}(A-6I) = 1$, $\mathbf{nullity}(A-5I) = 1$. There is only one eigenvector for $\lambda = 5$, because there is only one free variable. Conclusion: A is not diagonalizable.

82. Verify $A = \begin{pmatrix} 1 & 2 & 0 \\ -8 & 9 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ is not diagonalizable.

83. Invent a 3×3 matrix which has exactly one eigenpair.

Solution: Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The eigenvalues are $\lambda = 0, 0, 0$. Then $\mathbf{rank}(A-0I) = 2$ and $\mathbf{nullity}(A-0I) = 1$. There is only one free variable hence only one eigenvector.

84. Invent a 4×4 matrix which has exactly two eigenpairs.

Fourier's Heat Model

Define
$\vec{\mathbf{v}}_1 = \sin \pi x, \vec{\mathbf{v}}_2 = \sin 2\pi x, \vec{\mathbf{v}}_3 = \sin 3\pi x$
considered as vectors in the vector space V of twice continuously differentiable functions on $0 \le x \le 1$.

85. Verify that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are independent vectors in V.

Solution: Apply Theorem 6.11 page 453: a finite list of distinct Euler atoms is independent.

86. Verify that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ vanish at $x = 0$ and $x = 1$.

87. Define $u(x) = \sin \pi x$ (from $\vec{\mathbf{v}}_1$). Explain: Function u satisfies differential equation $\dfrac{d^2u}{dx^2} + \pi^2 u = 0$.

Solution: The general solution of the harmonic oscillator $x'' + \omega^2 x = 0$ is $x = c_1 \cos \omega t + c_2 \sin \omega t$. Choose $\omega = \pi$.

88. Write vector expression
$c_1 e^{-\pi^2 t}\vec{\mathbf{v}}_1 + c_2 e^{-4\pi^2 t}\vec{\mathbf{v}}_2$
$+c_3 e^{-9\pi^2 t}\vec{\mathbf{v}}_3$
as a scalar function $u(t,x)$. Find initial heat distribution $u(0,x)$. Explain how Fourier replacement (re-scaling) constructs future state $u(t,x)$ from initial state $u(0,x)$.

9.2 Eigenanalysis Applications

Discrete Dynamical Systems

Define matrix A via equation

$$\vec{\mathbf{y}} = \frac{1}{10}\begin{pmatrix} 5 & 1 & 0 \\ 3 & 4 & 3 \\ 2 & 5 & 7 \end{pmatrix}\vec{\mathbf{x}} \tag{1}$$

1. Find eigenpair packages of A.
Answers:
$D=\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$P=\begin{pmatrix} -1 & 1 & 1 \\ 0 & -4 & 5 \\ 1 & 3 & 9 \end{pmatrix}$
Solution:

```
# Exercise 1, Discrete Dynamical Systems
with(LinearAlgebra):
A:=(1/10)*Matrix([[5,1,0],[3,4,3],[2,5,7]]);
B:=A-lambda*IdentityMatrix(3);
DD,P:=Eigenvectors(A);
 # lambda = [1/2, 1, 1/10]
 # v1=[-1, 0, 1], v2=[1/9, 5/9, 1], v3=[1/3, -4/3, 1]
factor(Determinant(B));
 # -(1/20*(lambda-1))*(2*lambda-1)*(10*lambda-1)
```

2. Explain: A is a **transition matrix**.[1]

3. Assume $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ has period one year. Find the system state after two years.
Solution: $A\vec{\mathbf{x}}$ is the state after one year, $A^2\vec{\mathbf{x}}$ is after two years. Compute in `maple`:

$$A^2 = \begin{pmatrix} \frac{7}{25} & \frac{9}{100} & \frac{3}{100} \\ \frac{33}{100} & \frac{17}{50} & \frac{33}{100} \\ \frac{39}{100} & \frac{57}{100} & \frac{16}{25} \end{pmatrix}$$

[1]Perron-Frobenius theory extensions in the literature apply to transition matrices. See the Weierstrass Proof exercises.

```
# Exercise 3, Discrete Dynamical Systems
with(LinearAlgebra):
A:=(1/10)*Matrix([[5,1,0],[3,4,3],[2,5,7]]);
X:=<x1,x2,x3>;
B:=A^2;
y:=B . X;
```

4. Explain: $A^n\vec{\mathbf{x}}$ is the system state after n periods.

Market Shares

Define matrix A via equation

$$\vec{\mathbf{y}} = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix} \vec{\mathbf{x}} \tag{2}$$

5. Find with software the eigenpairs of A given by equation 2.

Solution: The `maple` eigenvectors $v1, v2, v$ have fractions. Multiply to clear fractions. Then:

$$\vec{\mathbf{v}}_1 = 13 * v1 = \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = v2 = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = v3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

```
# Exercise 5, Market shares
A:=(1/10)*Matrix([[5,4,0],[3,5,3],[2,1,7]]);
Eigenvectors(A);
 # lambda = [1, 1/5, 1/2]
 # v1=[12/13, 15/13, 1], v2=[-4, 3, 1], v3=[-1, 0, 1]
```

6. Compute A^2, A^3, A^4 using software. Predict the limit of A^n as n approaches infinity.

7. Compute with software (rounded)

$$A^{10} = \begin{pmatrix} .30 & .30 & .30 \\ .37 & .38 & .37 \\ .32 & .32 & .33 \end{pmatrix} \tag{3}$$

Solution:

$$A^{10} = \begin{pmatrix} 0.3001953637 & 0.3005207480 & 0.2992188012 \\ 0.3749999616 & 0.3750000640 & 0.3749999616 \\ 0.3248046747 & 0.3244791880 & 0.3257812372 \end{pmatrix}$$

```
# Exercise 7, Market shares
A:=(1/10)*Matrix([[5,4,0],[3,5,3],[2,1,7]]);
B:=A^10;
evalf(B);# rounded to default number of digits
```

8. Let $\vec{\mathbf{x}}=\frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$. Compute

$A^{10}\vec{\mathbf{x}} = \begin{pmatrix}0.30\\0.37\\0.33\end{pmatrix}$ (rounded)

in two ways by calculator:
(1) Fourier replacement (3).
(2) Matrix multiply using (17).

Stochastic Matrices

Reference: Perron-Frobenius proof on page 715.

9. Establish the identity $|A - \lambda I| = |A^T - \lambda I|$.

Solution: Determinant theory provides $|B| = |B^T|$ for any square matrix B and $(C + D)^T = C^T + D^T$ for any two square matrices C,D.

Let $B = A - \lambda I$. Then $B^T = (A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$. Apply the identity $|B| = |B^T|$. ■

10. Explain why A and A^T have the same eigenvalues but not necessarily the same eigenvectors.

11. Verify $\mathbf{max_r}(A) = \langle \vec{\mathbf{w}}\,|\vec{\mathbf{w}}\,|\cdots|\vec{\mathbf{w}}\,\rangle$, where $\vec{\mathbf{w}}$ has components $w_i = \max\{a_{ij}, 1 \le j \le n\}$.

Solution: Let $B = \mathbf{max_r}(A)$, which is the $n \times n$ matrix formed by replacing a_{ij} by the largest element in row i, for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

We are given vector $\vec{\mathbf{w}}$ with components equal to the largest element in each row:

$$\vec{\mathbf{w}} = \begin{pmatrix}\max\{a_{1j}, 1 \le j \le n\}\\ \vdots \\ \max\{a_{nj}, 1 \le j \le n\}\end{pmatrix}$$

By definition, $b_{ij} = \max\{a_{ij}, 1 \le j \le n\} = w_i$. Therefore, each column of B is a copy of vector $\vec{\mathbf{w}}$. ■

12. Verify $\mathbf{max_r}(A) = D\mathcal{O}$, where D is the diagonal matrix of row maxima and $\mathcal{O}$ is the matrix of all ones.

Perron-Frobenius Theorem

Let $A > 0$ be $n \times n$ stochastic with unique eigenpair $(1, \vec{\mathbf{w}})$, all $w_i > 0$ and $\sum_{i=1}^n w_i = 1$. Assume $\vec{\mathbf{v}} \geq \vec{\mathbf{0}}$, $\sum_{i=1}^n v_i = 1$ and $\delta = \min_{i,j} a_{ij}$.

13. Apply inequality $\mathbf{min_r}(A^n)\vec{\mathbf{v}} \leq A^n\vec{\mathbf{v}} \leq \mathbf{max_r}(A^n)\vec{\mathbf{v}}$ to prove $\lim_{n\to\infty} A^n\vec{\mathbf{v}} = \left(\sum_{i=1}^n v_i\right)\vec{\mathbf{w}} = \vec{\mathbf{w}}$.

Solution:

Proof from Perron-Frobenius:
Part (a) of Perron-Frobenius Theorem 9.13 page 705 concludes $\lim_{n\to\infty} A^n = \left\langle \vec{\mathbf{w}}|\cdots|\vec{\mathbf{w}}\right\rangle$. Uniqueness was used to draw the conclusion. The definition of matrix multiply gives

$$\lim_{n\to\infty} A^n\vec{\mathbf{v}} = \left\langle \vec{\mathbf{w}}|\cdots|\vec{\mathbf{w}}\right\rangle\vec{\mathbf{v}} = \sum_{i-1}^{n} v_i\vec{\mathbf{w}} = \vec{\mathbf{w}}$$

Proof using the inequality:
Apply proof details in Lemma 5a in the Perron-Frobenius proof: $\mathbf{min_r}(A^n)$ and $\mathbf{max_r}(A^n)$ converge by the calculus squeeze theorem to some matrix P.

Limit as $n \to \infty$ across the inequality $\mathbf{min_r}(A^n)\vec{\mathbf{v}} \leq A^n\vec{\mathbf{v}} \leq \mathbf{max_r}(A^n)\vec{\mathbf{v}}$ to obtain inequality $P\vec{\mathbf{v}} \leq \lim_{n\to\infty} A^n\vec{\mathbf{v}} \leq P\vec{\mathbf{v}}$, which implies $\lim_{n\to\infty} A^n\vec{\mathbf{v}} = P\vec{\mathbf{v}}$.

Matrix P has identical elements in each row which means $P = \left\langle \vec{\mathbf{y}}|\cdots|\vec{\mathbf{y}}\right\rangle$ for some vector $\vec{\mathbf{y}}$. Argue as in the proof of Lemma 5a that $\vec{\mathbf{y}} = A\vec{\mathbf{y}}$ and $\vec{\mathbf{y}} > 0$. So $(1, \vec{\mathbf{y}})$ is an eigenpair of A. In summary:

$$\lim_{n\to\infty} A^n\vec{\mathbf{v}} = P\vec{\mathbf{v}} = \left(\sum_{i=1}^{n} v_i\right)\vec{\mathbf{y}} = \vec{\mathbf{y}}$$

It remains to prove $\vec{\mathbf{y}} = \vec{\mathbf{w}}$ by uniqueness. First, $\vec{\mathbf{y}} > \vec{\mathbf{0}}$ was argued above. Second, relation $\sum_{i=1}^n v_i = 1$ implies $\sum_{i=1}^n y_i = 1$ by Stochastic Matrix Properties Theorem 9.12 page 705 and limiting. Because $(1, \vec{\mathbf{y}})$ is an eigenpair of A with properties $\vec{\mathbf{y}} > 0$ and $\sum_{i=1}^n y_i = 1$ then $\vec{\mathbf{y}} = \vec{\mathbf{w}}$ by uniqueness. ■

Brief Proof:
Because $0 \leq A^{k+1}\vec{\mathbf{v}} - A^{k+1+p}\vec{\mathbf{v}} \leq \mathbf{max_r}(A^{k+1})\vec{\mathbf{v}} - \mathbf{min_r}(A^{k+1+p})\vec{\mathbf{v}} \leq \mathbf{max_r}(A^{k+1})\vec{\mathbf{v}} - \mathbf{min_r}(A^{k+1})\vec{\mathbf{v}} \leq (1-\delta)^k\mathcal{O}\vec{\mathbf{v}}$, then in the limit as $p \to \infty$ $A^{k+1}\vec{\mathbf{v}} - (\sum_{i=1}^n v_i)\vec{\mathbf{w}} \leq (1-\delta)^k\mathcal{O}\vec{\mathbf{v}}$. Because $\sum_{i=1}^n v_i = 1$, simplification shows that each vector entry on the left is no greater than $(1-\delta)^k$, which implies the result. ■

14. Verify Euclidean norm inequality
$\|A^{k+1}\vec{\mathbf{v}} - \vec{\mathbf{w}}\| \leq \sqrt{n}\,(1-\delta)^k$

Weierstrass Proof

These exercises establish existence of an eigenpair $(1, \vec{\mathbf{v}})$ for stochastic matrix A having only nonnegative entries.

Weierstrass Compactness Theorem

A sequence of vectors $\{\vec{\mathbf{v}}_i\}_{i=1}^{\infty}$ contained in a closed, bounded set K in $\mathcal{R}^n$ has a subsequence converging in the vector norm of $\mathcal{R}^n$ to some vector $\vec{\mathbf{v}}$ in K.

Define set K to be all vectors $\vec{\mathbf{v}}$ with nonnegative components adding to 1. Let $\vec{\mathbf{v}}_0$ be any element of K. Assume stochastic A with $a_{ij} \geq 0$ and define $\vec{\mathbf{v}}_N = \frac{1}{N}\sum_{j=0}^{N-1} A^j \vec{\mathbf{v}}_0$.

15. Verify K is closed and bounded in $\mathcal{R}^n$. Then prove $\lambda\vec{\mathbf{x}} + (1-\lambda)\vec{\mathbf{y}}$ is in K for $0 \leq \lambda \leq 1$ and $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ in K.

Solution:

Closed and Bounded.

Let $\vec{\mathbf{u}}$ be the vector of all ones. The first requirement can be written as $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 1$. A convergent sequence whose terms $\vec{\mathbf{v}}$ have nonnegative entries and satisfy $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 1$ gives two relations for the sequence limit $\vec{\mathbf{v}}^*$:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}^* = 1 \quad \text{and} \quad v_i^* \geq 0$$

So K is closed.

Set K is norm-bounded because $\|\vec{\mathbf{v}}\|^2 = \sum_{i-1}^{n} v_i^2 \leq \sum_{i-1}^{n} v_i = 1$, due to inequality $0 \leq v_i \leq 1$.

Convexity.

Let $0 \leq \lambda \leq 1$. Let $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ be in K. Let $\vec{\mathbf{z}} = \lambda\vec{\mathbf{x}} + (1-\lambda)\vec{\mathbf{y}}$. To be proved:

(1) $\vec{\mathbf{z}} \geq \vec{\mathbf{0}}$
(2) $\sum_{i=1}^{n} z_i = 1$

Item (1): Because $\vec{\mathbf{x}} \geq \vec{\mathbf{0}}$ and $\vec{\mathbf{y}} \geq \vec{\mathbf{0}}$ then

$\vec{\mathbf{z}} = \lambda\vec{\mathbf{x}} + (1-\lambda)\vec{\mathbf{y}} \geq \lambda\vec{\mathbf{0}} + (1-\lambda)\vec{\mathbf{0}} = \vec{\mathbf{0}}$

Item (2):

$$\begin{aligned}\sum_{i=1}^{n} z_i &= \sum_{i=1}^{n} (\lambda x_i + (1-\lambda) y_i) \\ &= \lambda \sum_{i=1}^{n} x_i + (1-\lambda)\sum_{i=1}^{n} y_i \\ &= \lambda(1) + (1-\lambda)(1) = 1\end{aligned}$$

Convexity verified. ■

16. Prove identity
$\vec{\mathbf{v}}_{N+1} = \lambda\vec{\mathbf{v}}_N + (1-\lambda)A^N\vec{\mathbf{v}}_0$
where $\lambda = \frac{N}{N+1}$ and then prove by induction that $\vec{\mathbf{v}}_N$ is in K.

17. Verify all hypotheses in the Weierstrass theorem applied to $\{\vec{\mathbf{v}}_N\}_{N=0}^{\infty}$. Applying the theorem produces a subsequence $\{\vec{\mathbf{v}}_{N_p}\}_{p=1}^{\infty}$ limiting to some $\vec{\mathbf{v}}$ in K.

Solution: To prove: set K is closed and bounded and all sequence elements are in K.

Closed and Bounded.
Exercise 15 establishes K as closed and bounded.

Sequence element $\vec{\mathbf{v}}_N$ is in K.

Vector $\vec{\mathbf{v}}_0$ is given to belong to K. Matrix A is stochastic with nonnegative elements. Stochastic Matrix Properties Theorem 9.12 page 705 shows that $A\vec{\mathbf{v}}_0$ is in K. Several applications imply $A^i\vec{\mathbf{v}}_0$ is in K for $i \geq 0$. Exercise 15 shows directly that $\vec{\mathbf{v}}_2 = \frac{1}{2}(\vec{\mathbf{v}}_0 + A\vec{\mathbf{v}}_0)$ is in K. Induction on the convexity result of Exercise 15 provides: if $\vec{\mathbf{y}}_i$ is in K and $\lambda_i \geq 0$ with $\sum_{i-1}^{k} \lambda_i = 1$ then $\sum_{i=1}^{k} \lambda_i \vec{\mathbf{y}}_i$ is in K. This is called **generalized convexity**. Apply generalized convexity with $\lambda_i = (i+1)/N$, $0 \leq i \leq N-1$. Then $\sum_{i=0}^{N-1} \lambda_i = 1$ and $\vec{\mathbf{v}}_N = \frac{1}{N}\sum_{i=0}^{N-1} A^i\vec{\mathbf{v}}_0 = \sum_{i=0}^{N-1} \lambda_i A^i \vec{\mathbf{v}}_0$. Because $A^i\vec{\mathbf{v}}_0$ is in K then $\vec{\mathbf{v}}_N$ is in K. $\blacksquare$

18. Verify identity
$\vec{\mathbf{v}}_N - A\vec{\mathbf{v}}_N = \frac{1}{N}(\vec{\mathbf{v}}_0 - A^N\vec{\mathbf{v}}_0)$.

19. Explain why $A\vec{\mathbf{v}} = \lim_{p\to\infty} A\vec{\mathbf{v}}_{N_p}$. Then prove $\vec{\mathbf{v}} = A\vec{\mathbf{v}}$.

Solution:
Proof:
The limit is $A\vec{\mathbf{v}}$ because function $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is continuous from $\mathcal{R}^n$ to $\mathcal{R}^n$. Applied is the advanced calculus theorem which says that all subsequences of a convergent sequence are also convergent. Also used: continuity is equivalent to sequential continuity.

Because $\vec{\mathbf{v}}_0$ and $A^N\vec{\mathbf{v}}_0$ are in bounded set K, then their Euclidean norms are bounded by some number $M > 0$. Compute:

$$\begin{aligned}\vec{\mathbf{v}}_N - A\vec{\mathbf{v}}_N &= \frac{1}{N}\left(\sum_{i=0}^{N-1} A^i\vec{\mathbf{v}}_0 - \sum_{i=0}^{N-1} A^{i+1}\vec{\mathbf{v}}_0\right) \\ &= \frac{1}{N}\left(\vec{\mathbf{v}}_0 - A^N\vec{\mathbf{v}}_0\right)\end{aligned}$$

The triangle inequality gives $\|\vec{\mathbf{v}}_N - A\vec{\mathbf{v}}_N\| \leq \frac{1}{N}\left(\|v_0\| + \|A^N\vec{\mathbf{v}}_0\|\right) \leq 2M/N$. Replace N by subsequence values N_p, $p = 1, \ldots, \infty$ and limit $p \to \infty$ to conclude $\vec{\mathbf{v}} = \lim_{p\to\infty} \vec{\mathbf{v}}_{N_p} = A\vec{\mathbf{v}}$. $\blacksquare$

20. The claimed eigenpair $(1, \vec{\mathbf{v}})$ has been found, provided $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$. Explain why $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$.

Coupled Systems

Find the coefficient matrix A. Identify as coupled or uncoupled and explain why.

21. $x' = 2x + 3y$, $y' = x + y$

Solution: Coupled. The matrix $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ is not diagonal.

22. $x' = 3y$, $y' = x$

23. $x' = 3x$, $y' = 2y$

Solution: Uncoupled. The matrix of coefficients is diagonal.

24. $x' = 3x$, $y' = 2y$, $z' = z$

Solving Uncoupled Systems

Solve for the general solution.

25. $x' = 3x$, $y' = 2y$

Solution: No linear algebra required. Both equations are growth-decay equations $u' = au$ with solution $u = u_0 e^{at}$. Then: $x = x_0 e^{3t}$, $y = y_0 e^{2t}$.

26. $x' = 3x$, $y' = 2y$, $z' = z$

Change of Coordinates

Given the change of coordinates $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$, find the matrix B for the inverse change $\vec{\mathbf{x}} = B\vec{\mathbf{y}}$.

27. $\vec{\mathbf{y}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{\mathbf{x}}$

Solution: The matrix is $B = A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

```
# Exercise 27, Change of coordinates
A:=Matrix([[1,0,0],[1,0,1],[0,1,0]]);
B:=1/A;
 # [[1, 0, 0], [0, 0, 1], [-1, 1, 0]]
```

28. $\vec{\mathbf{y}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\mathbf{x}}$

Constructing Coupled Systems

Given the uncoupled system and change of coordinates $\vec{\mathbf{y}} = P\vec{\mathbf{x}}$, find the coupled system.

29. $x_1' = 2x_1$, $x_2' = 3x_2$, $P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

Solution: Let $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ = matrix of coefficients for the uncoupled system.

The coupled system is $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$ where $\vec{\mathbf{u}} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $AP = PD$. Then $A = PDP^{-1} = \begin{pmatrix} \frac{8}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{7}{3} \end{pmatrix}$ and the coupled system is

$$\begin{cases} x' &= & \dfrac{8}{3}x & - & \dfrac{1}{3}y \\ y' &= & -\dfrac{2}{3}x & + & \dfrac{7}{3}y \end{cases}$$

```
# Exercise 29, Constructing Coupled Systems
P:=<1,-1|2,1>^+;
DD:=<2,0|0,3>^+;
A:=P.DD.(1/P);
<D(x),D(y)> = A . <x,y>;
```

30. $x_1' = x_1$, $x_2' = -x_2$, $P = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

Uncoupling a System

Change the given coupled system into an uncoupled system using the eigenanalysis change of variables $\vec{\mathbf{y}} = P\vec{\mathbf{x}}$.

31. $x_1' = 2x_1$, $x_2' = x_1 + x_2$, $x_3' = x_3$

Ans: $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $y_1' = 2y_1$, $y_2' = y_2$, $y_3' = y_3$

Solution: The eigenvectors of coefficient matrix A are the columns of P, reported above. The corresponding eigenvalues are $2, 1, 1$.

```
# Exercise 31, Uncoupling a System
A:=<2,0,0|1,1,0|0,0,1>^+;;
Evalues,P:=Eigenvectors(A);
 # Evalues = [2, 1, 1]
 # P = [[1, 0, 0], [1, 0, 1], [0, 1, 0]]
DD:=(1/P).A.P;# Solve AP=PD for D
 # DD = [[2, 0, 0], [0, 1, 0], [0, 0, 1]]
<D(y1),D(y2),D(y3)> = DD . <y1,y2,y3>;
 # y_'=2y1, y2'=y2, y3'=y3
```

32. $x_1' = x_1 + x_2$, $x_2' = x_1 + x_2$, $x_3' = x_3$

Ans: $P = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $y_1' = 0$, $y_2' = 2y_2$, $y_3' = y_3$

Solving Coupled Systems

Report the answers for $x(t), y(t)$.

33. $x' = -x - 2y$, $y' = -4x + y$

Solution: Eigenanalysis of $A = \begin{pmatrix} -1 & -2 \\ -4 & 1 \end{pmatrix}$ is required. Use $AP = PD$.

Matrix $P = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix}$. Matrix $D = \textbf{diag}(-3, 3)$.

Then $\vec{\mathbf{X}} = P\vec{\mathbf{Y}}$, $Y_1 = ae^{-3t}$, $Y_2 = be^{3t}$, $x = X_1 = ae^{-3t} - \frac{1}{2}b\,e^{3t}$, $y = X_2 = a\,e^{-3t} + b\,e^{3t}$.

```
# Exercise 33, Solving Coupled Systems
A:=<-1,-2|-4,1>^+;
Lambda,P:=Eigenvectors(A);
DD:=DiagonalMatrix(Lambda);
 # DD = [[-3, 0], [0, 3]]
 # P = [[1, -1/2], [1, 1]]
 # Warning: eigenpair order can change!
Y:=<a*exp(Lambda[1]*t),b*exp(Lambda[2]*t)>;
X:=P.Y;
# ans check
de:=diff(x(t),t)=A[1].vars,diff(y(t),t)= A[2].vars;
dsolve([de],[x(t),y(t)]);
q:=subs(x(t)=X[1],y(t)=X[2],[de]);
simplify(q);
```

34. $x' = 8x - y$, $y' = -2x + 7y$

Eigenanalysis and Footballs

The exercises study the ellipsoid
$17x^2 + 8y^2 - 12xy + 80z^2 = 80$.

35. Let $A = \begin{pmatrix} 17 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 80 \end{pmatrix}$. Expand equation $\vec{\mathbf{W}}^T A \vec{\mathbf{W}} = 80$, where $\vec{\mathbf{W}}$ has components x, y, z.

Solution: Answer: $17x^2 - 12xy + 8y^2 + 80z^2 = 80$

```
Exercise 35, Eigenanalysis and Footballs
A:=Matrix([[17,-6,0],[-6,8,0],[0,0,80]]);
W:=<x,y,z>;
expand(W^+ . A . W=80);
 # 7*x^2-12*x*y+8*y^2+80*z^2 = 80
```

36. Find the eigenpairs of
$A = \begin{pmatrix} 17 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 80 \end{pmatrix}$.

37. Verify the semi-axis lengths $4, 2, 1$.

Solution: The standard ellipsoid equation is $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} + \dfrac{Z^2}{c^2} = 1$. Numbers a, b, c are the semiaxis lengths in coordinate system X, Y, Z.

Use relations $\lambda_1 = \dfrac{80}{a^2}$, $\lambda_2 = \dfrac{80}{b^2}$, $\lambda_3 = \dfrac{80}{c^2}$. The order a, b, c depends on the eigenpairs used to create the new X, Y, Z coordinate system.

The computations are possible by hand, but labor intensive. Computer algebra system `maple` will be used.

```
# Exercise 37, Eigenanalysis and Footballs
# Equation 7*x^2-12*x*y+8*y^2+80*z^2 = 80
# Standard form (X^2/16+Y^2/4+Z^2 = 1
A:=Matrix([[17,-6,0],[-6,8,0],[0,0,80]]);
Lambda,P:=Eigenvectors(A);
 # Lambda=[[5, 20, 80]] or some other order
 # P = [[1/2, -2, 0], [1, 1, 0], [0, 0, 1]]
# warning: eigenpairs can be in any order!
semiAxisLength:=proc(eqnRHS,LAMBDA) sqrt(eqnRHS/LAMBDA);end proc;
a:=semiAxisLength(80,Lambda[1]);
b:=semiAxisLength(80,Lambda[2]);
c:=semiAxisLength(80,Lambda[3]);
# Semiaxis lengths a,b,c = 4,2,1 for eigenvalue order 5,20,80
DD:=DiagonalMatrix(Lambda);
W:=<'X','Y','Z'>;
stdForm:=(W^+ . DD . W )/80 = 1;
```

38. Verify that the ellipsoid has semi-axis unit directions
$\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\\0\end{pmatrix}$

The Ellipse and Eigenanalysis

The exercises study the ellipse
$2x^2 + 4xy + 5y^2 = 24$.

39. Let $A = \begin{pmatrix}2 & 2\\2 & 5\end{pmatrix}$. Expand equation $\vec{\mathbf{W}}^T A\vec{\mathbf{W}} = 24$, where $\vec{\mathbf{W}} = \begin{pmatrix}x\\y\end{pmatrix}$.

Solution:

$$\begin{aligned}
\vec{\mathbf{W}}^T A\vec{\mathbf{W}} &= \begin{pmatrix}x\\y\end{pmatrix}^T \begin{pmatrix}2 & 2\\2 & 5\end{pmatrix} \begin{pmatrix}x\\y\end{pmatrix} \\
&= \begin{pmatrix}x\\y\end{pmatrix}^T \begin{pmatrix}2x+2y\\2x+5y\end{pmatrix} \\
&= x2x + 2y + y(2x + 5y) \\
&= 2x^2 + 2xy + 2xy + 5y^2 \\
&= 2x^2 + 4xy + 5y^2
\end{aligned}$$

40. Find the eigenpairs of $A = \begin{pmatrix}2 & 2\\2 & 5\end{pmatrix}$.

41. Verify the semi-axis lengths $2, 2\sqrt{6}$.

Solution: The eigenpairs of A are $\left(6, \begin{pmatrix}1\\2\end{pmatrix}\right)$, $\left(1, \begin{pmatrix}-2\\1\end{pmatrix}\right)$. Let $D =$ **diag**$(5, 1)$. The same equation $2x^2 + 4xy + 5y^2 = 24$ in new coordinates X, Y is

$$\begin{aligned}
&\begin{pmatrix}X\\Y\end{pmatrix}^T \begin{pmatrix}6 & 0\\0 & 1\end{pmatrix} \begin{pmatrix}X\\Y\end{pmatrix} = 24 \\
\rightsquigarrow \quad &\begin{pmatrix}X\\Y\end{pmatrix}^T \begin{pmatrix}6X\\Y\end{pmatrix} = 24 \\
\rightsquigarrow \quad &6X^2/24 + Y^2/24 = 1
\end{aligned}$$

Then $\sqrt{4}, \sqrt{24}$ are the semiaxis lengths.

42. Verify that the ellipse has semi-axis unit directions
$\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}$.

Orthogonal Triad Computation

The exercises fill in details from page 711.

The ellipsoid equation: $x^2{+}4y^2{+}xy{+}4z^2{=}16$ or $\vec{\mathbf{x}}^T A\vec{\mathbf{x}}{=}16$,

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

43. Find the characteristic equation of A. Then verify the roots are 4, $5/2 + \sqrt{10}/2$, $5/2 - \sqrt{10}/2$.

Solution: Characteristic equation $A - \lambda I \ = (4-\lambda)(((1-\lambda)(4-\lambda) - 1/4)$ by cofactor expansion along row 3. Expand and factor:

$$\begin{aligned}(4-\lambda)(((1-\lambda)(4-\lambda)-1/4) &= (4-\lambda)(4-5\lambda+\lambda^2-\tfrac{1}{4}) \\ &= (4-\lambda)(4-5\lambda+\lambda^2-\tfrac{1}{4}) \\ &= \tfrac{1}{4}(\lambda-4)(4\lambda^2-20\lambda+15)\end{aligned}$$

One root is $\lambda = 4$. Quadratic formula roots are $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{10}$.

44. Show the steps from **rref** to second eigenvector $\vec{\mathbf{x}}_2$:

$$\mathbf{rref} = \begin{pmatrix} 1 & 3{-}\sqrt{10} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\vec{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{10}{-}3 \\ 1 \\ 0 \end{pmatrix}$$

9.3 Advanced Topics in Linear Algebra

Diagonalization

Find the eigenpair packages P and D in the relation $AP = PD$.

1. $A = \begin{pmatrix} -4 & 2 \\ 0 & -1 \end{pmatrix}$

Solution:

$D = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$, $P = \begin{pmatrix} 2/3 & 1 \\ 1 & 0 \end{pmatrix}$

```
# Exercise 1, Diagonalization
A:=<-4,2|0,-1>^+;Lambda,P:=Eigenvectors(A);
DD:=DiagonalMatrix(Lambda);
 # Lambda = [-1, -4]
 # P = [[2/3, 1], [1, 0]]
```

2. $A = \begin{pmatrix} 7 & 5 \\ 10 & -7 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution:

$D = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$, $P = \begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix}$

```
# Exercise 3, Diagonalization
A:=<1,2|2,4>^+;Lambda,P:=Eigenvectors(A);
DD:=DiagonalMatrix(Lambda);
 # Lambda = [0, 5]
 # P = [[-2, 1/2], [1, 1]]
```

4. $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

5. $A = \begin{pmatrix} -1 & 0 & 3 \\ 3 & 4 & -9 \\ -1 & 0 & 3 \end{pmatrix}$

Solution:

$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

```
# Exercise 5, Diagonalization
A:=<-1,0,3|3,4,-9|-1,0,3>^+;
Lambda,P:=Eigenvectors(A);
DD:=DiagonalMatrix(Lambda);
 # Lambda = [4, 2, 0]
 # P = [[0, 1, 3], [1, 3, 0], [0, 1, 1]]
```

6. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

7. $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Solution:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \; P = \begin{pmatrix} -1 & 1 & -1/3 & 0 \\ 1 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
# Exercise 7, Diagonalization
A:=<1,1,0,1|1,1,0,1|0,0,-3,0|0,0,0,-1>^+;
Lambda,P:=Eigenvectors(A);
DD:=DiagonalMatrix(Lambda);
 # Lambda = [0, 2, -1, -3]
 # P = [[-1,1,-1/3,0],[1,1,-1/3,0],[0,0,0,1],[0,0,1,0]]
```

8. $A = \begin{pmatrix} 4 & 0 & 0 & 1 \\ 12 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 21 & -6 & 1 & 0 \end{pmatrix}$

Jordan's Theorem

Given matrices P and T, verify Jordan's relation $AP = PT$.

9. $A = \begin{pmatrix} -4 & 2 \\ 0 & -1 \end{pmatrix}$, $P = I$, $T = A$.

Solution: $AP = AI = A = IA = PT$

10. $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

Cayley-Hamilton Theorem

11. Verify that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies

$A^2 = (a+d)A - (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution:
$$\begin{aligned}
\text{LHS} &= A^2 \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= (a+d)A - (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix} - \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}
\end{aligned}$$

Then LHS = RHS. ■

```
# Exercise 11, Diagonalization
A:=<a,b|c,d>^+;
A.A;
(a+d)*A-(a*d-b*c)*IdentityMatrix(2);
p:=simplify(%);
simplify(A.A-p);
```

12. Verify $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{20} = \begin{pmatrix} 1 & 0 \\ 40 & 1 \end{pmatrix}$ by induction using Cayley-Hamilton.

Gram-Schmidt Process

Find the Gram–Schmidt orthonormal basis from the given independent set.

13. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Ans: Columns of I.

Solution: Notation is important for use of the Gram-Schmidt formulas. Follow notation $\vec{\mathbf{x}}_j$, $\vec{\mathbf{y}}_k$ developed in subsection **The Gram-Schmidt process**. After the $\vec{\mathbf{y}}_k$ are found then the final answer will be orthogonal unit vectors $\vec{\mathbf{y}}_k/\|\vec{\mathbf{y}}_k\|$.

Let

$$\vec{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{x}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{x}}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The first two vectors are already of unit length and orthogonal. Therefore Gram-Schmidt gives $\vec{\mathbf{y}}_1 = \vec{\mathbf{x}}_1$ and $\vec{\mathbf{y}}_2 = \vec{\mathbf{x}}_2$. It remains to find

$$\vec{\mathbf{y}}_3 = \vec{\mathbf{x}}_3 - \sum_{k=1}^{2} (\text{vector shadow projection of } \vec{\mathbf{x}}_3 \text{ onto } \vec{\mathbf{y}}_k)$$

The shadow projection formula is

$$\text{Shadow projection of } \vec{\mathbf{X}} \text{ onto a line with direction } \vec{\mathbf{Y}} = \frac{\vec{\mathbf{X}} \cdot \vec{\mathbf{Y}}}{\vec{\mathbf{Y}} \cdot \vec{\mathbf{Y}}} \vec{\mathbf{Y}}$$

The two shadow projections to be inserted into the answer for $\vec{\mathbf{y}}_3$ are:

$$\text{Shadow projection of } \vec{\mathbf{x}}_3 \text{ onto } \vec{\mathbf{y}}_1 = \frac{\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{y}}_1}{\vec{\mathbf{y}}_1 \cdot \vec{\mathbf{y}}_1} \vec{\mathbf{y}}_1 = \frac{-1}{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Shadow projection of } \vec{\mathbf{x}}_3 \text{ onto } \vec{\mathbf{y}}_2 = \frac{\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{y}}_2}{\vec{\mathbf{y}}_2 \cdot \vec{\mathbf{y}}_2} \vec{\mathbf{y}}_2 = \frac{0}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then

$$\vec{\mathbf{y}}_3 = \vec{\mathbf{x}}_3 - \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Because $\vec{\mathbf{y}}_3$ has unit length then $\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3$ are **orthonormal**. The Gram-Schmidt orthonormal basis constructed from $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3$ is the list of columns of the 3×3 identity matrix. ■

```
# Exercise 13, Gram-Schmidt answer check
x1:=<1,0,0>;x2:=<0,1,0>;x3:=<-1,0,1>;
GramSchmidt([x1,x2,x3]);
```

14. $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$.

15. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

Solution: The answer is the list of columns of the 4×4 identity matrix. The paper and pencil solution is computation-intensive, but possible in 5-10 minutes.

```
# Exercise 15, Gram-Schmidt answer check
V:=[<1,0,0,0>,<1,1,0,0>,<1,1,1,0>,<1,1,1,1>];
GramSchmidt(V);
```

16. $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$

Ans: Columns of I.

Gram-Schmidt on Polynomials

Define $V = \mathbf{span}(1, x, x^2)$ with inner product $\int_0^1 f(x)g(x)dx$. Find a Gram–Schmidt orthonormal basis.

17. 1, $1+x$, x^2

Solution: Answer: $1, x - 1/2, x^2 - x + 1/6$

Details for Gram-Schmidt.
Let $\vec{\mathbf{x}}_1 = 1$, $\vec{\mathbf{x}}_2 = 1+x$, $\vec{\mathbf{x}}_3 = x^2$(an abuse of notation, but faster communication). Then $\vec{\mathbf{y}}_1 = 1$ in Gram-Schmidt. Compute $\vec{\mathbf{y}}_2$:

$$\begin{aligned}
\vec{\mathbf{y}}_2 &= \vec{\mathbf{x}}_2 - (\vec{\mathbf{x}}_2 \cdot \vec{\mathbf{y}}_1)\vec{\mathbf{y}}_1/(\vec{\mathbf{y}}_1 \cdot \vec{\mathbf{y}}_1) \\
&= 1 + x - (\textstyle\int_0^1 (1+x)(1)dx)(1)(\int_0^1 (1)^2 dx) \\
&= 1 + x - \left(x + x^2/2 \big|_{x=0}^{1} \right)(1)(1) \\
&= 1 + x - (3/2)(1)(1) \\
&= x - 1/2
\end{aligned}$$

Check: $\vec{\mathbf{y}}_1 = 1$ and $\vec{\mathbf{y}}_2 = x - 1/2$ are independent and $\int_0^1 \vec{\mathbf{y}}_1 \vec{\mathbf{y}}_2 dx = \int_0^1 (1)(x-1/2)dx = 0$: they are orthogonal.

Then $\vec{\mathbf{y}}_1 = 1$, $\vec{\mathbf{y}}_2 = x - 1/2$ in Gram-Schmidt. Compute $\vec{\mathbf{y}}_3$:

$$\begin{aligned}
\vec{\mathbf{y}}_3 &= \vec{\mathbf{x}}_3 - (\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{y}}_1)\vec{\mathbf{y}}_1/(\vec{\mathbf{y}}_1 \cdot \vec{\mathbf{y}}_1) - (\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{y}}_2)\vec{\mathbf{y}}_2/(\vec{\mathbf{y}}_2 \cdot \vec{\mathbf{y}}_2) \\
&= x^2 - (\textstyle\int_0^1 \vec{\mathbf{x}}_3 \vec{\mathbf{y}}_1 dx)\vec{\mathbf{y}}_1/(\vec{\mathbf{y}}_1 \cdot \vec{\mathbf{y}}_1) - (\int_0^1 \vec{\mathbf{x}}_3 \vec{\mathbf{y}}_2 dx)\vec{\mathbf{y}}_2/(\int_0^1 \vec{\mathbf{y}}_2^2 dx) \\
&= x^2 - (\textstyle\int_0^1 x^2 dx)(1)/(1) - (\int_0^1 x^2(x-1/2)dx)(x-1/2)/(\int_0^1 (x-1/2)^2 dx) \\
&= x^2 - (1/3)(1)/(1) - (1/4 - 1/6)(x - 1/2)/(1/12) \\
&= x^2 - 1/3 - (x - 1/2) \\
&= x^2 - x + 1/6
\end{aligned}$$

Check: $\int_0^1 \vec{\mathbf{y}}_1 \vec{\mathbf{y}}_3 dx = \int_0^1 (x^2 - x + 1/6)dx = 1/3 - 1/2 + 1/6 = 0,$

$\int_0^1 \vec{\mathbf{y}}_2\vec{\mathbf{y}}_3 dx = \int_0^1 (x-1/2)(x^2-x+1/6)dx = 0$. Then $\vec{\mathbf{y}}_3$ is orthogonal to $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{y}}_2$. Conclusion: $1, x-1/2, x^2-x+1/6$ are pairwise orthogonal vectors in V.

A Warning about method.
A commonly attempted technique uses the mapping

$$T \;:\; a+bx+cx^2 \to \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The mapping allows the polynomial computation to be computerized, but it also allows efficient hand computation in low dimensions. The mapping is called an **isomorphism**, meaning T is a one-to-one linear map from V onto $\mathcal{R}^3$. A basis constructed from the images

$$T(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T(1+x) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad T(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

will map by the inverse T^{-1} into a basis in V. Because Gram-Schmidt vectors $\vec{\mathbf{y}}_k$ are orthonormal then they form a basis of $\mathcal{R}^3$. Therefore, the inverse mapping provides a basis for V.

Normal Gram-Schmidt on paper will produce vectors $\vec{\mathbf{y}}_1$, $\vec{\mathbf{y}}_2$, $\vec{\mathbf{y}}_3$ from

$$\vec{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{x}}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{x}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The vectors $\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3$ are the columns of the identity matrix. The inverse images are the polynomials

$$p_1 = 1 = T^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = x = T^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad p_3 = x^2 = T^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The process looks like it worked. **But it fails**. The polynomials p_1, p_2, p_3 are independent but **they fail to be orthogonal** with inner product $\left\langle f,\, g \right\rangle = \int_0^1 fg dx$.

18. $1-x$, $1+x$, $1+x^2$

Gram-Schmidt: Coordinate Map

Define $V = \mathbf{span}(1, x, x^2)$ with inner product $\int_0^1 f(x)g(x)dx$. The coordinate map is

$$T:\; c_1 + c_2x + c_3x^2 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

19. Find the images of $1-x$, $1+x$, $1+x^2$ under T.

Solution: See Exercise 17 for a warning using the coordinate map with Gram-Schmidt.

The images are

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

20. Assume column vectors $\vec{\mathbf{x}}_1$, $\vec{\mathbf{x}}_2$, $\vec{\mathbf{x}}_3$ in $\mathcal{R}^3$ orthonormalize under Gram-Schmidt to $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, $\vec{\mathbf{u}}_3$. Are the pre-images $T^{-1}(\vec{\mathbf{u}}_1)$, $T^{-1}(\vec{\mathbf{u}}_2)$, $T^{-1}(\vec{\mathbf{u}}_3)$ orthonormal in V?

Solution: Hint: Read the solution to Exercise 17.

Shadow Projection

Compute shadow vector $(\vec{\mathbf{x}} \cdot \vec{\mathbf{u}})\vec{\mathbf{u}}$ for direction $\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$. Illustrate with a hand–drawn figure.

21. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Ans: $-\frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Solution: Compute $\vec{\mathbf{u}} = \vec{\mathbf{v}}/\|\vec{\mathbf{v}}\| = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then the shadow projection $=$ $(\vec{\mathbf{x}} \cdot \vec{\mathbf{u}})\vec{\mathbf{u}} = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \dfrac{1}{\sqrt{5}}\dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \dfrac{-1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

22. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

23. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Ans: $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Solution: Compute $\vec{\mathbf{u}} = \vec{\mathbf{v}}/\|\vec{\mathbf{v}}\| = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Then the shadow projection $= (\vec{\mathbf{x}} \cdot \vec{\mathbf{u}})\vec{\mathbf{u}} = \left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right) \dfrac{1}{\sqrt{5}}\dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \dfrac{5}{5}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

24. $\vec{\mathbf{x}} = \begin{pmatrix}1\\1\\2\\1\end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix}1\\0\\2\\1\end{pmatrix}$

Orthogonal Projection

Find an orthonormal basis $\{\vec{\mathbf{u}}_k\}_{k=1}^n$ for $V = \mathbf{span}(1+x, x, x+x^2)$, inner product $\int_0^1 f(x)g(x)dx$. Then compute the orthogonal projection $\vec{\mathbf{p}} = \sum_{k=1}^n (\vec{\mathbf{x}} \cdot \vec{\mathbf{u}}_k)\vec{\mathbf{u}}_k$.

25. $\vec{\mathbf{x}} = 1 + x + x^2$

Solution: Vector $\vec{\mathbf{x}}$ is in V so $\vec{\mathbf{p}} = \vec{\mathbf{x}}$.

The basis computation follows. We'll answer-check $\vec{\mathbf{p}} = 1 + x + x^2 = \vec{\mathbf{x}}$.

Exercise 17 gives orthogonal basis $1, x - 1/2, x^2 - x + 1/6$ for $\mathbf{span}(1, x, x^2)$. Because $V = \mathbf{span}(1 + x, x, x + x^2) = \mathbf{span}(1, x, x^2)$ then the same basis can be used in this exercise. An orthonormal basis is required, so unitize the three vectors to obtain $\{\vec{\mathbf{u}}_k\}_{k=1}^3$:

$$\vec{\mathbf{u}}_1 = \frac{1}{\sqrt{\int_0^1 1^2 dx}} = 1$$

$$\vec{\mathbf{u}}_2 = \frac{x - 1/2}{\sqrt{\int_0^1 (x - 1/2)^2 dx}} = \sqrt{12}(x - 1/2)$$

$$\vec{\mathbf{u}}_3 = \frac{x^2 - x + 1/6}{\sqrt{\int_0^1 (x^2 - x + 1/6)^2 dx}} = \sqrt{180}(x^2 - x + 1/6)$$

To compute the orthogonal projection of $\vec{\mathbf{x}} = 1+x+x^2$ requires three inner products to be computed.

$\vec{\mathbf{x}} \cdot \vec{\mathbf{u}}_1 = \int_0^1 (1 + x + x^2)(1)\, dx = 1 + 1/2 + 1/3 = 11/6$

$\vec{\mathbf{x}} \cdot \vec{\mathbf{u}}_2 = \int_0^1 (1 + x + x^2)(x - 1/2)\sqrt{12}\, dx = 1/\sqrt{3}$

$\vec{\mathbf{x}} \cdot \vec{\mathbf{u}}_3 = \int_0^1 (1 + x + x^2)(\sqrt{180}(x^2 - x + 1/6))\, dx = \sqrt{5}/30$

Then $\vec{\mathbf{p}} = (11/6)\vec{\mathbf{u}}_1 + (1/\sqrt{3})\vec{\mathbf{u}}_2 + (\sqrt{5}/30)\vec{\mathbf{u}}_3 = 1 + x + x^2 = \vec{\mathbf{x}}$. ■

26. $\vec{\mathbf{x}} = 1 + 2x + x^2 + x^3$

Orthogonal Projection: Theory

27. Prove that the orthogonal projection $\mathbf{Proj}_V(\vec{\mathbf{x}})$ on $V = \mathbf{span}\{\vec{\mathbf{Y}}\}$ is the vector shadow projection $\mathbf{proj}_{\vec{\mathbf{Y}}}(\vec{\mathbf{x}})$.

Solution: Let $\vec{\mathbf{u}} = \vec{\mathbf{Y}}/\|\vec{\mathbf{Y}}\|$, a unit vector. Then V has orthonormal basis $\{\vec{\mathbf{u}}\}$ and $\mathbf{Proj}_V(\vec{\mathbf{x}}) = (\vec{\mathbf{x}} \cdot \vec{\mathbf{u}})\vec{\mathbf{u}}$. The shadow projection of $\vec{\mathbf{x}}$ onto the line determined by $\vec{\mathbf{Y}}$ is:

$$\begin{aligned}
\mathbf{proj}_{\vec{Y}}(\vec{\mathbf{x}}) &= \frac{\vec{\mathbf{x}} \cdot \vec{\mathbf{Y}}}{\vec{\mathbf{Y}} \cdot \vec{\mathbf{Y}}}\vec{\mathbf{Y}} \\
&= \left(\vec{\mathbf{x}} \cdot \frac{\vec{\mathbf{Y}}}{\|\vec{\mathbf{Y}}\|}\right)\frac{\vec{\mathbf{Y}}}{\|\vec{\mathbf{Y}}\|} \\
&= (\vec{\mathbf{x}} \cdot \vec{\mathbf{u}})\vec{\mathbf{u}} \\
&= \mathbf{Proj}_V(\vec{\mathbf{x}}) \quad \blacksquare
\end{aligned}$$

28. (Gram-Schmidt Construction)
Define $\vec{\mathbf{x}}_j^{\perp} = \vec{\mathbf{x}}_j - \mathbf{Proj}_{W_{j-1}}(\vec{\mathbf{x}}_j)$,
and $W_{j-1} = \mathbf{span}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{j-1})$.
Prove these properties.

(a) Subspace W_{j-1} is equal to the Gram-Schmidt $V_{j-1} = \mathbf{span}(\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_j)$.

(b) Vector $\vec{\mathbf{x}}_j^{\perp}$ is orthogonal to all vectors in W_{j-1}.

(c) The vector $\vec{\mathbf{x}}_j^{\perp}$ is not zero.

(d) The Gram-Schmidt vector is

$$\vec{\mathbf{u}}_j = \frac{\vec{\mathbf{x}}_j^{\perp}}{\|\vec{\mathbf{x}}_j^{\perp}\|}.$$

Near Point Theorem

Find the near point to the subspace V.

29. $\vec{\mathbf{x}} = \begin{pmatrix}1\\1\end{pmatrix}$, $V = \mathbf{span}\left(\begin{pmatrix}1\\2\end{pmatrix}\right)$

Solution: The near point is the orthogonal projection of $\vec{\mathbf{x}}$ onto V, which is the same as the shadow projection of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{Y}} = \begin{pmatrix}1\\2\end{pmatrix}$. Compute $\|\vec{\mathbf{Y}}\| = \sqrt{5}$ then the near point $= \dfrac{\vec{\mathbf{x}} \cdot \vec{\mathbf{Y}}}{\|\vec{\mathbf{Y}}\|^2}\vec{\mathbf{Y}} = \dfrac{3}{5}\begin{pmatrix}1\\2\end{pmatrix}$.

30. $\vec{\mathbf{x}} = \begin{pmatrix}1\\1\end{pmatrix}$, $V = \mathbf{span}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$

31. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, V = \mathbf{span}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$

Solution: The near point is $\frac{1}{9}\begin{pmatrix} 7 \\ 10 \\ 2 \end{pmatrix}$. Computation follows.

Compute an orthonormal basis $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$ for V spanned by $\vec{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\vec{\mathbf{x}}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Details:

$$\vec{\mathbf{y}}_1 = \vec{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\vec{\mathbf{u}}_1 = \frac{\vec{\mathbf{y}}_1}{\|\vec{\mathbf{y}}_1\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\vec{\mathbf{y}}_2 &= \vec{\mathbf{x}}_2 - \frac{\vec{\mathbf{x}}_2 \cdot \vec{\mathbf{y}}_1}{\vec{\mathbf{y}}_1 \cdot \vec{\mathbf{y}}_1}\vec{\mathbf{y}}_1 \\
&= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 4/5 \\ -2/5 \\ 1 \end{pmatrix}
\end{aligned}$$

$$\vec{\mathbf{u}}_2 = \frac{\vec{\mathbf{y}}_2}{\|\vec{\mathbf{y}}_2\|} = \frac{\sqrt{5}}{15}\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$$

Then the near point is the orthogonal projection

$$\begin{aligned}
\mathbf{Proj}_V(\vec{\mathbf{x}}) &= (\vec{\mathbf{x}} \cdot \vec{\mathbf{u}}_1)\vec{\mathbf{u}}_1 + (\vec{\mathbf{x}} \cdot \vec{\mathbf{u}}_2)\vec{\mathbf{u}}_2 \\
&= \begin{pmatrix} 7/9 \\ 10/9 \\ 2/9 \end{pmatrix}
\end{aligned}$$

```
# Exercise 31, Near Point Theorem
x1:=<1,2,0>;x2:=<1,0,1>;
y1:=x1;
u1:=y1/sqrt(y1.y1);
y2:=x2-(x2.y1)*y1/(y1.y1);
u2:=y2/sqrt(y2.y2);
u1.u2;u1.u1;u2.u2;# check orthonormal
X:=<1,1,0>;NearPoint:=(X.u1)*u1+(X.u2)*u2;
 # NearPoint=[7/9, 10/9, 2/9]
```

32. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, V = \mathbf{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$

QR-Decomposition

Give A, find an orthonormal matrix Q and an upper triangular matrix R such that $A = QR$.

33. $A = \begin{pmatrix} 5 & 9 \\ 1 & 7 \\ 1 & 5 \\ 3 & 5 \end{pmatrix}$, Ans: $R = \begin{pmatrix} 6 & 12 \\ 0 & 6 \end{pmatrix}$

Solution: $Q = \begin{pmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ 1/6 & 1/2 \\ 1/2 & -1/6 \end{pmatrix}$

The details by pencil and paper involve Gram-Schmidt on the columns of A. The k columns of A must be independent for success. The method first writes the Gram-Schmidt identities for $\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_k$ in terms of $\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k$. Symbol $\vec{\mathbf{u}}_i = \vec{\mathbf{y}}_i/\|\vec{\mathbf{y}}_i\|$. Then find (see Theorem 9.26, Matrices Q and R in A = QR)

$$R = \begin{pmatrix} \|y_1\| & \vec{\mathbf{u}}_1 \cdot \vec{\mathbf{x}}_2 & \vec{\mathbf{u}}_1 \cdot \vec{\mathbf{x}}_3 & \cdots & \vec{\mathbf{u}}_1 \cdot \vec{\mathbf{x}}_n \\ 0 & \|y_2\| & \vec{\mathbf{u}}_2 \cdot \vec{\mathbf{x}}_3 & \cdots & \vec{\mathbf{u}}_2 \cdot \vec{\mathbf{x}}_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \|y_n\| \end{pmatrix}.$$

Equation $A = QR$ is solved for $Q = AR^{-1}$. Compute R^{-1} with row operations, then find Q by matrix multiply.

```
# Exercise 33, QR-Decomposition
A:=< 5,9 | 1,7 | 1,5 | 3,5 >^+;
Q,R:=QRDecomposition(A);
```

34. $A = \begin{pmatrix} 2 & 1 \\ 2 & 0 \\ 2 & 0 \\ 2 & 1 \end{pmatrix}$, Ans: $R = \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix}$

35. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, Ans: $R = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Solution: $Q = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$

```
# Exercise 35, QR-Decomposition
A:=< 1,0,0 | 1,1,0 | 1,1,0 | 1,0,0 >^+;
Q,R:=QRDecomposition(A);
```

36. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, Ans: $R = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Linear Least Squares: 3×2

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$.

37. Find the normal equations for $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Solution: The **normal equation** refers to $A^T A\vec{\mathbf{x}} = A^T\vec{\mathbf{b}}$. Applications generally require the columns of A to be independent. The $m \times n$ matrix A generally has m is much larger than n: there are more rows than columns. In applications, the number of rows can be the number of data samples and the number of columns can represent a short list of parameters.

$$A^T A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$$

$$A^T\vec{\mathbf{b}} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

The normal equation:

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \vec{\mathbf{x}} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

```
# Exercise 37, Linear Least Squares: 3 x 2
A:=<2,0 | 0,2 | 1,1>^+;
A^+ . A;
b:=<1,0,5>;
A^+ . b;
```

38. Solve $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ by least squares.

Linear Least Squares: 4×3

Let $A = \begin{pmatrix} 4 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

39. Find the normal equations for $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Solution: $$\begin{pmatrix} 18 & 1 & 6 \\ 1 & 2 & 1 \\ 6 & 1 & 3 \end{pmatrix} \vec{\mathbf{x}} = \begin{pmatrix} 12 \\ 0 \\ 3 \end{pmatrix}$$

```
# Exercise 39, Linear Least Squares: 4 x 3
A:=<4,0,1 | 1,0,1 |0,1,0 | 1,1,1>^+;
A^+ . A;
b:=<3,0,0,0>;
A^+ . b;
LeastSquares(A,b);# Answer check to Exercise 40
```

40. Solve $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ by least squares.

Orthonormal Diagonal Form

Let $A = A^T$. The **spectral theorem** implies $AQ = QD$ where D is diagonal and Q has orthonormal columns. Find Q and D.

41. $A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$

Solution: $D = \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 2 & -1/2 \\ 1 & 1 \end{pmatrix}$, $Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$.

The plan: find D and P for matrix A by eigenanalysis. Matrix D is the correct diagonal matrix of eigenvalues. Check that A is symmetric: $A^T - A$. Then the spectral theorem applies and success is guaranteed to find the orthonormal matrix Q from P. Matrix Q contains the orthonormal vectors

$\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$ constructed by Gram-Schmidt from the columns $\vec{\mathbf{x}}_1$, $\vec{\mathbf{x}}_2$ of matrix P.

Details.
Computation by hand or computer gives $D = \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & -1/2 \\ 1 & 1 \end{pmatrix}$. Then $\vec{\mathbf{x}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{\mathbf{x}}_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ are the columns of P to which Gram-Schmidt will be applied. We check first that $\vec{\mathbf{x}}_1 \cdot \vec{\mathbf{x}}_2 = 0$, then $\vec{\mathbf{y}}_1 = \vec{\mathbf{x}}_1$, $\vec{\mathbf{y}}_2 = \vec{\mathbf{x}}_2$ and Gram-Schmidt has no details except to unitize the two answers, making $\vec{\mathbf{u}}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{\mathbf{u}}_2 = \frac{1}{\sqrt{5/4}}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Then

$$Q = \left\langle \vec{\mathbf{u}}_1 | \vec{\mathbf{u}}_2 \right\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

```
# Exercise 41, Orthonormal Diagonal Form
A:=<7,2|2,4>^+;
Lambda,P:=Eigenvectors(A);
 # Lambda = [8,3]
 # P = [[2, -1/2], [1, 1]]
DD:=DiagonalMatrix(Lambda);
L:=[Column(P,1),Column(P,2)];
q:=GramSchmidt(L,normalized);
Q:=Matrix(q);
Q^+ . Q; # Check orthogonal Q: Q^TQ=I
A.Q-Q.DD; # Check identity: AQ=QD
```

42. $A= \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$

43. $A= \begin{pmatrix} 1 & 5 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
Ans: Eigenvalues $-4, 2, 6$, orthonormal eigenvectors
$\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$,

$minicolvectorC001$, $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Solution: $D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, $Q = \sqrt{2} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$.

```
# Exercise 43, Orthonormal Diagonal Form
A:=<1,5,0|5,1,0|0,0,2>^+;
Lambda,P:=Eigenvectors(A);
 # Lambda = [-4, 2, 6]
 # P = Matrix([[-1, 0, 1], [1, 0, 1], [0, 1, 0]])
DD:=DiagonalMatrix(Lambda);
L:=[Column(P,1),Column(P,2)];
q:=GramSchmidt(L,normalized);
Q:=Matrix(q);
Q^+ . Q; # Check orthogonal Q: Q^TQ=I
A.Q-Q.DD; # Check identity: AQ=QD
```

44. $A = \begin{pmatrix} 1 & 5 & 0 \\ 5 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Eigenpairs of Symmetric Matrices: **Spectral Theorem**.

45. Let $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$. Eigenvalues are $2, 2, 5$. Find three orthonormal eigenpairs.

Solution: Eigenanalysis of A gave eigenpair packages $P = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. The orthonormal eigenvectors are found by Gram-Schmidt from the columns of P, reported below as the columns of Q:

$$Q = \begin{pmatrix} -\frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{3}\sqrt{6} & -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{3} \end{pmatrix}$$

```
# Exercise 45, Eigenpairs of Symmetric Matrices
A:=<3,-1,1|-1,3,-1|1,-1,3>^+;
Lambda,P:=Eigenvectors(A);
 # Lambda = [2, 2, 5]
 # P = Matrix([[-1, 1, 1], [0, 1, -1], [1, 0, 1]])
DD:=DiagonalMatrix(Lambda);
L:=[seq( Column(P,j), j=1..RowDimension(A) )];
q:=GramSchmidt(L,normalized);
Q:=Matrix(q);
Q^+ . Q; # Check orthogonal Q: Q^TQ=I
A.Q-Q.DD; # Check identity: AQ=QD
```

46. Let $A=\begin{pmatrix} 5 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}$. Then $|A-\lambda I|=(4-\lambda)^2(7-\lambda)$. Find three orthonormal eigenpairs.

47. Let $A=\begin{pmatrix} 6 & -1 & 1 \\ -1 & 6 & -1 \\ 1 & -1 & 6 \end{pmatrix}$. Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are for $\lambda = 5, 5, 8$. Illustrate $AQ = QD$ with D diagonal and Q orthogonal.

Solution: The plan is similar to Exercise 45: apply Gram-Schmidt to find orthonormal eigenvectors, then insert the answers into matrix Q. Then

$$D = \textbf{diag}(5,5,8) \text{ and } Q = \left(\begin{matrix} -\frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{3}\sqrt{6} & -\frac{1}{3}\sqrt{3} \; array \\ \frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{3} \end{matrix} \right)$$

```
# Exercise 47, Eigenpairs of Symmetric Matrices
A:=<6,-1,1|-1,6,-1|1,-1,6>^+;
Lambda,P:=Eigenvectors(A);
 # Lambda = [5, 5, 8]
 # P = Matrix([[-1,1,1],[0,1,-1],[1,0,1]])
DD:=DiagonalMatrix(Lambda);
L:=[seq( Column(P,j), j=1..RowDimension(A) )];
q:=GramSchmidt(L,normalized);
Q:=Matrix(q);
Q^+ . Q; # Check orthogonal Q: Q^TQ=I
A.Q-Q.DD; # Check identity: AQ=QD
```

48. Matrix A for $\lambda = 1, 1, 4$ has orthogonal eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
Find A and directly verify $A = A^T$.

Singular Value Decomposition

Find the SVD $A = U\Sigma V^T$.

49. $A=\begin{pmatrix} -1 & 1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$.
Ans: U=3×3, V=2×2. Matrix
$\Sigma=\begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$=$3 \times 2$, the size of A.

Solution: Details:

Let $A = \begin{pmatrix} -1 & 1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$.

Compute

$B = A^T A$

$$= \begin{pmatrix} -1 & -2 & 2 \\ 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}.$$

The eigenvalues of B are $0, 18$. Then the **singular values** are $0, 3\sqrt{2}$, to be reordered largest to smallest: $3\sqrt{2}, 0$. The size of Σ is 3×2:

$$\Sigma = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenpairs of $B = A^T A = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$ are

$$\left(0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad \left(18, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$$

The eigenvectors are orthogonal. Unitize them to obtain

$$\vec{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Define $\vec{\mathbf{u}}_1 = \frac{1}{\sqrt{18}} A \vec{\mathbf{v}}_1 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$. Define $C = \left\langle \vec{\mathbf{u}}_1 | I \right\rangle$ where I is the 3×3 identity matrix. Find $\mathbf{rref}(C)$ and select the pivot columns of C as the columns of U:

$$U = \begin{pmatrix} 1/3 & 1 & 0 & 0 \\ 2/3 & 0 & 1 & 0 \; array \\ -2/3 & 0 & 0 & 1 \end{pmatrix}$$

To check the answers, compute $U\Sigma V^T$, which should equal A.

$$U\Sigma V^T = \begin{pmatrix} 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \\ -2/3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

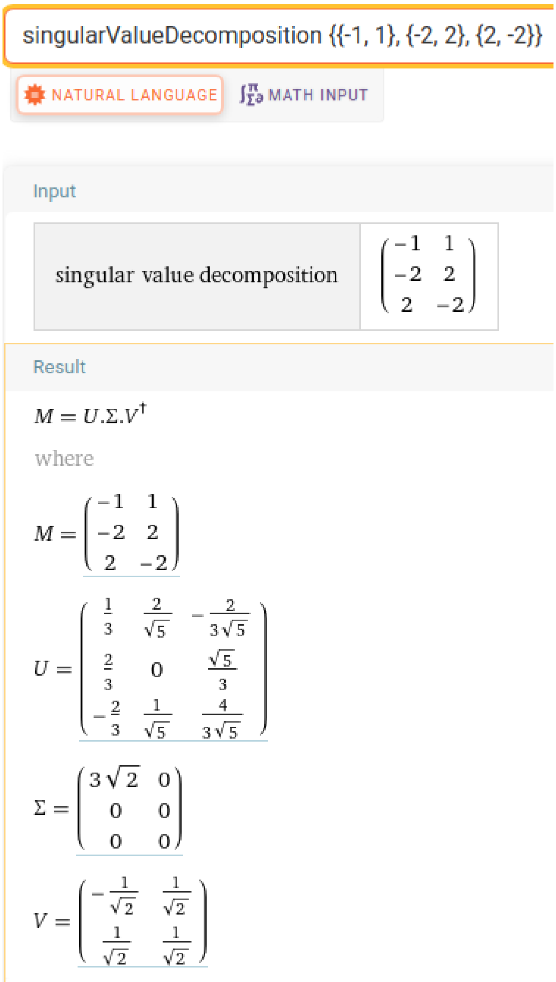
$$= \begin{pmatrix} -1 & 1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} = A$$

```
# Exercise 49, Singular Value Decomposition
A:=<-1,1|-2,2|2,-2>^+;
B:=A^+ . A;
Lambda,P:=Eigenvalues(B);map(sqrt,Lambda);
# reorder singular values to 3*sqrt(2), 0,
# then make Sigma 3x2
Sigma:=Matrix([[3*sqrt(2),0],[0,0],[0,0]]);
#
# Compute orthogonal matrix V
v2:=(1/sqrt(2))*<1,1>;v1:=(1/sqrt(2))*<-1,1>;
V:=<v1 | v2>;
#
# Compute orthogonal matrix U
u1:=(1/sqrt(18))*A.v1;
C:=<u1 | IdentityMatrix(3)>;
ReducedRowEchelonForm(C);
U:=C[1..3,1..3];# Select pivot columns of C
L:=[seq(Column(U,j),j=1..3)];
q:=GramSchmidt(L);
U:=Matrix(q);# Columns of U are orthonormal
#
# Answer check
A-U.Sigma.V^+;# Expect zero
#
# Answer check with package LinearAlgebra[SingularValues]
# Warning: answers are floats, not symbolic
S:=SingularValues(A);# answer check singular values
U, Vt := SingularValues(A, output = ['U', 'Vt']);# float answers
```

Solution:

The support in `maple` for the svd has computation limited to floating point and limited symbolic support, whereas `mathematica` has full support. The list of singular values returned by `maple` is in the wrong order, causing manual construction of Σ.
Access to `mathematica` in 2022 is free via

https://www.wolframalpha.com/



50. $A = \begin{pmatrix} -1 & 1 \\ -2 & 2 \\ 1 & 1 \end{pmatrix}$.

Ans: $\sigma_1 = \sqrt{10}, \sigma_2 = \sqrt{2}$.

51. $A = \begin{pmatrix} -3 & 3 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$.

Solution: The method follows Exercise 49.

Let $B = A^T A = \left(\begin{array}{cc} 10 & -8 \\ -8 & 10 \end{array}\right)$. The eigenpairs of B are

$$\left(18, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

The eigenvalues of B are $18, 2$. Then the **singular values** are $3\sqrt{2}, \sqrt{2}$, to be reordered largest to smallest: $3\sqrt{2}, \sqrt{2}$. Define 3×2 matrix

$$\Sigma = \left(\begin{array}{cc} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{array}\right)$$

The eigenvectors of B are orthogonal. Unitize them to obtain

$$\vec{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} -1 & 1 \\ 1 & 1 \end{array}\right)$$

Define $\vec{\mathbf{u}}_1 = \frac{1}{\sqrt{18}} A \vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Define $\vec{\mathbf{u}}_2 = \frac{1}{\sqrt{2}} A \vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Define $C = \left\langle \vec{\mathbf{u}}_1 | \vec{\mathbf{u}}_2 | I \right\rangle$ where I is the 3×3 identity matrix. Find $\mathbf{rref}(C)$ and identify the pivot columns 1,2,4. These columns of C are the columns of U:

$$U = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$$

To check the answers, compute $U \Sigma V^T$, which should equal A.

$$\begin{aligned} U \Sigma V^T &= \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right) \left(\begin{array}{cc} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{array}\right) \frac{1}{\sqrt{2}} \left(\begin{array}{cc} -1 & 1 \\ 1 & 1 \end{array}\right) \\ &= \left(\begin{array}{cc} -3 & 3 \\ 0 & 0 \\ 1 & 1 \end{array}\right) = A \end{aligned}$$

```
# Exercise 51, Singular Value Decomposition
A:=<-3,3|0,0|1,1>^+;
B:=A^+ . A;
Lambda,P:=Eigenvalues(B);map(sqrt,Lambda);
# reorder singular values to 3*sqrt(2), sqrt(2),
Sigma:=Matrix([[3*sqrt(2),0],[0,sqrt(2)],[0,0]]);# size 3x2
# Compute orthogonal matrix V
v2:=(1/sqrt(2))*<1,1>;v1:=(1/sqrt(2))*<-1,1>;
V:=<v1 | v2>;
# Compute orthogonal matrix U
1:=(1/sqrt(18))*A.v1;u2:=(1/sqrt(2))*A.v2;
C:=<u1 | u2 | IdentityMatrix(3)>;
ReducedRowEchelonForm(C);# pivots 1,2,4
U:=C[1..3, [1,2,4] ];# Select pivot columns 1,2,4 of C
 # Cols are already orthonormal, no Gram-Schmidt!
A-U.Sigma.V^+;# Expect zero
```

52. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$.

Ellipse and the SVD

Repeat Example 9.17, page 736 for the given ellipse equation.

53. $50x^2 - 30xy + 10y^2 = 2500$

Solution: Let $B = \begin{pmatrix} 50 & -15 \\ -15 & 10 \end{pmatrix}$. Then $BP = PD$ with eigenpair packages

$D = \begin{pmatrix} 5 & 0 \\ 0 & 55 \end{pmatrix}$, $P = \begin{pmatrix} 1/3 & -3 \\ 1 & 1 \end{pmatrix}$

Unitize the orthogonal eigenvectors in P and define

$Q = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$

Let $\vec{\mathbf{u}} = \begin{pmatrix} x \\ y \end{pmatrix} = Q\vec{\mathbf{w}}$ with $\vec{\mathbf{w}} = \begin{pmatrix} X \\ Y \end{pmatrix}$. Then

$50x^2 - 30xy + 10y^2 = 2500$

$\rightsquigarrow \vec{\mathbf{w}}^T D\vec{\mathbf{w}} = 2500$

$\rightsquigarrow \langle X|Y\rangle \begin{pmatrix} 5 & 0 \\ 0 & 55 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 2500$

$\rightsquigarrow \frac{1}{500}X^2 + \frac{11}{500}Y^2 = 1$

The semiaxis lengths are $10\sqrt{5}$ and $10\sqrt{55}/11$. ■

54. $40x^2 - 16xy + 10y^2 = 2500$

Mapping and the SVD

Reference: Example 9.18, page 738.

Let $\vec{\mathbf{w}} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2$,

$U = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$,

$A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$. Then $A = U\Sigma V^T$.

55. Verify $\|\vec{\mathbf{w}}\|^2 = \vec{\mathbf{w}} \cdot \vec{\mathbf{w}} = c_1^2 + c_2^2$.

Solution: $\|\vec{\mathbf{w}}\|^2 = \vec{\mathbf{w}} \cdot \vec{\mathbf{w}} = (c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2) \cdot (c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2) = c_1^2\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1 + 2c_1c_2\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_1 + c_2^2\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2 = c_1^2(1) + 2c_1c_2(0) + c_2^2(1)$ due to $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ given orthonormal (unit vector, pairwise orthogonal). Then $\|\vec{\mathbf{w}}\|^2 = \vec{\mathbf{w}} \cdot \vec{\mathbf{w}} = c_1^2 + c_2^2$. ■

56. Verify $V^T\vec{\mathbf{w}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ from the general identity $V^TV = I$. Then show that $\Sigma V^T\vec{\mathbf{w}} = \begin{pmatrix} 10c_1 \\ 5c_2 \end{pmatrix}$.

Therefore, coordinate map $\vec{\mathbf{w}} \to \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ undergoes re-scaling by 10 in direction $\vec{\mathbf{v}}_1$ and 5 in direction $\vec{\mathbf{v}}_2$.

57. Find the angle θ of rotation for V^T and the reflection axis for U.

Solution:

The angle θ of rotation for V^T.

Angle θ must satisfy

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = V^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

A clever shortcut is to use the isomorphism between 2×2 matrices and complex numbers:

$$a + bi \to \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Then $\cos\theta = 1/\sqrt{5}$, $\sin\theta = 2/\sqrt{5}$ or $\tan\theta = 2$ with θ in quadrant 1. Conclusion: $\theta = \arctan(2) = 1.107148718$ radians $= 63.43494883$ degrees, proper rotation about the origin clockwise.

Line of reflection for U.

Let $R = U = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$. Then $|R| = -1$ and $R^TR = I$, so it is an improper rotation, which is a reflection across a line of symmetry. The line of reflection has equation $y = mx + b$ for some slope m and intercept b.

Points of the line do not move under the action R. Choose two points on the line, say $x = 1$ and $x = 2$. Then the following equations hold:

$$\vec{\mathbf{w}} = \begin{pmatrix} 1 \\ m+b \end{pmatrix} \quad \text{for } x = 1$$

$$\begin{aligned}
\vec{\mathbf{w}} &= R\vec{\mathbf{w}} \\
\begin{pmatrix} 1 \\ m+b \end{pmatrix} &= R\begin{pmatrix} 1 \\ m+b \end{pmatrix} \\
&= \tfrac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ m+b \end{pmatrix} \\
&= \tfrac{1}{\sqrt{5}}\begin{pmatrix} 1+2m+2b \\ 2-m-b \end{pmatrix}
\end{aligned}$$

The vectors must match entries, giving two equations in two unknowns:

$$\begin{cases} \sqrt{5} &= 1+2m+2b \\ \sqrt{5}(m+b) &= 2-m-b \end{cases}$$

$$\vec{\mathbf{w}} = \begin{pmatrix} 1 \\ m+b \end{pmatrix} \quad \text{for } x = 2$$

$$\begin{aligned}
\vec{\mathbf{w}} &= R\vec{\mathbf{w}} \\
\begin{pmatrix} 2 \\ 2m+b \end{pmatrix} &= R\begin{pmatrix} 2 \\ 2m+b \end{pmatrix} \\
&= \tfrac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 2m+b \end{pmatrix} \\
&= \tfrac{1}{\sqrt{5}}\begin{pmatrix} 1+4m+2b \\ 4-2m-b \end{pmatrix}
\end{aligned}$$

Match vector entries to give two equations in two unknowns:

$$\begin{cases} 2\sqrt{5} &= 1+2m+2b \\ \sqrt{5}(2m+b) &= 4-2m-b \end{cases}$$

Solve the four equations for $m = \dfrac{-1+\sqrt{5}}{2}$, $b = 0$. The line of reflection is $y = mx + 0$, which is $y = (-1+\sqrt{5})x/2$. ■

```
# Exercise 57, Mapping and the SVD
R:=(1/sqrt(5))*Matrix([[1,2],[2,-1]]);
w1:=<1,m+b>;A1:=R.w1-w1;
w2:=<2,2*m+b>;A2:=R.w2-w2;
solve({A1[1]=0,A1[2]=0,A2[1]=0,A2[2]=0},[m,b]);
# reflection line: y = mx+b, m=(-1+sqrt(5))/2, b=0
```

58. Assume $|\vec{\mathbf{w}}\| = 1$, a point on the unit circle. Is $A\vec{\mathbf{w}}$ on an ellipse with semi-axes 10 and 5? Justify your answer geometrically, no proof expected. Check your answer with a computer plot.

Solution:
Proof: Let $A = U\Sigma V^T$. Let vector $\vec{\mathbf{w}}$ be given. Then $V^T\vec{\mathbf{w}}$ rotates $\vec{\mathbf{w}}$ by angle θ, so the image remains on the unit circle. Then Σ scales the axes. Finally, U reflects $\Sigma V^T\vec{\mathbf{w}}$ across the line of symmetry found in Exercise 57. ■

Four Fundamental Subspaces

Compute matrices S_1, S_2 such that the column spaces of S_1, S_2 are the nullspaces of A and A^T. Verify the two orthogonality relations of the four subspaces page 739 from the matrix identities $AS_1 = 0$, $A^TS_2 = 0$.

59. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$. Answer:

$$S_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; S_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Solution: Details require finding the nullspace of A and the nullspace of A^T. The calculations can be done on paper or by computer. The orthogonality tests are by matrix multiply, which should return a matrix with columns all zero.

```
# Exercise 59, Four Fundamental Subspaces
A:=<1,0,0|1,1,0|2,1,0>^+;
S1:=Matrix(convert(NullSpace(A), 'list' ));
A.S1;# check A perp cols of S1
B:=A^+;
S2:=Matrix(convert(NullSpace(B), 'list' ));
B.S2;# check A^T perp cols of S2
```

60. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}$. Answer:

$$S_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; S_2 = \begin{pmatrix} -1 & -1 \\ -2 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

61. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \end{pmatrix}$ Answer:

$$S_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

62. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ Answer:

$$S_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 2 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Fundamental Theorem of Linear Algebra

Strang's Theorem says that the four subspaces built from $n \times m$ matrix A and $m \times n$ matrix A^T satisfy

$$\begin{aligned} &\mathbf{colspace}(A^T) \perp \mathbf{nullspace}(A), \\ &\mathbf{colspace}(A) \perp \mathbf{nullspace}(A^T). \end{aligned}$$

Let $r = \mathbf{rank}(A) = \mathbf{rank}(A^T)$. The four subspace dimensions are:

$$\begin{aligned} &\dim(\mathbf{colspace}(A)) = r, \\ &\dim(\mathbf{nullspace}(A)) = n - r, \\ &\dim(\mathbf{colspace}(A^T)) = r, \\ &\dim(\mathbf{nullspace}(A^T)) = m - r. \end{aligned}$$

63. Explain why $\dim(\mathbf{colspace}(A)) = \dim(\mathbf{colspace}(A^T)) = r$ from the Pivot Theorem.

Solution: Let $r_1 = \dim(\mathbf{colspace}(A))$ and $r_2 = \dim(\mathbf{colspace}(A^T))$. To prove: $r_1 = r_2 = r = \mathbf{rank}(A)$. First, r_1 is the number of pivot columns of A by the **pivot theorem**, which the theorem states is equal the number of independent columns of A. Second, r_2 is the number of independent columns of A^T, which equals the number independent rows of A. Because rank = number of independent columns of A = number of independent rows of A, by the theorem **row rank = column rank**, then $r = r_1 = r_2$. ■

64. Suppose A is 10×4. What are the dimensions of the four subspaces?

65. Invent a 4×4 matrix A where one of the four subspaces is the zero vector alone.

Solution: Let A be a 4×4 invertible matrix, like the identity matrix. Then the nullspace of A is the zero vector. ■

66. Prove that the only vector in common with **rowspace**(A) and **nullspace**(A) is the zero vector.

67. Prove that each vector $\vec{\mathbf{x}}$ in $\mathcal{R}^n$ can be uniquely written as $\vec{\mathbf{x}} = \vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2$ where $\vec{\mathbf{x}}_1$ is in **colspace**(A^T) and $\vec{\mathbf{x}}_2$ is in **nullspace**(A). See **direct sum** in exercise page 428.

Solution: Assume A is $m \times n$, $r = \mathbf{rank}(A)$, $s = \mathbf{nullity}(A)$. Then $r+s=n$ by the rank-nullity theorem, which in simpler language says the number of lead variables plus the number of free variables equals n = number of variables.

Let $S_1 = \mathbf{colspace}(A^T) = \mathbf{rowspace}(A)$ and $S_2 = \mathbf{nullspace}(A)$. Let $V = \mathcal{R}^n$. Assemble these facts:

> Both S_1 and S_2 are subspaces.
> Exercise 66 provides $S_1 \cap S_2 = \{\vec{\mathbf{0}}\}$.
> Subspace S_1 has a basis $\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_r$ where $r = \mathbf{rank}(A)$.
> Subspace S_2 has basis $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_s$ where $s = \mathbf{nullity}(A) = n - r$.
> Let $W = \{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_r, \vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_s\}$. Then W contains n independent vectors because $S_1 \cap S_2 = \{\vec{\mathbf{0}}\}$ (independence proof omitted).
> Set W is a basis for $V = \mathcal{R}^n$. See Theorem 5.40 page 406.

The proof:
Let $\vec{\mathbf{x}}$ be any vector in $\mathcal{R}^n$. Expand $\vec{\mathbf{x}}$ with basis W. Then

$\vec{\mathbf{x}} = \sum_{i=1}^{r} a_i \vec{\mathbf{u}}_i + \sum_{j=1}^{s} b_j \vec{\mathbf{v}}_j$

Let $\vec{\mathbf{x}}_1 = \sum_{i=1}^{r} a_i \vec{\mathbf{u}}_i$ and $\vec{\mathbf{x}}_2 = \sum_{j=1}^{s} b_j \vec{\mathbf{v}}_j$. Then $\vec{\mathbf{x}} = \vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2$ with $\vec{\mathbf{x}}_1$ in S_1 and $\vec{\mathbf{x}}_2$ in S_2. Existence established.

It remains to prove uniqueness. Suppose $\vec{\mathbf{x}} = \vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 = \vec{\mathbf{y}}_1 + \vec{\mathbf{y}}_2$ with $\vec{\mathbf{x}}_1, \vec{\mathbf{y}}_1$ in $S+1$ and $\vec{\mathbf{x}}_2, \vec{\mathbf{y}}_2$ in S_2. To prove: $\vec{\mathbf{x}}_1 = \vec{\mathbf{y}}_1$ and $\vec{\mathbf{x}}_2 = \vec{\mathbf{y}}_2$. Rearrange the equation for $\vec{\mathbf{x}}$: $\vec{\mathbf{x}}_1 - \vec{\mathbf{y}}_1 = \vec{\mathbf{y}}_2 - \vec{\mathbf{x}}_2$. Then the LHS is in S_1 and the RHS is in S_2. Because $S_1 \cap S_2 = \{\vec{\mathbf{0}}\}$ then $\vec{\mathbf{x}}_1 - \vec{\mathbf{y}}_1 = \vec{\mathbf{y}}_2 - \vec{\mathbf{x}}_2 = \vec{\mathbf{0}}$, which proves uniqueness. ■

68. Prove that each vector $\vec{\mathbf{y}}$ in $\mathcal{R}^m$ can be uniquely written as $\vec{\mathbf{y}} = \vec{\mathbf{y}}_1 + \vec{\mathbf{y}}_2$ where $\vec{\mathbf{y}}_1$ is in **colspace**(A) and $\vec{\mathbf{y}}_2$ is in **nullspace**(A^T).

Chapter 10

Phase Plane Methods

Contents

10.1 Planar Autonomous Systems

Autonomous Planar Systems.

Consider

(1)
$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

1. (Vector-Matrix Form) System (1) can be written in vector-matrix form

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u}(t)).$$

Display formulas for $\vec{u}$ and $\vec{F}$.

Solution: The formulas:

$\vec{\mathbf{u}} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \begin{pmatrix} x+y \\ 1-x^2 \end{pmatrix}$

Computer implementations use strict rules with code only similar to the mathematics, not the same.

```
# Exercise 1, Autonomous Planar Systems
PDEtools[declare]((x, y)(t), prime = t);
 ## x(t), y(t) are displayed as x, y
 ## diff(f(t),t) displayed in prime notation f'
u:=t-><x(t),y(t)>;
F0:=(x,y)-><x+y,1-x^2>;
F:=w->F0(w[1],w[2]):
F(u(t));## = < x(t)+y(t), 1-x(t)^2 >
```

2. **(Picard's Theorem)** Picard's vector existence-uniqueness theorem applies to system (1) with initial data $x(0) = x_0$, $y(0) = y_0$. Show the details.

Solution: Expected details are hypothesis checks: $\vec{\mathbf{F}}$ and $.F_y$ continuous with initial data in domain $\mathcal{D}$.

Trajectories Don't Cross.

3. **(Theorem 10.1 Details)** Show $\frac{dy}{dt} = g(x_1(t+c), y_1(t+c))$, then show that $y'(t) = g(x(t), y(t))$ in the proof of Theorem 10.1.

Solution:

$$\begin{aligned}\frac{dy}{dt} &= \tfrac{d}{dt}\, y_1(t+c) \\ &= g(x_1(t+c), y_1(t+c)) \\ &= g(x(t), y(t)) \quad \blacksquare\end{aligned}$$

4. **(Orbits Can Cross)** The example

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3y^{2/3}$$

has infinitely many orbits crossing at $x = y = 0$. Exhibit two distinct orbits which cross at $x = y = 0$. Does this example contradict Theorem 10.1?

Equilibria. A point (x_0, y_0) is called an **Equilibrium** provided $x(t) = x_0$, $y(t) = y_0$ is a solution of the dynamical system.

5. Justify that $(1, -1), (-1, 1)$ are the only equilibria for the system $x' = x + y$, $y' = 1 - x^2$.

Solution: For (x_0, y_0) to be an equilibrium the following equations must hold:

$0 = x_0' = x_0 + y_0, \quad 0 = y_0' = 1 - x_0^2$

The second equation gives $x_0 = 1$ or $x_0 = -1$. The first equation provides

$y_0 = -x_0$. Then the two solution pairs are $(x_0, y_0) = (1, -1)$ and $(x_0, y_0) = (-1, 1)$. ■

6. Display the details which justify that $(0,0), (90,0), (0,60), (80,20)$ are all equilibria for the system $x'(t) = x(-2x - y + 180)$, $y'(t) = y(-x - 2y + 120)$.

Practical Methods for Computing Equilibria.

7. **(Murray System)** The biological system

$$x' = x(6 - 2x - y), y' = y(4 - x - y)$$

has equilibria $(0,0)$, $(3,0)$, $(0,4)$, $(2,2)$. Justify the four answers.

Solution:

Instead of using symbols x_0, y_0 let's use x, y. Equilibrium (x, y) satisfies

$0 = x(6 - 2x - y), \quad 0 = y(4 - x - y)$

First equation: $x = 0$ or $2x + y = 6$.

Second equation: $y = 0$ or $x + y = 4$.

There are 4 possibilities:

$$\begin{array}{ll} x = 0, & y = 0 \\ x = 0, & x + y = 4 \\ 2x + y = 6, & y = 0 \\ 2x + y = 6, & x + y = 4 \end{array}$$

The first three possibilities give three equilibria: $(0,0)$, $(0,4)$, $(3,0)$. The last possibility requires a solving with the linear algebra toolkit or Cramer's rule to find the unique solution $(2,2)$.

8. **(Nullclines)** Curves along which either $x' = 0$ or $y' = 0$ are called **nullclines**. The biological system

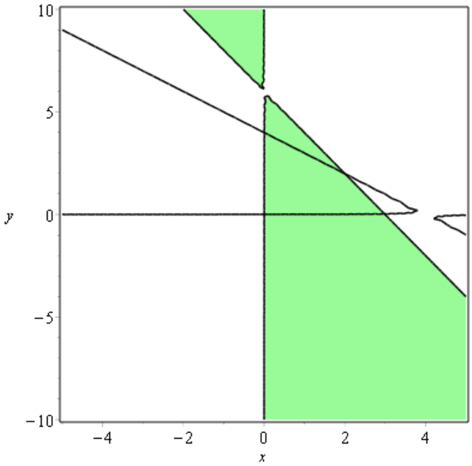
$$x' = x(6 - 2x - y), y' = y(4 - x - y)$$

has nullclines $x = 0$, $y = 0$, $6 - 2x - y = 0$, $4 - x - y = 0$. Justify the four answers.

9. **(Nullclines by Computer)** Produce a graphical display of the nullclines of the Murray System above. Maple code below makes a plot from equations $x(6 - 2x - y) = 0$, $y(4 - x - y) = 0$.

```
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=-5..5,y=-10..10;
opts:=wind,contours=[0];
plots[contourplot](eqns,opts);
```

Solution: Plot options can improve a nullcline plot. System `mathematica` offers a nullcline plot demonstration
https://demonstrations.wolfram.com/NullclinePlot/
that uses the free WolframPlayer. System `maple` produced the plot below using the code that follows the plot.




```
# Exercise 9, Autonomous planar systems: nullclines
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=-5..5,y=-10..10;
opts:=contours=[0],filledregions = true,
coloring = ["White", "PaleGreen"];
plots[contourplot](eqns,wind,opts);
```

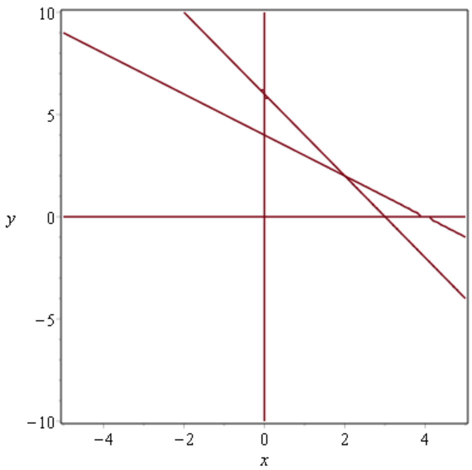
10. (Isoclines by Computer) Level curves $f(x,y) = c$ are called **Isoclines**.
`Maple` will plot level curves $f(x,y) = -2$, $f(x,y) = 0$, $f(x,y) = 2$ using the nullcline code above, with replacement `contours=[-2,0,2]`. Produce an isocline plot for the Murray System above with these same contours.

11. (Implicit Plot) Equilibria can be found graphically by an implicit plot.

```
# MAPLE implicit plot
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=-5..5,y=-10..10;
plots[implicitplot](eqns,wind);
```

Produce the implicit plot. Is it the same as the nullcline plot?

Solution:




```
# Exercise 11, Autonomous systems: implicit plot, equilibria
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=-5..5,y=-10..10;
plots[implicitplot](eqns,wind,gridrefine=2);
solve(eqns,[x,y]);
 # equilibria: [x=0, y=0],[x=0, y=4],[x=3, y=0],[x=2, y=2]
```

12. **(Implicit Plot)** Find the equilibria graphically by an implicit plot. Then find the equilibria exactly.

$$\begin{cases} x'(t) &= x(t) + y(t), \\ y'(t) &= 4 - x^2(t). \end{cases}$$

Rabbit-Fox System.

13. **(Predator-Prey)** Consider a rabbit and fox system

$$\begin{aligned} x' &= \frac{1}{200}x(30-y), \\ y' &= \frac{1}{100}y(x-40). \end{aligned}$$

Argue why extinction of the rabbits ($x = 0$) implies extinction of the foxes ($y = 0$).

Solution: When $x = 0$ then $y' = -40y$ which is a decay equation. Then $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y_0 e^{-40t} = 0$, which is extinction.

14. (Predator-Prey) The rabbit and fox system

$$\begin{aligned} x' &= \frac{1}{200}x(40-y), \\ y' &= \frac{1}{100}y(x-40), \end{aligned}$$

has extinction of the foxes ($y=0$) implying Malthusian population explosion of the rabbits ($\lim_{t=\infty} x(t) = \infty$). Explain.

Trout System. Consider

$$\begin{aligned} x'(t) &= x(-2x-y+180), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

15. (Carrying Capacity) Show details for calculation of the equilibrium $x = 80$, $y = 20$, which is **co-existence**.

Solution: Equilibria (x, y) are found from the equations

$$\begin{aligned} 0 &= x(-2x-y+180), \\ 0 &= y(-x-2y+120). \end{aligned}$$

Follow the method in Exercise 7 to solve for

$$(x, y) = (0,0), \quad (0,60), \quad (90,0), \quad (80,20)$$

The first three equilibria involve at least one extinction state $x = 0$ or $y = 0$. Equilibrium $x = 80$, $y = 20$ is co-existence, analagous to co-habitation for foxes and rabbits.

16. (Stability) Equilibrium point $x = 80$, $y = 20$ is stable. Explain this statement using geometry from Figure 10 and the definition of stability.

Phase Portraits. Consider

$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

17. (Equilibria) Solve for x, y in the system

$$\begin{aligned} 0 &= x + y, \\ 0 &= 1 - x^2, \end{aligned}$$

for equilibria $(1,-1)$, $(-1,1)$. Explain why $|x| \le 2$, $|y| \le 2$ is a suitable graph window.

Solution: Equilibrium (x, y) is a solution of the system of equations

$$\begin{aligned} 0 &= x + y, \\ 0 &= (1 - x)(1 + x)2, \end{aligned}$$

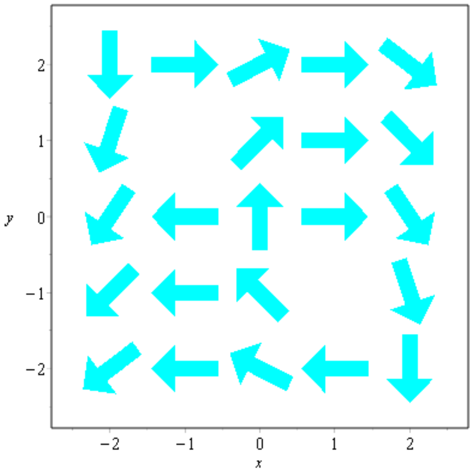
which arises by factoring the second equation. Follow the method in Exercise 7 to solve for

$$(x, y) = (1, -1), \quad (-1, 1)$$

18. (Grid Points) Draw a 5×5 grid on the graph window $|x| \le 2, |y| \le 2$. Label the equilibria.

19. (Direction Field) Draw direction field arrows on the 5×5 grid of the previous exercise. They coincide with the tangent direction $\vec{v} = x'\vec{\imath} + y'\vec{\jmath} = (x + y)\vec{\imath} + (1 - x^2)\vec{\jmath}$, where (x, y) is the grid point. The arrows may not touch.

Solution: Expected is a plot made on paper using graph paper or similar. There should be 5 lines of 5 grid points, a uniform grid. The arrows have tail or midpoint at a grid point and head pointing in the direction of the tangent vector $\vec{\mathbf{v}}$. The arrow length is by trial and error. The result should look like the figure below.




```
# Exercise 19, Direction field
de1:=diff(x(t),t)=x(t)+y(t);
de2:=diff(y(t),t)=1-x(t)*x(t);
trange:=t=-10..10:xrange:=x=-2..2:yrange:=y=-2..2:
vars:=[x(t),y(t)];opts1:=trange,xrange,yrange:
opts2:=arrows=large,color=cyan,dirfield=[5,5]:
DEtools[dfieldplot]([de1,de2],vars,opts1,opts2);
```

20. **(Threaded Orbits)** On the direction field of the previous exercise, draw orbits (*threaded solution curves*), using the rules:

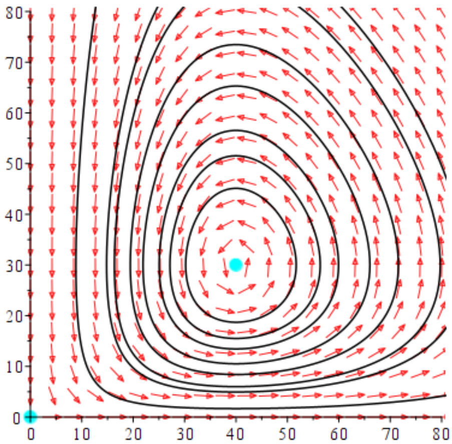
1. Orbits don't cross.
2. Orbits pass direction field arrows with nearly matching tangent.

Phase Plot by Computer. Use a computer algebra system or a numerical workbench to produce phase portraits for the given dynamical system. A graph window should contain all equilibria.

21. **(Rabbit-Fox System I)**

$$\begin{aligned} x' &= \frac{1}{200}x(30-y), \\ y' &= \frac{1}{100}y(x-40). \end{aligned}$$

Solution:



The solution uses `maple`. Instructions:

> To open the Phase portrait task, click **Tools** => **Tasks menu** => **Browse**, then **Differential Equations** => **ODEs** => **Phase portrait - Autonomous Systems**. Click on **Insert Minimal Content**, which inserts the template into the worksheet. For safety, save the **WorkSheet** as soon as the template loads.
>
> There are input boxes to fill. Careful: any click error or keyboard error may destroy the template, resulting in all data and images lost. The most common error is the **RETURN** key. Don't use it. If an error stops you then the only alternative is to exit `maple`, start again and load the **WorkSheet** saved earlier. Do not save a worksheet that has a template error!

To begin a new example, click button **Erase Data**. Use a mouse click to start filling a data box, then **Backspace** and **Delete** keys for correction. It is OK to copy text from an editor and paste it.

After all data is entered, then click button **Enter Data**. To add threaded curves to the plot, click on the plot where you want the curve to start. If no action, then right-click on the plot to bring up the **Plot Menu**. In the menu select **Manipulator** => **Click and Drag**. Then try clicking on the plot to generate a threaded curve.

```
# Exercise 21, Rabbit-Fox System I
# Launch Task: Phase portrait - Autonomous Systems
# Save the task (ctrl-S).
  # Click button: Erase Data
  # Keyboard data into 5 text boxes
  Box 1: x: 0 to 80, y: 0 to 80
  Box 2: (1/200)*x*(30-y)
  Box 3: (1/100)*y*(x-40)
  Box 4: [0, 0], [40, 30]
  Box 5: t: -10 to 20
  # Click button: Enter Data
  # Click image: Thread a solution curve
```

22. (Rabbit-Fox System II)

$$\begin{aligned} x' &= \frac{1}{100}x(50-y), \\ y' &= \frac{1}{200}y(x-40). \end{aligned}$$

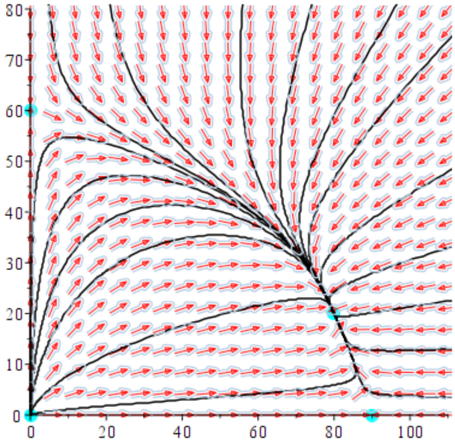
Solution:

```
# Exercise 22, Rabbit-Fox System II
# Tools => Task: Phase portrait - Autonomous Systems
# Save the task (ctrl-S).
  # Click button: Erase Data
  # Keyboard data into 5 text boxes
  Box 1: x: 0 to 60, y: 0 to 80
  Box 2: (1/100)*x*(50-y)
  Box 3: (1/200)*y*(x-40)
  Box 4: [0, 0], [40, 50]
  Box 5: t: -10 to 20
  # Click button: Enter Data
  # Click image: Thread a solution curve
```

23. (Trout System I)

$$\begin{aligned} x'(t) &= x(-2x-y+180), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

Solution:
The solution uses `maple`. Follow instructions in Exercise 21.




```
# Exercise 23, Trout System
# Launch Task: Phase portrait - Autonomous Systems
# Save the task (ctrl-S).
  # Click button: Erase Data
  # Keyboard data into 5 text boxes
  Box 1: x: 0 to 110, y: 0 to 80
  Box 2: x*(-2*x-y+180)
  Box 3: y*(-x-2*y+120)
  Box 4: [0, 0], [0,60], [90,0], [80, 20]
  Box 5: t: -10 to 20
  # Click button: Enter Data
  # Click image: Thread a solution curve
```

24. (Trout System II)

$$\begin{aligned} x'(t) &= x(-2x - y + 200), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

Stability Conditions. Consider equilibrium point $(0,0)$ and nearby solution curves $x(t), y(t)$ with $(x(0), y(0))$ near $(0,0)$.

25. (Instability: Repeller) Prove: If for every $\delta > 0$ there is one solution with $|x(0)^2 + y(0)^2| < \delta^2$ such that $\lim_{t\to\infty} |x(t)| + |y(t)| = \infty$ then equilibrium $(0,0)$ is unstable.

Solution: Let $\epsilon > 0$. Consider a disk $\mathcal{D} = \{(x,y) : x^2 + y^2 < \epsilon^2\}$. Stability means that for all $\delta > 0$, with $\delta < \epsilon$, a solution with $\sqrt{x(0)^2 + y(0)^2} < \delta$ is required to satisfy $\sqrt{x(t)^2 + y(t)^2} < \epsilon$, i.e., the solution remains in $\mathcal{D}$ for

$t \geq 0$. Limit condition $\lim_{t\to\infty} |x(t)| + |y(t)| = \infty$ causes this requirement to fail. Therefore, equilibrium $(0, 0)$ is unstable. ■

26. (Stability: Attractor) Prove that $x'(t) < 0$ and $y'(t) < 0$ for all nearby solutions implies stability at $(0, 0)$, but not asymptotic stability.

Solution: Hint: Look at the geometry.

27. (Instability in x) Prove that $\lim_{t\to\infty} |x(t)| = \infty$ implies instability at $(0, 0)$.

Solution: Then $\lim_{t\to\infty} |x(t)| + |y(t)| = \infty$, Apply Exercise 25.

28. (Instability in y) Prove that $\lim_{t\to\infty} |y(t)| = \infty$ implies instability at $(0, 0)$.

Geometric Stability.

29. (Attractor) Imagine a dust particle in a fluid draining down a funnel, whose trace is a space curve. Assume fluid drains at $x = 0$, $y = 0$ and the funnel centerline is along the z-axis. Project the space curve onto the xy-plane. Is this planar orbit stable at $(0, 0)$ in the sense of the definition?

Solution: Maybe yes, maybe no. Exercise 25 requires $\lim_{t\to\infty} |x(t)|+|y(t)| = 0$. That may not happen, due to gravity effects altering the path of the dust particle. What does happen: the dust particle moves closer to $(0, 0)$ as $t \to \infty$. Layman conclusion: stable.

30. (Repeller) Imagine a paint droplet from a paint spray can, pointed downward, which traces a space curve. Project the space curve onto the xy-plane orthogonal to the spray nozzle direction, centerline along the z-axis. Is this planar orbit stable at $(0, 0)$ in the sense of the definition?

Solution: Maybe yes, maybe no. Tracing the droplet to $t = \infty$ reveals a path that moves away from the centerline (z-axis). Gravity effects may keep the xy-plane projected droplet near $(0, 0)$, which means a test like $\lim_{t\to\infty} |x(t)| = \infty$ or $\lim_{t\to\infty} |y(t)| = \infty$ can fail. The key issue is the path of a droplet from near the center of the nozzle: it likely follows the z-axis due to gravity effects. Layman conclusion: undecided.

Geometric Stability: Phase Portrait.

31. (Rabbit–Fox I Stability) Plot a phase portrait for system

$$\begin{aligned} x' &= \frac{1}{200}x(30 - y), \\ y' &= \frac{1}{100}y(x - 40). \end{aligned}$$

Provide geometric evidence for stability of equilibrium $x = 40$, $y = 30$.

Solution: The figure in Exercise 21 shows stability but not asymptotic stability. In this chapter see the discussions of **center** and **spiral**. Theorems later in the chapter allow from calculus calculations a prediction of center or spiral, either stable or unstable. Conclusion: the phase portrait really helps to classify stability at an equilibrium.

32. **(Rabbit–Fox II Instability)** Plot a phase portrait for system

$$\begin{aligned} x' &= \frac{1}{100}x(50-y), \\ y' &= \frac{1}{200}y(x-40). \end{aligned}$$

Provide geometric evidence for instability of equilibrium $x = 0$, $y = 0$ and stability of equilibrium $x = 40$, $y = 50$.

Solution: See Exercise 22.

10.2 Planar Constant Linear Systems

Planar Constant Linear Systems

1. (Picard's Theorem) Explain why planar solutions don't cross, by appeal to Picard's existence-uniqueness theorem for $\frac{d}{dt}\vec{u}=A\vec{u}$.

Solution: Function $\vec{\mathbf{u}} \to A\vec{\mathbf{u}}$ is continuously differentiable for all $\vec{\mathbf{u}}$. Picard-Lindelöf applies: solutions to initial value problems are locally unique.

If two solutions $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ cross or touch then there are times t_1 and t_2 such that $\vec{\mathbf{u}}_1(t_1) = \vec{\mathbf{u}}_2(t_2) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Define $\vec{\mathbf{v}}_1(t) = \vec{\mathbf{u}}_1(t+t_1)$, $\vec{\mathbf{v}}_2(t) = \vec{\mathbf{u}}_2(t+t_2)$. Then $\vec{\mathbf{v}}_1' = A\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2' = A\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1(0) = \vec{\mathbf{v}}_2(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Picard's theorem says $\vec{\mathbf{v}}_1(t) = \vec{\mathbf{v}}_2(t)$ which implies that trajectories $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ coalesce locally near the contact point. To cross means the curves don't coalesce at the contact point. ■

2. (Equilibria) System $\frac{d\vec{u}}{dt} = A\vec{u}$ always has solution $\vec{u}(t) = \vec{0}$, so there is always one equilibrium point. Give an example of a matrix A for which there are infinitely many equilibria.

Putzer's Formula

3. (Cayley-Hamilton) Define matrices $\vec{\mathbf{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\vec{\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Given matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, expand left and right sides to verify the **Cayley-Hamilton identity**
$A^2-(a+d)A+(ad-bc)\vec{\mathbf{I}} = \vec{\mathbf{0}}$.

Solution: Expand LHS:

$$\begin{aligned}
\text{LHS} &= A^2-(a+d)A+(ad-bc)\vec{\mathbf{I}} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (c+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^2-(a+d)\,a+ad & ab+bd-(a+d)\,b \\ ca+dc-(a+d)\,c & d^2-(a+d)\,d+ad \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

```
# Exercise 3, Cayley-Hamilton
A:=<a,b|c,d>^+;
LHS:=A^2-(a+d)*A+(a*d-b*c)*<1,0|0,1>;
simplify(LHS);
```

4. (Complex Roots) Verify the Putzer solution $\vec{u} = \Phi(t)\vec{u}(0)$ of $\vec{u}' = A\vec{u}$ for complex roots $\lambda_1 = \overline{\lambda_2} = a + bi$, $b > 0$, where $\Phi(t)$ is

$$e^{at}\left(\cos(bt)\, I + (A - aI)\frac{\sin(bt)}{b}\right).$$

5. (Distinct Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \vec{u}.$$

Solution: Let's apply Theorem 10.2 page 767. The real eigenvalues of $A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$ are $\lambda_1 = -1$, $\lambda_2 = 2$. Then

$$\begin{aligned}
\Phi(t) &= e^{\lambda_1 t}\, I + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(A - \lambda_1 I) \\
&= e^{-t}\, I + \frac{e^{2t} - e^{-t}}{2 - (-1)}(A + I) \\
&= e^{-t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{2t} - e^{-t}}{3}\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} + \frac{e^{2t} - e^{-t}}{3}\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} e^{-t} & \frac{1}{3}(e^{2t} - e^{-t}) \\ 0 & e^{2t} \end{pmatrix}
\end{aligned}$$

6. (Real Equal Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix} \vec{u}.$$

7. (Complex Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \vec{u}.$$

Solution: Let's apply Theorem 10.2 page 767. The complex eigenvalues of $A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$ are $2 \pm 3i$. Let $a = 2$, $b = 3$. Then

$$\begin{aligned}
\Phi(t) &= e^{at}\left(\cos(bt)\, I + (A - aI)\frac{\sin(bt)}{b}\right) \\
&= e^{2t}\left(\cos(3t)\, I + (A - 2I)\frac{\sin(3t)}{3}\right)
\end{aligned}$$

$$= e^{2t}\left(\cos(3t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2-2 & 3 \\ -3 & 2-2 \end{pmatrix}\frac{\sin(3t)}{3}\right)$$
$$= e^{2t}\left(\begin{pmatrix} \cos(3t) & 0 \\ 0 & \cos(3t) \end{pmatrix} + \begin{pmatrix} 0 & \sin(3t) \\ -\sin(3t) & 0 \end{pmatrix}\right)$$
$$= e^{2t}\begin{pmatrix} \cos(3t) & \sin(3t) \\ -\sin(3t) & \cos(3t) \end{pmatrix}$$

8. (Purely Complex Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\vec{u}.$$

Solution: Eigenvalues are $\pm 3i$. Then

$\Phi(t) = \begin{pmatrix} \cos(3t) & \sin(3t) \\ -\sin(3t) & \cos(3t) \end{pmatrix}$ by details in Exercise 7.

Continuity and Redundancy

9. (Real Equal Eigenvalues) Show that limiting $\lambda_2 \to \lambda_1$ in the Putzer formula for distinct eigenvalues gives Putzer's formula for real equal eigenvalues.

Solution: Limiting $\lambda_2 \to \lambda_1$ is done on the formula

$\Phi(t) = e^{\lambda_1 t} I + \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(A - \lambda_1 I)$

Quotient $Q = \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}$ can be written as

$Q = \dfrac{f(x_0 + h) - f(x_0)}{h}$, where $f(x) = e^{xt}$, $x_0 = \lambda_1$ and $h = \lambda_2 - \lambda_1$

Then Q is a Newton quotient for $f'(x_0)$. Because f is differentiable with $f'(x) = t\,e^{xt}$ then $\lim_{h\to 0} Q = f'(x_0) = t\,e^{x_0 t} = te^{\lambda_1 t}$

These details prove:

$$\begin{aligned}\lim_{\lambda_2\to\lambda_1}\Phi(t) &= e^{\lambda_1 t} I + \lim_{\lambda_2\to\lambda_1}\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(A - \lambda_1 I) \\ &= e^{\lambda_1 t} I + (\lim_{h\to 0} Q)(A - \lambda_1 I) \\ &= e^{\lambda_1 t} I + t\,e^{\lambda_1 t}(A - \lambda_1 I)\end{aligned}$$

10. (Complex Eigenvalues) Assume $\lambda_1 = \overline{\lambda_2} = a + ib$ with $b > 0$. Then Putzer's first formula holds. Show the third formula details for $\Phi(t)$:

$$e^{at}\left(\cos(bt)\,I + (A - aI)\frac{\sin(bt)}{b}\right).$$

Illustrations

11. (Distinct Eigenvalues) Show the details for the solution of

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix} \vec{u}.$$

Solution: Let's apply to $A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}$ the Putzer formula for distinct eigenvalues $\lambda_1 = 5$, $\lambda_2 = 2$.

$$\begin{aligned}
\Phi(t) &= e^{\lambda_1 t}\, I + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}\,(A - \lambda_1 I) \\
&= e^{5t}\, I + \frac{e^{5t} - e^{2t}}{5 - 2}\,(A - 5I) \\
&= e^{5t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{5t} - e^{2t}}{3} \begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}
\end{aligned}$$

Then $\vec{\mathbf{u}}(t) = \Phi(t)\vec{\mathbf{u}}(0)$, verifying the illustration answer. Let's continue to simplify the answer:

$$\begin{aligned}
\Phi(t) &= e^{5t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{5t} - e^{2t}}{1} \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{5t} \end{pmatrix} + \begin{pmatrix} -2e^{5t} + 2e^{2t} & e^{5t} - e^{2t} \\ -2e^{5t} + 2e^{2t} & e^{5t} - e^{2t} \end{pmatrix} \\
&= \begin{pmatrix} -e^{5t} + 2e^{2t} & e^{5t} - e^{2t} \\ -2e^{5t} + 2e^{2t} & 2e^{5t} - e^{2t} \end{pmatrix}
\end{aligned}$$

```
# Exercise 11, Illustrations: distinct eigenvalues, Ans Check
A:=<-1,3|-6,8>^+;
MatrixExponential(A,t);
 # Matrix([[2*exp(2*t)-exp(5*t), exp(5*t)-exp(2*t)],
 # [-2*exp(5*t)+2*exp(2*t), -exp(2*t)+2*exp(5*t)]]);
```

12. (Complex Eigenvalues) Show the details for the solution of

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix} \vec{u}.$$

Isolated Equilibria

13. (Determinant Expansion) Verify that $|A - \lambda I|$ equals

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

Solution: Because $|A + xI|$ is a quadratic polynomial with roots r_1, r_2 then $|A + xI| = (x - r_1)(x - r_2)$. Then

$$\begin{aligned} |A - \lambda I| &= |A + xI|\,|_{x=-\lambda} \\ &= (-\lambda - \lambda_1)(-\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

14. (Infinitely Many Equilibria) Explain why $A\vec{u} = \vec{0}$ has infinitely many solutions when $\det(A) = 0$.

Classification of Equilibria

15. (Rotating Figures) When sines and cosines appear in the Euler atoms, the phase portrait at $(0, 0)$ rotates around the origin. Explain precisely why this is true.

Solution: Sines and cosines appear because of complex eigenvalues $a \pm bi$ with $b > 0$. The phase portrait is realized as a choice of several (x_0, y_0) initial conditions, from which threaded solution curves are added to the portrait.

Matrix A satisfies $AP = PD$ where $D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is formed from the complex eigenvalues $a \pm bi$. Geometrically, matrix D is a rotation matrix. Matrix P is invertible and real: it is a change of coordinates. The columns of P are the real and imaginary parts of an eigenvector $\vec{\mathbf{v}}$ of A. A threaded curve $(x(t), y(t))$ starting at (x_0, y_0) has a simpler expression in terms of the coordinate system (X, Y) defined by the columns of P.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \quad \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = P^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = e^{at} \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$$

Choosing a starting point (x_0, y_0) amounts to choosing (X_0, Y_0). The impact of factor e^{at} is scaling, nothing to do with rotation. Matrix factor $\Psi(t) = \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}$ for fixed t is itself a rotation matrix. Then

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = e^{at}\Psi(t) \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$$

is rotation $\Psi(t)$ followed by scaling $e^{at}I$.

There are five values of t that provide the most insight about rotation of the threaded curves. They are $bt = 0, \pi/2, \pi, 3\pi/2, 2\pi$. Then the five rotation matrices $\Psi(t)$ are

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, R_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, R_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let's ignore the scale factor e^{at} for the moment and examine the position of initial point $\vec{\mathbf{v}} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$ at the five times. For the discussion, assume $\vec{\mathbf{v}}$ is in quadrant I, both coordinates positive. At $t = 0$ and $t = 2\pi/b$, point $\vec{\mathbf{v}}$ is multiplied by $R_1 = R_5 = I$: the point is stationary. At $t = \dfrac{\pi}{2b}$ the geometric result is

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} X \\ Y \end{pmatrix} = R_1 \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} Y_0 \\ -X_0 \end{pmatrix}$$

which is a 90 degree counter-clockwise rotation of $\vec{\mathbf{v}}$. Target $\vec{\mathbf{v}}_1$ is in quadrant II.

The analysis continues with rotation matrices R_2, R_3 resulting in vectors $\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ having the same length as $\vec{\mathbf{v}}$, each a 90 degree counter-clockwise rotation. Target $\vec{\mathbf{v}}_2$ is in quadrant III and target $\vec{\mathbf{v}}_3$ is in quadrant IV.

The snapshot analysis of vector $\vec{\mathbf{v}}$ rotation at the five times from quadrant I counter-clockwise through the four quadrants is the rotation evidence sought. ■

16. (Non-Rotating Figures) When sines and cosines do not appear in the Euler atoms, the phase portrait at $(0, 0)$ has no rotation. Give a precise explanation.

Attractor and Repeller

17. (Classification) Which of spiral, center, saddle, node can be an attractor or a repeller?

Solution: Spiral and node.

18. (Attractor) Prove that $(0, 0)$ is an attractor if and only if the Euler atoms have limit zero at $t = \infty$.

19. (Repeller) Prove that $(0, 0)$ is a repeller if and only if the Euler atoms have limit zero at $t = -\infty$.

Solution: Definition: A repeller is an equilibrium point (x_0, y_0) such that all nearby solutions limit to (x_0, y_0) as t tends to negative infinity.

A solution $\vec{\mathbf{v}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of $\vec{\mathbf{v}}' = A\vec{\mathbf{v}}$ can be written in terms of the Euler atoms A_1, A_2 as $\vec{\mathbf{v}} = \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ for some constants a_1, a_2, b_2, b_2. Let $B = \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}$. Then $\vec{\mathbf{v}} = B \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ is a solution of $\vec{\mathbf{v}}' = A\vec{\mathbf{v}}$ for any constant matrix B.

Assume the origin is a repeller.
To prove: the Euler atoms have limit zero at $t = -\infty$. Choose matrix $B = I$. Then $\vec{\mathbf{v}} = B \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ has limit zero at $t = -\infty$. This proves the Euler atoms have limit zero at $t = -\infty$.

Assume the Euler atoms have limit zero at $t = -\infty$.
To prove: the origin is a repeller, i.e., the limit of any nearby solution $\vec{\mathbf{v}} = B \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ is zero at $t = -\infty$. Because B is a constant matrix then

$$\lim_{t=-\infty} \vec{\mathbf{v}}(t) = B \lim_{t=-\infty} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = B \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{\mathbf{0}}$$

■

20. (Center) A center is neither an attractor nor a repeller. Explain, using Euler atoms.

Phase Portrait Linear

Show the classification details for spiral, center, saddle, proper node, improper node. Include for saddle and node a drawing which shows eigenvector directions. Notation: $' = \frac{d}{dt}$.

21. (Spiral)

$$\begin{aligned} x' &= 2x + 3y, \\ y' &= -3x + 2y. \end{aligned}$$

Solution: Eigenvalues $2\pi 3i$, atoms $e^{2t}\cos 3t$, $e^{2t}\sin 3t$. Sines and cosines are present and also scale factor e^{2t}. It is a rotating figure. It is a center or spiral. Center eliminated by the scale factor, which limits to infinity at $t = \infty$. It is an unstable spiral.

22. (Center)

$$\begin{aligned} x' &= 3y, \\ y' &= -3x. \end{aligned}$$

Solution: Purely complex eigenvalues $\pm 3i$ with atoms $\cos 3t$, $\sin 3t$. It is a rotating figure, center or spiral. No exponential scale factor in the atoms implies it is a center.

23. (Saddle)

$$\begin{aligned} x' &= 3x, \\ y' &= -5y. \end{aligned}$$

Solution: Eigenvector directions are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. A solution $\vec{\mathbf{u}}$ of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ can be expanded as $\vec{\mathbf{u}} = c_1\vec{\mathbf{v}}_1 e^{\lambda_1 t} + c_2\vec{\mathbf{v}}_2 e^{\lambda_2 t} = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{2t} + c_2\begin{pmatrix}0\\1\end{pmatrix}e^{-5t}$. The eigenpairs $\lambda_1, \vec{\mathbf{v}}_1)$, $\lambda_2, \vec{\mathbf{v}}_2)$ determine the asymptotes of the phase portrait as $t \to \infty$ or $t \to -\infty$. This is because an exponential with negative exponent λt limits to zero. In this example, the asymptote at $t = \infty$ is along $\vec{\mathbf{v}}_1 = \begin{pmatrix}1\\0\end{pmatrix}$: the x-axis. Eigenvectors do not have to be orthogonal, therefore do not expect asymptote directions to be orthogonal.

The eigenvalues of $A = \begin{pmatrix}3 & 0\\0 & -5\end{pmatrix}$ are $3. - 5$. The Euler atoms are e^{3t}, e^{-5t}. No sines or cosines so it is a non-rotating phase portrait: saddle or node. Because one atom limits to infinity as $t \to \infty$ and the other limits to minus infinity it must be a saddle. Reminder: Atoms are formed from the eigenvalues by strict rules: leading coefficient 1, zero is not an atom.

The test for a saddle or node:

$L_1 = \lim_{t=\infty}$ (Atom 1), $L_2 = \lim_{t=\infty}$ (Atom 2)
Extended limit values of $\pm\infty$ allowed.

Saddle: $L_1 \neq L_2$.
If you trace a threaded curve in the phase portrait then $x \to 0$, $y \to \pm\infty$ or $y \to 0$, $x \to \pm\infty$. The phase portrait asymptotes are eigenvector directions.

Node: $L_1 = L_2$.
If you trace a threaded curve in the phase portrait then $\lim x = \lim y = 0$ at $t = \infty$ or at $t = -\infty$. An asymptote except for a star node follows an eigenvector in the phase portrait.

24. (Proper Node)

$$\begin{aligned} x' &= 2x, \\ y' &= 2y. \end{aligned}$$

Solution: Atoms are e^{2t}, te^{2t}. This is a star node. There are no asymptotes to report.

25. (Improper Node: Degenerate)

$$\begin{aligned} x' &= 2x + y, \\ y' &= 2y. \end{aligned}$$

Solution: Repeated eigenvalue 2 produces atoms e^{2t}, te^{2t}. No sines or cosines means it is a non-rotating figure: saddle or node. Find limits L_1, L_2 of the two atoms. Then $L_1 = L_2$ so it is a node.

Degenerate means equal eigenvalues but only one real eigenvector.

Asymptotes are not found like in Exercise 23 because there is only one eigenvector $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The solution can be written

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix} \right) e^{2t}$$

The solution follows $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ when $c_2 = 0$. This is the asymptote direction, which is the eigenvector $\vec{\mathbf{v}}_1$ found for $\lambda = 2$.

26. **(Improper Node:** $\lambda_1 \neq \lambda_2$**)**

$$\begin{aligned} x' &= 2x + y, \\ y' &= 3y. \end{aligned}$$

Solution: Eigenvalues $2, 3$. Atoms e^{2t}, e^{3t} with equal limits L_1, L_2 at infinity. No sines and cosines means a non-rotating figure: node or saddle. Equal limits $L_1 = L_2$ eliminates the saddle: it is a node.

An **improper node** is distinguished from a **proper node** and a **degenerate node** by having distinct eigenvalues. The classification terminology has only limited use in the literature: all are called nodes, ignoring the delicate distinctions. **Degenerate** means equal eigenvalues but only one real eigenvector.

Asymptotes are found found using ideas in Exercise 23. The eigenpairs are $(2, \vec{\mathbf{v}}_1)$, $(3, \vec{\mathbf{v}}_2)$ where $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The solution can be written

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{3t}$$

The solution follows $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ because of the decay factor e^{-t}. The asymptote direction is eigenvector $\vec{\mathbf{v}}_2$ found for $\lambda = 3$, the larger eigenvalue.

10.3 Planar Almost Linear Systems

Almost Linear Systems. Find all equilibria (x_0, y_0) of the given nonlinear system. Then compute the Jacobian matrix $A = J(x_0, y_0)$ for each equilibria.

1. (Spiral and Saddle)

$$\begin{aligned} \tfrac{d}{dt}x &= x + 2y, \\ \tfrac{d}{dt}y &= 1 - x^2. \end{aligned}$$

Solution: Jacobian $J(x,y) = \begin{pmatrix} 1 & 2 \\ -2x & 0 \end{pmatrix}$.

Unstable spiral at $(1, -1/2)$, $J(1, -1/2) = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$.

Unstable saddle at $(-1, 1/2)$, $J(-1, 1/2) = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$.

```
# Exercise 1, Spiral and Saddle
f:=(x,y)->x+2*y;g:=(x,y)->1-x^2;
p:=solve({f(x,y)=0,g(x,y)=0},{x,y});
 # p = {x = 1, y = -1/2}, {x = -1, y = 1/2}
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
A1:=subs(p[1],J(x,y));Eigenvectors(A1);# unstable spiral
A2:=subs(p[2],J(x,y));Eigenvectors(A2);# unstable saddle
```

2. (Two Improper Nodes, Spiral)

$$\begin{aligned} \tfrac{d}{dt}x &= x - 3y + 2xy, \\ \tfrac{d}{dt}y &= 4x - 6y - xy - x^2. \end{aligned}$$

Solution:

$J(x,y) = \begin{pmatrix} 2y+1 & 2x-3 \\ -2x-y+4 & -x-6 \end{pmatrix}$

```
# Exercise 2, Two Improper Nodes, Spiral
f:=(x,y)->x-3*y+2*x*y;g:=(x,y)->4*x-6*y-x*y-x^2;
q:=solve({f(x,y)=0,g(x,y)=0},{x,y});
 # q := {x=0, y=0},
 #    {x=RootOf(_Z^2-6*_Z+3), y=-(1/5)*RootOf(_Z^2-6*_Z+3)+2/5}
r:=[allvalues(RootOf(_Z^2-6*_Z+3))];
 # r = 3-sqrt(6), 3+sqrt(6)
p:=[ {x=0,y=0}, {x=r[1], y=-(1/5)*r[1]},
     {x=r[2], y=-(1/5)*r[2] }] ;
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
J(x,y);
A1:=subs(p[1],J(x,y));Eigenvectors(A1);# stable improper node
A2:=subs(p[2],J(x,y));Eigenvectors(A2);# stable improper node
A3:=subs(p[3],J(x,y));Eigenvectors(A3);# stable spiral
```

3. (Proper Node, Saddle)

$$\begin{aligned}\tfrac{d}{dt}x &= 3x - 2y - x^2 - y^2,\\ \tfrac{d}{dt}y &= 2x - y.\end{aligned}$$

Solution:
Jacobian $J(x,y) = \begin{pmatrix} -2x+3 & -2y-2 \\ 2 & -1 \end{pmatrix}$

Equilibria at $x = 0, y = 0$ and $x = -1/5, y = -2/5$.

$J(0,0) = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$, unstable star node (a proper node)

$J(-1/5,-2/5) = \begin{pmatrix} 17/5 & -6/5 \\ 2 & -1 \end{pmatrix}$, unstable saddle

```
# Exercise 3, Spiral, Saddle
f:=(x,y)->3*x-2*y-x^2-y^2;g:=(x,y)->2*x-y;
p:=solve({f(x,y)=0,g(x,y)=0},{x,y});
 # p : {x = 0, y = 0}, {x = -1/5, y = -2/5}
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
A1:=subs(p[1],J(x,y));Eigenvectors(A1);# unstable spiral
A2:=subs(p[2],J(x,y));Eigenvectors(A2);# unstable saddle
```

4. (Center and Three Saddles)

$$\begin{aligned}\tfrac{d}{dt}x &= x - y + x^2 - y^2,\\ \tfrac{d}{dt}y &= 2x - y - xy.\end{aligned}$$

Solution: $J(x,y) = \begin{pmatrix} 2x+1 & -1-2y \\ -y+2 & -x-1 \end{pmatrix}$

```
# Exercise 4, Center and Three Saddles
f:=(x,y)->x-y+x^2-y^2;g:=(x,y)->2*x-y-x*y;
p:=solve({f(x,y)=0,g(x,y)=0},{x,y});
 # p := {x = 0, y = 0}, {x = 1, y = 1},
 #   {x = -1-RootOf(_Z^2-2*_Z-2), y = RootOf(_Z^2-2*_Z-2)}
r:=allvalues(RootOf(_Z^2-2*_Z-2));
p:=[{x = 0, y = 0}, {x = 1, y = 1}, {x = -1-r[1], y = r[1]},
    {x = -1-r[2], y = r[2]}];
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
A1:=subs(p[1],J(x,y));Eigenvectors(A1);# stable center
A2:=subs(p[2],J(x,y));Eigenvectors(A2);# unstable saddle
A3:=subs(p[3],J(x,y));Eigenvectors(A3);# unstable saddle
A4:=subs(p[4],J(x,y));Eigenvectors(A4);# unstable saddle
```

5. (Proper Node and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= y - xy. \end{aligned}$$

Solution: Jacobian $J(x,y) = \begin{pmatrix} 2x+1 & -1-2y \\ -y & -x+1 \end{pmatrix}$.

Equilibria:
$x = 0, y = 0, \quad x = -1, y = 0, \quad x = 1, y = -2, \quad x = 1, y = 1$

Jacobians in order:
$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & -3 \\ -1 & 0 \end{pmatrix}$.

```
# Exercise 5, Proper Node and Three Saddles
f:=(x,y)->x-y+x^2-y^2;g:=(x,y)->y-x*y;
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];
 # p := [ {x=0, y=0}, {x=-1, y=0}, {x=1, y=-2}, {x=1, y=1} ]
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
A1:=subs(p[1],J(x,y));Eigenvectors(A1);# star node
A2:=subs(p[2],J(x,y));Eigenvectors(A2);# unstable saddle
A3:=subs(p[3],J(x,y));Eigenvectors(A3);# unstable saddle
A4:=subs(p[4],J(x,y));Eigenvectors(A4);# unstable saddle
```

6. (Degenerate Node, Spiral and Two Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3 + y^3, \\ \frac{d}{dt}y &= y + 3xy. \end{aligned}$$

Solution: Jacobian $J(x, y) = \begin{pmatrix} 3x^2+1 & 3y^2-1 \\ 3y & 3x+1 \end{pmatrix}$

```
# Exercise 6, Degenerate Node, Spiral and Two Saddles
f:=(x,y)->x-y+x^3+y^3;g:=(x,y)->y+3*x*y;
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];# Ignore RootOf(_Z^2+1)
r:=[allvalues(RootOf(_Z^2-2*_Z-5))];
p := [{x=0, y=0}, {x=-1/3, y=-2/3}, {x=-1/3, y=(1/3)*r[1]},
      {x=-1/3, y=(1/3)*r[2]}];
 # p := [{x=0, y=0}, {x=-1/3, y=-2/3},
 #       {x=-1/3, y=1/3+(1/3)*sqrt(6)},
 #       {x=-1/3, y=1/3-(1/3)*sqrt(6)}]
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
A1:=subs(p[1],J(x,y));Eigenvectors(A1);# degenerate node
A2:=subs(p[2],J(x,y));Eigenvectors(A2);# unstable spiral
A3:=subs(p[3],J(x,y));Eigenvectors(A3);# unstable saddle
A4:=subs(p[4],J(x,y));Eigenvectors(A4);# unstable saddle
```

7. (Improper Node, Saddle)

$$\begin{aligned} \tfrac{d}{dt}x &= x - y + x^3, \\ \tfrac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Solution: Jacobian $J(x, y) = \begin{pmatrix} 3x^2+1 & -1 \\ 3y & 3x+2 \end{pmatrix}$

Equilibria: $x = 0, y = 0, \quad x = -2/3, y = -26/27$

Jacobians in order:

$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 7/3 & -1 \\ -26/9 & 0 \end{pmatrix}$

```
# Exercise 7, Improper Node, Saddle
f:=(x,y)->x-y+x^3;g:=(x,y)->2*y+3*x*y;
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];# Ignore RootOf(_Z^2+1)
p := [{x = 0, y = 0}, {x = -2/3, y = -26/27}];
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
A1:=subs(p[1],J(x,y));Eigenvectors(A1);# improper node
A2:=subs(p[2],J(x,y));Eigenvectors(A2);# unstable saddle
```

8. (Proper Node and a Saddle)

$$\begin{aligned} \tfrac{d}{dt}x &= 2x + y^3, \\ \tfrac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Solution: Jacobian $J(x,y) = \begin{pmatrix} 2x+1 & -1-2y \\ -y & -x+1 \end{pmatrix}$

```
# Exercise 8, Proper Node and a Saddle
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];
restPointClassify:=proc(valueSet)# arg = {x=a,y=b}
 global f,g; local J,A,Lambda,P,jacobian,L;
 with(LinearAlgebra):L:=subs(valueSet,[x,y]);
 jacobian:=Student[MultivariateCalculus][Jacobian]:
 J:=LL->jacobian([f(x,y),g(x,y)],[x,y]=LL);
 A:=J(L);Lambda,P:=Eigenvectors(A);
 RETURN (A,Lambda,evalf(Lambda),P,evalf(P));
end proc:
fmt:="%a: A,Lambda,evalf(Lambda),P,evalf(P)\n":
for i from 1 to nops(p)
do printf(fmt,p[i]); restPointClassify(p[i]); od;
```

Phase Portrait Almost Linear. Linear library phase portraits can be locally pasted atop the equilibria of an almost linear system, with limitations. Apply the theory for the following examples. Complete the phase diagram by computer, thereby resolving the possible mutation of a center or node into a spiral. Label eigenvector directions where it makes sense.

9. (Center and Three Saddles)

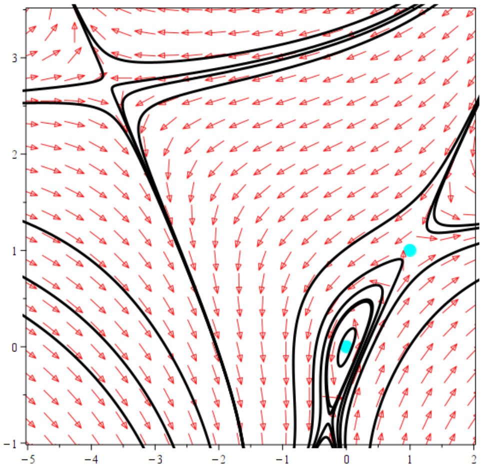
$$\begin{aligned} \tfrac{d}{dt}x &= x - y + x^2 - y^2, \\ \tfrac{d}{dt}y &= 2x - y - xy. \end{aligned}$$

Solution:
Equilibria:
$x = 0, y = 0, \quad x = 1, y = 1.0,$
$x = -3.73, y = 2.73, \quad x = -.2679, y = -.732$
Eigenvalue pairs:
`[I,-I], [2.3,-1.3], [3.2,-6.95], [1.14,-1.41]`
Jacobian $J(x,y) = \begin{pmatrix} 2x+1 & -1-2y \\ -y+2 & -x-1 \end{pmatrix}$
xy-Window: $-5 < t < 2$, $-1 < y < 3.5$

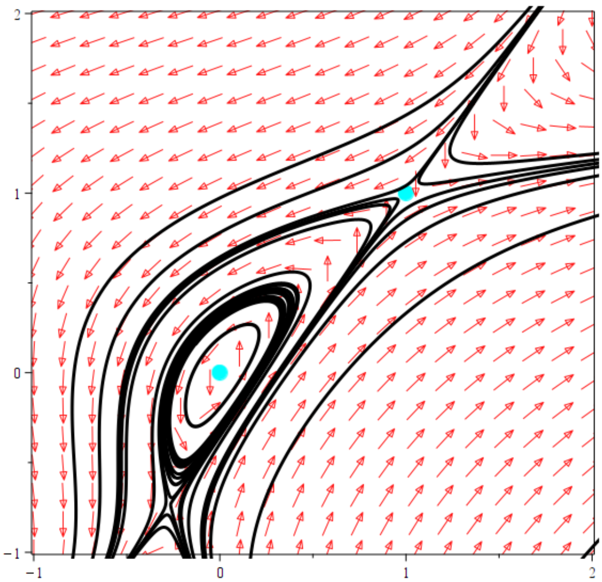
WolframAlpha offers a free online phase plane plotter. It is basic without enough features to be confusing and limited in what it can display. For Exercise 9 the graph window has to be limited to a region around one equilibrium in order to show adequate detail. At the online Wolfram site below, search for string **phase plane**:

https://www.wolframalpha.com/widgets/gallery/?category=math

The phase plot in `maple` is a challenge due to proximity of the equilibria. The `maple` phase portrait task needed a resolution change to 800x800 from the default 400x400. Changes were made to line width. The graph window was first estimated then corrected by trial and error. Changing the graph window to focus on three of the four equilibria made it possible to see most details. There was no facility to add z separatrix to saddles and nodes. To do that, print the graphic on paper and draw the lines by hand.



Exercise 9: center, 3 saddles [0, 0], [1, 1], [−3.73, 2.73], [−0.268, −0.732]



Exercise 9: center, 2 saddles [0, 0], [1, 1], [−0.268, −0.732]

```
# Exercise 9, Center and Three Saddles
f:=(x,y)->x-y+x^2-y^2;g:=(x,y)->2*x-y-x*y;
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];
r:=allvalues(RootOf(_Z^2-2*_Z-2));
p := [{x = 0, y = 0}, {x = 1, y = 1},
  {x = -1-r[1], y = r[1]}, {x = -1-r[2], y = r[2]}];
 # p := [{x = 0, y = 0}, {x = 1, y = 1},
 #  {x=-2-sqrt(3), y=1+sqrt(3)}, {x=-2+sqrt(3), y=1-sqrt(3)}]
restPointClassify:=proc(valueSet)# arg = {x=a,y=b}
 global f,g; local J,A,Lambda,P,jacobian,L;
 with(LinearAlgebra):L:=subs(valueSet,[x,y]);
 jacobian:=Student[MultivariateCalculus][Jacobian]:
 J:=LL->jacobian([f(x,y),g(x,y)],[x,y]=LL);
 A:=J(L);Lambda,P:=Eigenvectors(A);
 RETURN (A,Lambda,evalf(Lambda),P,evalf(P));
end proc:
fmt:="%a: A,Lambda,evalf(Lambda),P,evalf(P)\n":
for i from 1 to nops(p)
do printf(fmt,p[i]); restPointClassify(p[i]); od;
 # center, three saddles
```

10. **(Degenerate Node, Three Saddles)**

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= y - xy. \end{aligned}$$

11. **(Degenerate Node, Spiral, Two Saddles)**

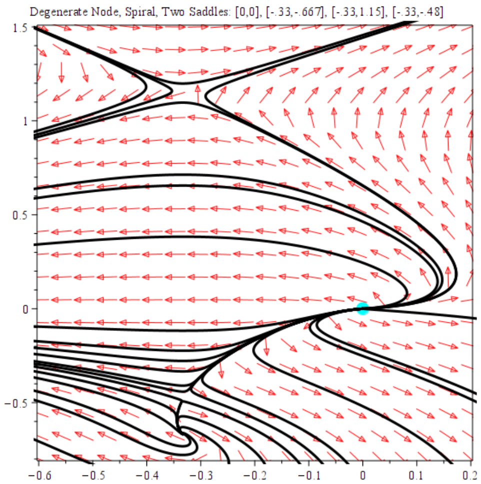
$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3 + y^3, \\ \frac{d}{dt}y &= y + 3xy. \end{aligned}$$

Solution: Jacobian $J(x,y) = \begin{pmatrix} 3x^2+1 & 3y^2-1 \\ 3y & 3x+1 \end{pmatrix}$

Equilibria:
$x = 0, y = 0, \quad x = -0.33, y = -0.667, \quad x = -0.33, y = 1.15,$
$x = -0.33, y = -0.48$
For `maple`: `[0,0], [-.33,-.667], [-.33,1.15], [-.33,-.48]`

xy-Window: $-0.6 < x < 0.2$, $-0.8 < y < 1.5$

The graphic obtained from the Phase Portrait Task in `maple` lacks important detail near equilibria. One fix is to make 4 plots, each focused on an equilibrium. Then plot on a full size window to show the global behavior.




```
# Exercise 11, Degenerate Node, Spiral, Two Saddles
f:=(x,y)->x-y+x^3+y^3;g:=(x,y)->y+3*x*y;
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];
r:=allvalues(RootOf(_Z^2-2*_Z-5));
p := [{x = 0, y = 0}, {x = -1/3, y = -2/3},
  {x = -1/3, y = (1/3)*r[1]}, {x = -1/3, y = (1/3)*r[2]}];
 #  p := [{x=0,y=0},{x=-1/3,y=-2/3},
 # {x=-1/3,y=1/3+(1/3)*sqrt(6)},{x=-1/3,y=1/3-(1/3)*sqrt(6)}]
restPointInfo:=proc(valueSet)# arg = {x=a,y=b}
 global f,g,J; local A,Lambda,P,jacobian,L;
 with(LinearAlgebra):
 jacobian:=Student[MultivariateCalculus][Jacobian]:
 J:=LL->jacobian([f(x,y),g(x,y)],[x,y]=LL);
 L:=subs(valueSet,[x,y]);A:=J(L);Lambda,P:=Eigenvectors(A);
 RETURN (A,Lambda,evalf(Lambda),P,evalf(P));
end proc:
JACOBIAN:=J(['x','y']);
fmt:="%a: A,Lambda,evalf(Lambda),P,evalf(P)\n":
for i from 1 to nops(p)
do printf(fmt,p[i]); restPointInfo(p[i]); od;
 # unstable degenerate node, stable spiral, two saddles
```

12. **(Improper Node, Saddle)**

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Solution:

```
# Exercise 12, Improper Node, Saddle
f:=(x,y)->x-y+x^3;g:=(x,y)->2*y+3*x*y;
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];
p:=[{x=0,y=0},{x=-2/3,y=-26/27}];
restPointInfo:=proc(valueSet)# arg = {x=a,y=b}
 global f,g,J; local A,Lambda,P,jacobian,L;
 with(LinearAlgebra):
 jacobian:=Student[MultivariateCalculus][Jacobian]:
 J:=LL->jacobian([f(x,y),g(x,y)],[x,y]=LL);
 L:=subs(valueSet,[x,y]);A:=J(L);Lambda,P:=Eigenvectors(A);
 RETURN (A,Lambda,evalf(Lambda),P,evalf(P));
end proc:
JACOBIAN:=J(['x','y']);
fmt:="%a: A,Lambda,evalf(Lambda),P,evalf(P)\n":
for i from 1 to nops(p)
do printf(fmt,p[i]); restPointInfo(p[i]); od;
 # Improper node, saddle
```

13. (Proper Node, Saddle)

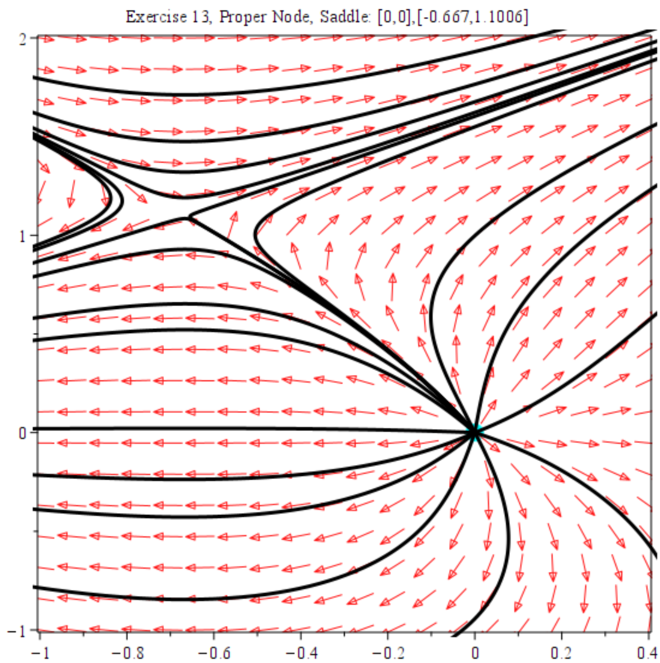
$$\begin{aligned} \frac{d}{dt}x &= 2x + y^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Solution:

Jacobian $J(x,y) = \begin{pmatrix} 2 & 3y^2 \\ 3y & 3x+2 \end{pmatrix}$

Equilibria for `maple`: `[0,0],[-0.667,1.1006]`

xy-Window: $-1 < x < 0.4$, $-1 < y < 2$




```
# Exercise 13, Proper Node, Saddle
f:=(x,y)->2*x+y^3;g:=(x,y)->2*y+3*x*y;
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];
r:=allvalues(RootOf(3*_Z^3-4));
p:=[{x=0,y=0},{x=-2/3,y=r[1]}];# ignore y complex
restPointInfo:=proc(valueSet)# arg = {x=a,y=b}
 global f,g,J; local A,Lambda,P,jacobian,L;
 with(LinearAlgebra):
 jacobian:=Student[MultivariateCalculus][Jacobian]:
 J:=LL->jacobian([f(x,y),g(x,y)],[x,y]=LL);
 L:=subs(valueSet,[x,y]);A:=J(L);Lambda,P:=Eigenvectors(A);
 RETURN (A,Lambda,evalf(Lambda),P,evalf(P));
end proc:
JACOBIAN:=J(['x','y']);
fmt:="%a: A,Lambda,evalf(Lambda),P,evalf(P)\n":
for i from 1 to nops(p)
do printf(fmt,p[i]); restPointInfo(p[i]); od;
 # Proper Node, saddle
```

14. (Two Improper Nodes and Two Saddles)

$$\begin{aligned}\frac{d}{dt}x &= (120-4x-2y)x,\\ \frac{d}{dt}y &= (60-x-2y)y\end{aligned}$$

Solution:

```
# Exercise 14, Two Nodes and Two Saddles
# Peaceful co-existence, Rabbit-Gerbil system
f:=(x,y)->(120-4*x-2*y)*x;g:=(x,y)->(60-x-2*y)*y;
solve({f(x,y)=0,g(x,y)=0},{x,y});
 # {x=0,y=0},{x=0,y=30},{x=30,y=0},{x=20,y= 20}
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
A:=J(0,0);Eigenvectors(A);# unstable node
A:=J(0,30);Eigenvectors(A);# unstable saddle
A:=J(30,0);Eigenvectors(A);;# unstable saddle
A:=J(20,20);Eigenvectors(A);evalf(%);# stable node
```

Classification of Almost Linear Equilibria. With computer assist, find and classify the nonlinear equilibria.

15. (Co-existing Species)

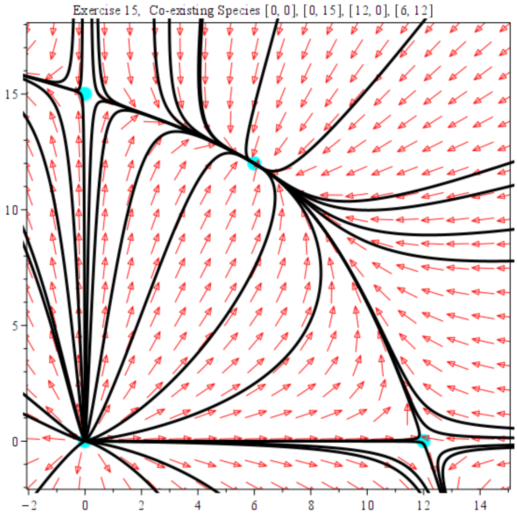
$$\begin{aligned} x'(t) &= x(t)(24 - 2x(t) - y(t)), \\ y'(t) &= y(t)(30 - 2y(t) - x(t)). \end{aligned}$$

Solution:

Jacobian $J(x,y) = \begin{pmatrix} 24 - 4x - y & -x \\ -y & 30 - 4y - x \end{pmatrix}$

Equilibria for `maple`: `[0,0],[0,15],[12,0],[6,12]`

xy-Window: $-1 < x < 15$, $-1 < y < 18$



```
# Exercise 15, Co-existing Species
f:=(x,y)->x*(24-2*x-y);g:=(x,y)->y*(30-2*y-x);
p:=[solve({f(x,y)=0,g(x,y)=0},{x,y})];
# p:=[{x=0,y=0},{x=0,y=15},{x=12,y=0},{x=6,y=12}]
restPointInfo:=proc(valueSet)# arg = {x=a,y=b}
 global f,g,J; local A,Lambda,P,jacobian,L;
 with(LinearAlgebra):
 jacobian:=Student[MultivariateCalculus][Jacobian]:
 J:=LL->jacobian([f(x,y),g(x,y)],[x,y]=LL);
 L:=subs(valueSet,[x,y]);A:=J(L);Lambda,P:=Eigenvectors(A);
 RETURN (A,Lambda,evalf(Lambda),P,evalf(P));
end proc:
JACOBIAN:=J(['x','y']);
fmt:="%a: A,Lambda,evalf(Lambda),P,evalf(P)\n":
for i from 1 to nops(p)
do printf(fmt,p[i]); restPointInfo(p[i]); od;
 # unstable node, saddle, saddle, stable node
```

16. (Doomsday-Extinction)

$$\begin{aligned} x'(t) &= x(t)(x(t) - y(t) - 4), \\ y'(t) &= y(t)(x(t) + y(t) - 8). \end{aligned}$$

Almost Linear Geometry. A separatrix S is a union of curves and equilibria. Ideally, orbits limit to S. With computer assist, make a plot of threaded curves which identify one or more separatrices near the equilibrium.

17. (Saddle $(-1, 1)$)

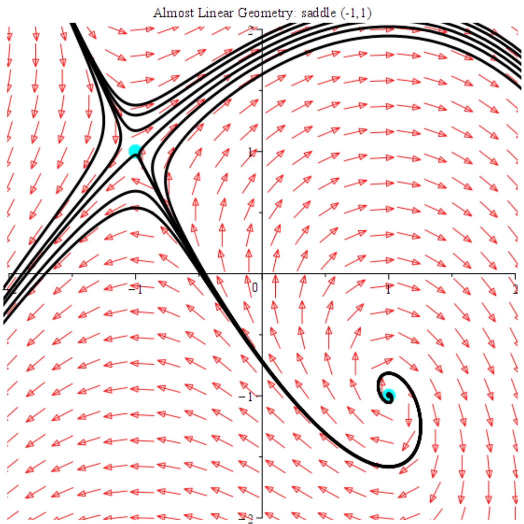
$$\begin{aligned} \frac{d}{dt}x &= x + y, \\ \frac{d}{dt}y &= 1 - x^2. \end{aligned}$$

Solution:

Jacobian $J(x, y) = \begin{pmatrix} 1 & 1 \\ -2x & 0 \end{pmatrix}$

Equilibria for `maple`: `[1,-1],[-1,1]`

xy-Window: $-2 < x < 2$, $-2 < y < 2$



18. **(Saddle** $(-1/5, -2/5)$**)**

$$\begin{aligned} \frac{d}{dt}x &= 3x - 2y - x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y. \end{aligned}$$

19. **(Saddle** $(-2/3, \sqrt[3]{4/3})$**)**

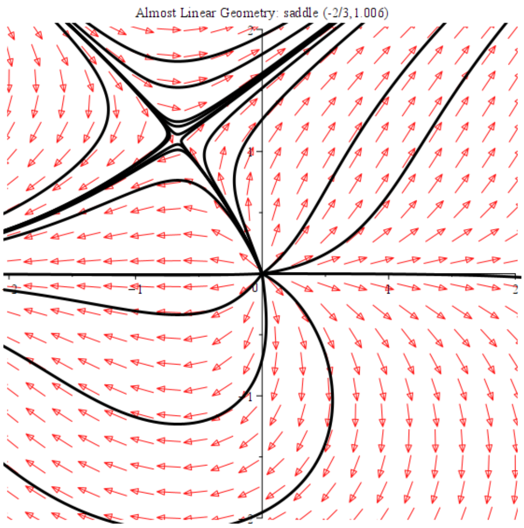
$$\begin{aligned} \frac{d}{dt}x &= 2x + y^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Solution:

Jacobian $J(x, y) = \begin{pmatrix} 2 & 3y^2 \\ 3y & 2 + 3x \end{pmatrix}$

Equilibria for `maple`: `[0,0],[-0.667,1.100642416]`

xy-Window: $-2 < x < 2$, $-2 < y < 2$



20. **(Degenerate Improper Node** $(0,0)$**)**

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3 + y^3, \\ \frac{d}{dt}y &= y + 3xy. \end{aligned}$$

Rayleigh and van der Pol. Each example below has a unique periodic orbit surrounding an equilibrium point that is the limit at $t = \infty$ of any other orbit. Discuss the spiral repeller at $(0,0)$ in the attached figure, from the linearized problem at $(0,0)$ and **Paste Theorem** 10.4. Create a phase portrait with computer assist for the nonlinear problem.

21. **(Lord Rayleigh 1877, Clarinet Reed Model)**

$$\begin{aligned} \frac{d}{dt}x &= y, \\ \frac{d}{dt}y &= -x + y - y^3. \end{aligned}$$

Solution:

Jacobian: $J(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & -3y^2+1 \end{pmatrix}$, $J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

The eigenvalues of $J(0,0)$ are $\frac{1}{2} \pm \frac{1}{2}i$. The linearized problem $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ at $x = y = 0$ is an unstable center. However, the Paste Theorem 10.4 does not predict the phase portrait near $(0,0)$ for the nonlinear problem: it is a center or spiral. Stability is also not inherited: the nonlinear phase portrait can be stable or unstable at $(0,0)$.

Graphing the nonlinear phase portrait reveals $(0,0)$ is unstable, a repeller.

Orbits that start far from $(0,0)$ wind around the origin but never reach it: they limit to a cycle as shown in the figure.

```
# Exercise 21, Clarinet Reed Model, Lord Rayleigh 1877
f:=(x,y)->y; g:=(x,y)->-x+y-y^3;
JJ:=(a,b)->Student[MultivariateCalculus][Jacobian]
    ([f(x,y),g(x,y)],[x,y]=[a,b]);
JACOBIAN:=JJ('x','y');A:=JJ(0,0);
Lambda,P:=LinearAlgebra[Eigenvectors](A);
ic1:=[x(0)=0,y(0)=0.5],[x(0)=0,y(0)=-0.5],
     [x(0)=0.5,y(0)=0],[x(0)=-0.5,y(0)=0];
ic2:=[x(0)=0,y(0)=1.6],[x(0)=0,y(0)=-1.6],
     [x(0)=1.6,y(0)=0],[x(0)=-1.6,y(0)=0];
des:=diff(x(t),t)=f(x(t),y(t)),diff(y(t),t)=g(x(t),y(t)):
wind:=x=-3..3,y=-3..3:Times:=t=-15..15:
opts:=axes=none,thickness=2,arrows=small,color=blue,
  linecolor=black,numpoints=500,stepsize=0.05:
ics:=[ic1,ic2]:
DEtools[DEplot]([des],[x(t),y(t)],Times,ics,wind,opts);
```

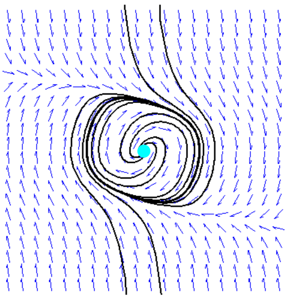


Figure 1. Clarinet Reed.

22. (van der Pol 1924, Radio Oscillator Circuit Model)

$$\begin{aligned} \frac{d}{dt}x &= y, \\ \frac{d}{dt}y &= -x + (1 - x^2)y. \end{aligned}$$

Solution: Details follow Exercise 21.

```
# Exercise 22, van der Pol 1924, Radio Oscillator Circuit
f:=(x,y)->y; g:=(x,y)->-x + (1-x^2)*y;
JJ:=(a,b)->Student[MultivariateCalculus][Jacobian]
    ([f(x,y),g(x,y)],[x,y]=[a,b]);
JACOBIAN:=JJ('x','y');A:=JJ(0,0);
Lambda,P:=LinearAlgebra[Eigenvectors](A);
ic1:=[x(0)=0,y(0)=0.5],[x(0)=0,y(0)=-0.5],
     [x(0)=0.5,y(0)=0],[x(0)=-0.5,y(0)=0];
ic2:=[x(0)=0,y(0)=2.8],[x(0)=0,y(0)=-2.3],
     [x(0)=2.4,y(0)=0],[x(0)=-2.8,y(0)=0];
des:=diff(x(t),t)=f(x(t),y(t)),diff(y(t),t)=g(x(t),y(t)):
wind:=x=-3..3,y=-3..3:Times:=t=-15..15:
opts:=axes=none,thickness=2,arrows=small,color=blue,
  linecolor=black,numpoints=500,stepsize=0.05:
ics:=[ic1,ic2]:
DEtools[DEplot]([des],[x(t),y(t)],Times,ics,wind,opts);
```

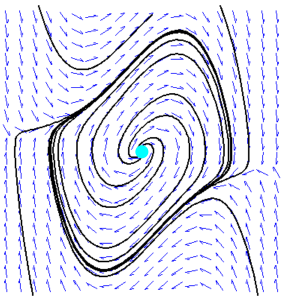


Figure 2. Oscillator Circuit.

10.4 Biological Models

Predator-Prey Models.

Consider the system

$$\begin{aligned} x'(t) &= \frac{1}{250}(1 - 2y(t))x(t), \\ y'(t) &= \frac{3}{500}(2x(t) - 1)y(t). \end{aligned}$$

1. (System Variables) The system has vector-matrix form

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u}(t)).$$

Display formulas for $\vec{u}$ and $\vec{F}$.

Solution: The formulas:

$\vec{\mathbf{u}} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \begin{pmatrix} \frac{1}{250}x(1-2y) \\ \frac{3}{500}y(2x-1) \end{pmatrix}$

Computer implementations are not the same:

```
# Exercise 1, Predator-Prey, System Variables
PDEtools[declare]((x, y)(t), prime = t);
 ## x(t), y(t) are displayed as x, y
 ## diff(f(t),t) displayed in prime notation f'
u:=t-><x(t),y(t)>;
F0:=(x,y)-><x*(1-2*y)/250,3*y*(2*x-1)/500>;
F:=w->F0(w[1],w[2]):
F(u(t));
 ## <(1/250)*x(t)*(1-2*y(t)),(3/500)*y(t)*(2*x(t)-1)>
```

2. (System Parameters) Identify the values of a, b, c, d, p, q, as used in the textbook's predator-prey system.

3. (Identify Predator and Prey) Which of $x(t), y(t)$ is the predator?

Solution: When the number of predators is near zero then the number of prey explodes: think rabbits and foxes. In the model, $y \approx 0$ in the first differential equation reduces to $x' = cx$ with $c = 1/250$ positive. The model is Malthusian population growth: population x explodes. Therefore, $x =$ prey, $y =$ predator.

4. (Switching Predator and Prey) Give an example of a predator-prey system in which $x(t)$ is the predator and $y(t)$ is the prey.

Implicit Solution Predator-Prey. These exercises prove equation

$$a \ln|y| + b \ln|x| - q\,x - p\,y = C$$

for predator-prey system

$$\begin{aligned} x'(t) &= (a - p\,y(t))x(t), \\ y'(t) &= (q\,x(t) - b)y(t). \end{aligned}$$

5. (First Order Equation) Verify from the chain rule of calculus the first order equation

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{y}{x}\frac{qx - b}{a - py}.$$

Solution: Details:

$$\begin{aligned} \tfrac{dy}{dx} &= \frac{y'(t)}{x'(t)} \\ &= \frac{(q\,x(t) - b)y(t)}{(a - p\,y(t))x(t)} \\ &= \frac{y(t)}{x(t)}\frac{(q\,x(t) - b)}{(a - p\,y(t))} \\ &= \frac{y}{x}\frac{q\,x - b}{a - p\,y} \quad \blacksquare \end{aligned}$$

6. (Separated Variables) Verify

$$\left(\frac{a}{y} - p\right) dy = \left(q - \frac{b}{x}\right) dx.$$

7. (Quadrature) Integrate the equation of Exercise 6 to obtain

$$a \ln|y| - p\,y = q\,x - b \ln|x| = C.$$

Then re-arrange to obtain the reported implicit solution.

Solution: Details:

$\int \left(\frac{a}{y} - p\right) dy = a \ln|y| - py + c_1$

$\int \left(q - \frac{b}{x}\right) dx = qx - b \ln|x| + c_2$

Equate the two answers above. Move constants to the right and all other terms to the left. Let $C = c_2 - c_1$. $\blacksquare$

8. (Energy Function) Define $E(t) = a \ln|u| - pu$. Show that $dE/du = (a - pu)/u$. Then show that $dE/du < 0$ for $a > 0$, $p > 0$ and $a/p < u < \infty$.

Linearized Predator-Prey System. Consider

$$\begin{aligned} x'(t) &= (100 - 2y(t))x(t), \\ y'(t) &= (2x(t) - 160)y(t). \end{aligned}$$

9. (Find Equilibria) Verify equilibria $(0,0), (80,50)$.

Solution: An equilibrium point (x,y) satisfies the equations

$$\begin{aligned} 0 &= (100 - 2y)x, \\ 0 &= (2x - 160)y. \end{aligned}$$

If either $x = 0$ or $y = 0$ then the other variable is zero, giving equilibrium $(0,0)$. If both $x \neq 0$ and $x \neq 0$ then $100 - 2y = 0$ and $2x - 160 = 0$, giving equilibrium $(80,50)$. ■

10. (Jacobian Matrix) Compute $J(x,y)$ for each x, y. Then find $J(0,0)$ and $J(80,50)$.

Solution:

Jacobian $J(x,y) = \begin{pmatrix} 100 - 2y & -2x \\ 2y & 2x - 160 \end{pmatrix}$

$J(80,50) = \begin{pmatrix} 0 & -160 \\ 100 & 0 \end{pmatrix}.$

```
# Exercise 10, Jacobian Matrix J(x,y)
f:=(x,y)->(100-2*y)*x;g:=(x,y)->(2*x-160)*y;
p:=solve({f(x,y)=0,g(x,y)=0},{x,y});
 # p = {x = 1, y = -1/2}, {x = -1, y = 1/2}
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
J(x,y);J(0,0);J(80,50);
```

11. (Transit Time) Find the transit time of an orbit for one loop about $(0,0)$ for system $\frac{d}{dt}\vec{v} = \begin{pmatrix} 0 & -160 \\ 100 & 0 \end{pmatrix} \vec{v}$, the linearization about $(80,50)$.

Solution: The transit time T for one loop is $T = 0.04967294134$ seconds.

```
# Exercise 11, Transit time for one loop about (0,0)
f:=(x,y)->(100-2*y)*x;g:=(x,y)->(2*x-160)*y;
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
A:=J(80,50);
LinearAlgebra[Eigenvectors](A);
 # eigenvalues = [ (40*I)*sqrt(10),-(40*I)*sqrt(10) ]
omega:=40*sqrt(10);
# One period of cosine and sine of omega*t is 2*Pi/omega
T:=evalf(2*Pi/omega);
 # T = 0.04967294134
```

12. **(Paste Theorem)** Describe the local figures expected near equilibria in the nonlinear phase portrait.

Solution: Saddle at $(0,0)$. At $(80,50)$ either a spiral or a center.

Rabbits and Foxes. Consider

$$\begin{aligned} x'(t) &= \frac{1}{200}\,x(t)(50-y(t)), \\ y'(t) &= \frac{1}{100}\,y(t)(x(t)-40). \end{aligned}$$

13. **(Equilibria)** Verify equilibria $(0,0)$, $(40,50)$, showing all details.

Solution: Equilibria (x,y) satisfy

$$\begin{aligned} 0 &= \frac{1}{200}\,x(50-y), \\ 0 &= \frac{1}{100}\,y(x-40). \end{aligned}$$

If $x=0$ or $y=0$ then both $x=y=0$ and the equilibrium is $(0,0)$. Otherwise, $x\neq 0$ and $y\neq 0$ and then $50-y=0$, $x-40=0$ giving equilibrium $(40,50)$. ■

14. **(Jacobian)** Compute Jacobian $J(x,y)$, then $J(0,0)$ and $J(40,50)$.

Solution:

$$J(x,y)=\begin{pmatrix} 1/4-y/200 & -x/200 \\ y/100 & x/100-2/5 \end{pmatrix}$$

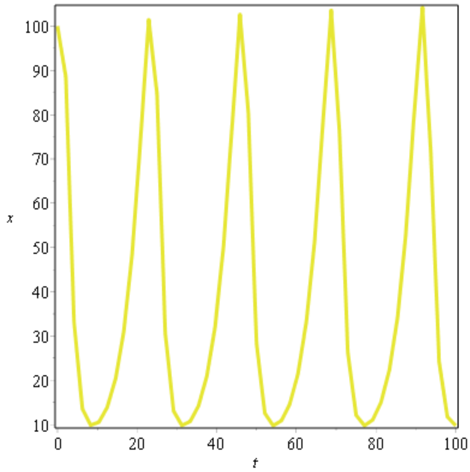
$$J(0,0)=\begin{pmatrix} 1/4 & 0 \\ 0 & -2/5 \end{pmatrix}$$

$$J(40,50)=\begin{pmatrix} 0 & -1/5 \\ 1/2 & 0 \end{pmatrix}$$

```
# Exercise 14, Jacobian, Rabbits and Foxes
f:=(x,y)->(50-y)*x/200;g:=(x,y)->(x-40)*y/100;
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
   ([f(x,y),g(x,y)],[x,y]=[a,b]);
J(x,y);J(0,0);J(40,50);
```

15. **(Rabbit Oscillation)** Find a graphical estimate for the period of oscillation of the rabbit population $x(t)$ for the nonlinear system, given $x(0)=100$, $y(0)=60$ and t is in weeks. Answer: about 23 weeks.

Solution: The plan is to graph the solution of the nonlinear Lotka-Volterra system with initial data $x(0)=100$, $y(0)=30$ and then estimate the period from the plot. Expected is a plot that looks like sine curve. The period is the t-range between two consecutive local maxima. Because the curve fails to be exactly periodic, several pairs of maxima are tested to arrive at an estimate for the period.



In `maple` the maxima can be located by mouse probe. Right-click on the plot, then click on menu item **Manipulator** $\rightsquigarrow$ *Point Probe*. Right-click again on the plot and click on menu item **Probe Info** $\rightsquigarrow$ *Cursor position*. Next step: hover the mouse over several consecutive maxima in the plot. Write on paper the t-value displayed by `maple`. Subtract t-values to get an estimate for the period:

$$46.32 - 22.58, \quad 68.876 - 46.32, \quad 91.735 - 68.876$$

The three values are all different, but approximately equal to 23 weeks. ■

```
# Exercise 15, Rabbit Estimates, Rabbits and Foxes
f:=(x,y)->(50-y)*x/200;g:=(x,y)->(x-40)*y/100;
ic:=[x(0)=x0,y(0)=y0];
de:=diff(x(t),t)=f(x(t),y(t)),diff(y(t),t)=g(x(t),y(t));
DEtools[DEplot]([de],[x(t),y(t)],t=0..100,
  [[x(0)=100,y(0)=30]],scene=[t,x]);
 # See solution text for mouse hover to display t-values at maxima.
```

16. (Rabbit-Gerbil Competing Species) Consider system

$$\begin{aligned} x' &= \left(\frac{5}{4} - \frac{x}{160} - \frac{3y}{1000}\right) x, \\ y' &= \left(3 - \frac{3y}{500} - \frac{3x}{160}\right) y. \end{aligned}$$

Verify equilibria $(0,0)$, $(0,500)$, $(200,0)$, $(80,250)$. Show the first three are nodes and the last is a saddle.

Pesticides. Consider the system

$$\begin{aligned} x'(t) &= (10 - y(t))x(t) - s_1 x(t), \\ y'(t) &= (x(t) - 20)y(t) - s_2 y(t). \end{aligned}$$

17. (Average Populations) Explain: A field biologist should count, on the average, populations of about $20 + s_2$ prey and $10 - s_1$ predators.

Solution: Equilibrium $(20 + s_2, 10 - s_1)$ is an attractor, which means $\lim_{t\to\infty} x(t) = 20 + s_2$ and $\lim_{t\to\infty} y(t) = 10 - s_1$. On a given day after the populations have oscillated sufficiently long ($t \to \infty$) the limiting populations should be observed on the average by a field biologist.

18. (Equilibria) Show details for computing the pesticide system equilibria $(0, 0)$, $(20 + s_2, 10 - s_1)$, where s_1, s_2 are the pesticide death rates.

Survival of One Species. Consider

$$\begin{aligned} x'(t) &= x(t)(24 - x(t) - 2y(t)), \\ y'(t) &= y(t)(30 - y(t) - 2x(t)). \end{aligned}$$

19. (Equilibria) Find all equilibria.

Solution: Equilibria (x, y) satisfy the equations

$$\begin{aligned} 0 &= x(24 - x - 2y), \\ 0 &= y(30 - y - 2x). \end{aligned}$$

Solve for x, y: $x = 0$ and $y = 0$ or else $x = 0$ and $y = 30$ or else $y = 0$ and $x = 24$ or else $24 - x - 2y = 0$ and $30 - y - 2x = 0$. The last equilibrium is the unique solution of the linear algebraic system of equations. Elimination gives $x = 12$, $y = 6$.

```
# Exercise 19, Survival of One Species, equilibria
x:='x':y:='y':
eqs:=24-x-2*y=0, 30-y-2*x=0;
solve([eqs],[x,y]);
 # [[x = 12, y = 6]]
```

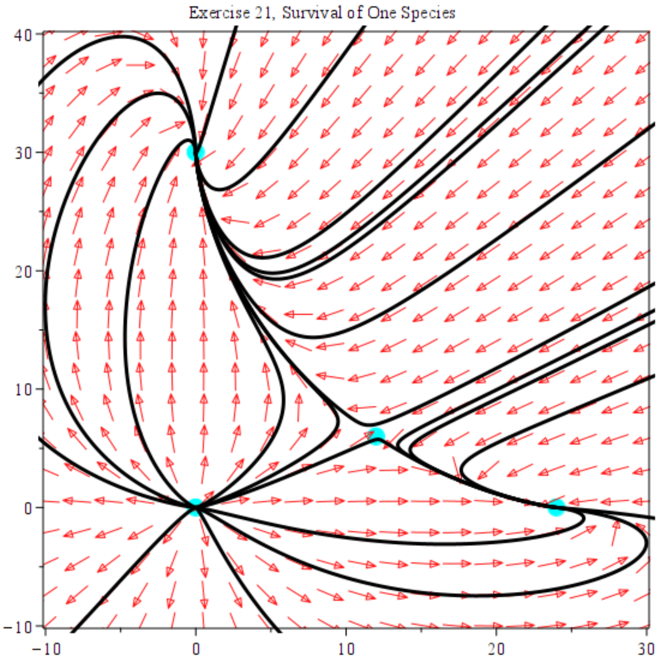
20. (Interactions) Show that doubling either x or y causes the interaction term $2xy$ to double.

21. (Nonlinear Classification) Classify each equilibrium point (x_0, y_0) as center, spiral, node, saddle, using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

Solution:
Equilibria: `[0, 0], [0, 30], [24, 0], [12, 6]`

(0,0) unstable improper node (repeller)
(0,30) stable improper node (attractor)
(24,0) stable improper node (attractor)
(12,6) unstable saddle

The Paste Theorem says that the linear classification and stability are inherited to the nonlinear phase portrait.



```
# Exercise 21, Survival of One Species, Paste Theorem
f:=(x,y)->(24-x-2*y)*x;g:=(x,y)->(30-y-2*x)*y;
p:=solve([f(x,y)=0,g(x,y)=0],[x,y]);
 # [0, 0], [0, 30], [24, 0], [12, 6]
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
    ([f(x,y),g(x,y)],[x,y]=[a,b]);
JACOBIAN:=J('x','y');
q:=seq(subs(p[i],JACOBIAN),i=1..nops(p));
seq(print(LinearAlgebra[Eigenvectors](q[i])),i=1..nops(p));
 # (0,0)  unstable improper node (repeller)
 # (0,30) stable improper node (attractor)
 # (24,0) stable improper node (attractor)
 # (12,6) unstable saddle
# Used MAPLE task Phase Portrait
# Window: -10<x<30, -10<y<40
# F=(24-x-2*y)*x, G=(30-y-2*x)*y
# Equilibria: [0, 0], [0, 30], [24, 0], [12, 6]
# Time: -20 to 20
```

22. (Extinction and Competing Species) Equilibria for which either $x = 0$ or $y = 0$ signal extinction states. Discuss how the phase portrait of the nonlinear system shows extinction of one species but not both.

Co-existence

Find the equilibria, then classify them as node, saddle, spiral, center using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

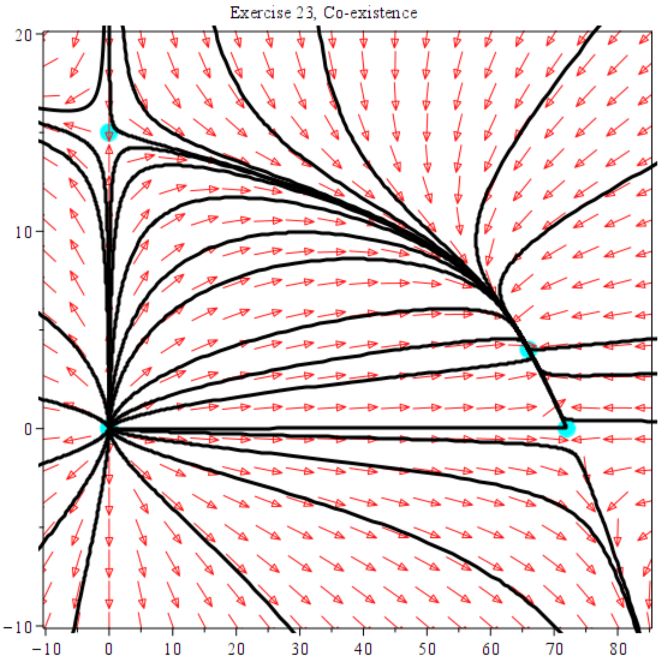
23. (Node, Saddle, Saddle, Node)

$$\begin{aligned} x' &= (144 - 2x - 3y)x, \\ y' &= (90 - 6y - x)y. \end{aligned}$$

Solution:
Equilibria: `[0,0], [0,15], [72,0], [66,4]`

(0,0) unstable improper node (repeller)
(0,15) unstable saddle
(24,0) unstable saddle
(12,6) stable improper node (attractor)

The Paste Theorem says that the linear classification and stability are inherited to the nonlinear phase portrait.




```
# Exercise 23, Co-existence, Paste Theorem
f:=(x,y)->(144-2*x-3*y)*x;g:=(x,y)->(90-6*y-x)*y;
p:=solve([f(x,y)=0,g(x,y)=0],[x,y]);
 #  [0,0], [0,15], [72,0], [66,4]
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
    ([f(x,y),g(x,y)],[x,y]=[a,b]);
JACOBIAN:=J('x','y');
q:=seq(subs(p[i],JACOBIAN),i=1..nops(p));
seq(print(LinearAlgebra[Eigenvectors](q[i])),i=1..nops(p));
 # (0,0)  unstable improper node (repeller)
 # (0,15) unstable saddle
 # (72,0) unstable saddle
 # (66,4) stable improper node (attractor)
# Used MAPLE task Phase Portrait
# Window: -10<x<85, -10<y<20
# F=(144-2*x-3*y)*x  G=(90-6*y-x)*y
# Equilibria: [0,0], [0,15], [72,0], [66,4]
# Time: -20 to 20
```

24. (Node, Saddle, Saddle, Node)

$$\begin{aligned} x' &= (120 - 4x - 2y)x, \\ y' &= (60 - x - 2y)y. \end{aligned}$$

Solution: Equilibria $(0,0)$, $(30,0)$, $(0,30)$, $(20,20)$. The nodes are stable.

Explosion and Extinction

Find the equilibria, then classify them as node, saddle, spiral, center using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

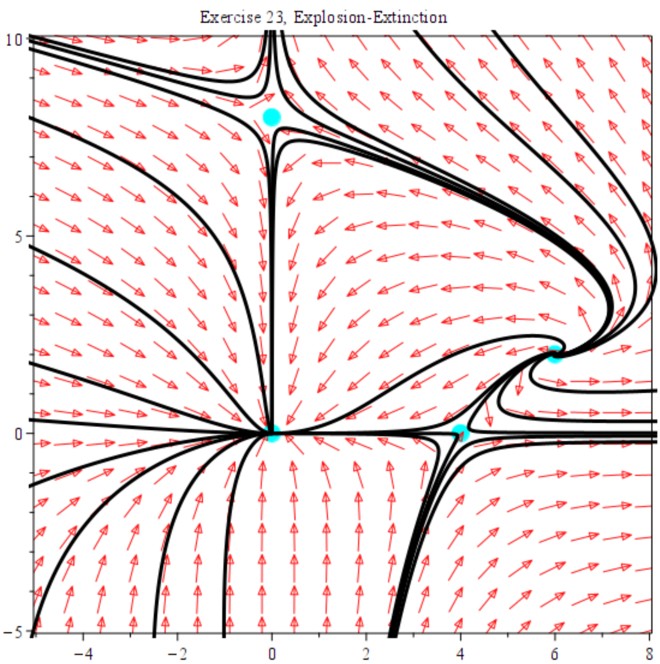
25. (Node, Saddle, Saddle, Spiral)

$$\begin{aligned} x' &= x(x-2y-4), \\ y' &= y(x+2y-8). \end{aligned}$$

Solution:
Equilibria: `[0,0], [0,8], [4,0], [6,2]`

(0,0) stable improper node (attractor)
(0,8) unstable saddle
(4,0) unstable saddle
(6,2) unstable spiral (repeller)

The Paste Theorem says that the linear classification and stability are inherited to the nonlinear phase portrait.



```
# Exercise 25, Explosion and Extinction, Paste Theorem
f:=(x,y)->(x-y-4)*x;g:=(x,y)->(x+y-8)*y;
p:=solve([f(x,y)=0,g(x,y)=0],[x,y]);
 #  [0,0], [0,8], [4,0], [6,2]
J:=(a,b)->Student[MultivariateCalculus][Jacobian]
    ([f(x,y),g(x,y)],[x,y]=[a,b]);
JACOBIAN:=J('x','y');
q:=seq(subs(p[i],JACOBIAN),i=1..nops(p));
seq(print(LinearAlgebra[Eigenvectors](q[i])),i=1..nops(p));
 # (0,0) stable improper node (attractor)
 # (0,8) unstable saddle
 # (4,0) unstable saddle
 # (6,2) unstable spiral (repeller)
# Used MAPLE task Phase Portrait
# Window: -5<x<8, -5<y<10
# F=((x-y-4)*x   G=(x+y-8)*y
# Equilibria: [0,0], [0,8], [4,0], [6,2]
# Time: -20 to 20
```

26. (Node, Saddle, Saddle, Spiral)

$$\begin{aligned} x' &= x(x-y-4), \\ y' &= y(x+y-6). \end{aligned}$$

10.5 Mechanical Models

Linear Mechanical Models

Consider the unforced linear model $mx'' + cx' + kx = 0$, where m, c, k are positive constants: m=mass, c=dashpot constant, k=Hooke's constant.

1. **(Dynamical System Form)** Write the scalar problem as $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$. Explicit definitions of $\vec{\mathbf{u}}(t)$ and A are expected.

   Solution:

$$\vec{\mathbf{u}} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$$

```
# Exercise 1, Linear Mechanical Model, Dynamical System Form
PDEtools[declare]((x, y)(t), prime = t);
 ## x(t), y(t) are displayed as x, y
 ## diff(f(t),t) displayed in prime notation f'
A:=Matrix([[0, 1], [-k/m, -c/m]]);
u:=t-><x(t),y(t)>;
# Check dynamical system form
map(diff,u(t),t) = A.u(t);
```

2. **(Attractor to $\vec{\mathbf{u}} = \vec{\mathbf{0}}$)** Explain why $\lim_{t\to\infty} \vec{\mathbf{u}}(t) = \vec{\mathbf{0}}$, giving citations to theorems in this book.

   Solution: Hint: Theorem 6.21 (Transient Solution) page 530.

3. **(Isolated Equilibrium)** Prove that $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$ has a unique equilibrium at $\vec{\mathbf{u}} = \vec{\mathbf{0}}$. Then explain why the equilibrium is isolated.

   Solution: Matrix $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$ has nonzero determinant, therefore it is invertible. If $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ then $\vec{\mathbf{x}} = A^{-1}A\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{0}} = \vec{\mathbf{0}}$. This proves the equilibrium $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ is unique.

   Isolated means there is a disk $\|\vec{\mathbf{x}}\| < r$ which excludes all solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ except $\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Let $r = 1$ and assume a solution $\vec{\mathbf{x}}$ of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ with $\|\vec{\mathbf{x}}\| < r$. Multiply $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ by A^{-1} to obtain $\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Then this disk contains no other solution of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ except $\vec{\mathbf{x}} = \vec{\mathbf{0}}$. ■

4. **(Phase Plots)** Classify the cases of **over-damped** and **under-damped** as a stable node or a stable spiral for $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$ at equilibrium $\vec{\mathbf{u}} = \vec{\mathbf{0}}$. Why are classifications *center* and *saddle* impossible?

Nonlinear Spring-Mass System

Consider the general model $x'' + F(x) = 0$ with the assumptions on page 804.

5. **(Harmonic Oscillator)** Let $F(x) = \omega^2 x$ with $\omega > 0$. Show F is odd and $F(0) = 0$. Then find the general solution $x(t)$ for $x'' + F(x) = 0$.

Solution: The general solution is $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$.

Odd means $F(-x) = -F(x)$. Calculate $F(-x) = \omega^2(-x) = -\omega^2 x = -F(x)$. ■

6. **(Taylor Series)** Show that an odd function $F(x)$ with Maclaurin series $\sum_{n=0}^{\infty} a_n x^n$ has all even order terms zero, that is, $a_n = 0$ for n even.

Soft and Hard Springs

Classify as a hard or soft spring. Then write the conservation law for the equation.

7. $x'' + x + x^3 = 0$

Solution: Hard spring $mx'' + F(x) = 0$: $F(x) = kx + \beta x^3$ with $m = k = \beta = 1$.

8. $x'' + x - x^3 = 0$

Hard spring

9. Prove that a hard spring has exactly one equilibrium $x = y = 0$.

Solution: The dynamical system is $x' = y$, $y' = -F(x)$. Solve equations $0 = y$, $0 = -F(x)$ for x, y. Answer: $y = 0$ and $\frac{1}{m}\left(kx + \beta x^3\right) = 0$. Solve for (x, y): $y = 0$ and $x = 0$ or $1 + \beta x^2 = 0$. Equation $1 + \beta x^2 = 0$ has no real solution x, because $\beta > 0$. The only equilibrium is $x = y = 0$.

10. Substitute $x = x(t), y = x'(t)$ into $z = y^2 + x^2 + x^4$ to obtain $z(t)$. Function $z(t)$ has a minimum when $\frac{dz}{dt} = 0$. Reduce this equation to $x'' + x + 2x^3 = 0$.

Soft Spring

Consider soft spring $x'' + kx - \beta x^3 = 0$, $k > 0$, $\beta > 0$.

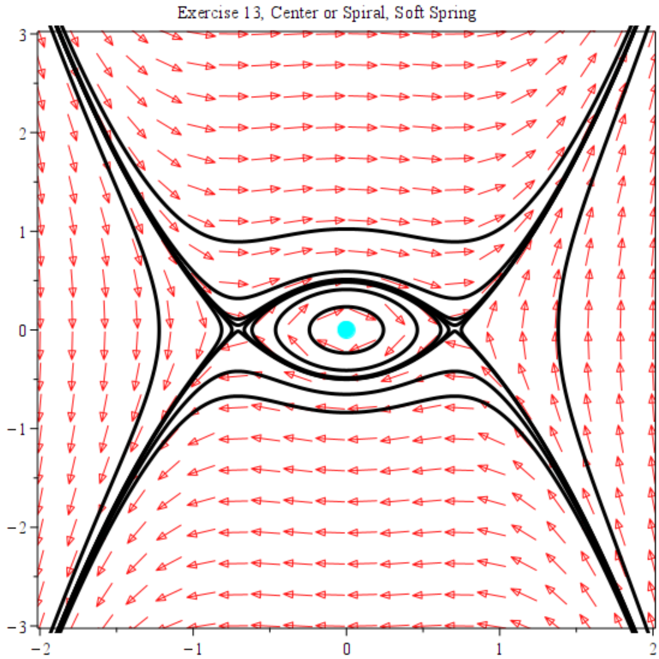
11. **(Equilibria)** Verify the three equilibria $(0, 0), (0, \sqrt{k}\beta), (0, -\sqrt{k}\beta)$.

Solution: The dynamical system is $x' = y$, $y' = -F(x) = -kx + \beta x^3$. Solve equations $0 = y$, $0 = -F(x)$ for x, y. Answer: $y = 0$ and $kx - \beta x^3 = 0$. Solve for (x, y): $y = 0$ and $x = 0$ or $1 - \beta x^2 = 0$. Then the equilibria are $(0, 0)$, $(1/\sqrt{\beta}, 0)$, $(-1/\sqrt{\beta}, 0)$.

12. **(Saddles)** Verify by linearization and the **Paste Theorem** that nonlinear equilibria $(0, \sqrt{k}\beta), (0, -\sqrt{k}\beta)$ are saddles.

13. **(Center or Spiral)** The **Paste Theorem** says that equilibrium $(0, 0)$ of the nonlinear system is a center or spiral. Verify by computer phase portrait $m = k = 1$ and $\beta = 2$ Figure 37, page 807.

Solution: The phase portrait at $(0, 0)$ is a center.

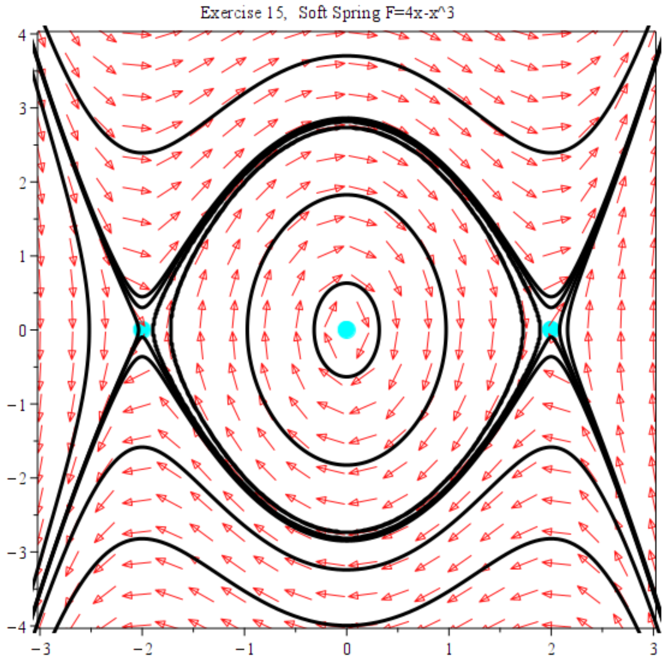



```
# Exercise 13, Center or Spiral, phase portrait soft spring
beta:=1/sqrt(2.);# 0.7071067814
# Used MAPLE task Phase Portrait
# Window: -2<x<2, -3<y<3
# F=y    G=-x-2*x^3
# Equilibria: [0,0], [0.707,0], [-0.707,0]
# Time: -20 to 20
```

14. **(Mass at Rest)** Verify that the only solutions with the mass at rest are the equilibria. **Mass at rest** means velocity zero: $\vec{\mathbf{u}}'(t_0) = \vec{\mathbf{0}}$ for some t_0, vector notation from Exercise 1.

15. **(Phase Portrait)** Solve for the equilibria of $x'' + 4x - x^3 = 0$. Draw a phase portrait similar to Figure 37, page 807.

Solution: Equilibria: `[0,0], [2,0], [-2,0]`.

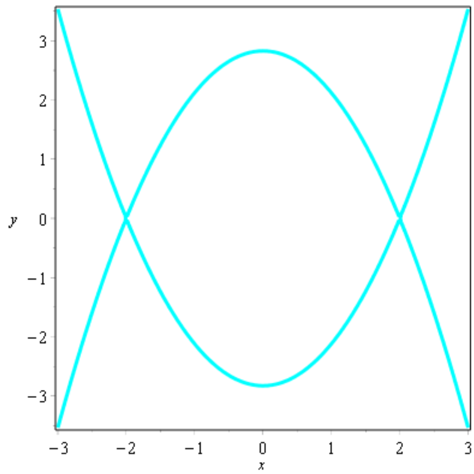



```
# Exercise 15, Soft Spring phase portrait
# Used MAPLE task Phase Portrait
# Window: -3<x<3, -4<y<4
# F=y   G=-4*x-x^3
# Equilibria: [0,0], [2,0], [-2,0]
# Time: -20 to 20
```

16. **(Separatrix)** The energy equation for $x'' + 4x - x^3 = 0$ is $\frac{1}{2}y^2 + 2x^2 - \frac{1}{4}x^4 = E$. Substitute the saddle equilibria to find $E = 4$. Plot implicitly the energy equation curve. A separatrix is the union of the two saddle equilibria and this implicit curve.

Solution:

```
# Exercise 16, Separatrix
Energ:=(1/2)*y^2+2*x^2-(1/4)*x^4;
plots[implicitplot](Energ=4,x=-3..3,y=-4..4,gridrefine=3);
```



Damped Nonlinear Pendulum

Consider $\frac{d^2\theta(t)}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{L}\sin(\theta(t)) = 0$, which has vector-matrix form $\vec{\mathbf{u}}' = \vec{\mathbf{G}}(\vec{\mathbf{u}}(t))$.

17. Display both $\vec{\mathbf{u}}$ and $\vec{\mathbf{G}}$.

Solution: Let $\vec{\mathbf{u}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix}$. Then

$$\vec{\mathbf{G}}(\vec{\mathbf{u}}) = \begin{pmatrix} \theta'(t) \\ -c\theta'(t) - \frac{g}{L}\sin\theta(t) \end{pmatrix} = \begin{pmatrix} u_2 \\ -cu_2 - \frac{g}{L}\sin(u_1) \end{pmatrix}$$

18. Find the Jacobian matrix of $\vec{\mathbf{G}}$ with respect to $\vec{\mathbf{u}}$.

Undamped Nonlinear Pendulum

Consider $\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin(\theta(t)) = 0$, having vector-matrix form $\vec{\mathbf{u}}' = \vec{\mathbf{F}}(\vec{\mathbf{u}}(t))$.

19. Find the Jacobian matrix of $\vec{\mathbf{F}}$ with respect to $\vec{\mathbf{u}}$.

Solution: Apply Exercise 17 with $c = 0$, $u_1 = x$, $u_2 = y$:

$$\vec{\mathbf{G}}(\vec{\mathbf{u}}) == \begin{pmatrix} y \\ -\frac{g}{L}\sin(x) \end{pmatrix}$$

Then

$$J(x, y) = \text{Jacobian matrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos(x) & 0 \end{pmatrix}$$

$$J(\theta, \theta') = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos(\theta) & 0 \end{pmatrix}$$

20. Solve $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \vec{\mathbf{0}}$ for $\vec{\mathbf{u}}$, showing all details.

Solution: Equation $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \vec{\mathbf{0}}$ means $\vec{\mathbf{G}}(\vec{\mathbf{u}}) = \begin{pmatrix} u_2 \\ -cu_2 - \frac{g}{L}\sin(u_1) \end{pmatrix} = \vec{\mathbf{0}}$ in Exercise 17. Let $c = 0$. Details omitted to find equilibria $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$, which are points along the abscissa equally spaced by pi units.

21. Evaluate the Jacobian matrix at the roots of $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \vec{\mathbf{0}}$.

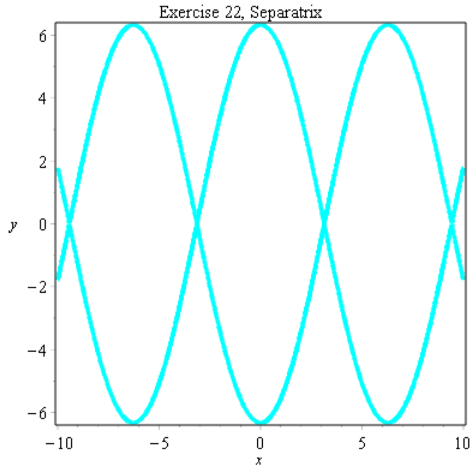
Solution: By Exercise 20 the equilibria are $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$. By Exercise 19:

$$J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos(n\pi) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(-1)^n g/L & 0 \end{pmatrix}$$

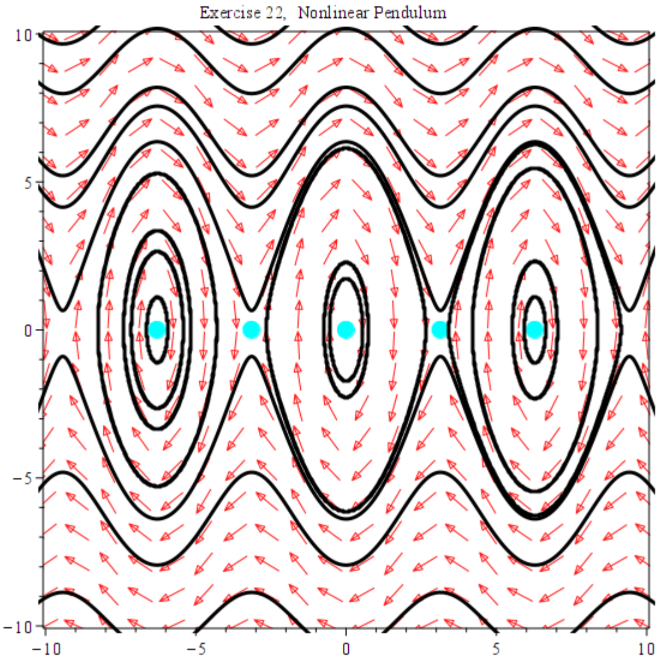
22. Plot $y^2 + \frac{4g}{L}\sin^2(x/2) = 4\frac{g}{L}$ implicitly for $\frac{g}{L} = 10$. The separatrix is this curve plus equilibria.

Solution:

```
# Exercise 22, Separatrix, Nonlinear Pendulum
Energ:=y^2 + 40*sin(x/2)^2;
plots[implicitplot](Energ=40,x=-10..10,y=-10..10,gridrefine=4);
```



The phase portrait below shows the separatrix location.


Exercise 22, Nonlinear Pendulum
10
5
0
−5
−10
−10
−5
0
5
10

Chapter 11

Systems of Differential Equations

Contents

11.1 Examples of Systems

There are no exercises for this section of examples. Later sections use this section for definitions, equations and key examples.

11.2 Fundamental System Methods

Solving 2×2 Systems

1. Solve $x_1' = 2x_1 + x_2$, $x_2' = x_2$. Ans: $x_1 = c_1\,\mathrm{e}^{2\,t} - c_2\,\mathrm{e}^t$, $x_2 = c_2\,\mathrm{e}^t$

Solution: Solve growth-decay equation $x_2' = x_2$ by the Growth-Decay shortcut for $u' = ku$:. Then $x_2 = c_2 e^t$. Insert this answer into the first equation. Then $x_1' = 2x_1 + c_2 e^t$. Write it in standard form $x_1' + (-2)x_1 = c_2 e^t$. Solve by the linear integrating factor method.

$x_1' + (-2)x_1 = c_2 e^t$

$\dfrac{\left(x_1 e^{-2t}\right)'}{e^{-2t}} = c_2 e^t$ Replace LHS $x' + px$ by $\left(x\, e^{\int p dt}\right)' / e^{\int p dt}$.

$\left(x_1 e^{-2t}\right)' = c_2 e^t\, e^{-2t}$

$x_1 e^{-2t} = \int c_2 e^t\, e^{-2t} dt$ Quadrature method.

$x_1 = -c_2\,\mathrm{e}^t + c_1\,\mathrm{e}^{2\,t}$

2. Discuss how to solve $\vec{\mathbf{x}}' = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \vec{\mathbf{x}}$.

Triangular 2×2 Matrix A

3. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \vec{\mathbf{x}}$.

Solution: Answer: $x_1 = c_1 e^{2t} + c_2 e^{3t}$, $x_2 = c_2 e^{3t}$

Use the scalar method in Exercise 1. First step: convert the matrix form to scalar form.

Given $x_1' = 2x_1 + x_2$, $x_2' = 3x_2$, solve for x_2 then x_1.

$x_2' = 3x_2$ Second DE, solve for x_2.

$x_2 = c_2 e^{3t}$ Growth-decay shortcut for $u' = ku$.

$x_1' = 2x_1 + x_2$ First DE, solve for x_1.

$x_1' = 2x_1 + c_2 e^{3t}$ Substitute $x_2 = c_2 e^{3t}$ into the first DE.

$(e^{-2t} x_1)'/e^{-2t} = c_2 e^{3t}$ Linear integrating factor method.

$e^{-2t} x_1 = \int c_2 e^{3t} e^{-2t}\, dt$ Quadrature method.

$e^{-2t} x_1 = c_1 + c_2 e^t$

$x_1 = c_1 e^{2t} + c_2 e^{3t}$

4. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} \vec{\mathbf{x}}$.

Non-Triangular 2×2 Matrix A

5. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \vec{\mathbf{x}}$.

Solution: Answer: $x_1 = c_1 e^{4t} + c_2 e^{-2t}$, $x_2 = c_1 e^{4t} - c_2 e^{-2t}$

Details:
Characteristic equation: $r^2 - \mathbf{trace}(A) r + |A| = r^2 - 2r - 8 = 0$. Roots: $r = 4, -2$. Atoms: e^{4t}, e^{-2t}. The scalar system is $x_1' = x_1 + 3x_2$, $x_2' = 3x_1 + x_2$. The Cayley-Hamilton-Ziebur Theorem page 840 applies:

$x_1 = c_1 e^{4t} + c_2 e^{-2t}$.

Solve the first differential equation for x_2:

$x_2 = \frac{1}{3}(x_1' - x_1)$

Insert equation $x_1 = c_1 e^{4t} + c_2 e^{-2t}$ and simplify:

$$\begin{aligned} x_2 &= \tfrac{1}{3}(x_1' - x_1) \\ &= \tfrac{1}{3}(4c_1 e^{4t} - 2c_2 e^{-2t} - c_1 e^{4t} - c_2 e^{-2t}) \\ &= c_1 e^{4t} - c_2 e^{-2t} \end{aligned}$$

6. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \vec{\mathbf{x}}$.

Method for $n \times n$ Diagonal A

7. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Solution: Answer: $\vec{\mathbf{x}} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ c_2 e^{3t} \\ c_3 e^{2t} \end{pmatrix}$

8. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Method for $n \times n$ Lower Triangular

9. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Solution: Answer:

By `maple`:

$x_1(t) = d_1\, \mathrm{e}^t,$
$x_2(t) = -\frac{1}{2}\, d_1\, \mathrm{e}^t + d_2\, \mathrm{e}^{3\,t},$
$x_3(t) = -d_1\, \mathrm{e}^t + d_3\, \mathrm{e}^{2\,t}$

Vector-Matrix form of the answer: $\vec{\mathbf{x}} = \begin{pmatrix} e^t & 0 & 0 \\ -\frac{1}{2}e^t & e^{3t} & 0 \\ -e^t & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

Pencil and paper: solve the differential equations top to bottom by either the shortcut for $u' = ku$ or else the linear integrating factor method for $u' + pu = q$. Involved in the preparation to solve is substitution of previously known functions, order x_1, x_2, x_3 for lower triangular matrices.

Start: $x_1 = c_1 e^t$.

Second equation: $x_2' = x_1 + 3x_2 = c_1 e^t + 3x_2$. To standard form:

$x_2' - 3x_2 = c_1 e^t$.

$x_2 = -\frac{1}{2} c_1 e^t + c_2 e^{3t}$ By the linear integrating factor method.

Third equation: $x_3' = x_1 + 2x_3$

$x_3' - 2x_3 = c_1 e^t$

$x_3 = -c_1 e^t + c_3 e^{2t}$ By the Linear integrating factor method.

Vector-Matrix Answer obtained by the linear integrating factor method matches the `maple dsolve` answer:

$$\vec{\mathbf{x}} = \begin{pmatrix} e^t & 0 & 0 \\ -\frac{1}{2}e^t & e^{3t} & 0 \\ e^t & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

```
# Exercise 9, 3x3 lower triangular
A:=Matrix([[1 , 0 , 0],[1 , 3 , 0],[1 , 0 , 2]]);
sys:=[diff(x(t),t)=x(t),
      diff(y(t),t)=x(t)+3*y(t),
      diff(z(t),t)=x(t)+2*z(t)];
vars:=[x(t),y(t),z(t)];
dsolve(sys,vars);
```

10. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Method for $n \times n$ Upper Triangular

11. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Solution: The scalar equations:

$$x_1' = x_1 + x_3, \quad x_2' = 3x_2 + x_3, \quad x_3' = 2x_3$$

`maple` **answer**:

$x_1(t) = d_3\, e^{2t} + d_1 e^t$,
$x_2(t) = -d_3 e^{2t} + d_2 e^{3t}$,
$x_3(t) = d_3 e^{2t}$

Vector-Matrix form: $\vec{\mathbf{x}} = \begin{pmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & -e^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

Pencil and paper answer:
Solve the differential equations bottom to top by either the shortcut for $u' = ku$ or else the linear integrating factor method for $u' + pu = q$. Involved in the preparation to solve is substitution of previously known functions, order x_3, x_2, x_1 for upper triangular matrices.

Start: $x_3 = c_3 e^{2t}$.

Second equation:
$x_2' = 3x_2 + x_3 = 3x_2 + c_3 e^{2t}$

$x_2' - 3x_2 = c_3 e^{2t}$ Standard form.

$x_2 = -c_3 e^{2t} + c_2 e^{3t}$ By the linear integrating factor method.

Third equation:
$x_1' = x_1 + x_3$

$x_1' - x_1 = c_3 e^{2t}$

$x_1 = c_1 e^t + c_3 e^{2t}$ By the Linear integrating factor method.

Vector-Matrix Answer obtained by the linear integrating factor method matches the `maple dsolve` answer:

$$\vec{\mathbf{x}} = \begin{pmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & -e^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

```
# Exercise 11, 3x3 upper triangular
A:=Matrix([[1 , 0 , 1],[0 , 3 , 1],[0 , 0 , 2]]);
sys:=[diff(x(t),t)=x(t)+z(t),
      diff(y(t),t)=3*y(t)+z(t),
      diff(z(t),t)=2*z(t)];
vars:=[x(t),y(t),z(t)];
p:=dsolve(sys,vars);
```

12. Solve $\vec{\mathbf{x}}\,' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Jordan's $n \times n$ Variable Change

Let $A = PTP^{-1}$ with T upper triangular and P invertible. Define change of variable $\vec{\mathbf{x}}\,(t) = P\vec{\mathbf{y}}\,(t)$. Prove these results:

13. If $\vec{\mathbf{x}}\,(t)$ solves $\vec{\mathbf{x}}\,'(t) = A\vec{\mathbf{x}}\,(t)$, then $\vec{\mathbf{y}}\,(t) = P^{-1}\vec{\mathbf{x}}\,(t)$ solves $\vec{\mathbf{y}}\,'(t) = T\vec{\mathbf{y}}\,(t)$.

Solution: Let $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$, Then $AP\vec{\mathbf{y}} = A\vec{\mathbf{x}} = \vec{\mathbf{x}}\,' = P\vec{\mathbf{y}}\,'$. Reverse to $P\vec{\mathbf{y}}\,' = AP\vec{\mathbf{y}}$, then multiply by P^{-1} and use $AP = PT$:

$\vec{\mathbf{y}}\,' = P^{-1}AP\vec{\mathbf{y}} = P^{-1}PT\vec{\mathbf{y}}T\vec{\mathbf{y}}$. ■

14. If $\vec{\mathbf{y}}\,'(t) = T\vec{\mathbf{y}}\,(t)$, then $\vec{\mathbf{x}}\,(t) = P\vec{\mathbf{y}}\,(t)$ solves $\vec{\mathbf{x}}\,'(t) = A\vec{\mathbf{x}}\,(t)$.

Convert Scalar Linear 2nd Order to $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}\,(t)$

15. $x'' + 2x' + x = \sin t$

Solution: Let $\vec{\mathbf{u}} = \begin{pmatrix} x \\ x' \end{pmatrix}$. Then

$$\begin{aligned}
\vec{\mathbf{u}}\,' &= \begin{pmatrix} x' \\ x'' \end{pmatrix} \\
&= \begin{pmatrix} x' \\ -2x' - x + \sin t \end{pmatrix} \\
&= \begin{pmatrix} x' \\ -2x' - x \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \vec{\mathbf{u}} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}
\end{aligned}$$

Then $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$. $\vec{\mathbf{F}} = \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$. ■

16. $2x'' + 3x' + 8x = 4\cos t$

Convert Second Order Scalar System to $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$

17. $x'' = x + y$, $y'' = x - y$

Solution: Let $\vec{\mathbf{u}} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$. Then

$$\vec{\mathbf{u}}\,' = \begin{pmatrix} x' \\ x'' \\ y' \\ y'' \end{pmatrix}$$

$$= \begin{pmatrix} x' \\ x+y \\ y' \\ x-y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$$

Then $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$ and $\vec{\mathbf{F}} = \vec{\mathbf{0}}$. ■

18. $x'' = x + y + \sin t$, $y'' + y = x + \cos t$

Convert Coupled Spring-Mass System to $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}$

19. $\vec{\mathbf{x}}\,'' = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$

Solution: Assume $' = \frac{d}{dt}$. Let $\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{\mathbf{u}} = \begin{pmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{pmatrix}$. Then

$$\vec{\mathbf{u}}\,' = \begin{pmatrix} x_1' \\ x_2' \\ x_1'' \\ x_2'' \end{pmatrix}$$

$$= \begin{pmatrix} x_1' \\ x_2' \\ -2x_1 + x_2 \\ x_1 - x_2 + \sin t \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{pmatrix}$$

Then $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$ and $\vec{\mathbf{F}}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin t \end{pmatrix}$. ■

20. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \vec{\mathbf{x}}$

Convert Higher Order Linear Equations to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$

21. $x''' = x$

Solution: Let $\vec{\mathbf{u}} = \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}$, then

$$\vec{\mathbf{u}}' = \begin{pmatrix} x' \\ x'' \\ x''' \end{pmatrix} = \begin{pmatrix} x' \\ x'' \\ x \end{pmatrix}$$

$$\vec{\mathbf{u}}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}$$

Then $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. ■

22. $\dfrac{d^4y}{dx^4} + 16y = 0$

Convert Scalar Continuous-Coefficient Equation to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$

23. $x^2y'' + 3xy' + 2y = 0$

Solution: Let $\vec{\mathbf{u}} = \begin{pmatrix} y \\ y' \end{pmatrix}$, then

$$\vec{\mathbf{u}}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -3y'/x - 2y/x^2 \end{pmatrix}$$

$$\vec{\mathbf{u}}' = \begin{pmatrix} 0 & 1 \\ -2/x^2 & -3/x \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

Then $A = \begin{pmatrix} 0 & 1 \\ -2/x^2 & -3/x \end{pmatrix}$. ■

24. $y''' + xy'' + x^2y + y = 0$

Convert Forced Higher Order Equation to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$

25. $\dfrac{d^4y}{dx^4} = y''' + y + \sin x$

Solution: Variable t is the same as variable x and $' = \frac{d}{dx} = \frac{d}{dt}$.

Let $\vec{\mathbf{u}} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}$, then

$$\begin{aligned}
\vec{\mathbf{u}}' &= \begin{pmatrix} y' \\ y'' \\ y''' \\ y'''' \end{pmatrix} \\
&= \begin{pmatrix} y' \\ y'' \\ y''' \\ y''' + y + \sin x \end{pmatrix} \\
&= \begin{pmatrix} y' \\ y'' \\ y''' \\ y''' + y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin x \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin x \end{pmatrix}
\end{aligned}$$

Then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \vec{\mathbf{F}}(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin x \end{pmatrix}. \quad \blacksquare$$

26. $\dfrac{d^6y}{dx^6} = \dfrac{d^4y}{dx^4} + y + \cos t$

Convert 2nd Order System to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{G}}(t)$

27. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution: Assume $' = \frac{d}{dt}$.

Let $\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{\mathbf{u}} = \begin{pmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{pmatrix}$. Then:

$$\vec{\mathbf{u}}' = \begin{pmatrix} x_1' \\ x_2' \\ x_1'' \\ x_2'' \end{pmatrix}$$

$$= \begin{pmatrix} x_1' \\ x_2' \\ -2x_1 + x_2 + 1 \\ x_1 - x_2 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1' \\ x_2' \\ -2x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Then

$$\vec{\mathbf{u}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \vec{\mathbf{G}}(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \quad \blacksquare$$

28. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \vec{\mathbf{x}} + e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Convert Damped 2nd Order System to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{G}}(t)$

29. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{\mathbf{x}}' + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution: Assume $' = \frac{d}{dt}$. Let $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ x_2 \end{pmatrix}$. Let $\vec{\mathbf{u}} = \begin{pmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{pmatrix}$.

Write the differential equation in terms of components:

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 + x_2' + 1 \\ x_1 - x_2 + x_1' - 1 \end{pmatrix}$$

Then

$$\begin{aligned} \vec{\mathbf{u}}' &= \begin{pmatrix} x_1' \\ x_2' \\ x_1'' \\ x_2'' \end{pmatrix} \\ &= \begin{pmatrix} x_1' \\ x_2' \\ -2x_1 + x_2 + x_2' + 1 \\ x_1 - x_2 + x_1' - 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

Then $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$ and $\vec{\mathbf{G}}(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$. ■

30. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \vec{\mathbf{x}} + \vec{\mathbf{x}}' + e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

11.3 Structure of Linear Systems

Linear Systems

Convert to matrix notation $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$.

1. $x_1' = 2x_1 + x_2 + e^t$,
$x_2' + x_1 - 2x_2 = \sinh(t)$

Solution: Answer: $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, $\vec{\mathbf{F}}(t) = \begin{pmatrix} e^t \\ \sinh(t) \end{pmatrix}$. ■

2. $x_1' = x_1 + x_2 + x_3$,
$x_2' + x_1 - 2x_2 + x_3 = \ln|1 + t^2|$,
$x_3' = x_2 + x_3 + \cosh(t)$

Existence-Uniqueness

3. Apply Gronwall's inequality to
$|y(t)| \le 4 + \int_0^t (1 + r^2)|y(r|\,dr$, $t \ge 0$.

Solution: Gronwall's inequality assumes for t for $t_0 \le t \le t_0 + H$

$$u(t) \le c + \int_{t_0}^t u(r)v(r)dr$$

and concludes

$$u(t) \le c + e^{-\int_{t_0}^t v(r)dr}$$

Let $u(t) = |y(t)|$, $v(r) = 1+r^2$, $c = 4$ and $t_0 = 0$. Then Gronwall's inequality concludes

$$|y(t)| \le 4 + \int_0^t |y(r)|(1 + r^2)dr$$

■

4. Solve with $x_1(0) = x_2(0) = 0$:
$x_1' = e^t x + e^{-t}x_2$,
$x_2' = \ln|1 + \sinh^2(t)|\,x_1 + x_2$

5. Find the interval on which the solution is defined:
$x_1' = tx_1 + x_2$, $x_2' = x_1 + \tan(t)\,x_2$

Solution: Answer: $-\pi/2 < t < \pi/2$. Reason: It is a linear system $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$ with $A(t) = \begin{pmatrix} t & 1 \\ 1 & \tan(t) \end{pmatrix}$ continuous on domain $-\pi/2 < t < \pi/2$, $\vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere continuous. Map $(t, \vec{\mathbf{u}}) \to A(t)\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$ is therefore continuous in $(t, \vec{\mathbf{u}})$ and continuously differentiable in $\vec{\mathbf{u}}$ on the domain $|t| < \pi/2$, $\vec{\mathbf{u}}$ in $\mathcal{R}^2$. Picard's Theorem page 851 says that the solution is defined **on the entire domain**, initial value problems uniquely solvable.

Remark on Global Existence
Picard's theorem for nonlinear problems $\vec{\mathbf{u}}' = \vec{\mathbf{G}}(t, \vec{\mathbf{u}})$ **does not** claim global definition of solutions. For instance, scalar problem $y' = 1 + y^2$, $y(0) = 0$ has solution $y(t) = \tan t$, which is only defined on $|t| < \pi/2$, even though the map $(t, y) \to 1 + y^2$ is continuous in t and infinitely continuously differentiable in y.

6. Let matrix A be 2×2 constant. Find A, given $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has general solution $x_1 = c_1 e^t + c_2 e^{2t}$, $x_2 = 5c_1 2e^t + 4c_2 e^{2t}$.

Solution: Hint: Write the general solution as $\vec{\mathbf{u}} = \Phi(t)\begin{pmatrix} c1 \\ c_2 \end{pmatrix}$ and insert this equation into $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ to get an equation for A.

7. Let $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ have two solutions : $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} e^t \\ e^t \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$.

Solution: Matrix $A(t)$ has to be 2×2. The solutions are independent, therefore $\vec{\mathbf{x}} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}$ is the general solution by Picard's Theorem.

It is possible to find $A(t)$ explicitly. All possible constants are allowed in the general solution. Insert $\vec{\mathbf{x}}$ into the equation $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$:

$$\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$$

$$c_1\tfrac{d}{dt}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\tfrac{d}{dt}\begin{pmatrix} e^t \\ e^t \end{pmatrix} = A(t)\left(c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}\right)$$

$$\tfrac{d}{dt}\begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A(t)\begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & e^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A(t)\begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\left(\begin{pmatrix} 0 & e^t \\ 0 & e^t \end{pmatrix} - A(t)\begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix}\right)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } c_1, c_2.$$

Choose $c_1 = 1$, $c_2 = 0$ and then $c_1 = 0$, $c_2 = 1$ to prove the coefficient matrix is zero:

$$\begin{pmatrix} 0 & e^t \\ 0 & e^t \end{pmatrix} = A(t)\begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix}$$

The matrix multiplying $A(t)$ has determinant $-e^t$, so it is invertible for all t with inverse

$$\begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix}^{-1} = \frac{1}{-e^t}\begin{pmatrix} e^t & -e^t \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2e^{-t} & -e^{-t} \end{pmatrix}$$

Solve by matrix inversion

$$A(t) = \begin{pmatrix} 0 & e^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2e^{-t} & -e^{-t} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \quad \blacksquare$$

8. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

9. Let constant matrix A be 10×10. Two solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ have equal value at $t = 100$. Are they the same solution?

Solution: Yes. Initial value problems have unique solutions by Picard's Theorem.

10. Solutions y_1, y_2 of $y' + p(x)y = q(x)$ are zero at $x = -2$. What assumptions on p, q imply $y_1 \equiv y_2$?

Superposition

11. Explain: e^t is a solution of $y'' - y = 0$ because $\cosh(t)$, $\sinh(t)$ are a solution basis.

Solution: Function e^t is a linear combination of the solution basis: it is the sum $\cosh(t) + \sinh(t)$. Because linear combinations of solutions are solutions then e^t is a solution.

12. Explain: $e^t + 10$ is a solution of $y'' - y = -10$, therefore 10 is a particular solution.

13. The shortest solution of $y' + y = 100$ is $y = 100$. Explain why.

Solution: All solutions are $y = y_h + y_p$. Solution y_p can be taken to be $y_p = 100$, an equilibrium solution. The homogeneous solution is $y_h = c/\text{integrating factor} = ce^{-t}$. Let $z = 100$ and let y be any solution of $y' + y = 100$. Then $u = y - z$ is a solution of $u' + u = 0$, so it equals ce^{-t} for some c: $u = y - z = ce^{-t}$ implies $y = ce^{-t} + 100$. The shortest solution is when $c = 0$.

14. Let $x_1' = 2x_1$, $x_2' = -x_2$. Report the matrix form $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ and the vector general solution.

15. Let 2-dimensional $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have general solution $x_1 = c_1e^t + c_2e^{3t}$, $x_2 = (c_1 + c_2)e^t + 2c_2e^{3t} + \cos(t)$. Find formulas for vectors $\vec{\mathbf{x}}_h$ and $\vec{\mathbf{x}}_p$.

Solution: Homogeneous solution $\vec{\mathbf{x}}_h = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $\Phi(t) = \begin{pmatrix} e^t & e^{3t} \\ e^t & e^t + 2e^{3t} \end{pmatrix}$. The columns of $\Phi(t)$ are the partial derivatives of vector $\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ on symbols c_1, c_2. Particular solution $\vec{\mathbf{x}}_p = \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$, found from $c_1 = c_2 = 0$.

16. Let $\vec{\mathbf{x}}\,' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have two solutions $x_1 = e^t + e^{3t}$, $x_2 = 2e^t + \sin(t)$ and $x_1 = e^{3t}$, $x_2 = e^{3t} + \sin(t)$. Find a solution of $\vec{\mathbf{x}}\,' = A\vec{\mathbf{x}}$.

Superposition $\vec{\mathbf{x}}\,' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

17. Let $\vec{\mathbf{u}}_1(t), \ldots, \vec{\mathbf{u}}_k(t)$ be solutions of $\vec{\mathbf{x}}\,' = A(t)\vec{\mathbf{x}}$. Let $c_1, \ldots, c_k$ be constants. Prove: $\vec{\mathbf{u}}(t) = \sum_{i=1}^k c_i\vec{\mathbf{u}}_i(t)$ is a solution of $\vec{\mathbf{x}}\,' = A(t)\vec{\mathbf{x}}$.

Solution: // **Proof**:

$$\begin{aligned}\vec{\mathbf{u}}\,'(t) &= \textstyle\sum_{i=1}^k c_i\vec{\mathbf{u}}_i'(t)\\ &= \textstyle\sum_{i=1}^k c_i A(t)\vec{\mathbf{u}}_i(t)\\ &= A(t)\left(\textstyle\sum_{i=1}^k c_i\vec{\mathbf{u}}_i(t)\right)\\ &= A(t)\vec{\mathbf{u}}(t) \quad \blacksquare\end{aligned}$$

18. Find the **standard basis** $\vec{\mathbf{w}}_1(t), \vec{\mathbf{w}}_2(t)$:

$$\vec{\mathbf{x}}\,' = \begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{pmatrix}\vec{\mathbf{x}}$$

19. Let matrix A be 2×2. For $\vec{\mathbf{x}}\,' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$, find $\vec{\mathbf{x}}_h(t)$, $\vec{\mathbf{x}}_p(t)$: $x_1 = c_1 + c_2t + e^t$, $x_2 = (c_1 - c_2)t + e^{2t}$

Solution: Let $c_1 = c_2 = 0$ to find particular solution $\vec{\mathbf{x}}_p = \begin{pmatrix} e^t \\ e^{2t}\end{pmatrix}$.

Subtract the particular solution from the general solution to find the homogeneous solution

$$\vec{\mathbf{x}}_h = \begin{pmatrix} c_1 + c_2t + e^t \\ (c_1 - c_2)t + e^{2t}\end{pmatrix} - \vec{\mathbf{x}}_p = \begin{pmatrix} c_1 + c_2t \\ (c_1 - c_2)t\end{pmatrix} = \begin{pmatrix}1 & t\\ t & -t\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

20. Let matrix $A(t)$ be 2×2. Let $\vec{\mathbf{x}}\,' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have two solutions $\begin{pmatrix}1 + e^t\\ 1\end{pmatrix}$, $\begin{pmatrix}1 + e^{-t}\\ -1\end{pmatrix}$. Find a solution of $\vec{\mathbf{x}}\,' = A(t)\vec{\mathbf{x}}$.

General Solution

21. Assume A is 2×2 and $\vec{\mathbf{x}}\,' = A\vec{\mathbf{x}}$ has solutions $e^t\begin{pmatrix}1\\1\end{pmatrix}$, $e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$. Find the general solution and explain.

Solution: The two solutions are independent. The solution space is 2-dimensional for a 2×2 constant matrix, by Picard's Theorem. Therefore the two solutions are a basis for the solution space and every solution is a linear combination of the two solutions. $\blacksquare$

22. Assume $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \begin{pmatrix}1\\1\end{pmatrix}$. Prove that zero is not a solution.

23. Assume $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \begin{pmatrix}1\\1\end{pmatrix}$ and $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_0 =$ constant. Find an equation for $\vec{\mathbf{x}}_0$.

Solution: The equation: $\vec{\mathbf{0}} = A\vec{\mathbf{x}}_0 + \begin{pmatrix}1\\1\end{pmatrix}$. ■

24. Find the vector general solution:
$\vec{\mathbf{x}}' = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\vec{\mathbf{x}} + \begin{pmatrix}1\\1\end{pmatrix}$.

25. Given 3 $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ with scalar general solution $x_1 = c_1 + c_2 t + c_3 t^2$, $x_2 = c_2 + c_3 t$, $x_3 = c_3$, find the vector general solution.

Solution: There are two forms of the answer:

$$\vec{\mathbf{x}} = \begin{pmatrix} c_1 + c_2 t + c_3 t^2 \\ c_2 + c_3 t \\ c_3 \end{pmatrix} \text{ or } \vec{\mathbf{x}} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Matrix $\Phi(t)$ has columns equal to the partial derivatives on symbols c_1, c_2, c_3 of the vector general solution $\begin{pmatrix} c_1 + c_2 t + c_3 t^2 \\ c_2 + c_3 t \\ c_3 \end{pmatrix}$. We compute by calculus:

$$\Phi(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ■$$

26. Given 3 $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ with scalar general solution $x_1 = c_1 + c_2 t + c_3 t^2$, $x_2 = c_2 + c_3 t$, $x_3 = c_3$, find $A(t)$.

27. Find the vector general solution:
$\vec{\mathbf{x}}' = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 0\end{pmatrix}\vec{\mathbf{x}} + \begin{pmatrix}1\\1\\0\end{pmatrix}$.

Solution: Let's find a valid $\vec{\mathbf{x}}_p$ by seeking a constant solution $\vec{\mathbf{x}} = \vec{\mathbf{x}}_0$. Then

$$\vec{\mathbf{0}} = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 0\end{pmatrix}\vec{\mathbf{x}}_0 + \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Then $\vec{\mathbf{x}}_0 = \begin{pmatrix}-1\\-\frac{1}{2}\\c\end{pmatrix}$ for any constant c. Choose $c = 0$ and $\vec{\mathbf{x}}_p = \begin{pmatrix}-1\\-\frac{1}{2}\\0\end{pmatrix}$.

Let's find a $\vec{\mathbf{x}}_h$ by solving $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\mathbf{x}}$ or scalar equations $\begin{cases} x_1' = x_1 \\ x_2' = 2x_2 \\ x_3' = 0 \end{cases}$ Then $x_1 = c_1 e^t$, $x_2 = c_2 e^{2t}$, $x_3 = c_3$.

The vector general solution is

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 0 \end{pmatrix} \quad \blacksquare$$

28. Find the vector general solution:
$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Independence

29. Assume A is 2×2 and $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has solutions $e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Prove they are independent directly from the definition.

Solution: Let $\vec{\mathbf{v}}_1 = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The vectors are functions from $-\infty < t < \infty$ to $\mathcal{R}^2$, which is a known vector space V. To prove independence in V form the equation

$c_1 \vec{\mathbf{v}}_2 + c_2 \vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$

and solve for c_1, c_2. The vectors are independent in V if the only solution is $c_1 = c_2 = 0$. The equation formed means

$$c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } t.$$

Vector equality gives two equations:

$c_1 e^t + c_2 e^{-t} = 0, \quad c_1 e^t - c_2 e^{-t} = 0$

Add the two equations to get $c_1 = 0$. Subtract the two equations to get $c_2 = 0$. Then the only solution i $c_1 = c_2 = 0$ and vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are independent. $\blacksquare$

30. Compute the Wronskian:
$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Abel-Liouville Formula

31. Apply Abel's Independence Test:
$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution: Compute the Wronskian of the two vector functions:
$W(t) = \det \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$

Then $W(0) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$ implies the two vector functions are independent. ■

32. Let $\Phi(t)$ an invertible matrix satisfying $\Phi'(t) = A\Phi(t)$. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

33. Let $\Phi(t)$ an invertible matrix satisfying $\Phi'(t) = A\Phi(t)$. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

Solution: Multiply $\Phi'(t) = A\Phi(t)$ by a column $\vec{\mathbf{e}}$ of the identity matrix. This proves each column of $\Phi(t)$ is a solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Independence follows from Abel's Wronskian Independence Test. ■

34. Let $\Phi(t)$ any matrix satisfying $\Phi'(t) = A\Phi(t)$. Assume the determinant of $\Phi(t_0)$ is nonzero. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

35. Let $\Phi(t)$ any matrix satisfying $\Phi'(t) = A\Phi(t)$. Let C be a constant matrix. Prove that the columns of $\Phi(t)C$ are solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

Solution: A column of $\Phi(t)C$ is formally $\Phi(t)C\vec{\mathbf{e}}$ for a column $\vec{\mathbf{e}}$ of the identity matrix. Product $\vec{\mathbf{v}} = C\vec{\mathbf{e}}$ is a column vector of constants. Matrix multiply $\Phi(t)\vec{\mathbf{v}}$ is a linear combination of the columns of $\Phi(t)$, which is a linear combination of solutions to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Therefore, each column of $\Phi(t)C$ is a solution to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. ■

36. Assume continuous coefficients:
$y^{(n)}+p_{n-1}y^{(n-1)}+\cdots+p_0y=0$
Prove from the Abel-Liouville formula for the companion system that the Wronskian $W(t)$ of solutions $y_1, \ldots, y_n$ satisfies
$W' + p_{n-1}(t)W = 0$.

Initial Value Problem

37. Let matrix A be 3×3. Assume $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ has scalar general solution $x_1 = c_1e^t + c_2e^{-t} + t$, $x_2 = (c_1 + c_2)e^t + c_3e^{2t}$, $x_3 = (c_1 + c_2)e^t -$

$2c_2e^{-t} + c_3e^{2t} + t$. Given initial conditions $x_1(0) = x_2(0) = 0$, $x_3(0) = 1$, solve for c_1, c_2, c_3.

Solution: Answers: $c_1 = 1/2$, $c_2 = -1/2$, $c_3 = 0$.

There are two common solution methods:

> (1) Scalar equations and linear algebra without matrices, finding the scalar answers using college algebra.
>
> (2) Vector-matrix notation and linear algebra, obtaining both the scalar answers and a vector-matrix representation of the solution $\vec{\mathbf{x}}(t)$.

Only the second solution method (2) is offered. Let

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} & 0 \\ e^t & e^t & e^{2t} \\ e^t & e^t - 2e^{-t} & e^{2t} \end{pmatrix}, \vec{\mathbf{v}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \vec{\mathbf{x}}_p = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$$

Then $\vec{\mathbf{x}} = \Phi(t)\vec{\mathbf{v}} + \vec{\mathbf{x}}_p$ is the vector general solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$.

The initial value problem is solved by finding c_1, c_2, c_3 satisfying

$$\vec{\mathbf{x}}(0) = \Phi(0)\vec{\mathbf{v}} + \vec{\mathbf{x}}_p(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Because $\vec{\mathbf{x}}_p(0) = \vec{\mathbf{0}}$ then there is the simplified equation

$$\Phi(0)\vec{\mathbf{v}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Constants c_1, c_2, c_3 are found on paper from augmented matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

The reduced row-echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $c_1 = 1/2$, $c_2 = -1/2$, $c_3 = 0$. The solution found:

$$\vec{\mathbf{x}}(t) = \begin{pmatrix} e^t & e^{-t} & 0 \\ e^t & e^t & e^{2t} \\ e^t & e^t - 2e^{-t} & e^{2t} \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}. \quad \blacksquare$$

38. Let matrix A be 3×3. Assume $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ has scalar general solution $x_1 = c_1 + c_2t + c_3t^2 + e^t$, $x_2 = c_2 + c_3t + e^{2t}$, $x_3 = c_3$. Find the **vector** particular solution $\vec{\mathbf{x}}$ for initial conditions $x_1(0) = x_2(0) = 0$, $x_3(0) = 1$.

Equilibria

39. Find all equilibria:
$\vec{\mathbf{x}}' = \begin{pmatrix} \cos(t) & \cos(t) \\ 2 & 2 \end{pmatrix} \vec{\mathbf{x}}$

Solution: Solve $\vec{\mathbf{0}} = \begin{pmatrix} \cos(t) & \cos(t) \\ 2 & 2 \end{pmatrix} \vec{\mathbf{x}}_0$ for constant $\vec{\mathbf{x}}_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then $\cos(t)(x_1+x_2) = 0$, $2(x_1+x_2) = 0$ for all t with x_1, x_2 constant. The solution is all points on the line $y = -x$ in the xy-plane, equivalently, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for all constants c. There are infinitely many equilibria, which are constant solutions to the equation $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$. ■

40. Find all equilibria:
$\vec{\mathbf{x}}' = \begin{pmatrix} \sin(t) & \sin^2(t) \\ 2 & 2 \end{pmatrix} \vec{\mathbf{x}}$

11.4 Matrix Exponential

Matrix Exponential.

1. **(Picard)** Let A be real 2×2. Write out the two initial value problems which define the columns $\vec{\mathbf{w}}_1(t)$, $\vec{\mathbf{w}}_2(t)$ of e^{At}.

Solution:

(1) $\vec{\mathbf{w}}_1' = A\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(2) $\vec{\mathbf{w}}_2' = A\vec{\mathbf{w}}_2$, $\vec{\mathbf{w}}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ■

2. **(Picard)** Let A be real 3×3. Write out the three initial value problems which define the columns $\vec{\mathbf{w}}_1(t)$, $\vec{\mathbf{w}}_2(t)$, $\vec{\mathbf{w}}_3(t)$ of e^{At}.

3. Let A be real 2×2. Show that $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{u}}_0$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{u}}_0$.

Solution: Let $\vec{\mathbf{u}}_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. By definition, $e^{At} = \left\langle \vec{\mathbf{w}}_1(t) | \vec{\mathbf{w}}_2(t) \right\rangle$. Uniqueness in Picard's Theorem implies $\vec{\mathbf{x}} = c_1\vec{\mathbf{w}}_1 + c_2\vec{\mathbf{w}}_2$, because

$$\vec{\mathbf{x}}(0) = \vec{\mathbf{u}}_0 = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1\vec{\mathbf{w}}_1(0) + c_2\vec{\mathbf{w}}_2(0)$$

Then:

$$\begin{aligned}
\vec{\mathbf{x}}'(t) &= \left(e^{At}\right)' \vec{\mathbf{u}}_0 \\
&= \left(\left\langle \vec{\mathbf{w}}_1(t) | \vec{\mathbf{w}}_2(t) \right\rangle\right)' \vec{\mathbf{u}}_0 \\
&= \left\langle \vec{\mathbf{w}}_1'(t) | \vec{\mathbf{w}}_2'(t) \right. \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= c_1\vec{\mathbf{w}}_1'(t) + c_2\vec{\mathbf{w}}_2'(t) && \text{matrix multiply is a linear combination of columns} \\
&= c_1 A\vec{\mathbf{w}}_1(t) + c_2 A\vec{\mathbf{w}}_2(t) && \text{because } \vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2 \text{ are solutions of } \vec{\mathbf{x}}' = A\vec{\mathbf{x}} \\
&= A(c_1\vec{\mathbf{w}}_1(t) + c_2\vec{\mathbf{w}}_2(t)) \\
&= A\vec{\mathbf{x}}(t)
\end{aligned}$$

Proved: $\vec{\mathbf{x}}(0) = \vec{\mathbf{u}}_0$ and $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. ■

4. Let A be real $n \times n$. Show that $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$.

Matrix Exponential 2×2. Find e^{At} from representation $e^{At} = \left\langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \right\rangle$. Use first-order scalar methods.

5. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution: Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$: $x_1 = c_1 e^t$, $x_2 = c_2 e^{2t}$. Then the vector general solution is $\vec{\mathbf{x}} = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}$

Vector $\vec{\mathbf{w}}_1$ satisfies $\vec{\mathbf{w}}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 e^0 \\ c_2 e^{2(0)} \end{pmatrix}$, resulting in $c_1 = 1$, $c_2 = 0$ and $\vec{\mathbf{w}}_1 = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$. Similarly, $\vec{\mathbf{w}}_2 = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$. Then

$e^{At} = \left\langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \right\rangle = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$. ■

6. $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$.

7. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Solution: Matrix A is upper triangular. The vector general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is found by scalar methods applied the scalar system

$x_1' = x_1 + x_2$, $x_2' = 0$

Then

$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^t - c_2 \\ c_2 \end{pmatrix}$

$\vec{\mathbf{w}}_1 = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$

$\vec{\mathbf{w}}_2 = \begin{pmatrix} e^t - 1 \\ 1 \end{pmatrix}$

$e^{At} = \left\langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \right\rangle = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}$. ■

8. $A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.

Matrix Exponential Identities. Verify from exponential identities.

9. $e^A e^{-A} = I$

Solution: Matrices A and $B = -A$ commute: $AB = BA$. Apply identity $e^{At} e^{Bt} = e^{(A+B)t}$, valid for $AB = BA$. Then $e^A e^{-A} = e^{(A-A)t} = e^{\mathbf{0}t} = \left\langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \right\rangle$ where $\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_2$ are solutions of system

$$\vec{\mathbf{x}}' = (A - A)\vec{\mathbf{x}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{\mathbf{x}}$$

with initial data equal to the columns of the identity matrix. Solutions of this system are constant vectors. Therefore $\vec{\mathbf{w}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{w}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by the definitions of $\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_2$. Then

$$e^A e^{-A} = \left\langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \right\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I. \quad \blacksquare$$

10. $e^{-A} = \left(e^A\right)^{-1}$

11. $A = \dfrac{d}{dt} e^{At}$ evaluated at $t = 0$

Solution: Let $\Phi(t) = e^{At} = \left\langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \right\rangle$. The exercise can be re-phased as $A = \Phi'(0)$. The columns of Φ are the special solutions $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2$ of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Definitions of $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2$ give identity $\Phi(0) = I$. The proof is completed from equation $\Phi'(t) = A\Phi(t)$ by substitution of $t = 0$: $\Phi'(0) = A\Phi(0) = AI = A$.

Proof of $\Phi'(t) = A\Phi(t)$:

$$\begin{aligned} \Phi'(t) &= \left\langle \vec{\mathbf{w}}_1' | \vec{\mathbf{w}}_2' \right\rangle \\ &= \left\langle A\vec{\mathbf{w}}_1 | A\vec{\mathbf{w}}_2 \right\rangle \\ &= A\left\langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \right\rangle \\ &= A\Phi(t) \quad \blacksquare \end{aligned}$$

12. If $A^3 = \mathbf{0}$, then $e^A = I + A + \frac{1}{2}A^2$.

13. Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Verify $N^2 = \mathbf{0}$ and $e^{At+Nt} = e^{At}(I + Nt)$.

Solution:

(1) Verify $N^2 = \mathbf{0}$:

$$N^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(2) Verify $e^{At+Nt} = e^{At}(I + Nt)$:

$$e^{At} = \begin{pmatrix} e^{at} 0 & \\ 0 & e^{at} \end{pmatrix} = e^{at} I$$

$$I + Nt = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} e^{At+Nt} &= e^{at}I + te^{at}(A+N-aI) \text{ by Putzer's formula for } \lambda_1=\lambda_2=a \\ &= e^{at}I + te^{at}N \\ &= e^{at}I(I+tN) \\ &= e^{At}(I+tN) \quad \blacksquare \end{aligned}$$

14. Let A be 3×3 diagonal and $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Prove $N^3 = \mathbf{0}$ and

$$e^{At+Nt} = e^{At}(I + Nt + N^2\frac{t^2}{2}).$$

15. $e^{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} t} = \begin{pmatrix} e^t & e^{2t} - e^t \\ 0 & e^{2t} \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. Apply Putzer's formula for $\lambda_1 = 1$, $\lambda_2 = 2$.

$$\begin{aligned} e^{At} &= e^tI + \frac{e^t - e^{2t}}{1-2}(A-(1)I) \\ &= e^tI + (e^{2t}-e^t)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} + \begin{pmatrix} 0 & e^{2t}-e^t \\ 0 & e^{2t}-e^t \end{pmatrix} \\ &= \begin{pmatrix} e^t & e^{2t}-e^t \\ 0 & e^{2t} \end{pmatrix} \quad \blacksquare \end{aligned}$$

16. $e^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$

Putzer's Spectral Formula.

17. Apply Picard-Lindelöf theory to conclude that $r_1, \ldots, r_n$ are everywhere defined,

Solution: Growth-decay differential equation $r_1' = \lambda_1 r_1$ has solution $r_1 = e^{\lambda_1 t}$, continuous everywhere. Substitute into the the second differential equation to get $r_2' + pr_2 = q$ where p is constant and $q(t)$ is everywhere continuous. Picard's Theorem says that $r_2(t)$ is everywhere continuous. Cascade each r_i into the next differential equation and apply Picard's Theorem repeatedly to conclude $r_1, \ldots, r_n$ are everywhere continuous. $\blacksquare$

18. Prove that $P_1, \ldots, P_k$ commute.

Putzer's Formula 2×2 .

19. Find a formula for $\frac{d}{dt}e^{At}$ for a 2×2 matrix A with eigenvalues $1, 2$.

Solution: Use

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$$

with $\lambda_1 = 1$, $\lambda_2 == 2$. Then

$$e^{At} = e^t I + \frac{e^t - e^{2t}}{1-2}(A - I).$$

Differentiate on t:

$$\frac{d}{dt}e^{At} = e^t I + \frac{e^t - 2e^{2t}}{1-2}(A - I).$$

20. Let 2×2 matrix A have duplicate eigenvalues $0, 0$. Compute r_1, r_2 and then report e^{At}.

Putzer: Real Distinct. Find the matrix exponential.

21. $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ in Exercise 19. Then

$$\begin{aligned} e^{At} &= e^t I + \tfrac{e^t - e^{2t}}{1-2}(A - I) \\ &= e^t I + \left(-e^t + e^{2t}\right)(A - I) \\ &= e^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(-e^t + e^{2t}\right) \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} + \begin{pmatrix} 0 & -2e^t + 2e^{2t} \\ 0 & -e^t + e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} e^t & -2e^t + 2e^{2t} \\ 0 & e^{2t} \end{pmatrix} \end{aligned}$$

```
# Exercise 21: Matrix exponential
A:=<1,2|0,2>^+;
with(LinearAlgebra):
MatrixExponential(A,t);
```

22. $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

Putzer: Real Equal. Find the matrix exponential.

23. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

$e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t}(A - \lambda_1 I)$

$e^{At} = e^t I + t e^t (A - I)$

$e^{At} = e^t I$

24. $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Putzer: Complex Eigenvalues. Find the matrix exponential.

25. $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Solution: Eigenvalues $1 \pm i$. Let $a = 1$, $b = 1$. Then:

$$\begin{aligned} e^{At} &= e^{at}\left(\cos bt\, I + \tfrac{\sin bt}{b}(A - aI)\right) \\ &= e^t\left(\cos t\, I + \tfrac{\sin t}{1}(A - I)\right) \\ &= e^t\left(\cos(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \\ &= e^t\begin{pmatrix} \cos t & \sin t \\ -\sin t & cost \end{pmatrix}. \end{aligned}$$

26. $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

How to Remember Putzer's 2×2 Formula.

27. Find $\lim_{\lambda \to \lambda_1} \dfrac{e^{\lambda t} - e^{\lambda_1 t}}{\lambda - \lambda_1}$.

Solution: L'Hôpital's Rule applies.

28. Let matrix A be 2×2 real. Take the real part: $e^{At} = I + \dfrac{e^{it} - e^{-it}}{2i} A$.

Classical $n \times n$ Spectral Formula. Find e^{At}.

29. $A = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution: The basic idea is to use the formula for a block diagonal matrix, Theorem 11.19. Then $e^{At} =$ **diag** $\left(e^{Bt}, e^{Ct}\right)$ where $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and $C = (1)$

$= 1 \times 1$ matrix.

$$e^{At} = \begin{pmatrix} \cos(2t) & \sin(2t) & 0 \\ -\sin(2t) & \cos(2t) & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

```
# Exercise 29: Matrix exponential
A:=<0,2,0|-2,0,0|0,0,1>^+;
with(LinearAlgebra):
MatrixExponential(A,t);
```

30. $A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Proofs of Matrix Exponential Properties.

31. Let $A\vec{\mathbf{u}} = B\vec{\mathbf{u}}$ for all vectors $\vec{\mathbf{u}}$. Prove $A = B$.

Solution: Let $\vec{\mathbf{u}}$ equal the first column of the identity matrix I. Then $A\vec{\mathbf{u}} = B\vec{\mathbf{u}}$ says the first column of A equals the first column of B. Repeat for all columns of I to proved A, B have exactly the same entries. ■

32. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Compute the first four Picard iterates for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$.

Special Cases e^{At}.

33. Show the details to solve
$x_1' = 2x_1 + x_3$,
$x_2' = 3x_2 + x_3$,
$x_3' = 4x_3$,
$x_1(0) = 1, x_2(0) = x_3(0) = 0$.

Solution: Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$. Compute

$$e^{At} = \begin{pmatrix} e^{2t} & 0 & -\frac{1}{2}e^{2t} + \frac{1}{2}e^{4t} \\ 0 & e^{3t} & e^{4t} - e^{3t} \\ 0 & 0 & e^{4t} \end{pmatrix}$$

then

$$\vec{\mathbf{x}}(t) = e^{At}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix}.$$

```
# Exercise 33: Matrix exponential
A:=<2,0,1|0,3,1|0,0,4>^+;
with(LinearAlgebra):MatrixExponential(A,t);
```

34. Let $A = \textsf{\textbf{diag}}(1, 2, 3, 4)$. Find e^{At}.

35. Let $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A = \textsf{\textbf{diag}}(B, B)$. Find e^{At}.

Solution: Theorem 11.19 gives
$e^{At} = \textsf{\textbf{diag}}\left(e^{Bt}, e^{Bt}\right)$.

Putzer's identity gives

$$\begin{aligned}
e^{Bt} &= e^{0t}I + \frac{e^{0t} - e^{1t}}{0 - 1}(B - 0I) \\
&= I + (-1 + e^t)B \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-1 + e^t)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
e^{At} &= \textsf{\textbf{diag}}\left(e^{Bt}, e^{Bt}\right) \\
&= \textsf{\textbf{diag}}\left(\begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} e^t & e^t - 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^t & e^t - 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

```
# Exercise 35: Matrix exponential
with(LinearAlgebra):
B:=<1,1|0,0>^+;MatrixExponential(B,t);
Z:=Matrix(2,2);A:=< <B,Z> | <Z,B> >;
MatrixExponential(A,t);
```

36. Let $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and
$A = \textsf{\textbf{diag}}(B, B)$. Find e^{At}.

11.5 Eigenanalysis, Spectral, CHZ

Determinant $|A - rI|$

Justify these statements.

1. Subtract r from the diagonal of A to form $|A - rI|$.

Solution: Matrix A with r subtracted from the diagonal is matrix $B = A - rI$. The determinant of B is $|A - rI|$, the characteristic equation of A.

2. If A is 2×2, then $|A - rI|$ is a quadratic.

3. If A is 3×3, then $|A - rI|$ is a cubic.

Solution: Symbol r appears three times in $A - rI$, once per row. One term in the determinant expansion is the product of the diagonal elements, which is

$$(a_{11} - r)(a_{22} - r)(a_{33} - r).$$

Then $(-r)^3$ appears as the largest power of r in the determinant expansion: $|A - rI|$ is a cubic polynomial in variable r.

4. Expansion of $|A - rI|$ by the cofactor rule often preserves factorizations.

5. If A is triangular, then $|A - rI|$ is the product of diagonal entries.

Solution: Apply the triangular rule for determinants, because $A - rI$ is also triangular.

6. The *combo, mult* and *swap* rules for determinants are generally counter-productive for expansion of $|A - rI|$.

Characteristic Polynomial

Show expansion details for $|A - rI|$.

7. $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$.

Ans: $(2 - r)(4 - r)$

Solution: $|A - rI| = \begin{vmatrix} 2 - r & 3 \\ 0 & 4 - r \end{vmatrix} = (2 - r)(4 - r).$

8. $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$.

Ans: $(2 - r)(5 - r)(7 - r)$

Eigenanalysis Method: 2×2

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

9. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Solution: The eigenvalues of A are $1, 2$. Two eigenpairs are $\left(1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $\left(2, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. Then Theorem 11.21 gives

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

10. $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

Eigenanalysis Method: 3×3

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

11. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution: The eigenvalues are $3, 0, 1$ with matching eigenvectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The general solution using Theorem 11.22:

$$\vec{\mathbf{x}}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2 e^{0t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

```
# Exercise 11, Eigenanalysis method 3x3
with(LinearAlgebra):
A:=<1,1,0|2,2,0|0,0,1>^+;Eigenvectors(A);
```

12. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Eigenanalysis Method: $n \times n$

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

13. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Solution: The eigenvalues are $0, 1, 1, 3$ with matching eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. The general solution using Theorem 11.23:

$$\vec{\mathbf{x}}(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 e^{3t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

```
# Exercise 13, Eigenanalysis method 4x4
with(LinearAlgebra):
A:=<1,1,0,0|2,2,1,0|0,0,1,0|0,0,0,1>^+;Eigenvectors(A);
```

14. $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

e^{At} for Simple Eigenvalues

Find a^{At} using classical spectral theory. Check by computer.

15. $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

Solution: The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$. Define $Q_1 = (A - 0I)/(3 - 0) = \frac{1}{3}A$, $Q_2 = (A - 3I)/(0 - 3) = -\frac{1}{3}A - I$. Theorem 11.24 gives

$$\begin{aligned} e^{At} &= e^{\lambda_1 t} Q_1 + e^{\lambda_2 t} Q_2 \\ &= \tfrac{1}{3}\, e^{\lambda_1 t} A + e^{\lambda_2 t}(-\tfrac{1}{3}A - I) \\ &= \tfrac{1}{3}\, e^{0t} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + e^{3t} \left(-\tfrac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= \tfrac{1}{3}\, e^{0t} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + e^{3t} \begin{pmatrix} -\frac{4}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{5}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3} + \frac{1}{3}e^{3t} & \frac{1}{3}e^{3t} - \frac{1}{3} \\ \frac{2}{3}e^{3t} - \frac{2}{3} & \frac{1}{3} + \frac{2}{3}e^{3t} \end{pmatrix} \end{aligned}$$

```
# Exercise 15, Classical spectral theory
with(LinearAlgebra):
A:=<1,1|2,2>^+;Eigenvectors(A);
MatrixExponential(A,t);
```

16. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

e^{At} for Multiple Eigenvalues

Find a^{At} using classical spectral theory. Check by computer.

17. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Solution: The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$. The characteristic polynomial is $p(\lambda) = (1-\lambda)^2$. Then $a_1(\lambda) = 1$, $a_2(\lambda) = 0$, $m_1 = 2$, $m_2 = 0$. Define $Q_1 = a_1(A)(A-I)^0 = I$, $Q_2 = 0$. Theorem 11.25 gives

$$\begin{aligned} e^{At} &= e^{\lambda_1 t} Q_1 (I + (A-I)t) \\ &= e^{1t} I(I + (A-I)t) \\ &= e^{1t} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \\ &= e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \end{aligned}$$

```
# Exercise 17, Classical spectral theory
with(LinearAlgebra):
A:=<1,1|0,1>^+;Eigenvectors(A);
MatrixExponential(A,t);
```

18. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Cayley-Hamilton Theorem

Prove the identity by applying the Cayley-Hamilton Theorem.

19. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a_0 = |A| = ad - bc$, $a_1 = \textbf{trace}(A) = a + d$. Then $A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Solution: Compute $|A - rI| = r^2 - (a+d)r + ad - bc$. Cayley-Hamilton says

$$A^2 - (a+d)A + (ad - bc)I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then

$$A^2 + a_1(-A) + a_0 I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

■

20. Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$. Then:

$(2I - A)(5I - A)(7I - A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

CHZ Theorem: Scalar Form

21. Write Theorem 11.27 proof missing details for $n = 3$.

Solution: The book's proof is routine up to the point of multiplying by rows of the identity matrix. Start with

$\vec{\mathbf{x}}'' + a_1\vec{\mathbf{x}}' + a_0\vec{\mathbf{x}} = \vec{\mathbf{0}}$

then multiply left by row vector $\vec{\mathbf{e}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T$, which is the first row of I. Left multiply by $\vec{\mathbf{e}}$ is the same as taking the dot product of the equation with vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \vec{\mathbf{x}}'' + a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \vec{\mathbf{x}}' + a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \vec{\mathbf{0}}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \vec{\mathbf{0}}$

$x_1'' + a_1 x_1' + a_0 x_1 = 0$

The second equation is obtained similarly by using $\vec{\mathbf{e}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T$, which is the second row of I. ■

22. Write Theorem 11.27 proof missing details for any n.

CHZ Theorem: Vector Form

23. Write Theorem 11.28 proof details for $n = 2$.

Solution: Let $n = 2$. Expand $|A - rI| = r^2 + a_1 r + a_0$. The roots of the quadratic generate Euler atoms $A_1(t)$, $A_2(t)$. Theorem 11.27 implies components x_1, x_2 of solution $\vec{\mathbf{x}}(t)$ to system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ satisfy the $2nd$ order scalar differential equation $u'' + a_1 u' + a_0 u = 0$. Then each of x_1, x_2 is a linear combination of atoms A_1, A_2. Assume $x_1 = c_{11}A_1 + c_{12}A_2$ and $x_2 = c_{21}A_1 + c_{22}A_2$. Then

$$\begin{aligned} \vec{\mathbf{x}} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} c_{11}A_1 + c_{12}A_2 \\ c_{21}A_1 + c_{22}A_2 \end{pmatrix} \\ &= \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} A_1 + \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} A_2 \end{aligned}$$

which is a vector linear combination of atoms A_1, A_2. ■

24. Write Theorem 11.28 proof details for $n = 3$.

CHZ Identity: Vandermonde

Find matrix $D = \left\langle \vec{\mathbf{d}}_1 | \cdots | \vec{\mathbf{d}}_n \right\rangle$ using Theorems 11.29, 11.31, given $\vec{\mathbf{x}}(0) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

25. $A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$. Ans: $W(0)^T, D =$ $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & c_1 \\ 2c_1 + c_2 & -2c_1 \end{pmatrix}$

Solution: The eigenvalues are $2, 1$. The atoms are e^{2t}, e^t. The Wronskian matrix of the atoms is $W(t) = \begin{pmatrix} e^{2t} & e^t \\ 2e^{2t} & e^t \end{pmatrix}$. Then $W(0)^T = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^T =$ $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ = the Vandermonde matrix for list $2, 1$. See `maple` help for the definition of the Vandermonde matrix: $\begin{pmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{pmatrix}$ = Vandermonde matrix for list $[x_1, x_2]$. The literature has more than one definition, the currently accepted definition matching `maple` help, agreeing with Wikipedia:

https://en.wikipedia.org/wiki/Vandermonde_matrix

Form the equation in Theorem 11.29 and solve for matrix $\left\langle \vec{\mathbf{d}}_1 | \vec{\mathbf{d}}_2 \right\rangle$.

$$\begin{aligned} \left\langle \vec{\mathbf{d}}_1 | \vec{\mathbf{d}}_2 \right\rangle &= \left\langle \vec{\mathbf{x}}_0 | A\vec{\mathbf{x}}_0 \right\rangle \left(W(0)^T \right)^{-1} \\ &= \begin{pmatrix} c_1 & c_1 \\ c_2 & 2c_1 + 2c_2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_1 \\ c_2 + 2c_1 & -2c_1 \end{pmatrix} \end{aligned}$$

```
# Exercise 25, Vandermonde identity
with(LinearAlgebra):
A:=<1,2|0,2>;
C:=x-><x|A.x>;
EV:=convert(Eigenvalues(A),list);
V:=VandermondeMatrix(EV);
x0:=<c1,c2>;
<d1|d2> = C(x0).(1/V);
```

26. $A=\begin{pmatrix}1&0&0\\2&2&0\\0&0&3\end{pmatrix}$. Ans: $W(0)^T, D=$
$\begin{pmatrix}1&1&1\\1&2&4\\1&3&9\end{pmatrix}$, $\begin{pmatrix}c_1&0&0\\-2c_1&2c_1+c_2&0\\0&0&c_3\end{pmatrix}$

CHZ and Eigenvectors

Supply details for the following.

27. Find a scalar 3rd order linear differential equation that has e^t, e^{2it}, e^{-2it} as solutions. Apply theorems to conclude that the Wronskian of the exponentials is invertible for every t.

Solution: The plan: apply Theorem 11.13 (Abel-Liouville Formula). Matrix A will be the companion matrix for the characteristic polynomial $(1-r)(r^2+4)$, the latter constructed from roots $1, 2i, -2i$ of atoms $e^t, \cos 2t, \sin 2t$ extracted from e^t, e^{2it}, e^{-2it}.

28. Assume $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ are independent exponentials . Apply theorems to conclude that the Wronskian of the exponentials is invertible for every t.

29. If $\vec{\mathbf{d}}_1 e^t + \vec{\mathbf{d}}_2 e^{-t} + \vec{\mathbf{d}}_3 e^{2t} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, then $\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_2 = \vec{\mathbf{d}}_3 = \vec{\mathbf{0}}$.

Solution: Let $\vec{\mathbf{e}}$ = a column of the identity matrix. Take the dot product of $\vec{\mathbf{e}}$ across the equation. The result is a scalar linear combination of distinct Euler atoms equal to zero. Independence of atoms implies all the constants are zero. The constants for all choices of $\vec{\mathbf{e}}$ exhaust the components of $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, $\vec{\mathbf{d}}_3$. Therefore, the three vectors $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, $\vec{\mathbf{d}}_3$ have all zero components. ■

30. Independence of atoms applied to the n-vector equation $\vec{\mathbf{d}}_1 e^t + \vec{\mathbf{d}}_2 e^{-t} = c_1\vec{\mathbf{v}}_1 e^t + c_2\vec{\mathbf{v}}_2 e^{-t}$ implies $\vec{\mathbf{d}}_1 = c_1\vec{\mathbf{v}}_1$ and $\vec{\mathbf{d}}_2 = c_2\vec{\mathbf{v}}_2$.

31. There is a 2×2 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ for which CHZ vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ are not eigenvectors of A.

Solution: Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which is non-diagonalizable with eigenvalues $1, 1$. Theorem 11.31 (Vandermonde Matrix and ...) does not apply because the eigenvalues are not distinct. Equation $|A - rI| = (1-r)^2$ has associated atoms e^t, te^t (double root case). Theorem 11.29 (Cayley-Hamilton-Ziebur Identity: Real) applies:

$\left\langle \vec{\mathbf{d}}_1 | \vec{\mathbf{d}}_2 \right\rangle = \left\langle \vec{\mathbf{x}}_0 | A\vec{\mathbf{x}}_0 \right\rangle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ Then for $\vec{\mathbf{x}}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we compute $\vec{\mathbf{d}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{d}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. But $A\vec{\mathbf{d}}_2 \neq A\vec{\mathbf{d}}_2$. Then $\vec{\mathbf{d}}_2$ fails to equal an eigenvector of A.

32. Let A be the 3×3 identity matrix. For $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, two of the CHZ vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3$ are zero.

Eigenvectors by Matrix Multiply Find the eigenvectors of A by Theorem 11.33. Report the choice of $\vec{\mathbf{U}}$.

33. $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Ans: $\vec{\mathbf{U}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

34. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Ans: $\vec{\mathbf{U}} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

CHZ 2×2 Matrix Shortcut Find the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ using Theorem 11.36.

35. $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $r = -2, 4$

Solution: Follow Example 11.8. Let $y_1 = e^{-2t}$, $y_2 = e^{4t}$, $\vec{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then

$$\begin{aligned} \vec{\mathbf{y}}' &= \begin{pmatrix} -2e^{-2t} \\ 4e^{4t} \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} e^{-2t} \\ e^{4t} \end{pmatrix} \end{aligned}$$

Let $B = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$, which is the diagonal matrix of eigenvalues $-2, 4$. Then

$$\begin{aligned} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} &= \tfrac{1}{3}(B^T - I) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \tfrac{1}{3} \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -c_1 \\ c_2 \end{pmatrix}$$

Conclusion:

$x_1 = c_1 e^{-2t} + c_2 e^{4t}$,
$x_2 = k_1 y_1 + k_2 y2 = -c_1 e^{-2t} + c_2 e^{4t}$

Remark.
The algorithm is designed to generate a solution in the correct form using a computer workbench or a computer algebra system. This example can be used to write the code for a subroutine that solves $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ for a non-diagonal matrix A.

36. $A = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$, $r = 1 \pm 3i$

CHZ Scalar 2×2 Shortcut Find the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ using Theorem 11.35.

37. $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$, $r = -3, 5$

Solution: The scalar equations are $x_1' = x_1 + 4x_2$, $x_2' = 4x_1 + x_2$. To apply Theorem 11.35, define $x_1 = c_1 e^{-3t} + c_2 e^{5t}$. Solve the first differential equation $x_1' = x_1 + 4x_2$ for $4x_2 = x_1' - x_1 = (c_1 e^{-3t} + c_2 e^{5t})' - c_1 e^{-3t} - c_2 e^{5t} = -4c_1 e^{-3t} + 4e^{5t}$. Then

$x_1 = c_1 e^{-3t} + c_2 e^{5t}$
$x_2 = -c_1 e^{-3t} + e^{5t}$ ■

38. $A = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$, $r = 1 \pm 4i$

Putzer's 2×2 Spectral Formula Verify the identity.

39. $A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}$

$$e^{At} = e^{5t} I + \frac{e^{5t} - e^{2t}}{3} \begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$$

Solution: Factor $|A - rI| = r^2 - \mathbf{trace}()r + |A|$ as $r^2 - 7r + 10 = (r-2)(r-5)$. The eigenvalues of A are $5, 2$. Apply Putzer's formula for distinct real roots:

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(A - \lambda_1 I)$$

$$e^{At} = e^{5t} I + \frac{e^{2t} - e^{5t}}{2 - 5}(A - 5I)$$

$$e^{At} = e^{5t}I + \frac{e^{5t} - e^{2t}}{3}\begin{pmatrix} -1-5 & 3 \\ -6 & 8-5 \end{pmatrix}$$

$$e^{At} = e^{5t}I + \frac{e^{5t} - e^{2t}}{3}\begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$$

40. $A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$

$$e^{At} = e^{-2t}I + \frac{e^{3t} - e^{-2t}}{5}\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$$

41. $A = \begin{pmatrix} 0 & 1 \\ -16 & 8 \end{pmatrix}$

$$e^{At} = e^{4t}I + te^{4t}\begin{pmatrix} -4 & 1 \\ -16 & 4 \end{pmatrix}$$

Solution: The eigenvalues of A are $4, 4$. Apply Putzer's formula for a double root:

$$\begin{aligned} e^{At} &= e^{\lambda_1 t}I + te^{\lambda_1 t}(A - \lambda_1 I) \\ &= e^{4t}I + te^{4t}(A - 4I) \\ &= e^{4t}I + te^{4t}\begin{pmatrix} 0-4 & 1 \\ -16 & 8-4 \end{pmatrix} \\ &= e^{4t}I + te^{4t}\begin{pmatrix} -4 & 1 \\ -16 & 4 \end{pmatrix} \end{aligned}$$

42. $A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$, $e^{At} =$

$$e^{3t}\cos(2t)I + e^{3t}\sin(2t)\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

11.6 Jordan Form and Eigenanalysis

Jordan block definition. Write out the Jordan form matrix explicitly.

1. $\mathbf{diag}(B(7,2), B(5,3))$

Answer: $\begin{pmatrix} 7&1&0&0&0\\ 0&7&0&0&0\\ 0&0&5&1&0\\ 0&0&0&5&1\\ 0&0&0&0&5 \end{pmatrix}$

Solution: By definition page 894,

$$B(7,2) = \begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix},$$

$$B(5,3) = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}.$$

Then

$$\mathbf{diag}(B(7,2), B(5,3)) = \begin{pmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

2. $\mathbf{diag}(B(0,2), B(4,3))$

3. $\mathbf{diag}(B(-1,1), B(-1,2), B(5,3))$

Solution: Jordan matrix $\mathbf{diag}(B(-1,1), B(-1,2), B(5,3)) =$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

4. $\mathbf{diag}(B(1,1), B(5,2), B(5,3))$

Jordan form definition. Which are Jordan forms and which are not? Explain.

5. $\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&5&1&0\\ 0&0&0&5&1\\ 0&0&0&0&5 \end{pmatrix}$

Solution: Jordan form $\mathbf{diag}(B(0,2), B(5,3))$

6. $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{pmatrix}$

Solution: Not a Jordan form because of entry 5 below the diagonal.

8. $\begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$

Decoding $A = PJP^{-1}$. Decode $A = PJP^{-1}$ in each case, displaying explicitly the Jordan chain relations and their solutions.

9. $A = \begin{pmatrix} 4 & 8 & 0 & 0 & -8 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & -8 \\ 0 & 20 & 0 & 2 & -12 \\ 0 & 8 & 0 & 0 & -4 \end{pmatrix}$,

$J = \textbf{diag}(-4, 2, 2, 4, 4)$

Solution: The eigenvalues of A are $-4, 2, 2, 4, 4$, found by computer. What is not known initially is the block sizes for the repeated eigenvalues. Additional information supplied says all blocks have size one. All Jordan chain relations have the form $(A - \lambda I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$, the classical eigenvalue problem.

The 5 Jordan blocks in J correspond to 1-chains. Each block decodes into one vector equation. All vectors $\vec{\mathbf{v}}$ below are in $\mathcal{R}^5$, because the row dimension of A is 5. The ordering of the blocks is not important as long as eigenvalues and columns of P are paired. Recorded below is the Jordan Form computed by `maple` from the Frobenius Form. The `maple` answer for P is not used, because it comes from the Frobenius Form, having little in common with hand solution details. Details by hand usually differ because eigenvectors are not unique. Additional differences arise because of the free choice of two independent eigenvectors for both $\lambda = 2$ and $\lambda = 4$.

Block $B(-4, 1)$, $\lambda = -4$: The 1-chain $A\vec{\mathbf{v}}_1 = -4\vec{\mathbf{v}}_1$ is solved for $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ from homogeneous problem $(A + 4I)\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$. Simple eigenvalues always generate a 1-chain solved by classical eigenanalysis.

Two Blocks $B(2,1)$, $B(2,1)$ for $\lambda = 2$: Given in the exercise: there are no 2-chains. The task remaining: find two independent eigenvectors $\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ for problem $A\vec{\mathbf{v}} = 2\vec{\mathbf{v}}$. To expedite the computation we use `maple`, details below.

Two Blocks $B(4,1)$, $B(4,1)$ for $\lambda = 4$: Given is there are no 2-chains. The task remaining: find two independent eigenvectors $\vec{\mathbf{v}}_4, \vec{\mathbf{v}}_5$ for problem $A\vec{\mathbf{v}} = 4\vec{\mathbf{v}}$. Following the case for $\lambda = 2$, we use `maple`, details below..

The answers:

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \vec{\mathbf{v}}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{\mathbf{v}}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix}.$$

Let P be the augmented matrix of $\vec{\mathbf{v}}_1$ to $\vec{\mathbf{v}}_5$ and let $J = \mathbf{diag}(-4, 2, 2, 4, 4)$. Then

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Using `maple` we check the equation $AP = PJ$, verifying the Jordan Decomposition found by hand. The details of this example follow exactly the details for equation $AP = PD$ for a diagonalizable matrix A. ■

```
# Exercise 9, Decoding and solving 1-chains
A:=Matrix([
[  4,   8,   0,   0,  -8],
[  0,   4,   0,   0,   0],
[  2,   8,   2,   0 , -8],
[  0,  20,   0,   2, -12],
[  0,   8,   0,   0,  -4]]);
J:=JordanForm(A);Q:=JordanForm(A,output='Q');# Automated by maple
A.Q - Q.J; # Check maple answer, should be zero
# Proceed to find chains manually
ZV:=ZeroMatrix(5,1);
# Eigenvalue -4
N:=A-2*IdentityMatrix(5);
ZV:=ZeroMatrix(5,1);LinearSolve(N,ZV,free='s');
ReducedRowEchelonForm(N);
# v1:=<1,0,1,2,1>;
# Eigenvalue 2
N:=A-2*IdentityMatrix(5);
ZV:=ZeroMatrix(5,1);LinearSolve(N,ZV,free='ss');
ReducedRowEchelonForm(N);
# v2:=<0,0,1,0,0>;v3:=<0,0,0,1,0>;
# Eigenvalue 4
N:=A-4*IdentityMatrix(5);LinearSolve(N,ZV,free='sss');
ReducedRowEchelonForm(N);
# v4:=<1,0,1,0,0>;v5:=<0,1,0,4,1>;
v1:=<1,0,1,2,1>;
v2:=<0,0,1,0,0>;v3:=<0,0,0,1,0>;
v4:=<1,0,1,0,0>;v5:=<0,1,0,4,1>;
P:=<v1|v2|v3|v4|v5>;# pair eigenvalues and eigenvectors
JJ:=DiagonalMatrix([-4,2,2,4,4]);
A.P-P.JJ;# Should be zero
```

10. $A = \begin{pmatrix} -4 & -4 & -12 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 4 & -12 & 16 & 0 \\ -8 & 4 & -16 & 20 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix}$,

$J = \textbf{diag}(-4, 4, 4, 0, 0)$

Geometric and algebraic multiplicity.

Determine **GeoMult**(λ) and **AlgMult**(λ).

11. $A = \begin{pmatrix} 4 & 8 & 0 & 0 & -8 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & -8 \\ 0 & 20 & 0 & 2 & -12 \\ 0 & 8 & 0 & 0 & -4 \end{pmatrix}$, $\lambda = 4$

Solution: Answer: **GeoMult**(A) = **AlgMult**$(A) = 2$ because there are two eigenpairs for $\lambda = 4$.

```
# Exercise 11, GeoMult and AlgMult
A:=Matrix([[4,8,0,0,-8],[0,4,0,0,0],[2,8,2,0,-8],
          [0,20,0,2,-12],[0,8,0,0,-4]]);
Eigenvectors(A);
```

12. $A = \begin{pmatrix} -4 & -4 & -12 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 4 & -12 & 16 & 0 \\ -8 & 4 & -16 & 20 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix}$, $\lambda = 4$

Generalized eigenvectors. Find all generalized eigenvectors and represent $A = PJP^{-1}$. Check the answer in a computer algebra system.

13. $A = \begin{pmatrix} 4 & 8 & 0 & 0 & -8 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & -8 \\ 0 & 20 & 0 & 2 & -12 \\ 0 & 8 & 0 & 0 & -4 \end{pmatrix}$,

Answer: $J = \textbf{diag}(-4, 4, 4, 2, 2)$,

$P = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

Solution: The matrix is diagonalizable. Generalized eigenvectors are eigenvectors. Use `maple` for the computation.

$$J = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 4 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

```
# Exercise 13, Diagonalizable matrix
A:=Matrix([[4,8,0,0,-8],[0,4,0,0,0],[2,8,2,0,-8],
          [0,20,0,2,-12],[0,8,0,0,-4]]);
JV,P:=Eigenvectors(A);J:=DiagonalMatrix(convert(JV,list));
A.P-P.J;
```

14. $A = \begin{pmatrix} -4 & -4 & -12 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 4 & -12 & 16 & 0 \\ -8 & 4 & -16 & 20 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix}$,

Answer: $J = \textbf{diag}(-4, 4, 4, 0, 0)$,

$P = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 1 & -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 3 & 0 \end{pmatrix}$

15. $A = \begin{pmatrix} 0 & 2 & -2 & -2 \\ 2 & 0 & -2 & -4 \\ 2 & 2 & -4 & -2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$,

Ans: $J = \textbf{diag}(0, -4, -2, -2)$,

$P = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & -4 & 0 \\ 1 & 0 & -3 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Solution: The matrix is diagonalizable. Generalized eigenvectors are eigenvectors. Use `maple` for the computation.

$$J = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
# Exercise 15, Diagonalizable matrix
A:=Matrix([[0,2,-2,-2],[2,0,-2,-4],[2,2,-4,-2],[0,0,0,-4]]);
JV,P:=Eigenvectors(A);J:=DiagonalMatrix(convert(JV,list));
A.P-P.J;
```

16. $A = \begin{pmatrix} -2 & 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(2, 2, B(2, 3))$,

$P = \begin{pmatrix} 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

17. $A = \begin{pmatrix} 2&1&0&1&0\\ 0&2&0&0&0\\ 0&1&2&0&0\\ 0&0&0&2&1\\ 0&0&0&0&2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(B(2,3), B(2,2))$,

$P = \begin{pmatrix} 1&2&1&2&1\\ 0&0&2&0&2\\ 0&2&1&2&1\\ 0&1&0&0&0\\ 0&0&1&0&0 \end{pmatrix}$

Solution: The matrix is not diagonalizable. Use `maple` for the computation, which reveals there is one eigenvalue $\lambda = 2$ and two eigenvectors. The eigenpairs are (which we do not use)

$$\left(2, \begin{pmatrix} 0\\0\\1\\0\\0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix}\right)$$

The possible chains are:

5-chain;
1-chain, 4-chain;
2-chain, 3-chain

Rank computations use the `maple` code below, following textbook Example 11.11, to find the possible block sizes. The result: there is a 2-chain and a 3-chain. Eliminated by the computation are three possibilities: no 1-chain, no 4-chain, no 5-chain.

Details: Record A, $N = A - 2I$, N^2 from computer assist:

$$A = \begin{pmatrix} 2&1&0&1&0\\ 0&2&0&0&0\\ 0&1&2&0&0\\ 0&0&0&2&1\\ 0&0&0&0&2 \end{pmatrix}, \quad N = \begin{pmatrix} 1&-1&1&0&0\\ 2&-2&1&1&0\\ 1&-1&0&1&0\\ -1&1&0&0&1\\ -3&3&0&-2&1 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 0&0&0&0&0\\ -2&2&0&-1&1\\ -2&2&0&-1&1\\ -2&2&0&-1&1\\ 2&-2&0&1&-1 \end{pmatrix}$$

The 3-chain. Let $m = 3$ (find a 3-chain). The plan is to find a vector $\vec{\mathbf{w}}$ with $N^3\vec{\mathbf{w}} = \vec{\mathbf{0}}$, $N^2\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and $N^2\vec{\mathbf{x}} = \vec{\mathbf{w}}$ has no solution $\vec{\mathbf{x}}$. Then $\vec{\mathbf{v}}_1 = N^2\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_2 = N\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_3 = \vec{\mathbf{w}}$ are the columns of P corresponding to Jordan block $B(\lambda, 3)$, to wit: columns 1,2,3 of P.

We will choose $\vec{\mathbf{w}}$ to be a basis element for the nullspace of $(N^2)^T$, following Table 2 and Proposition 11.9. This clever choice works because $N^m = 0$. We still have to check $N^2\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, as in Table 2, page 898. Employ `maple` to find the nullspace basis:

$$\mathbf{nullspace}((N^2)^T) = \mathbf{span}\left\{\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right\}$$

Choose vector $\vec{\mathbf{w}}$ to be the first basis vector above, that is, the vector with components $0, 0, 0, 0, 1$. Then (1) equation $N^2\vec{\mathbf{x}} = \vec{\mathbf{w}}$ is insolvable for $\vec{\mathbf{x}}$, (2) $N^2\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, (3) $N^3\vec{\mathbf{w}} = \vec{\mathbf{0}}$.

Columns 1,2,3 of P will be defined by equations

$$\vec{\mathbf{v}}_1 = N^2\vec{\mathbf{w}} = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \quad \vec{\mathbf{v}}_2 = N\vec{\mathbf{w}} = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \vec{\mathbf{w}} = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

The computation means that $AP = PJ$[1] where

$$P = \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 | \vec{\mathbf{0}} | \vec{\mathbf{0}} \right\rangle = \begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix}, \quad \left\{\begin{array}{l} N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}} \\ N\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1 \\ N\vec{\mathbf{v}}_3 = \vec{\mathbf{v}}_2 \end{array}\right.$$

The 2-chain. Let $m = 2$ (find a 2-chain). The plan is to find a vector $\vec{\mathbf{w}}$ with $N^2\vec{\mathbf{w}} = \vec{\mathbf{0}}$, $N\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and $N\vec{\mathbf{x}} = \vec{\mathbf{w}}$ has no solution $\vec{\mathbf{x}}$. Then $\vec{\mathbf{v}}_4 = N\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_5 = \vec{\mathbf{w}}$ are the columns of P corresponding to Jordan block $B(\lambda, 2)$, to wit: columns 4,5 of P.

We will choose $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ to be a vector in the nullspace of N^T, following Table 2 and Proposition 11.9. First, find a basis for the nullspace of N^T (see Proposition 11.9). Then write $\vec{\mathbf{w}}$ in terms of this basis:

$$\mathbf{nullspace}(N^T) = \mathbf{span}\left\{\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right\},$$

$$\vec{\mathbf{w}} = c_1\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}.$$

[1]Zero columns in P allow rapid testing of $AP = PJ$.

Next, we force $\vec{\mathbf{w}}$ to belong to the nullspace of $N^m = N^2$. Equation

$$N^2\vec{\mathbf{w}} = \begin{pmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{\mathbf{0}}$$

holds if and only if $c_1 = 0$. Choose $c_1 = 0$, $c_2 = 1$ to make it so, then compute

$$\vec{\mathbf{w}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad N\vec{\mathbf{w}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \neq \vec{\mathbf{0}}$$

Conclusions: (1) equation $N\vec{\mathbf{x}} = \vec{\mathbf{w}}$ is insolvable for $\vec{\mathbf{x}}$, (2) $N\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and (3) $N^2\vec{\mathbf{w}} = \vec{\mathbf{0}}$. Define

$$\vec{\mathbf{v}}_4 = N\vec{\mathbf{w}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_5 = \vec{\mathbf{w}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then

$$P = \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 | \vec{\mathbf{v}}_4 | \vec{\mathbf{v}}_5 \right\rangle = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Matrix multiply verifies $AP = PJ$, which means P is a matrix of generalized eigenvectors for A. ■

```
# Exercise 17, Generalized Eigenvectors, non-diagonalizable
with(LinearAlgebra):
getBlockCounts:=proc(A,lambda) local m,N,j,r,p,txt;
m:=RowDimension(A);
N:=A-lambda*IdentityMatrix(m);
for j from 1 to m do r[j]:=Rank(N^j); od:
for p from m to 2 by -1 do
if(r[p]<>r[p-1])then break;fi:od;
printf("lambda=%d, nilpotency=p=%d\n",lambda,p);
txt:=(j,x)-> printf("Blocks B(%a,%d):%d\n",lambda,j,x):
for j from p to 2 by -1 do
txt(j,-2*r[j]+r[j-1]+r[j+1]):
od:end proc:
A:=Matrix([[2,1,0,1,0],[0,2,0,0,0],[0,1,2,0,0],
           [0,0,0,2,1],[0,0,0,0,2]]);
Eigenvectors(A);lambda:=2;
getBlockCounts(A,\lambda);
J:=JordanBlockMatrix([[lambda,3],[lambda,2]]);
N:=A-2:print("N, N^2 N^3=",N,N^2,N^3);

# Exercise 17, Find the 3-chain
m:=3;B:=N^(m-1):B_transpose:=B^+;
NullSpace(B_transpose);
w:=<0,0,0,0,1>:v1:=N^2 .w: v2:=N.w:
v3:=w: print("v1,v2,v3=",v1,v2,v3);
Z:=ZeroMatrix(5,1):P:=<v1|v2|v3|Z|Z>;print("AP-PJ=",A.P - P.J);

# Exercise 17, Find the 2-chain
 # Define v4=N.w, v5=w
 # Need: N^2 .w=0, N.w not zero, N.x=w has no solution x
 # Let w = linear combination in nullspace((N^(m-1))^T)
 # Choose linear combination weights so that N^2 .w=0
 # Then test N.w <> 0
NullSpace(N^+);c1:='c1':c2:='c2':
w:=c1*<0,0,0,0,1>+c2*<0,1,0,0,0>;
# Solve N^2 .w = 0 for c1,c2;
N^2 .w=Z;# Solve it for c1,c2;choose c1=0, c2=1
w:=<0,1,0,0,0>;
N.w; N^2 .w;# Check N.w <> 0, N^2 .w = 0
# Define chain vectors
v5:=w:v4:=N.v5:
print("v4=",v4,"v5=",v5);
P:=<v1|v2|v3|v4|v5>:print("J, P, AP-PJ=",J,P,A.P-P.J);
```

18. $A = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(B(2,4), 2)$,

$P = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Number of Jordan Blocks. Outlined here is the derivation of

$$s(j) = 2k(j-1) - k(j-2) - k(j).$$

Definitions:

- $s(j)$= number of blocks $B(\lambda, j)$
- $N = A - \lambda I$
- $k(j) = \dim(\textbf{kernel}(N^j))$
- $L_j = \textbf{kernel}(N^{j-1})^\perp$ relative to $\textbf{kernel}(N^j)$
- $\ell(j) = \dim(L_j)$
- p minimizes $\textbf{kernel}(N^p) = \textbf{kernel}(N^{p+1})$

19. Verify $k(j) \le k(j+1)$ from

$$\textbf{kernel}(N^j) \subset \textbf{kernel}(N^{j+1}).$$

Solution: Given $\textbf{kernel}(N^j) \subset \textbf{kernel}(N^{j+1})$ then the number of basis elements for subspace $\textbf{kernel}(N^j)$ is less than or equal to the number of basis elements for the containing subspace $\textbf{kernel}(N^{j+1})$. Therefore the dimensions of the two subspaces satisfy the inequality $k(j) \le k(j+1)$. ■

20. Verify the direct sum formula

$$\textbf{kernel}(N^j) = \textbf{kernel}(N^{j-1}) \oplus L_j.$$

Then $k(j) = k(j-1) + \ell(j)$.

Solution:
Symbol definition: $k(j) = \dim(\textbf{kernel}(N^j))$, $\ell(j) = \dim(L_j) = \dim(\textbf{kernel}(N^{j-1})^\perp$. Let's derive equation $k(j) = k(j-1) + \ell(j)$.

$$\begin{aligned}
k(j) &= \dim(\textbf{kernel}(N^j)) \\
&= \dim\left(\textbf{kernel}(N^{j-1})\right) \oplus \textbf{kernel}\left((N^{j-1})^{\perp}\right) \text{ by Exercise 20} \\
&= \dim\left(\textbf{kernel}(N^{j-1})\right) + \dim\left(\textbf{kernel}(N^{j-1})^{\perp})\right) \\
&= k(j-1) + \ell(j)
\end{aligned}$$

Other details of Exercise 20 are omitted.

21. Given $N^m\vec{\mathbf{w}} = \vec{\mathbf{0}}$, $N^{m-1}\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, define $\vec{\mathbf{v}}_i = N^{m-i}\vec{\mathbf{w}}$, $i = 1, \ldots, m$. Prove $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_m\}$ is independent and they satisfy Jordan chain relations $N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$, $N\vec{\mathbf{v}}_{i+i} = \vec{\mathbf{v}}_i$.

Solution:
Independence:
Assume $\sum_{i=1}^m c_1\vec{\mathbf{v}}_i = \vec{\mathbf{0}}$. We prove the weights are zero. Replace $\vec{\mathbf{v}}_i = N^{m-i}\vec{\mathbf{w}}$ then multiply N^{m-1} across the equation:

$\sum_{i=1}^m c_1 N^{m-1}\left(N^{m-i}\vec{\mathbf{w}}\right) = \vec{\mathbf{0}}$.

All terms are zero except for $i = m$ because $N^{m-1}\left(N^{m-i}\vec{\mathbf{w}}\right)\Big|_{i=m} = N^{m-1}\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ while all preceding terms contain factor $N^{m-1}N\vec{\mathbf{w}} = \vec{\mathbf{0}}$. The result is equation

$c_m N^{m-1}\vec{\mathbf{w}} = \vec{\mathbf{0}}$

from which we conclude $c_m = 0$. The argument repeats: multiply next by N^{m-2} and distill the equation to one term, showing $c_{m-1} = 0$. By induction all the weights are zero and the vectors are independent.

Chain Relations:
First, $N\vec{\mathbf{v}}_1 = N\,N^{m-1}\vec{\mathbf{w}} = N^m\vec{\mathbf{w}} = \vec{\mathbf{0}}$, so $\vec{\mathbf{v}}_1$ is an eigenvector. Next, $N\vec{\mathbf{v}}_{i+i} = NN^{m-i-1}\vec{\mathbf{w}} = N^{m-1}\vec{\mathbf{w}} = \vec{\mathbf{v}}_i$. ■

22. A block $B(\lambda, p)$ corresponds to a Jordan chain $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ constructed from the Jordan decomposition. Use $N^{p-1}\vec{\mathbf{v}}_p = \vec{\mathbf{v}}_1$ and $\textbf{kernel}(N^p) = \textbf{kernel}(N^{p+1})$ to show that the number of such blocks $B(\lambda, p)$ is $\ell(p)$. Then for $p > 1$, $s(p) = k(p) - k(p-1)$.

Solution: Some of the details can be found in the solution to Exercise 23 *infra*.

23. Show that $\ell(j-1) - \ell(j)$ is the number of blocks $B(\lambda, j)$ for $2 < j < p$. Then

$$s(j) = 2k(j-1) - k(j) - k(j-2).$$

Solution:
Part I.
Prove $s(j) = 2k(j-1) - k(j) - k(j-2)$, $j > 2$.

A Jordan block $B(\lambda, m)$ appearing in J is paired with a Jordan chain $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_m$ consisting of the matching columns in P. The first chain vector $\vec{\mathbf{v}}_1$ is an eigenvector.

Let W be the set of eigenvectors $\vec{\mathbf{v}}_1$ found from P, considering all Jordan blocks in J. The columns of P are independent, so W is an independent set (subsets of independent sets are independent). The eigenvectors in W form a basis for the kernel of $N = A - \lambda I$, but this basis is different from a standard basis found by solving equation $N\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

Summary: The dimension of the eigenspace $\mathbf{kernel}(A - \lambda I)$ tells you the exact number of Jordan blocks with λ on the diagonal. It tells you nothing about the sizes of these blocks.

It may help to have an example in mind when reading the rest of the Part I proof. Suggestion: compute basis vectors for the nullspaces of N, N^3, N^3 using the matrix A of example 11.12 page 900.

$$A = \begin{pmatrix} 3 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ -3 & 3 & 0 & -2 & 3 \end{pmatrix}$$

Fix eigenvalue λ and let $N = A - \lambda I$. If nilpotency $p = 1$ then every Jordan block has size 1, so assume $p > 1$. The subspaces $\mathbf{kernel}(N^j)$ grow in dimension as j increases. Let $\mathbf{kernel}(N) = \mathbf{span}(\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_r)$ and extend the basis to $\mathbf{kernel}(N^2)$, by adding basis vectors $\vec{\mathbf{z}}_1, \ldots, \vec{\mathbf{z}}_q$ for $\mathbf{kernel}(N)^\perp$ (used $\mathbf{kernel}(N^2) = \mathbf{kernel}(N) \oplus \mathbf{kernel}(N)^\perp$). Then $\mathbf{kernel}(N^2) = \mathbf{span}(\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_r, \vec{\mathbf{z}}_1, \ldots, \vec{\mathbf{z}}_q)$. The number q of basis vectors added to obtain $\mathbf{kernel}(N^2)$ is $q = \ell(2)$. Also, $q = \dim(\mathbf{kernel}(N^2)) - \dim(\mathbf{kernel}(N)) = k(2) - k(1)$. Number q is the count of Jordan blocks of size greater than 1, because each such block is paired with an m-chain that has a vector in $\mathbf{kernel}(N^2)$, but not in $\mathbf{kernel}(N)$. Independence of chain vectors is a key part of this argument. Conclusion: $\ell(2) = k(2) - k(1)$ is the number of Jordan blocks of size greater than 1. Then the number of Jordan blocks of exactly size 1 is $k(1) - (k(2) - k(1)) = 2k(1) - k(2)$.

In general, the number of Jordan blocks of size greater than k is $\ell(j+1) = \dim(\mathbf{kernel}(N^j)^\perp) = \dim(\mathbf{kernel}(N^{j+1})) - \dim(\mathbf{kernel}(N^j)) = k(j+1) - k(j)$. The logic used above applies: the number $s(j+1)$ of Jordan blocks of size exactly $j+1$ equals $k(j) - k(j-1) - (k(j+1) - k(j)) = 2k(j) - k(j+1) - k(j-1)$.

Replace j by $j-1$ to obtain the claimed identity

$s(j) = 2k(j-1) - k(j) - k(j-2), \quad j > 2$

Part II.

Show $\ell(j-1) - \ell(j)$ is the number of blocks $B(\lambda, j)$ for $2 < j < p$.

Direct sum decomposition $\mathbf{kernel}(N^j) = \mathbf{kernel}(N^{j-1}) \oplus \mathbf{kernel}(N^{j-1})^\perp$ implies identity $k(j) = k(j-1) + l(j)$. Then $\ell(j-1) - \ell(j) = k(j-1) - k(j-2) - (k(j) - k(j-1)) = 2k(j-1) - k(j-2) - k(j)$, which is the number of blocks $B(\lambda, j)$ for $2 < j < p$ by **Part I**. ■

24. Test the formulas above on the special matrices

$$A = \textbf{diag}(B(\lambda,1), B(\lambda,1), B(\lambda,1)),$$

$$A = \textbf{diag}(B(\lambda,1), B(\lambda,2), B(\lambda,3)),$$

$$A = \textbf{diag}(B(\lambda,1), B(\lambda,3), B(\lambda,3)),$$

$$A = \textbf{diag}(B(\lambda,1), B(\lambda,1), B(\lambda,3)),$$

Computing Jordan m-chains. Find the Jordan m-chain formulas for the given eigenvalue. Then solve them to find the generalized eigenvectors.

25. $A = \begin{pmatrix} 1&0&1&0&1 \\ 0&1&0&1&1 \\ 0&0&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{pmatrix}$

Solution: The eigenvalues: $1, 1, 1, 1, 1$. Let $\lambda = 1$, $N = A - (1)I$. Use `maple` (code *infra*) to decide on the number of Jordan blocks:

```
lambda=1, nilpotency=p=3
Blocks B(1,3):1
Blocks B(1,2):1
```

Find a 3-chain.
Compute a basis for the nullspace of $(N^2)^T$: columns 1,3,4,5 of the identity matrix I. Let $\vec{\mathbf{w}}$ be a linear combination of the basis vectors with weights c_1, c_2, c_3, c_4. Compute the expected chain $N^2\vec{\mathbf{w}}$, $N\vec{\mathbf{w}}$, $\vec{\mathbf{w}}$ to find a choice for the weights that makes $N^2\vec{\mathbf{w}}$, $N\vec{\mathbf{w}}$, $\vec{\mathbf{w}}$ a 3-chain: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 1$. Then a 3-chain is

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 0\\1\\0\\0\\0 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} 1\\1\\0\\1\\0 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix}.$$

The common shortcut of choosing an eigenvector to start a 3-chain fails in this example. This has to be frustrating, given the common advice in textbooks.

Find a 2-chain.
Compute a basis for the nullspace of N: columns 3, 5 of I. Write

$$\vec{\mathbf{w}} = \begin{pmatrix} 0 \\ 0 \\ c_1 \\ 0 \\ c_2 \end{pmatrix},$$

the plan being to find the weights c_1, c_2 so that $\vec{\mathbf{v}}_4 = N\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_5 = \vec{\mathbf{w}}$ form a 2-chain: $N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$, $N\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$. Solving, $c_1 = 1$, $c_2 = 0$ works and

$$\vec{\mathbf{v}}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$P = \begin{pmatrix} 0 & 1 & 1 & 7 & 1 \\ -2 & 2 & 0 & 7 & -1 \\ -2 & 1 & 0 & 0 & 5 \\ -2 & -1 & 0 & 0 & -2 \\ 2 & -3 & 0 & 0 & 2 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

```
# Exercise 25, computing Jordan m-chains
with(LinearAlgebra):
A:=Matrix([[ 1,0,1,0,1],[0,1,0,1,1],[0,0,1,0,0],
          [0,0,0,1,1],[0,0,0,0,1]]);
Eigenvectors(A);
getBlockCounts(A,1);N:=A-1;
# 3-chain
w:=<c1,0,c2,c3,c4>;N^2 .w,N.w,w;
w:=subs(c1=0,c2=0,c3=0,c4=1,<c1,0,c2,c3,c4>);N^2 .w,N.w,w;
NullSpace(N^+);
w:=<0,0,c1,0,c2>;N.w,w;
w:=subs(c1=1,c2=0,<0,0,c1,0,c2>);N.w,w;
P:=<v1|v2|v3|v4|v5>;
J:=JordanBlockMatrix([[1,3],[1,2]]);
```

26. $A = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$, $\lambda = 2$

Solution: There is a 4-chain and a 1-chain. For the 4-chain choose $\vec{\mathbf{w}}$ to be the first column of $(N^3)^T$. This shortcut works to find a 4-chain. The 1-chain starts with an eigenvector independent from the one used in the

4-chain. One possible answer:

$$P = \begin{pmatrix} 0&1&0&0&1\\ 1&0&0&0&0\\ 1&0&0&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{pmatrix}, \quad J = \begin{pmatrix} 2&1&0&0&0\\ 0&2&1&0&0\\ 0&0&2&1&0\\ 0&0&0&2&0\\ 0&0&0&0&2 \end{pmatrix}.$$

Generalized Eigenspace Basis.

Let A be $n \times n$ with distinct eigenvalues λ_i, $n_i = \textbf{AlgMult}(\lambda_i)$ and $E_i = \mathbf{kernel}((A-\lambda_i I)^{n_i})$, $i = 1, \ldots, k$. Assume a Jordan decomposition $A = PJP^{-1}$.

27. Let Jordan block $B(\lambda, m)$ appear in J. Prove that a Jordan chain corresponding to this block is a set of m independent columns of P.

Solution: The columns of P are independent because P is invertible. Subsets of independent sets are independent, therefore the Jordan chain columns isolated from P are independent. By swapping blocks in J and corresponding columns in P we can assume that block $B(\lambda, m)$ occupies columns 1 to m of J. Matrix products AP and PJ will be expanded as follows:

$$PJ = \left\langle \lambda\,\mathbf{col}(P,1) + \mathbf{col}(P,2) | \lambda\,\mathbf{col}(P,2) + \mathbf{col}(P,3) | \cdots | \lambda\,\mathbf{col}(P,m) | \cdots \right\rangle$$

$$AP = \left\langle A\,\mathbf{col}(P,1) | \cdots | A\,\mathbf{col}(P,m) | \cdots \right\rangle$$

Match the first m columns:

$$\begin{array}{rcl} \lambda\,\mathbf{col}(P(A,1) + \mathbf{col}(P,2) & = & A\,\mathbf{col}(P,1) \\ \lambda\,\mathbf{col}(P(A,2) + \mathbf{col}(P,3) & = & A\,\mathbf{col}(P,2) \\ & \vdots & \\ \lambda\,\mathbf{col}(P(A,m) & = & A\,\mathbf{col}(P,m) \end{array}$$

Define

$\vec{\mathbf{v}}_m = \mathbf{col}(P,1)$, ..., $\vec{\mathbf{v}}_1 = \mathbf{col}(P,m)$.

Write

$\lambda\,\mathbf{col}(P(A,j) + \mathbf{col}(P,j+1) = A\,\mathbf{col}(P,j)$ as $\mathbf{col}(P,j+1) = N\,\mathbf{col}(P,j)$.

Then

$$\begin{array}{rcl} \mathbf{col}(P,2) & = & N\,\mathbf{col}(P,1) \\ \mathbf{col}(P,3) & = & N\,\mathbf{col}(P,2) \\ & \vdots & \\ \vec{\mathbf{0}} & = & N\,\mathbf{col}(P,m) \end{array}$$

and

$$\begin{aligned}
\vec{\mathbf{v}}_{m-1} &= N\vec{\mathbf{v}}_m \\
\vec{\mathbf{v}}_{m-2} &= N\vec{\mathbf{v}}_{m-1} \\
&\vdots \\
\vec{\mathbf{0}} &= N\vec{\mathbf{v}}_1
\end{aligned}$$

These are the Jordan Chain Relations in reverse order. ■

28. Let $\mathcal{B}_\lambda$ be the union of all columns of P originating from Jordan chains associated with Jordan blocks $B(\lambda, j)$. Prove that $\mathcal{B}_\lambda$ is an independent set.

29. Verify that $\mathcal{B}_\lambda$ has **AlgMult**(λ) basis elements.

Solution: There are j columns of P in $\mathcal{B}_\lambda$ from block $B(\lambda, j)$. The block has λ on the diagonal exactly j times. So λ is a repeated eigenvalue of A, j repeats counted from the block. The algebraic multiplicity of λ is the number of times λ is a repeated eigenvalue. Adding the repeats j block-by-block has to add to the algebraic multiplicity. ■

30. Prove that $E_i = \mathbf{span}\,(\mathcal{B}_{\lambda_i})$ and $\dim(E_i) = n_i$, $i = 1, \ldots, k$.

Direct Sum Decomposition.

31. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Let $\lambda = 2$. Compute $k =$ **AlgMult**(λ) and a basis of generalized eigenvectors for the subspace $\mathbf{kernel}((A - \lambda I)^k)$.

Solution: Matrix A is a Jordan block $B(2, 3)$. The algebraic multiplicity of $\lambda = 2$ is power of factor $(\lambda - 2)$ in the characteristic polynomial $(2 - \lambda)^3$. Then $k = 3$. The nullspace of $(A-2I)^3$ is $\mathcal{R}^3$. There is exactly one eigenpair: $\left(2, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$.

Because $J = A$ then $P = I$, because $AP = PJ$. The columns of I are a basis for the generalized eigenspace. ■

32. Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$, $\vec{\mathbf{y}} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 9 \end{pmatrix}$.

Find $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2$ in decomposition $\vec{\mathbf{y}} = \vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2$ in Theorem 11.42.

Solution: It suffices to find $x_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, then define $\vec{\mathbf{x}}_2 = \vec{\mathbf{y}} - \vec{\mathbf{x}}_1$.

Exponential Matrices. Compute the exponential matrix e^{At} on paper. Check the answer using `maple`.

33. $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: Using the results for diagonal matrices, $e^{At} = \mathbf{diag}(e^{2t}, e^{3t}, 1)$.

34. $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Nilpotent matrices. Find the nilpotency of N.

35. $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: Nilpotency $p = 2$.

36. $N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: Nilpotency $p = 5$.

Real Jordan Decomposition

Find Jordan decomposition $A = PJP^{-1}$ where J and P are real matrices.

37. $A = \begin{pmatrix} -2 & 6 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$. Answer:

$\lambda = -2, 4 \pm i$,

$J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 4 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Solution: Matrix A is diagonalizable with a full set of complex eigenvectors:

$$\left(-2, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(4+i, \begin{pmatrix} i \\ i \\ 1 \end{pmatrix}\right), \quad \left(4-i, \begin{pmatrix} -i \\ -i \\ 1 \end{pmatrix}\right)$$

The Jordan Form is obtained by replacing the conjugate pair $4+i$, $4-i$ by matrix $\begin{pmatrix} 4 & 1 \\ -1 & 4 \end{pmatrix}$:

$$J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 4 \end{pmatrix}$$

Matrix P is obtained from the eigenvectors by replacing the two complex eigenvectors respectively by the real and imaginary parts of the first eigenvector (the second eigenvector is the conjugate of the first eigenvector).

Replace in the complex Jordan matrix P:

Pair $\begin{pmatrix} i \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ -i \\ 1 \end{pmatrix}$ is replaced by pair $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

The real and imaginary parts apply only to the first complex eigenvector, the second complex conjugate eigenvector is not used!

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad AP - PJ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

How to find the real and imaginary parts of a vector: replace complex unit i by 0 to find the real part, take the derivative on symbol i to find the imaginary part.

38. $A = \begin{pmatrix} -31 & -10 & 18 \\ -15 & -5 & 10 \\ -54 & -20 & 32 \end{pmatrix}$. Answer:

$\lambda = -4, \pm 5i$

$J = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -5 & 0 \end{pmatrix}, P = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & -1 \\ 3 & 4 & 0 \end{pmatrix}$

Solving $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Solve for $\vec{\mathbf{x}}$ in the differential equation.

39. $\vec{\mathbf{x}}' = \begin{pmatrix} -2 & 6 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \vec{\mathbf{x}}$.

Solution: By Exercise 37, the real Jordan decomposition is

$$J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

Then $AP = PJ$ implies from $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ the new equation $\vec{\mathbf{y}}' = J\vec{\mathbf{y}}$ where $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$. Let's solve $\vec{\mathbf{y}}' = J\vec{\mathbf{y}}$ in its scalar form

$$\begin{cases} y_1' &= -2y_1, \\ y_2' &= 4y_2 + y_3, \\ y_3' &= -y_2 + 4y_3 \end{cases}$$

First, $\vec{\mathbf{y}}_1 = c_1 e^{-2t}$. The last two differential equations are solved by the Cayley-Hamilton-Ziebur scalar 2×2 shortcut, Theorem 11.35 page 883.

$y_2 = c_2 e^{4t}\cos(t) + c_3 e^{4t}\sin(t)$
$y_3 = y_2' - 4y_2 = -c_2 e^{4t}\sin(t) + c_3 e^{4t}\cos(t)$

Then

$$\begin{aligned}\vec{\mathbf{x}} &= P\vec{\mathbf{y}} \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= P\vec{\mathbf{y}} \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{4t}\cos(t) + c_3 e^{4t}\sin(t) \\ -c_2 e^{4t}\sin(t) + c_3 e^{4t}\cos(t) \end{pmatrix} \quad \blacksquare\end{aligned}$$

40. $\vec{\mathbf{x}}' = \begin{pmatrix} -31 & -10 & 18 \\ -15 & -5 & 10 \\ -54 & -20 & 32 \end{pmatrix} \vec{\mathbf{x}}$.

Numerical Instability

Show directly that Jordan form J of A satisfies $\lim_{\epsilon \to 0+} J(\epsilon) \neq J(0)$.

41. $A = \begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix}$

Solution: Matrix A has for $\epsilon > 0$ two eigenpairs and Jordan form $J(\epsilon) = \mathbf{diag}(1+\sqrt{\epsilon}, 1-\sqrt{\epsilon})$. The limit of $J(\epsilon)$ as $\epsilon \to 0$ is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

However, $J(0)$ is the Jordan matrix for $A|_{\epsilon=0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which is itself. $\blacksquare$

42. $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & \epsilon & 1 \\ 0 & 0 & 0 \end{pmatrix}$

11.7 Nonhomogeneous Linear Systems

Variation of Parameters

Let $A(t) = \begin{pmatrix} 0 & 1 \\ -c(t)/a(t) & -b(t)/a(t) \end{pmatrix}$,
$\vec{\mathbf{F}}(t) = \dfrac{1}{a(t)} \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$, $\vec{\mathbf{x}} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$.

1. Verify equivalence of $a(t)u'' + b(t)u' + c(t)u = f(t)$ and $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$.

Solution:
Scalar implies vector-matrix:
Assume $a(t)u'' + b(t)u' + c(t)u = f(t)$, prove $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$.

$$\begin{aligned}
\vec{\mathbf{x}}' &= \tfrac{d}{dt}\begin{pmatrix} u \\ u' \end{pmatrix} \\
&= \begin{pmatrix} u' \\ u'' \end{pmatrix} \\
&= \begin{pmatrix} u' \\ (-bu' - cu + f)/a \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} + \begin{pmatrix} 0 \\ f/a \end{pmatrix} \\
&= A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)
\end{aligned}$$

Vector-matrix implies scalar:
Assume $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$, prove $a(t)u'' + b(t)u' + c(t)u = f(t)$.

Let $u(t) = x_1(t)$ where $x_1(t)$ is the first component of $\vec{\mathbf{x}}(t)$. Convert $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ to scalar form:

$$\begin{cases} x_1' &= x_2(t), \\ x_2' &= -cx_1/a - bx_2/a + f/a \end{cases}$$

The second scalar equation becomes

$x_2' = -cx_1/a - bx_2/a + f/a$

$x_1'' = -cx_1/a - bx_1'/a + f/a$ replace x_2 by x_1'

$u'' = -cu/a - bu'/a + f/a$ replace x_1 by u, x_1' by u'

$u'' + cu/a + bu'/a = f/a$ collect terms to the LHS

$au'' + cu + bu' = f$ multiply by $a(t)$ ■

2. For $u'' + 100u = \sin(t)$, find $A(t)$ and $\vec{\mathbf{F}}(t)$.

3. For $u'' = f(t)$, find $A(t)$ and $\vec{\mathbf{F}}(t)$.

Solution: Answer: $A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{\mathbf{F}}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$.

4. For $u'' = f(t)$, let $u_1 = 1$, $u_2 = t$, $\Phi(t) = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}$. Verify $|\Phi(t)| = 1$, then find $A(t) = \Phi'(t)\Phi^{-1}(t)$.

5. State Theorem 11.46 for $n = 2$, then explain how it applies to this special case.

Solution:
Theorem (Variation of Parameters: General Linear System)
Let $A(t)$ be a 2×2 matrix and $\vec{\mathbf{F}}(t)$ a vector function, both with continuous entries near $t = t_0$. Let $\Phi(t)$ be the 2×2 matrix solution of $\Phi'(t) = A(t)\Phi(t)$, $\Phi(t_0) = I$, established by the Picard-Lindelöf Theorem.
Then the unique solution $\vec{\mathbf{x}}(t)$ of the matrix initial value problem

$$\vec{\mathbf{x}}'(t) = A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$$

is given by

$$\vec{\mathbf{x}}(t) = \Phi(t)\vec{\mathbf{x}}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\vec{\mathbf{F}}(s)ds. \tag{1}$$

How it applies.
Matrix $A(t) = \begin{pmatrix} 0 & 1 \\ -c(t)/a(t) & -b(t)/a(t) \end{pmatrix}$ and column vector $\vec{\mathbf{F}}(t) = \dfrac{1}{a(t)}\begin{pmatrix} 0 \\ f(t) \end{pmatrix}$ have continuous entries, therefore both $A(t)$ and $\vec{\mathbf{F}}(t)$ are continuous near $t = t_0$. The theorem applies.

6. Prove Theorem 11.47 using the previous exercise.

Variation of Parameters: Scalar 2nd Order

Let $a(t)u'' + b(t)u' + c(t)u = 0$ have two independent solutions u_1, u_2. Define $\Psi(t) = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}$. Then:

7. Matrix $\Psi(t)$ has an inverse.

Solution: Independence means the Wronskian determinant does not vanish, which is $|\Psi(t|$, then $\Psi(t)$ is invertible.

8. Matrix $\Phi(t) = \Psi(t)\Psi^{-1}(t_0)$ is invertible and $\Phi(t_0) = I$.

9. Let $\Psi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Define

$\begin{pmatrix} u \\ v \end{pmatrix} = \Psi(t) \int_0^t \Psi^{-1}(s) f(s) ds.$

Then u is a particular solution of $u'' = f(t)$.

Solution: Combine Exercise 3 and Exercise 5.

10. Let $\Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$. Define

$\begin{pmatrix} u \\ v \end{pmatrix} = \Psi(t) \int_0^t \Psi^{-1}(s) f(s) ds.$

Then u is a particular solution of $u'' - u = f(t)$.

Variation of Parameters

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ using $\vec{\mathbf{x}}_p = \int_0^t e^{A(t-s)} \vec{\mathbf{F}}(s) ds$ and computer assist.

11. $\vec{\mathbf{F}}(t) = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{\mathbf{x}}_p = \begin{pmatrix} e^{2t} - e^t \\ e^{3t} - e^t \end{pmatrix}$

Solution:
```
# Exercise 11, Variation of Parameters
F:=t -> exp(t)*<1,2>;
A:=Matrix([[2,0],[0,3]]);
Phi:=s -> MatrixExponential(A,s);
map(int,Phi(t-s).F(s),s=0..t);
```

12. $\vec{\mathbf{F}}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$,

$\vec{\mathbf{x}}_p = \begin{pmatrix} e^{2t} - e^t \\ \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} \end{pmatrix}$

Undetermined Coefficients

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ by undetermined coefficients. Assume

$\vec{\mathbf{x}}_h(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

13. $\vec{\mathbf{F}}(t) = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

$\vec{\mathbf{x}}_p = \begin{pmatrix} e^{-t} + 3te^t - e^t \\ e^t - e^{-t} \end{pmatrix}$

Solution: The initial trial solution: $\vec{\mathbf{x}} = e^t \vec{\mathbf{c}}$. Substitute to get equation $e^t \vec{\mathbf{c}} = e^t A \vec{\mathbf{c}} + e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then cancel e^t and try to solve for $\vec{\mathbf{c}}$:

$$\vec{\mathbf{c}} = A\vec{\mathbf{c}} + \begin{pmatrix}1\\2\end{pmatrix}$$

$$(I - A)\vec{\mathbf{c}} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}0 & -2\\0 & -2\end{pmatrix}\vec{\mathbf{c}} = \begin{pmatrix}1\\2\end{pmatrix}$$

It failed. The trial solution must be modified.

Second attempt, trial solution

$$\vec{\mathbf{x}} = e^t\vec{\mathbf{c}}_1 + te^t\vec{\mathbf{c}}_2$$

$$\vec{\mathbf{x}} = e^t\begin{pmatrix}d_1\\d_2\end{pmatrix} + te^t\begin{pmatrix}d_3\\d_4\end{pmatrix}$$

Substitute into $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}$:

$$e^t\vec{\mathbf{c}}_1 + te^t\vec{\mathbf{c}}_2 + e^t\vec{\mathbf{c}}_2 = e^tA\vec{\mathbf{c}}_1 + te^tA\vec{\mathbf{c}}_2 + e^t\begin{pmatrix}1\\2\end{pmatrix}$$

Cancel e^t and match coefficients (method of atoms):

$$\vec{\mathbf{c}}_1 + t\vec{\mathbf{c}}_2 + \vec{\mathbf{c}}_2 = A\vec{\mathbf{c}}_1 + tA\vec{\mathbf{c}}_2 + \begin{pmatrix}1\\2\end{pmatrix}$$

$$\vec{\mathbf{c}}_1 + \vec{\mathbf{c}}_2 = A\vec{\mathbf{c}}_1 + \begin{pmatrix}1\\2\end{pmatrix} \quad , \vec{\mathbf{c}}_2 = A\vec{\mathbf{c}}_2$$

Solve for $\vec{\mathbf{c}}_2 = \begin{pmatrix}k\\0\end{pmatrix}$ and insert into the first vector equation:

$$\vec{\mathbf{c}}_1 + \begin{pmatrix}k\\0\end{pmatrix} = A\vec{\mathbf{c}}_1 + \begin{pmatrix}1\\2\end{pmatrix} \quad \text{, where } k \text{ is to be determined.}$$

$$(I - A)\vec{\mathbf{c}}_1 = \begin{pmatrix}1-k\\2\end{pmatrix}$$

$$\begin{pmatrix}0 & -2\\0 & 2\end{pmatrix}\vec{\mathbf{c}}_1 = \begin{pmatrix}1-k\\2\end{pmatrix} \quad , \vec{\mathbf{c}}_2 = \begin{pmatrix}k\\0\end{pmatrix}$$

Choose $k = 3$ to find solutions $\vec{\mathbf{c}}_1 = \begin{pmatrix}0\\1\end{pmatrix}$, $\vec{\mathbf{c}}_2 = \begin{pmatrix}3\\0\end{pmatrix}$. Then

$$\vec{\mathbf{x}} = e^t\vec{\mathbf{c}}_1 + te^t\vec{\mathbf{c}}_2 = \begin{pmatrix}3te^t\\e^t\end{pmatrix}$$

```
# Exercise 13, Undetermined Coefficients
F:=t->exp(t)*<1,2>;
A:=Matrix([[1,2],[0,-1]]);
# Undetermined coefficients
trial:=exp(t)*<d1,d2> + t*exp(t)*<d3,d4>;
p:=map(diff,trial,t)-F(t)-A.trial;# Should equal zero
 # Tools to match coefficients of atoms
 # p:=simplify((1/exp(t))*p);# Cancel exp(t)
 # q:=map(PolynomialTools[CoefficientList],p,t);convert(q,list);
solve([d3-1-2*d2, -2*d4, 2*d2+d4-2, 2*d4],{d1,d2,d3,d4});
 # d1 = d1, d2 = 1, d3 = 3, d4 = 0 (let d1=0)
X:=exp(t)*<0,1> + t*exp(t)*<3,0>;# particular solution
map(diff,X,t)=A.X+F(t);# Check the answer
# Answer check scalar methods
des:=diff(u1(t),t)=u1(t)+2*u2(t)+exp(t),
     diff(u2(t),t)= -u2(t)+2*exp(t);
dsolve({des},[u1(t),u2(t)]);
```

14. $\vec{\mathbf{F}}(t) = 2\begin{pmatrix}\cos t\\ e^t\end{pmatrix}$,

$\vec{\mathbf{x}}_p = \begin{pmatrix}2te^t + \sin(t) - \cos(t) + e^{-t}\\ e^t - e^{-t}\end{pmatrix}$

Undetermined Coefficients

Let $A = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ by undetermined coefficients. Assume $\vec{\mathbf{x}}_h(t) = \begin{pmatrix}c_1e^{2t}\\ c_2e^{3t}\end{pmatrix}$.

15. $\vec{\mathbf{F}}(t) = e^t\begin{pmatrix}1\\ 2\end{pmatrix}$, $\vec{\mathbf{x}}_p = e^t\begin{pmatrix}-1\\ -1\end{pmatrix}$

Solution: Trial solution $\vec{\mathbf{x}} = e^t\vec{\mathbf{c}}$. Follow Exercise 13.

16. $\vec{\mathbf{F}}(t) = 4\begin{pmatrix}e^t\\ e^{-t}\end{pmatrix}$, $\vec{\mathbf{x}}_p = e^{-t}\begin{pmatrix}-4\\ -1\end{pmatrix}$

17. $\vec{\mathbf{F}}(t) = 10\begin{pmatrix}\cos t\\ e^t\end{pmatrix}$,

$\vec{\mathbf{x}}_p = \begin{pmatrix}-4\cos(t) + 2\sin(t)\\ -5e^t\end{pmatrix}$

Solution: Trial solution $\vec{\mathbf{x}} = \cos(t)\vec{\mathbf{c}}_1 + \sin(t)\vec{\mathbf{c}}_2 + e^t\vec{\mathbf{c}}_3$, because the atoms for $\vec{\mathbf{F}}$ are $\cos t$, $\sin t$, e^t. Follow Exercise 13.

18. $\vec{\mathbf{F}}(t) = 2e^t \begin{pmatrix} \cos t \\ 1 \end{pmatrix}$,

$\vec{\mathbf{x}}_p = e^t \begin{pmatrix} -\cos(t) + \sin(t) \\ -1 \end{pmatrix}$

11.8 Second Order Systems

Euler's Substitution: $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$

1. Change variables: $\vec{\mathbf{u}} = e^{rt}\vec{\mathbf{w}}$. Answer: $\vec{\mathbf{w}}' = (C - rI)\vec{\mathbf{w}}$

Solution: Differentiate the change of variable equation:

$\vec{\mathbf{u}}' = re^{rt}\vec{\mathbf{w}} + e^{rt}\vec{\mathbf{w}}'$ by the product rule

Then $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ becomes

$re^{rt}\vec{\mathbf{w}} + e^{rt}\vec{\mathbf{w}}' = e^{rt}C\vec{\mathbf{w}}$

$r\vec{\mathbf{w}} + \vec{\mathbf{w}}' = C\vec{\mathbf{w}}$ divide by e^{rt}

Rearrange the equation

$\vec{\mathbf{w}}' = (C - rI)\vec{\mathbf{w}}$ ■

2. Prove: $(\lambda, \vec{\mathbf{v}})$ is an eigenpair of C if and only if $(0, \vec{\mathbf{v}})$ is an eigenpair of $C - \lambda I$.

3. Let $|C - \lambda I|$ have factor λ^2. Let $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ have solution $\vec{\mathbf{u}} = \vec{\mathbf{d}}_1 + t\vec{\mathbf{d}}_2$. Prove: $C\vec{\mathbf{d}}_2 = \vec{\mathbf{0}}$, $C\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_2$. Are $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ eigenvectors of C? Discuss.

Solution:
Substitute $\vec{\mathbf{u}} = \vec{\mathbf{d}}_1 + t\vec{\mathbf{d}}_2$ into the differential equation $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$:

$\vec{\mathbf{d}}_2 = C\vec{\mathbf{d}}_1 + tC\vec{\mathbf{d}}_2$

Match vector coefficients of the Euler solution atoms 1, t:

$\vec{\mathbf{d}}_2 = C\vec{\mathbf{d}}_1$ and $\vec{\mathbf{0}} = C\vec{\mathbf{d}}_2$

Vector $\vec{\mathbf{d}}_2$ is an eigenvector if not zero, because zero is an eigenvalue of C. Vector $\vec{\mathbf{d}}_1$ is computed from C and $\vec{\mathbf{d}}_2$ with combo, swap and mult operations. There is no reason to think $\vec{\mathbf{d}}_1$ is an eigenvector of C. ■

4. Let $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{\mathbf{u}} = \vec{\mathbf{d}}_1 + t\vec{\mathbf{d}}_2$. Let $\vec{\mathbf{u}}$ solve $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$. Find $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ in terms of arbitrary constants c_1, c_2.

Euler's Substitution: $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$

5. Change variables: $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{y}}$. Answer: $\vec{\mathbf{y}}'' + 2r\vec{\mathbf{y}}' = (A - r^2I)\vec{\mathbf{y}}$

Solution:
Differentiate the change of variable equation twice:

$\vec{\mathbf{x}}' = re^{rt}\vec{\mathbf{y}} + e^{rt}\vec{\mathbf{y}}'$.

$\vec{\mathbf{x}}'' = r^2e^{rt}\vec{\mathbf{y}} + 2re^{rt}\vec{\mathbf{y}}' + e^{rt}\vec{\mathbf{y}}''$.

Substitute into $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$:

$r^2 e^{rt}\vec{\mathbf{y}} + 2re^{rt}\vec{\mathbf{y}}\,' + e^{rt}\vec{\mathbf{y}}\,'' = e^{rt}A\vec{\mathbf{y}}$

Cancel e^{rt}:

$r^2\vec{\mathbf{y}} + 2r\vec{\mathbf{y}}\,' + \vec{\mathbf{y}}\,'' = A\vec{\mathbf{y}}$

Re-arrange the terms:

$\vec{\mathbf{y}}\,'' + 2r\vec{\mathbf{y}}\,' = (A - r^2 I)\vec{\mathbf{y}}$

6. Prove: $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{v}}$ is a nonzero solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ if and only if $(r^2, \vec{\mathbf{v}})$ is an eigenpair of A.

Solution: Suppose $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{v}}$ is a nonzero solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$. Apply the previous exercise with $\vec{\mathbf{y}} = \text{constant} = \vec{\mathbf{v}}$. The left side is zero: $\vec{\mathbf{0}} = (A - r^2 I)\vec{\mathbf{v}}$. Then $\vec{\mathbf{v}}$ is an eigenvector of A for eigenvalue $\lambda = r^2$. The second half of the proof is omitted.

Repeated Root: $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, eigenvalues $0, 0$.

7. Verify: Matrix A is a Jordan block with generalized eigenvectors the columns of I.

Solution: Let $J = A$, which is a Jordan block $B(\lambda, 2) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda = 0$. Let $P = I$. Then $AP = AI = A = J = IJ = PJ$, so the columns of P are generalized eigenvectors (A is not diagonalizable). ■

8. Prove: $x_1 = c_1 + c_2 t + c_3 \dfrac{t^2}{2} + c_4 \dfrac{t^3}{6}$, $x_2 = c_3 + c_4 t$ for arbitrary constants c_1 to c_4.

9. Prove: The solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ is a vector linear combination of atoms $1, t, t^2, t^3$.

Solution: The scalar equations are

$x_1'' = x_2$, $x_2'' = 0$. Then $x_2 = c_1 + c_2 t$ and $x_1 = \int(\int(c_1 + c_2 t)dt)dt = \int(c_1 t + c_2 t^2/2 + c_3)dt = c_1 t^2/2 + c_2 t^3/6 + c_3 t + c_4$. Therefore,

$$\vec{\mathbf{x}} = \begin{pmatrix} c_1 t^2/2 + c_2 t^3/6 + c_3 t + c_4 \\ c_1 + c_2 t \end{pmatrix}$$

$$= \begin{pmatrix} c_4 \\ c_1 \end{pmatrix} + t\begin{pmatrix} c_3 \\ c_2 \end{pmatrix} + t^2\begin{pmatrix} c_1/2 \\ 0 \end{pmatrix} + t^3\begin{pmatrix} c_2/6 \\ 0 \end{pmatrix}$$

which is a vector linear combination of $1, t, t^2, t^3$. ■

10. Let $\vec{\mathbf{x}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 t + \vec{\mathbf{d}}_3\frac{t^2}{2} + \vec{\mathbf{d}}_4\frac{t^3}{6}$. Assume $\vec{\mathbf{x}}$ solves $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$. Prove: $A\vec{\mathbf{d}}_3 = A\vec{\mathbf{d}}_4 = \vec{\mathbf{0}}$, $A\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_3$, $A\vec{\mathbf{d}}_2 = \vec{\mathbf{d}}_4$. These are generalized eigenvector chains for eigenvalue zero.

CHZ Method

11. Given a 3×3 matrix A, supply proof details for the Cayley-Hamilton-Ziebur structure theorem.

Solution:
To prove for a real 3×3 matrix A:

> The solution $\vec{\mathbf{x}}(t)$ of second order equation $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$ is a vector linear combination of Euler solution atoms corresponding to roots of the equation $\det(A - r^2 I) = 0$.

Details: Expand $|A - \lambda I| = 0$ to find the characteristic equation $(-\lambda)^3 + a(-\lambda)^2 + b(-\lambda) + c = 0$, for some constants a, b, c. The Cayley-Hamilton theorem says that $-A^3 + aA^2 - bA + c\mathbf{I} = \mathbf{0}$. Let $\vec{\mathbf{x}}$ be a solution of $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$. Multiply the Cayley-Hamilton identity by vector $\vec{\mathbf{x}}$ and simplify to obtain

$$A^2\vec{\mathbf{x}} + cA\vec{\mathbf{x}} + d\vec{\mathbf{x}} = \vec{\mathbf{0}},$$

$$-A^3\vec{\mathbf{x}} + aA^2\vec{\mathbf{x}} - bA\vec{\mathbf{x}} + c\vec{\mathbf{x}} = \mathbf{0}$$

Using equation $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$ backwards, we compute $A^3\vec{\mathbf{x}} = A^2(A\vec{\mathbf{x}}) = A^2\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}'''' = \vec{\mathbf{x}}^{(6)}$, $A^2\vec{\mathbf{x}} = A\vec{\mathbf{x}}'' = \vec{\mathbf{x}}''''$. Replace the terms of the displayed equation to obtain the relation

$$-\vec{\mathbf{x}}^{(6)} + a\vec{\mathbf{x}}'''' - b\vec{\mathbf{x}}'' + c\vec{\mathbf{x}} = \mathbf{0}$$

Each component y of vector $\vec{\mathbf{x}}(t)$ then satisfies the 6th order linear homogeneous equation $y^{(6)} + ay^{(4)} - by^{(2)} + cy = 0$, which has characteristic equation $-r^6 + ar^4 - br^2 + c = 0$. This equation is the expansion of determinant equation $|A - r^2 I| = 0$. Therefore y is a linear combination of the Euler solution atoms found from the roots of this equation. It follows then that $\vec{\mathbf{x}}(t)$ is a vector linear combination of the Euler solution atoms so identified. ■

12. Invent a non-diagonal 3×3 example $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ and solve it by CHZ.

13. Solve $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ by CHZ for any 2×2 diagonal matrix with negative diagonal elements.

Solution:
Expand $|A - r^2 I| = (a - r^2)(b - r^2)$ for diagonal elements $-a^2, -b^2$ with $a > 0$ and $b > 0$. By Theorem 11.52 (CHZ Method and Negative Eigenvalues), the Euler solution atoms are $\cos at$, $\sin at$, $\cos bt$. $\sin bt$ and $\vec{\mathbf{x}}(t)$ is a

vector linear combination of these four atoms:

$\vec{\mathbf{x}} = \vec{\mathbf{d}}_1 \cos(at) + \vec{\mathbf{d}}_2 \sin(at) + \vec{\mathbf{d}}_3 \cos(bt) + \vec{\mathbf{d}}_4 \sin(bt)$

Then $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ gives

$-a^2\vec{\mathbf{d}}_1 \cos(at) - a^2\vec{\mathbf{d}}_2 \sin(at) - b^2\vec{\mathbf{d}}_3 \cos(bt) - b^2\vec{\mathbf{d}}_4 \sin(bt) =$
$\quad A\vec{\mathbf{d}}_1 \cos(at) + A\vec{\mathbf{d}}_2 \sin(at) + A\vec{\mathbf{d}}_3 \cos(bt) + A\vec{\mathbf{d}}_4 \sin(bt)$

Match the coefficients of atoms left and right:

$-a^2\vec{\mathbf{d}}_1 = A\vec{\mathbf{d}}_1,\ -a^2\vec{\mathbf{d}}_2 = A\vec{\mathbf{d}}_2,\ -b^2\vec{\mathbf{d}}_3 = A\vec{\mathbf{d}}_3,\ -b^2\vec{\mathbf{d}}_4 = A\vec{\mathbf{d}}_4$

The eigenpairs of diagonal matrix A are:

$\left(-a^2, \begin{pmatrix}1\\0\end{pmatrix}\right), \quad \left(-b^2, \begin{pmatrix}0\\1\end{pmatrix}\right)$

Equation $-a^2\vec{\mathbf{d}}_1 = A\vec{\mathbf{d}}_1$ implies vector $\vec{\mathbf{d}}_1$ is a multiple of the first eigenvector, similarly for the other three equations. Then

$\vec{\mathbf{d}}_1 = c_1 \begin{pmatrix}1\\0\end{pmatrix}, \vec{\mathbf{d}}_2 = c_2 \begin{pmatrix}1\\0\end{pmatrix}, \vec{\mathbf{d}}_3 = c_3 \begin{pmatrix}0\\1\end{pmatrix}, \vec{\mathbf{d}}_4 = c_4 \begin{pmatrix}0\\1\end{pmatrix}$

$\vec{\mathbf{x}}(t) = (c_1 \cos(at) + c_2 \sin(at)) \begin{pmatrix}1\\0\end{pmatrix} + (c_3 \cos(bt) + c_4 \sin(bt)) \begin{pmatrix}0\\1\end{pmatrix} \quad \blacksquare$

14. Solve $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ by CHZ for any 3×3 diagonal matrix with negative diagonal elements.

Conversion

Given $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, let $\vec{\mathbf{u}} = \begin{pmatrix}\vec{\mathbf{x}}\\ \vec{\mathbf{x}}'\end{pmatrix}$. Display system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$.

15. $A = \begin{pmatrix}1 & 3\\ -1 & 2\end{pmatrix}$

Solution: Answer: $C = \begin{pmatrix}\mathbf{0} & \mathbf{I}\\ A & \mathbf{0}\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 3 & 0 & 0\\ -1 & 2 & 1 & 0\end{pmatrix}$

Details:

$$\begin{aligned}\vec{\mathbf{u}}' &= \begin{pmatrix}\vec{\mathbf{x}}'\\ \vec{\mathbf{x}}''\end{pmatrix}\\ &= \begin{pmatrix}\vec{\mathbf{x}}'\\ A\vec{\mathbf{x}}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{0} & \mathbf{I}\\ A & \mathbf{0}\end{pmatrix}\begin{pmatrix}\vec{\mathbf{x}}\\ \vec{\mathbf{x}}'\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{0} & \mathbf{I}\\ A & \mathbf{0}\end{pmatrix}\vec{\mathbf{u}}\end{aligned}$$

16. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

Eigenanalysis $\lambda \le 0$

Display the general solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

17. $A = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$

Solution:
Use Theorem 11.56 page 928:

Eigenpairs: $\left(-4, \begin{pmatrix} 3 \\ -1 \end{pmatrix}\right), \left(0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

Solution:

$\vec{\mathbf{x}}(t) = (c_1 + c_2 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c_3 \cos 2t + c_4 \sin 2t)\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ ■

```
# Exercise 17, Eigenanalysis nonpositive lambda
A:=<-3,3|1,-1>^+;
Eigenvectors(A);# lambda = 0,-4
Determinant(A-r^2); # r^4+4*r^2
 # atoms: 1, t, cos 2t, sin 2t
```

18. $A = \begin{pmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \\ 5 & 0 & -1 \end{pmatrix}$

Solution:
Use Theorem 11.56 page 928, $\lambda = 0, -1, -4$.

Earthquakes

Apply formulas from the *Earthquakes subsection* page 929 to find particular solution $\vec{\mathbf{x}}_p$, the natural frequencies ω_j and the amplitudes of $\vec{\mathbf{x}}_p(t)$ near the largest natural frequency. Assume $F(t) = F_0 \cos(\omega t)$.

19. Three-floor problem, $k/m = 10$.

Solution:

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{\mathbf{H}} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix},$$

where m_1, m_2, m_3 are the three masses of the floors at positions x_1, x_2, x_3. Symbol $E_j = m_j F'' = -m_j F_0 \omega^2 \cos(\omega t)$, $1 \le j \le 3$. The Hooke's matrix:

$$K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}$$

In the last row, $k_4 = 0$ reflects the absence of a floor above the third floor. The second order system:

$$M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t) + \vec{\mathbf{H}}(t)$$

Identical Floors

Assume that all floors have the same mass m and the same Hooke's constant k. Then $M = mI$ and $M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t) + \vec{\mathbf{H}}(t)$ becomes:

$$\text{(1)} \qquad \vec{\mathbf{x}}'' = \frac{1}{m}\begin{pmatrix} -2k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \vec{\mathbf{x}} - F_0\omega^2 \cos(\omega t) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Assume $k/m = 10$ then:

$$\text{(2)} \qquad \vec{\mathbf{x}}'' = \begin{pmatrix} -20 & 10 & 0 \\ 10 & -20 & 10 \\ 0 & 10 & -10 \end{pmatrix} \vec{\mathbf{x}} - F_0\omega^2 \cos(\omega t) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Particular Solution: Identical Floors

The method of undetermined coefficients predicts a trial solution $\vec{\mathbf{x}}(t) = \vec{\mathbf{c}} \cos \omega t$. Constant vector $\vec{\mathbf{c}}$ is found by trial solution substitution. After cancel of common factor $\cos \omega t$:

$$-F_0\omega^2 \vec{\mathbf{c}} = \begin{pmatrix} -20 & 10 & 0 \\ 10 & -20 & 10 \\ 0 & 10 & -10 \end{pmatrix} \vec{\mathbf{c}} - F_0\omega^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The solution by `maple`:

$$\vec{\mathbf{c}} = \frac{F0\,\omega^2}{\omega^6 - 50\,\omega^4 + 600\,\omega^2 - 1000} \begin{pmatrix} \left(\omega^2 - 30\right)\left(\omega^2 - 10\right) \\ \omega^4 - 50\,\omega^2 + 500 \\ \omega^4 - 50\,\omega^2 + 600 \end{pmatrix}$$

Natural Frequencies.

The frequencies are obtain by `maple`'s `fsolve` applied to $\omega^6 - 50\,\omega^4 + 600\,\omega^2 - 1000 = 0$, because

$$B = \begin{pmatrix} -20 & 10 & 0 \\ 10 & -20 & 10 \\ 0 & 10 & -10 \end{pmatrix} + \omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $|B| = \omega^6 - 50\,\omega^4 + 600\,\omega^2 - 1000$. The frequencies ω are:

$$\begin{array}{rrr} -5.698227447 & -3.943295744 & -1.407345957 \\ 5.698227447 & 3.943295744 & 1.407345957 \end{array}$$

Amplitudes for $\vec{\mathbf{x}}_p$:
The amplitudes are the components of vector $\vec{\mathbf{c}}$ near $\omega = 5.698227447$. We report

$$\vec{\mathbf{c}}\,|_{\omega=5.698227447} = F_0 \begin{pmatrix} 9.009688690\,10^8 \\ -1.123489801\,10^9 \\ 5.000000010\,10^8 \end{pmatrix}$$

Homogeneous Solution: Identical Floors
The equation to solve is

$$\vec{\mathbf{x}}'' = \begin{pmatrix} -20 & 10 & 0 \\ 10 & -20 & 10 \\ 0 & 10 & -10 \end{pmatrix} \vec{\mathbf{x}}$$

Theorem 11.56 provides:

$$\text{(3)} \qquad \vec{\mathbf{x}}_h(t) = \sum_{j=1}^{3} (a_j \cos \omega_j t + b_j \sin \omega_j t) \vec{\mathbf{v}}_j$$

where $r = \omega_j$ and $\vec{\mathbf{v}} = \vec{\mathbf{v}}_j \neq \vec{\mathbf{0}}$ satisfy the **eigenpair equation**:

$$\left(\frac{1}{m} K + r^2\, I\right) \vec{\mathbf{v}} = \vec{\mathbf{0}}$$

Eigenpairs can be found numerically, a suitable online resource at
https://matrixcalc.org/en/

The answers:

$\lambda_1 = -32.470,\ \vec{\mathbf{v}}_1 = \begin{pmatrix} 1.802 \\ -2.247 \\ 1 \end{pmatrix}$

$\lambda_2 = -15.550,\ \vec{\mathbf{v}}_2 = \begin{pmatrix} -1.247 \\ -0.555 \\ 1 \end{pmatrix}$

$\lambda_3 = -1.981,\ \vec{\mathbf{v}}_3 = \begin{pmatrix} 0.445 \\ 0.802 \\ 1 \end{pmatrix}$

Symbols ω_j use in equation 3 satisfy
$\omega_1 = \sqrt{32.470}$, $\omega_2 = \sqrt{15.550}$, $\omega_3 = \sqrt{1.981}$. ■

```
# Exercise 19, Earthquakes n=3
K1:=Matrix([[-20,10,0],[10,-20,10],[0,10,-10]]);
B:=K1+omega^2*IdentityMatrix(3);
ans1:=LinearSolve(B,F0*omega^2*<1,1,1>);
q:=Determinant(B);
fsolve(q=0,omega);# Frequencies
C:=K1-lambda*IdentityMatrix(3);Determinant(C);
subs(omega=5.698227447,ans1);
# eigenpairs of K1, calculator at https://matrixcalc.org/en/
lambda1:=-32.470; v1:=<1.802,-2.247,1>;
lambda2:=-15.550;v2:=<-1.247,-0.555,1>;
lambda3:=-1.981;v3:=<0.445,0.802,1>;
```

20. Four-floor problem, $k/m = 10$.

Two Masses

Assume MKS units. Let $m_1 = 2$, $m_2 = 0.5$, $k_1 = 75$, $k_2 = 25$ in system:

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2[x_2 - x_1] \\ m_2 x_2'' &= -k_2[x_2 - x_1] \end{aligned}$$

21. Convert the system to the form $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

Solution:

$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$,

$K = \begin{pmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 \end{pmatrix}$,

$M\vec{\mathbf{x}}' = K\vec{\mathbf{x}}$

$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \vec{\mathbf{x}}' = \begin{pmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 \end{pmatrix} \vec{\mathbf{x}}$

$\begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} \vec{\mathbf{x}}' = \begin{pmatrix} -100 & 25 \\ 25 & -25 \end{pmatrix} \vec{\mathbf{x}}$

■

22. Show details for finding the vector solution $\vec{\mathbf{x}}(t)$.

Three Rail Cars: $k=2m$

Assume MKS units. Consider

$$\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -2 \end{pmatrix} \vec{\mathbf{x}}$$

23. Show eigenpair details for the 3×3 matrix.

Solution:
Eigenvalues: -6, 0, 2

Eigenvectors: $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

```
# Exercise 23, Three Rail Cars
A:=Matrix([[-2,2,0],[2,-4,2],[0,2,-2]]);
EigVals,EigVecs:=Eigenvectors(A);
```

24. Find the vector solution $\vec{\mathbf{x}}(t)$.

Three Rail Cars: Disengagement

For $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, assume FPS units and

$$A = \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}$$

Suppose the springs disengage upon full expansion. Let the cars engage at $t = 0$ with $x_1 = x_2 = x_3 = 0$.

25. Verify A has eigenvalues $\lambda = -16, 0, -4$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Solution:

```
# Exercise 25, Three Rail Cars
A:=Matrix([[-4,4,0],[6,-12,6],[0,4,-4]]);
EigVals,EigVecs:=Eigenvectors(A);
```

26. For $x_1{=}x_2{=}x_3{=}0$ at $t{=}0$, verify:
$x_1(t){=}c_1t{+}c_2\sin(2t){-}c_3\sin(4t)$,
$x_2(t) = c_1t + 3c_3\sin(4t)$,
$x_3(t){=}c_1t{-}c_2\sin(2t){-}c_3\sin(4t)$

27. Let $x_1' = 48$, $x_2' = 0$, $x_3' = 0$ at $t = 0$. Verify disengagement time $t_1 = \pi/2$ and determine the car velocities thereafter.

Solution:
Exercise 26 provides equations for the solution:

$x_1(t) = c_1t{+}c_2\sin(2t){-}c_3\sin(4t)$,
$x_2(t) = c_1t + 3c_3\sin(4t)$,
$x_3(t) = c_1t{-}c_2\sin(2t){-}c_3\sin(4t)$

Differentiate the equations and set $t = 0$ to obtain linear algebraic equations for the constants c_1, c_2, c_3:

$$\begin{aligned}
0 &= x_1'(0) = c_1 + 2c_2\cos(0) - 4c_3\cos(0),\\
0 &= x_2'(0) = c_1 + 12c_3\cos(0)\\
48 &= x_3'(0) = c_1 - 2c_2\cos(0) - 4c_3\cos(0)
\end{aligned}$$

In matrix form after setting $\cos(0) = 1$:

$$\begin{pmatrix} 1 & 2 & -4 \\ 1 & 0 & 12 \\ 1 & -2 & -4 \end{pmatrix} \vec{\mathbf{c}} = \begin{pmatrix} 48 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Then

$$\vec{\mathbf{c}} = \begin{pmatrix} 18 \\ 12 \\ -3/2 \end{pmatrix} \text{ and components } x_1, x_2, x_3 \text{ satisfy}$$

$$\begin{aligned}
x_1(t) &= 18t + 12\sin(2t) + \tfrac{3}{2}\sin(4t),\\
x_2(t) &= 18t - \tfrac{9}{2}\sin(4t),\\
x_3(t) &= 18t - 12\sin(2t) + \tfrac{3}{2}\sin(4t)
\end{aligned}$$

$$\begin{aligned}
x_1'(t) &= 18 + 24\cos(2t) + 6\cos(4t),\\
x_2'(t) &= 18 - 18\cos(4t),\\
x_3'(t) &= 18 - 24\cos(2t) + 6\cos(4t)
\end{aligned}$$

Car 1 moves ($x_1'(0) = 48$) into contact with two stationary cars ($x_2'(0) = x_3'(0) = 0$) using equation $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ on $0 \le t \le t_1$. Model $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ is valid for values of t such that $x_1 - x_2$ and $x_2 - x_3$ are positive (no contact). To get intuition about disengagement, plot $x_1 - x_2$ and $x_2 - x_3$ on $0 \le t \le \pi$. The two graphs show a curve starting at $(0,0)$ with first crossing at $t = \pi/2$. To confirm the root $t = \pi/2$, solve $x_1 - x_2 = 0$ for $t = t_1 = \pi/2$ in `maple`.

The speeds at $t = t_1 = \pi/2$ are:

$$\begin{aligned}
x_1'(t_1) &= 18 + 24\cos(\pi) + 6\cos(2\pi),\\
x_2'(t_1) &= 18 - 18\cos(2\pi),\\
x_3'(t_1) &= 18 - 24\cos(\pi) + 6\cos(2\pi)
\end{aligned}$$

Simplify using $\cos(\pi) = -1$, $\cos(2\pi) = 1$:

$$\begin{aligned}
x_1'(t_1) &= 18 - 24 + 6 = 0,\\
x_2'(t_1) &= 18 - 18 = 0,\\
x_3'(t_1) &= 18 + 24 + 6 = 48
\end{aligned}$$

The meaning: car 3 after impact continues on at speed 48, while cars 1 and 2 stop.

```
# Exercise 25, Three Rail Cars
A:=Matrix([[-4,4,0],[6,-12,6],[0,4,-4]]);
EigVals,EigVecs:=Eigenvectors(A);
B:=Matrix([
[1,2*cos(2*t),-4*cos(4*t)],
[1,0,12*cos(4*t)],
[1,-2*cos(2*t),-4*cos(4*t)]]);
B1:=simplify(subs(t=0,B));
LinearSolve(B1,<48,0,0>);
x_1:= t-> 18*t + 12*sin( 2*t ) + 3/2*sin( 4*t );
x_2:=t -> 18 *t - 9/2*sin( 4* t );
x_3:=t -> 18* t -12*sin( 2* t ) + 3/2*sin( 4*t );
eq1:=x_1(t)-x_2(t);eq2:=x_2(t)-x_3(t);
plot(eq1,t=0..Pi);solve(eq1,t);
plot(eq2,t=0..Pi);solve(eq2,t);
```

28. Let $x_1'(0) = 144$, $x_2'(0) = 48$, $x_3'(0) = 48$. Verify disengagement time $t_1 = \pi/2$ and determine the car velocities thereafter.
Answer: Velocities 144, 48, 48 at $t = t_1$.

Dynamic Dashpot

Assume conventions for Figure 26 and dynamic dashpot system

$$\begin{aligned} m_s X'' &= -k_1 X - d_1 X' - k_2(Y - X) \\ &\quad - d_2(Y' - X') + F(t), \\ m_b Y'' &= k_2(Y - X) + d_2(Y' - X') \end{aligned}$$

See page 936.

29. Assume $Y = 0$, ideal suspension. Derive:

$$\begin{aligned} &m_s X'' = -k_1 X - d_1 X' + F(t), \\ &d_2 X' + k_2 X = 0 \end{aligned}$$

Solution:
Use the second differential equation $m_b Y'' = k_2(Y - X) + d_2(Y' - X')$. Set $Y = Y' = 0$. Then

$0 = -k_2 X - d_2 X'$

which verifies equation 2: $d_2 X' + k_2 X = 0$.

Use the first differential equation $m_s X'' = -k_1 X - d_1 X' - k_2(Y - X) - d_2(Y' - X') + F(t)$. Set $Y = Y' = 0$. Then

$m_s X'' = -k_1 X - d_1 X' + k_2 X - d_2 X' + F(t)$

Replace $d_2 X'$ in this result by $-k_2 X$ using $d_2 X' + k_2 X = 0$. Then

$m_s X'' = -k_1 X - d_1 X' + F(t)$

which claimed equation 1: $m_s X'' = -k_1 X - d_1 X' + F(t)$. ■

30. Assume $Y = 0$, ideal suspension and $X(0) = 0.015$ meters. Find $X(t)$ and $F(t)$.

11.9 Numerical methods for Systems

Planar Methods

Apply the Euler, Heun and RK4 methods. Compare with the exact solution in a table.

1. $x' = x$, $y' = -y$, $x(0) = 2$, $y(0) = 2$. $h = 0.1$, 10 steps

Solution:

Exact Solution.

The differential equations are growth-decay equations with solutions $x = 2e^t$, $y = 2e^{-t}$.

Numerical Solution.

Computation by `maple` following Example 11.20. The `maple` code *infra* implements the algorithms, no library functions used. Values are rounded to 6 digits. The answers:

$t - Euler$	$x(t)$	$y(t)$	$x - exact$	$y - exact$
0.000000	2.000000	2.000000	2.000000	2.000000
0.100000	2.200000	1.800000	2.210342	1.809675
0.200000	2.420000	1.620000	2.442806	1.637462
0.300000	2.662000	1.458000	2.699718	1.481636
0.400000	2.928200	1.312200	2.983649	1.340640
0.500000	3.221020	1.180980	3.297443	1.213061
0.600000	3.543122	1.062882	3.644238	1.097623
0.700000	3.897434	0.956594	4.027505	0.993171
0.800000	4.287178	0.860934	4.451082	0.898658
0.900000	4.715895	0.774841	4.919206	0.813139
1.000000	5.187485	0.697357	5.436564	0.735759

$t - Heun$	$x(t)$	$y(t)$	$x - exact$	$y - exact$
0.000000	2.000000	2.000000	2.000000	2.000000
0.100000	2.210000	1.810000	2.210342	1.809675
0.200000	2.442050	1.638050	2.442806	1.637462
0.300000	2.698465	1.482435	2.699718	1.481636
0.400000	2.981804	1.341604	2.983649	1.340640
0.500000	3.294894	1.214152	3.297443	1.213061
0.600000	3.640857	1.098807	3.644238	1.097623
0.700000	4.023147	0.994420	4.027505	0.993171
0.800000	4.445578	0.899951	4.451082	0.898658
0.900000	4.912364	0.814455	4.919206	0.813139
1.000000	5.428162	0.737082	5.436564	0.735759

$t-RK4$	$x(t)$	$y(t)$	$x-exact$	$y-exact$
0.000000	2.000000	2.000000	2.000000	2.000000
0.100000	2.210342	1.809675	2.210342	1.809675
0.200000	2.442805	1.637462	2.442806	1.637462
0.300000	2.699717	1.481637	2.699718	1.481636
0.400000	2.983648	1.340641	2.983649	1.340640
0.500000	3.297441	1.213062	3.297443	1.213061
0.600000	3.644236	1.097624	3.644238	1.097623
0.700000	4.027503	0.993171	4.027505	0.993171
0.800000	4.451079	0.898659	4.451082	0.898658
0.900000	4.919203	0.813140	4.919206	0.813139
1.000000	5.436559	0.735760	5.436564	0.735759

■

```
# Exercise 1, Planar Methods Euler
f:=(t,x,y) -> x;g:= (t,x,y) -> -y;
x_0:=2;y_0:=2;h:=0.1;n:=10;t_0:=0;L:=[t_0,x_0,y_0];
for i from 1 to n do
 X := x_0+h*f(t_0,x_0,y_0);
 Y := y_0+h*g(t_0,x_0,y_0);
 t_0:=t_0+h:x_0:=X:y_0:=Y:L:=L,[t_0,x_0,y_0];
od:
x_exact:=t->2*exp(t):y_exact:=t->2*exp(-t):
tbl:=seq([seq(L[i][j],j=1..3),
         x_exact(h*i-h),y_exact(h*i-h)],i=1..n+1);

# Exercise 1, Planar Methods Heun
f:=(t,x,y) -> x;g:= (t,x,y) -> -y;
x_0:=2;y_0:=2;h:=0.1;n:=10;t_0:=0;L:=[t_0,x_0,y_0];
for i from 1 to n do
 X1 := x_0+h*f(t_0,x_0,y_0);Y1:= y_0+h*g(t_0,x_0,y_0);
 X:= x_0+h*(f(t_0,x_0,y_0)+f(t_0+h,X1,Y1))/2;
 Y:= y_0+h*(g(t_0,x_0,y_0)+g(t_0+h,X1,Y1))/2;
 t_0:=t_0+h:x_0:=X:y_0:=Y:L:=L,[t_0,x_0,y_0];
od:
x_exact:=t->2*exp(t):y_exact:=t->2*exp(-t):
tbl:=seq([seq(L[i][j],j=1..3),
         x_exact(h*i-h),y_exact(h*i-h)],i=1..n+1);
```

```
# Exercise 1, Planar Methods RK4
f:=(t,x,y) -> x;g:= (t,x,y) -> -y;
x_0:=2;y_0:=2;h:=0.1;n:=10;t_0:=0;L:=[t_0,x_0,y_0];
for i from 1 to n do
  k_1 :=  h*f(t_0,x_0,y_0);
  m_1 :=  h*g(t_0,x_0,y_0);
  k_2 :=  h*f(t_0+h/2,x_0+k_1/2,y_0+m_1/2);
  m_2 :=  h*g(t_0+h/2,x_0+k_1/2,y_0+m_1/2);
  k_3 :=  h*f(t_0+h/2,x_0+k_2/2,y_0+m_2/2);
  m_3 :=  h*g(t_0+h/2,x_0+k_2/2,y_0+m_2/2);
  k_4 :=  h*f(t_0+h,x_0+k_3,y_0+m_3);
  m_4 :=  h*g(t_0+h,x_0+k_3,y_0+m_3);
  X   :=  x_0 + ( k_1+2*k_2+2*k_3+k_4 )/6;
  Y   :=  y_0 + ( m_1+2*m_2+2*m_3+m_4 )/6;
 t_0:=t_0+h:x_0:=X:y_0:=Y:L:=L,[t_0,X,Y];
od:
x_exact:=t->2*exp(t):y_exact:=t->2*exp(-t):
tbl:=seq([seq(L[i][j],j=1..3),
         x_exact(h*i-h),y_exact(h*i-h)],i=1..n+1);
```

2. $x' = -3x + y$, $y' = x - 3y$, $x(0) = 2$, $y(0) = 0$, $h = 0.1$, 10 steps

3. $x' = -x + y$, $y' = -x - y$, $x(0) = 0$, $y(0) = 3$, $h = 0.2$, 5 steps

Solution: The answers:

$t - Euler$	$x(t)$	$y(t)$	$x - exact$	$y - exact$
0.000000	0.000000	3.000000	0.000000	3.000000
0.100000	0.300000	2.700000	0.270999	2.700951
0.200000	0.540000	2.400000	0.487970	2.407232
0.300000	0.726000	2.106000	0.656780	2.123192
0.400000	0.864000	1.822800	0.783105	1.852217
0.500000	0.959880	1.554120	0.872359	1.596842
0.600000	1.019304	1.302720	0.929647	1.358861
0.700000	1.047646	1.070518	0.959727	1.139428
0.800000	1.049933	0.858701	0.966987	0.939152
0.900000	1.030810	0.667838	0.955431	0.758183
1.000000	0.994512	0.497973	0.928680	0.596298

$t - Heun$	$x(t)$	$y(t)$	$x - exact$	$y - exact$
0.000000	0.000000	3.000000	0.000000	3.000000
0.100000	0.270000	2.700000	0.270999	2.700951
0.200000	0.486000	2.405700	0.487970	2.407232
0.300000	0.653913	2.121390	0.656780	2.123192
0.400000	0.779447	1.850399	0.783105	1.852217
0.500000	0.868038	1.595209	0.872359	1.596842
0.600000	0.924803	1.357564	0.929647	1.358861
0.700000	0.954503	1.138576	0.959727	1.139428
0.800000	0.961525	0.938813	0.966987	0.939152
0.900000	0.949866	0.758394	0.955431	0.758183
1.000000	0.923135	0.597067	0.928680	0.596298

$t - RK4$	$x(t)$	$y(t)$	$x - exact$	$y - exact$
0.000000	0.000000	3.000000	0.000000	3.000000
0.100000	0.271000	2.700950	0.270999	2.700951
0.200000	0.487972	2.407230	0.487970	2.407232
0.300000	0.656782	2.123189	0.656780	2.123192
0.400000	0.783107	1.852213	0.783105	1.852217
0.500000	0.872361	1.596838	0.872359	1.596842
0.600000	0.929648	1.358856	0.929647	1.358861
0.700000	0.959728	1.139423	0.959727	1.139428
0.800000	0.966987	0.939146	0.966987	0.939152
0.900000	0.955431	0.758178	0.955431	0.758183
1.000000	0.928679	0.596293	0.928680	0.596298

■

```
# Exercise 3, Planar Methods Euler
f:=(t,x,y) -> -x+y;g:= (t,x,y) -> -x-y;
x_0:=0;y_0:=3;h:=0.1;n:=10;t_0:=0;L:=[t_0,x_0,y_0];
# Exact solution
des:=diff(x(t),t)=f(t,x(t),y(t)),diff(y(t),t)=g(t,x(t),y(t));
ics:=x(0)=0,y(0)=3;
dsolve([des,ics],[x(t),y(t)]);
x_exact:=t->3*exp(-t)*sin(t):y_exact:=t->3*exp(-t)*cos(t):
# Numerical solution Euler
for i from 1 to n do
 X := x_0+h*f(t_0,x_0,y_0);
 Y := y_0+h*g(t_0,x_0,y_0);
 t_0:=t_0+h:x_0:=X:y_0:=Y:L:=L,[t_0,x_0,y_0];
od:
tbl:=seq([seq(L[i][j],j=1..3),
         x_exact(h*i-h),y_exact(h*i-h)],i=1..n+1);
```

```
# Exercise 3, Planar Methods Heun
for i from 1 to n do
 X1 := x_0+h*f(t_0,x_0,y_0);Y1:= y_0+h*g(t_0,x_0,y_0);
 X:= x_0+h*(f(t_0,x_0,y_0)+f(t_0+h,X1,Y1))/2;
 Y:= y_0+h*(g(t_0,x_0,y_0)+g(t_0+h,X1,Y1))/2;
 t_0:=t_0+h:x_0:=X:y_0:=Y:L:=L,[t_0,x_0,y_0];
od:
tbl:=seq([seq(L[i][j],j=1..3),
         x_exact(h*i-h),y_exact(h*i-h)],i=1..n+1);
```

```
# Exercise 3, Planar Methods RK4
for i from 1 to n do
  k_1 :=  h*f(t_0,x_0,y_0);
  m_1 :=  h*g(t_0,x_0,y_0);
  k_2 :=  h*f(t_0+h/2,x_0+k_1/2,y_0+m_1/2);
  m_2 :=  h*g(t_0+h/2,x_0+k_1/2,y_0+m_1/2);
  k_3 :=  h*f(t_0+h/2,x_0+k_2/2,y_0+m_2/2);
  m_3 :=  h*g(t_0+h/2,x_0+k_2/2,y_0+m_2/2);
  k_4 :=  h*f(t_0+h,x_0+k_3,y_0+m_3);
  m_4 :=  h*g(t_0+h,x_0+k_3,y_0+m_3);
  X   :=  x_0 + ( k_1+2*k_2+2*k_3+k_4 )/6;
  Y   :=  y_0 + ( m_1+2*m_2+2*m_3+m_4 )/6;
 t_0:=t_0+h:x_0:=X:y_0:=Y:L:=L,[t_0,X,Y];
od:
tbl:=seq([seq(L[i][j],j=1..3),
         x_exact(h*i-h),y_exact(h*i-h)],i=1..n+1);
```

4. $x' = 2x - 4y$, $y' = x - 3y$, $x(0) = 4$, $y(0) = 0$, $h = 0.1$, 10 steps

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, 2×2

Apply vector Euler, Heun and RK4 methods for 10 steps with $h = 0.1$.

5. $\vec{\mathbf{u}}' = \begin{pmatrix} u_1 + u_2 \\ -u_1 + u_2 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

Solution: The answers at $t = 1$:

Euler: $x(1) = 7.58833504640000$, $y(1) = -0.924241606400000$

Heun: $x(1) = 7.53896384528113$, $y(1) = -1.61703014238029$

RK4: $x(1) = 7.51211873880302$, $y(1) = -1.637349161148920$

Exact: $x(1) = 7.512098455$, $y(1) = -1.637322695$

■

```
# Exercise 5, Vectors Methods Euler
A:=Matrix([[1, 1],[-1 , 1]]):
F0:=unapply(A.<x,y>,(t,x,y));
F0(t,x,y);# Scalar variables
F:=(t,X)->F0(t,X[1],X[2]);# Vector variables
U0:=<2,2>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
U:=U0+h*F(t0,U0);Vals:=Vals,U;
U0:=U;t0:=t0+h;
od:
ValsEuler:=Vals[n+1];
# Exact answer
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)= -x(t)+y(t);
ics:=x(0)=2,y(0)=2;
qexact:=dsolve([des,ics],[x(t),y(t)]);
evalf(subs(t=1,qexact));
```

```
# Exercise 5, Vectors Methods Heun
U0:=<2,2>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
w:=U0+h*F(t0,U0);
U:=U0+0.5*h*(F(t0,U0)+F(t0+h,w));
U0:=U;t0:=t0+h;Vals:=Vals,U0;
od:
ValsHeun:=Vals[n+1];
```

```
# Exercise 5, Vectors Methods RK4
U0:=<2,2>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
k1:=h*F(t0,U0);
k2:=h*F(t0+h/2,U0+k1/2);
k3:=h*F(t0+h/2,U0+k2/2);
k4:=h*F(t0+h,U0+k3);
U:=U0+(1/6)*(k1+2*k2+2*k3+k4);U0:=U;t0:=t0+h;Vals:=Vals,U0;
od:
ValsRK4:=Vals[n+1];
```

6. $\vec{\mathbf{u}}' = \begin{pmatrix} -3u_1 + u_2 \\ u_1 - 3u_2 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Solution:

```
# Exercise 6 Euler
A:=Matrix([[-3, 1],[1 , -3]]):
U0:=<2,0>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
# Use code from Exercise 5
```

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$

Apply vector Euler, Heun and RK4 methods for 10 steps with $t_0 = 0$, $h = 0.1$. Compare results for the last step.

7. $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, $\vec{\mathbf{F}} = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$,

$\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Ans Euler: $3.81, -5.33$

Solution: The answers at $t = 1$:

Euler: $x(1) = 3.81345311556651$, $y(1) = -5.32607258418454$

Heun: $x(1) = 2.62373309709154$, $y(1) = -5.60474421071964$

RK4: $x(1) = 2.57616625457178$, $y(1) = -5.52765661055646$

Exact: $x(1) = 2.576385623$, $y(1) = -5.527674160$

■

```
# Exercise 7, Vectors Methods Euler
A:=Matrix([[1, 1],[-1 , 1]]):
F0:=unapply(A.<x,y>+<exp(t),0>,(t,x,y));
F0(t,x,y);# Scalar variables
F:=(t,X)->F0(t,X[1],X[2]);# Vector variables
U0:=<1,1>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
U:=U0+h*F(t0,U0);Vals:=Vals,U;
U0:=U;t0:=t0+h;
od:
ValsEuler:=Vals[n+1];
# Exact answer
des:=diff(x(t),t)=x(t)+2*y(t)+exp(t),
     diff(y(t),t)= -2*x(t)+y(t);
ics:=x(0)=1,y(0)=1;
qexact:=dsolve([des,ics],[x(t),y(t)]);
evalf(subs(t=1,qexact));
```

```
# Exercise 7, Vectors Methods Heun
U0:=<1,1>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
w:=U0+h*F(t0,U0);
U:=U0+0.5*h*(F(t0,U0)+F(t0+h,w));
U0:=U;t0:=t0+h;Vals:=Vals,U0;
od:
ValsHeun:=Vals[n+1];
```

```
# Exercise 7, Vectors Methods RK4
U0:=<1,1>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
k1:=h*F(t0,U0);
k2:=h*F(t0+h/2,U0+k1/2);
k3:=h*F(t0+h/2,U0+k2/2);
k4:=h*F(t0+h,U0+k3);
U:=U0+(1/6)*(k1+2*k2+2*k3+k4);U0:=U;t0:=t0+h;Vals:=Vals,U0;
od:
ValsRK4:=Vals[n+1];
```

8. $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\vec{\mathbf{F}} = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}$,

$\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans RK4: $2.576, -5.528, 0.0$

Solution: Modify the `maple` code in Exercise 9.

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, 3×3

Apply vector Euler, Heun and RK4 methods for 10 steps with $h = 0.1$.

9. $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans Heun: $1.36, -3.67, 0.00$

Solution: The answers at $t = 1$:

Euler: $x(1) = 3.81345311556651$, $y(1) = -5.32607258418454$, $z(1) = 0$

Heun: $x(1) = 1.36191852014674$, $y(1) = -3.66635681255906$, $z(1) = 0$

RK4: $x(1) = 1.34036497702700$, $y(1) = -3.60288223776972$, $z(1) = 0$

Exact: $x(1) = 1.340522288$, $y(1) = -3.602931054$, $z(1) = 0$

■

```
# Exercise 9, Vectors Methods Euler
A:=Matrix([[1, 2,0],[-2 , 1,0],[0,0,5]]):
F0:=unapply(A.<x,y,z>,(t,x,y,z));
F0(t,x,y,z);# Scalar variables
F:=(t,X)->F0(t,X[1],X[2],X[3]);# Vector variables
U0:=<1,1,0>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
U:=U0+h*F(t0,U0);Vals:=Vals,U;
U0:=U;t0:=t0+h;
od:
ValsEuler:=Vals[n+1];
# Exact answer
des:=diff(x(t),t)=x(t)+2*y(t),diff(y(t),t)= -2*x(t)+y(t),
     diff(z(t),t)=5*z(t);
ics:=x(0)=1,y(0)=1,z(0)=0;
qexact:=dsolve([des,ics],[x(t),y(t),z(t)]);
evalf(subs(t=1,qexact));

# Exercise 9, Vectors Methods Heun
U0:=<1,1,0>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
w:=U0+h*F(t0,U0);
U:=U0+0.5*h*(F(t0,U0)+F(t0+h,w));U0:=U;t0:=t0+h;Vals:=Vals,U0;
od:
ValsHeun:=Vals[n+1];
```

```
# Exercise 9, Vectors Methods RK4
U0:=<1,1,0>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
k1:=h*F(t0,U0);
k2:=h*F(t0+h/2,U0+k1/2);
k3:=h*F(t0+h/2,U0+k2/2);
k4:=h*F(t0+h,U0+k3);
U:=U0+(1/6)*(k1+2*k2+2*k3+k4);U0:=U;t0:=t0+h;Vals:=Vals,U0;
od:
ValsRK4:=Vals[n+1];
```

10. $A = \begin{pmatrix} 1 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans RK4: $-2.307, -3.075, 0.00$

Solution: Because $z(t) = 0$ then the numerical solutions match Exercise 7. The `maple` code in Exercise 9 can be modified to attack the problem directly.

```
# Exercise 10, Vector Methods 3x3
A:=Matrix([[1, 3,0],[-3 , 1,0],[0,0,1]]):
F0:=unapply(A.<x,y,z>,(t,x,y,z));
F0(t,x,y,z);# Scalar variables
F:=(t,X)->F0(t,X[1],X[2],X[3]);# Vector variables
U0:=<1,1,0>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
# Exact answer
des:=diff(x(t),t)=x(t)+3*y(t),diff(y(t),t)= -3*x(t)+y(t),
     diff(z(t),t)=1*z(t);
ics:=x(0)=1,y(0)=1,z(0)=0;
qexact:=dsolve([des,ics],[x(t),y(t),z(t)]);
evalf(subs(t=1,qexact));
# {x(1) = -2.307474660, y(1) = -3.074682568, z(1) = 0.}
```

Chapter 12

Series Methods

Contents

12.1 Review of Calculus Topics

Series Convergence

Find R, the radius of convergence.

1. $\sum_{k=2}^{\infty} \frac{x^k}{k\ \ln(k)}$

Solution: The radius of convergence is $R = 1$.

Details.

Let $c_n = \dfrac{1}{n\ln(n)}$. Then

$$\frac{c_n}{c_{n+1}} = \frac{\dfrac{1}{n\ln(n)}}{\dfrac{1}{(n+1)\ln(n+1)}}$$

$$= \frac{(n+1)\ln(n+1)}{n\ln(n)}$$

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)}$$

$$= 1 \cdot 1 = 1$$

```
# Exercise 1, Series Convergence
c:=k -> 1/(k*ln(k));
limit(c(k)/c(k+1),k=infinity);
```

■

2. $\sum_{k=1}^{\infty} a_k\, x^k$, $a_{2n} = 2$, $a_{2n+1} = 4$.

Series Properties

Compute the series given by the indicated operation(s).

3. $\frac{d}{dx} \sum_{k=2}^{\infty} \frac{x^k}{k\ \ln(k)}$

Solution: Apply term-by-term differentiation. Let $S = \sum_{k=2}^{\infty} \frac{x^k}{k\ \ln(k)}$

$$\frac{dS}{dx} = \sum_{k=2}^{\infty} \frac{d}{dx}\left(\frac{x^k}{k\ \ln(k)}\right)$$

$$= \sum_{k=2}^{\infty} \frac{x^{k-1}}{\ln(k)}$$

■

4. $4\sum_{k=1}^{\infty} \frac{1}{1+k}\, x^k + \sum_{k=2}^{\infty} \frac{1}{1+k^2}\, x^k$

Maclaurin Series

Find the Maclaurin series expansion.

5. $f(x) = \frac{1}{1+x^3}$ for $|x| < 1$.

Solution: Answer: $\displaystyle\sum_{n=0}^{\infty} (-x^3)^n$ for $|x| < 1$.

The geometric series expansion $\displaystyle\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ is applied with $r = -x^3$. It is known that the radius of convergence is $|r| < 1$ $(R = 1)$. The series must match the Maclaurin series obtained from $f(x) = \displaystyle\frac{1}{1+x^3}$ with identical radius of convergence. Calculus texts discuss this shortcut in detail. ■

6. $f(x) = \arctan(x)$, using
$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$.

7. $f(x) = \left(\frac{3}{2}\right)^x$ for all x.

Solution: Write $f(x) = e^{ax}$ with $a = \ln(3/2)$. Then the Maclaurin expansion of e^x applies:

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
e^{ax} &= \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} \quad \text{substitute } ax \text{ for } x \\
&= \sum_{n=0}^{\infty} \frac{(x\ln(3/2))^n}{n!} \quad \text{substitute } a = \ln(3/2)
\end{aligned}
$$

■

8. $f(x) = \int_0^x \frac{\sin t}{t}\,dt$, called the **Sine Integral**.

9. $f(x)$ is the solution of $f' = 1 + xf$, $f(0) = 0$.

Solution: Computer assist is expected t0 confirm the answer $f(x) = (x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + O(x^6))$.

Details by hand.
Assume $f(x)$ is a Maclaurin series $f(x) = \sum_{n=0}^{\infty} c_n x^n$, which has to agree with the Taylor series at $x = 0$: $c_n = f^{(n)}(0)/n!$. Use the differential equation to find the constants c_n as follows.

$c_0 = f(0)/0! = 0$

$f'(0) = 1 + (0)f(0) = 1$ Substitute $x = 0$ in the differential equation.

Then $c_1 = f'(0)/1! = 1$.

$f''(0) = (1 + xf(x))'|_{x=0}$ Differentiate the equation and set $x = 0$.

$f''(0) = (0 + f(x) + xf'(x))|_{x=0} = 0$

Then $c_2 = f''(0)/2! = 0$.

The process continues to obtain

$f(x) = (x + \frac{1}{3}x^3 + \frac{1}{15}x^5 \cdots$

```
# Exercise 9, Maclaurin series answer check
dsolve([diff(f(x),x)=1+x*f(x),f(0)=0],f(x),series);
```

■

10. The first 4 terms, $f(x) = \tan x$.

Taylor Series

Find the series expansion about the given point.

11. $f(x) = \ln|1-x|$, at $x = 0$.

Solution: The plan: use the Taylor expansion of $\ln|1+u|$ at $u = 0$ then replace $u = -x$, because $x = 0$ gives $u = 0$

$$\ln|1+u| = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}u^n}{n}$$

$$\ln|1+(-x)| = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-x)^n}{n}$$

$$\ln|1-x| = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n x^n}{n}$$

$$\ln|1-x| = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad \text{Because } (-1)^n(-1)^n = 1.$$

■

12. $f(x) = \frac{1}{x^2}$, at $x = 1$.

12.2 Algebraic Techniques

Differentiation

Verify using term–by–term differentiation. Document all series and calculus steps.

1. $\frac{d}{dx}\sum_{n=1}^{\infty}\frac{1}{n}\,x^n = \sum_{n=0}^{\infty} x^n$.
Is this valid for $x = -1$?

Solution: The left side is differentiated term-by-temr:

$$\begin{aligned}\frac{d}{dx}\sum_{n=1}^{\infty}\frac{1}{n}\,x^n &= \sum_{n=1}^{\infty}\ x^{n-1} \\ &= \sum_{k=01}^{\infty}\ x^k \quad \text{using index change } k = n-1.\end{aligned}$$

The geometric series on the right side converges for $|x| < 1$. Substitution of $x = -1$ gives alternating terms for which the nth term $(-1)^n$ does not have limit zero at ∞, therefore the series does not converge at $x = -1$, violating the

Theorem. If a series $\sum_n c_n$ converges, then $\lim_{n\to\infty}|c_n| = 0$.

■

2. $\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\,x^{2n+1} =$
$\sum_{n=0}^{\infty}(-1)^n\,x^{2n}$.

Subscripts

Perform a change of variables to verify the identity.

3. $\sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{k=2}^{\infty} c_{k-2}\,x^k$

Solution: The change of index is determined by matching x^{n+2} and x^k: $n+2 = k$. Then $n = 0$ to $n = \infty$ becomes $k = 2$ to $k = \infty$. The other changes in the summation are made via the two equations $n + 2 = k$ and $n = k - 2$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} c_n x^{n+2} &= \sum_{k=2}^{\infty} c_n x^{n+2} \quad \text{Change summation limits.} \\ &= \sum_{k=2}^{\infty} c_{k-2} x^k \quad \text{Change inside the summation.}\end{aligned}$$

■

4. $\sum_{n=2}^{\infty} n(n-1)c_n(x-x_0)^{n-2} =$
$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}\,(x-x_0)^k$

5. $-1 + x + \sum_{n=2}^{\infty}(-1)^{n+1}\,x^n =$
$\sum_{k=0}^{\infty}(-1)^{k+1}\,x^k$

Solution: No change of index is needed inside the summations, because matching x^n and x^k is unnecessary. The orphan terms $-1 + x$ can be written as $\sum_{n=0}^{1}(-1)^{n+1}x^n$. Then

$$\begin{aligned}
\text{LHS} &= -1+x+\textstyle\sum_{n=2}^{\infty}(-1)^{n+1}\,x^n \\
&= \sum_{n=0}^{1}(-1)^{n+1}x^n + \sum_{n=2}^{\infty}(-1)^{n+1}\,x^n \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\,x^n \qquad \text{Collect summations into one sum.} \\
&= \sum_{k=0}^{\infty}(-1)^{k+1}\,x^k \qquad \text{Change index variable } n \to k. \\
&= \text{RHS}
\end{aligned}$$

■

6. $\sum_{n=0}^{\infty} \frac{1}{n+1}\,x^n + \sum_{m=1}^{\infty} \frac{1}{m+2}\,x^m = 1 + \sum_{k=1}^{\infty} \frac{2k+1}{(k+1)(k+2)}\,x^k$

Linearity

Find the power series of the given function.

7. $\cos(x) + 2\sin(x)$

Solution: Assemble series identities for $\cos x$ and $\sin x$ from the Library of Maclaurin Series page 951:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \text{ Converges for all } x.$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \text{ Converges for all } x.$$

Then

$$\begin{aligned}
\cos(x) + 2\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + 2\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} 2\frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
&= \sum_{k=0}^{\infty} c_k x^k, \text{ for } c_k \text{ defined by}
\end{aligned}$$

$$c_k = \begin{cases} \dfrac{(-1)^n}{(2n)!} & \text{when } k = 2n \text{ is even,} \\ \dfrac{2(-1)^{2n+1}}{(2n+1)!} & \text{when } k = 2n+1 \text{ is odd} \end{cases}$$

It is sometimes possible to find a compact formula for c_k, but in this case there is little to simplify. ■

8. $e^x + \sin(x)$

Cauchy Product

Find the power series.

9. $(1 + x)\sin(x)$

Solution: Let $S_1 = 1 + x$, $S_2 = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ from Exercise 7.
Both S_1 and S_2 are power series that converge for all x. Then

$$\begin{aligned} S_1 S_2 &= (1 + x)S_2 \\ &= S_2 + xS_2 \\ &= S_2 + x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= S_2 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}, \text{ Constant } x \text{ moves inside the summation.} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \\ &= \sum_{k=0}^{\infty} c_k x^k, \text{ where } c_k \text{ is defined by} \end{aligned}$$

$$c_k = \begin{cases} 0 & \text{when } k = 0 \\ \dfrac{(-1)^n}{(2n+2)!} & \text{when } k = 2n+2 \text{ is even,} \\ \dfrac{(-1)^n}{(2n+1)!} & \text{when } k = 2n+1 \text{ is odd} \end{cases}$$

■

10. $\frac{\sin(x)}{e^x}$

Recursion Relations

Solve the given recursion.

11. $x_{k+1} = 2x_k$

Solution:
Let's solve it by ad-hoc methods. For comparison, we will afterwards apply the general solution product formula for first order recursions found on page 957.

Ad-Hoc Method.
Let $k = 0$ in recursion $x_{k+1} = 2x_k$:

$x_{0+1} = 2x_0$ or $x_1 = 2x_0$.

Let $k = 1$ in recursion $x_{k+1} = 2x_k$:

$x_{1+1} = 2x_1$ or $x_2 = 2x_1$. Then

$x_2 = 2x_1 = 2(2x_0) = 2^2 x_0$.

Let $k = 2$ in recursion $x_{k+1} = 2x_k$:

$x_{2+1} = 2x_2$ or $x_3 = 2x_2$. Then

$x_3 = 2x_2 = 2(4x_0) = 2^3 x_0$.

Conclusion: $x_{k+1} = 2^{k+1} x_0$

Consider the recursion $x_{k+1} = 2x_k$ as the general recursion

$$x_{k+1} = a_k x_k + b_k, \quad k \geq 0$$

where $a_k = 2$ and $b_k = 0$. Then the textbook general solution is

$$\begin{aligned} x_{k+1} &= \left(\Pi_{r=0}^{k} a_r\right) x_0 + \sum_{n=0}^{k} \left(\Pi_{r=n+1}^{k} a_r\right) b_n \\ &= \left(\Pi_{r=0}^{k}(2)\right) x_0 + \sum_{n=0}^{k} \left(\Pi_{r=n+1}^{k} a_r\right)(0) \\ &= \left(\Pi_{r=0}^{k}(2)\right) x_0 \\ &= \left(2^{k+1}\right) x_0 \end{aligned}$$

■

12. $x_{k+1} = 2x_k + 1$

13. $x_{k+2} = 2x_k + 1$

Solution:
The ad-hoc method follows the ideas in Exercise 11 by dividing the problem into two first order recursions corresponding to $k = 2n$ and $k = 2n + 1$:

$x_{2n+2} = 2x_{2n} + 1$ and
$x_{2n+1+2} = 2x_{2n+1} + 1$

The textbook formulas for second order recursions win the contest of which method is easier and more accurate. First of all, symbols x_0 and x_1 act like the free variable symbols t_1, t_2 in linear algebra: the solution is in terms of these two symbols. Therefore, the recursion solution on page 957 only shows indices $k \geq 2$ ($k = 2n+2$, $k = 2n+3$ for $n \geq 0$).

Details:
Given: Recursion $x_{k+2} = 2x_k + 1$, to be solved.
Given: General recursion $x_{k+2} = a_k x_k + b_k, \quad k \geq 0$ (c_k replaced by x_k).

Let $a_k = 2$ and $b_k = 1$ in the general recursion to match $x_{k+2} = 2x_k + 1$. Then general recursion solution

$$\begin{aligned} x_{2n+2} &= \left(\Pi_{r=0}^{n} a_{2r}\right) x_0 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} a_{2r}\right) b_{2r}, \quad n \geq 0, \\ x_{2n+3} &= \left(\Pi_{r=0}^{n} a_{2r+1}\right) x_1 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} a_{2r+1}\right) b_{2r+1}, \quad n \geq 0 \end{aligned}$$

becomes for $a_r = 2$ and $b_r = 1$ the equations

$$\begin{aligned} x_{2n+2} &= \left(\Pi_{r=0}^{n} 2\right) x_0 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} 2\right) (1), \quad n \geq 0, \\ x_{2n+3} &= \left(\Pi_{r=0}^{n} 2\right) x_1 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} 2\right) (1), \quad n \geq 0 \end{aligned}$$

which simplify to

$$\begin{aligned} x_{2n+2} &= 2^{n+1} x_0 + \sum_{k=0}^{n} \frac{2^{n+1}}{2^{k+1}}, \quad n \geq 0, \\ x_{2n+3} &= 2^{n+1} x_1 + \sum_{k=0}^{n} \frac{2^{n+1}}{2^{k+1}}, \quad n \geq 0. \end{aligned}$$

A further simplification is $\sum_{k=0}^{n} \frac{2^{n+1}}{2^{k+1}} = -1 + 2^{n+1}$. ■

14. $x_{k+3} = 2x_k + 1$

12.3 Power Series Methods

First Order Series Method

Solve by power series.

1. $y' - 4y = 0$

Solution:

Trial solution: $y = \sum_{n=0}^{\infty} c_n x^n$, a Maclaurin series. Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} n\, c_n x^{n-1} \\
&= \sum_{k=0}^{\infty} (k+1)\, c_{k+1} x^k, \text{ using index change } k = n-1.
\end{aligned}
$$

Let LHS stand for the left side of differential equation $y' - 4y = 0$. Expand LHS with the trial solution series:

$$
\begin{aligned}
\text{LHS} &= y' - 4y \\
&= \sum_{k=0}^{\infty} (k+1)\, c_{k+1} x^k - 4 \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{k=0}^{\infty} (k+1)\, c_{k+1} x^k - 4 \sum_{k=0}^{\infty} c_k x^k, \text{ change index } k = n \\
&= \sum_{k=0}^{\infty} \left((k+1)\, c_{k+1} - 4c_k\right) x^k, \text{ add series}
\end{aligned}
$$

Then LHS = RHS = 0 means LHS is the zero Maclaurin series, so all coefficients are zero, giving the recursion relation $(k+1)\, c_{k+1} - 4c_k = 0$, $k \geq 0$

The recursion is solved by the general solution product formula for first order recursions found on page 957:

$$
c_{k+1} = \frac{4^k}{(k+1)!} c_0
$$

Then the trial solution becomes

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_0 \sum_{k=0}^{\infty} \frac{4^k}{(k+1)!} x^{k+1}, \text{ using index } n = k+1
\end{aligned}
$$

■

2. $y' - xy = 0$

Second Order Series Method

Solve by power series using the Airy equation example.

3. $y'' = 4y$

Solution:

Trial solution: $y = \sum_{n=0}^{\infty} c_n x^n$, a Maclaurin series. Following Exercise 1, or using formulas on page 954,

$$y' = \sum_{k=0}^{\infty} (k+1)\, c_{k+1} x^k$$

$$y'' = \sum_{k=0}^{\infty} (k+1)(k+2)\, c_{k+2} x^k$$

Write the differential equation as $y'' - 4y = 0$. Substitute the series formulas into the left side LHS of the differential equation. Then

$$\begin{aligned} \text{LHS} &= y'' - 4y \\ &= \sum_{k=0}^{\infty} (k+1)(k+2)\, c_{k+2} x^k - 4 \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} (k+1)(k+2)\, c_{k+2} x^k - 4 \sum_{k=0}^{\infty} c_k x^k, \text{ re-index } n = k \\ &= \sum_{k=0}^{\infty} \left((k+1)(k+2)\, c_{k+2} - 4c_k\right) x^k, \text{ add series} \end{aligned}$$

Then LHS = RHS = 0 gives the second order recursion relation

$$(k+1)(k+2)\, c_{k+2} - 4c_k = 0, \; k \ge 0$$

Formulas on page 957 give the recursion answers

$$c_{2k+2} = \left(\Pi_{r=0}^{k} \frac{4}{(2r+3)\,(2r+4)}\right) c_0,$$

$$c_{2k+3} = \left(\Pi_{r=0}^{k} \frac{4}{(2r+4)\,(2r+5)}\right) c_1.$$

The products can be written in terms of the **Gamma function**, $\Gamma(n+1) = n!$ for integers $n \ge 0$. For instance, $c_{2k+3} = c_1 \dfrac{2(4^{k+1})}{(2k+4)!}$.

4. $y'' + y = 0$

Taylor Series Method

Solve by Taylor series about $x = 0$, finding the first 8 terms.

5. $y' = 16y$

Solution: The exact solution is $y = y_0 e^{16x}$. Taylor expansion should give the series

$$y(x) = y_0 + 16\,y_0 x + 128\,y_0 x^2 + \frac{2048\,y_0}{3}x^3 + \frac{8192\,y_0}{3}x^4 + \cdots$$

Taylor method details.

$$y(x) = \sum_{k=0}^{n} c_k x^k + R_n \text{ where } c_k = \frac{f^{(k)}(0)}{k!}.$$

We find the first 8 terms, so $n = 7$.

$c_0 = y(0) = y_0$, symbol y_0 being the initial value of $y(x)$ at $x = 0$.

$c_1 = y'(0)/1! = 16y_0$

$c_2 = y''(0)/2! = 16y'(0)/2 = 16^2 y_0/2$

$c_3 = y'''(0)/3! = 16y''(0)/3! = 16^3 y_0/(3! \cdot 2!)$

Continue:

$$\begin{aligned} y(x) &= y_0 + 16y_0x + 128y_0x^2 + \tfrac{2048}{3}y_0x^3 + \tfrac{8192}{3}y_0x^4 \\ &\quad + \tfrac{131072}{15}y_0x^5 + \tfrac{1048576}{45}y_0x^6 + R_8 \end{aligned}$$

```
# Exercise 5, Tayor series method
de:=diff(y(x),x)=16*y(x);dsolve([de,y(0)=y[0] ],y(x));
dsolve([de,y(0)=y[0] ],y(x),series);
taylor(y[0]*exp(16*x), x=0, 7);
```

6. $y'' = y$

7. $y' = (1 + x)y$

Solution:

$$y(x) = y_0 + y_0x + y_0x^2 + \frac{2\,y_0}{3}x^3 + \frac{5\,y_0}{12}x^4 + \frac{13\,y_0}{60}x^5 + \frac{19\,y_0}{180}x^6 + \cdots$$

```
# Exercise 7, Tayor series method
de:=diff(y(x),x)=(1+x)*y(x);dsolve([de,y(0)=y[0] ],y(x));
dsolve([de,y(0)=y[0] ],y(x),series);
taylor(y[0]*exp((1/2)*x*(x+2)), x=0, 7);
```

8. $y'' = (2 + x)y$

12.4 Ordinary Points

Standard Form

Convert to form $y'' + p(x)y' + q(x)y = 0$. Find the singular points and ordinary points.

1. $(x+1)y'' + xy' + y = 0$

Solution:
Singular points: $x = -1$, because $a(x) = x+1$ is zero at $x = -1$, preventing division into standard form.

Ordinary points: all points $x \neq -1$ are ordinary points, because $p(x) = \dfrac{x}{x+1}$ and $q(x) = \dfrac{1}{x+1}$ have power series expansions about $x = x_0$ for $x_0 \neq -1$. ■

2. $x^2y'' + 3xy' + 4y = 0$

3. $x(1+x)y'' + xy' + (1+x)y = 0$

Solution:
Singular points: $x = -1$ and $x = 0$, because $a(x) = x(x+1)$ is zero at $x = -1$ or $x = 0$, preventing division into standard form.

Ordinary points: all points $x \neq -1$ and $x \neq 0$ are ordinary points, because $p(x) = \dfrac{1}{x+1}$ and $q(x) = \dfrac{1}{x}$ have power series expansions about $x = x_0$ for $x_0 \neq -1$ and $x \neq 0$. ■

4. $xy'' = (1+x)y' + e^x y$

Ordinary Point Method

Find a power series solution, following the method in the text for $y'' - 2xy' + y = 0$. Use a CAS or mathematical workbench to check the answer.

5. $y'' + xy' = 0$

Solution:
The series answers by `maple` answer check should be

$y_1 = 1$,

$y_2 = x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - \frac{1}{336}x^7 + \frac{1}{3456}x^9 + O\left(x^{10}\right)$.

Details.
Let LHS $= y'' + xy'$, RHS $= 0$. Assume trial solution $y = \sum_{n=0}^{\infty} c_n x^n$. Then formulas on pages 954 imply

$$\text{LHS} = y'' + xy'$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)c_{n+1}x^{n+1} \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n + \sum_{k=1}^{\infty}(k)c_kx^k && \text{Index change: } k=n+1. \\
&= 2c_2 + \sum_{k=1}^{\infty}(k+1)(k+2)c_{k+2}x^k + \sum_{k=1}^{\infty}(k)c_kx^k && \text{Split off term for } n=0. \text{ Re-index.} \\
&= 2c_2 + \sum_{k=1}^{\infty}\left((k+1)(k+2)c_{k+2}x^n + (k)c_k\right)x^k && \text{Add, then Collect on } x^k.
\end{aligned}
$$

Power series LHS equals RHS, the zero power series, which gives rise to the recursion relations $2c_2 = 0$, $(k+1)(k+2)c_{k+2} + (k)c_k = 0$, c_0 and c_1 given, $k \geq 1$, or more succinctly the two-termed second order recursion

$$c_{k+2} + \frac{-k}{(k+1)(k+2)}c_k, \quad k \geq 1, \quad c_2 = 0, \quad c_0, c_1 \text{ given.}$$

All even coefficients c_2, c_4, ... are zero because $c_2 = 0$. The odd coefficients are obtained from recursion

$$c_{k+2} + \frac{-k}{(k+1)(k+2)}c_k, \quad k \geq 0, \quad k \text{ odd}, \quad c_0, c_1 \text{ given,}$$

Using the formulas on page 957 with $a_k = \dfrac{-k}{(k+1)(k+2)}$, $b_k = 0$, then the recursion answers are

$$c_{2k} = 0, \quad k \geq 1,$$

$$c_{2k+3} = \left(\Pi_{r=0}^{k} a_{2r+1}\right)c_1 = \left(\Pi_{r=0}^{k}\frac{(-1)(2r+1)}{(2r+2)(2r+3)}\right)c_1, \quad k \geq 0.$$

Taking $c_0 = 1$, $c_1 = 0$ gives y_1 and taking $c_0 = 0$, $c_1 = 1$ gives y_2:

$$
\begin{aligned}
y_1(x) &= 1, \\
y_2(x) &= x + \sum_{k=0}^{\infty}\left(\Pi_{r=0}^{k}\frac{(-1)(2r+1)}{(2r+2)(2r+3)}\right)x^{2k+3} \\
&= x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - \frac{x^7}{336} + \frac{x^9}{3456} - \frac{x^{11}}{42240} + \cdots
\end{aligned}
$$

The two solutions have Wronskian 1 at $x = 0$: they are independent and form a basis for the solution space of the differential equation.

Coefficient c_{2k+3} can be simplified to

$$c_{2k+3} = \frac{(-1)^{k+1}}{2^{k+1}\,(2k+3)}\,\frac{1}{(k+2)!}$$

```
# Exercise 5, Ordinary points
de:=diff(y(x),x,x)=(-x)*diff(y(x),x);
dsolve([de,y(0)=y[0],D(y)(0)=y[1] ],y(x));
p:=dsolve([de,y(0)=y[0],D(y)(0)=y[1] ],y(x),series);
subs(y[0]=1,y[1]=0,p);subs(y[0]=0,y[1]=1,p);
# Simplification
q:=k->product((-1)*(2*r+1)/((2*r+2)*(2*r+3)),r=0..k):
q(k);simplify(q(k));
sum(q(k)*x^(2*k+3),k=0..8);
```

6. $y'' + x^2 y' + y = 0$

12.5 Regular Singular Points

Regular Singular Point

Test the equation for regular singular points.

1. $x^2y'' + xy' + y = 0$

Solution: Regular singular point at $x = 0$.

2. $x^2(x-1)y'' + \sin(x)y' + y = 0$

3. $x^3(x^2-1)y'' - x(x+1)y' + (1-x)y = 0$

Solution: Regular singular points at $x = 0, -1, 1$.

4. $x^3(x-1)y'' + (x-1)y' + 2xy = 0$

Indicial Equation

Each equation is an Euler differential equation $ax^2y'' + bxy' + cy = 0$ with a, b, c replaced by power series. Find the Euler differential equation and the indicial equation.

5. $x^2y'' - 2x(x+1)y' + (x-1)y = 0$
Ans: $x^2y'' - 2xy' - y = 0$, $r(r-1) - 2r - 1 = 0$.

Solution: The equation in standard Frobenius form is $ax^2y' + bxy' + cy = 0$ with power series $a = 1$, $b = -2 - 2x$, $c = 1 - x$, all with a finite number of power series terms.

The regular singular point is $x = 0$. Substitute $x = 0$ into a, b, c to get the Cauchy-Euler equation $(1)x^2y''+(-2)xy'+(1)y = 0$. The indicial equation is the characteristic equation for the associated constant-coefficient equation $(1)(D-1)Dz + (-2)Dz + (1)z = 0$. Simplify the constant equation to $z'' - 3z' + z = 0$ and report indicial equation $r^2 - 3r + 1 = 0$.

Shortcut: Report indicial equation $(1)(r-1)r+(-2)r+(1) = 0$ by replacing D by r. ■

6. $x^2y'' - 2xy' + y = 0$
Ans: The same equation, $r(r-1) - 2r + 1 = 0$.

7. $xy'' + (1-x)y' + 2y = 0$

Solution: Multiply by x to get a Frobenius equation $x^2y''+x(1-x)y'+2xy = 0$. The associated Cauchy-Euler equation is $x^2y'' + x(1)y' + (0)y = 0$. The indicial equation $(r-1)r + r = 0$ is obtained from the constant-coefficient operator form $(D-1)Dz + Dz = 0$ by replacement $D \to r$. ■

8. $x^2y'' - 2xy' + (2 + \sin x)y = 0$

Solution:
Let $a = 1$, $b = -2$, $c = 2 + \sin x$ to identify the Frobenius equation $ax^2y'' + bxy' + cy = 0$ with associated Cauchy-Euler equation $x^2y'' - 2xy' + 2y = 0$ (replace $x = 0$ in a, b, c). Remaining details and answers omitted.

Frobenius Solutions

Find two linearly independent solutions. Follow Examples 1, 2, 3 for cases (a), (b), (c) in the Frobenius Theorem page 970. Examples: **(a)** page 971, **(b)** page 973, **(c)** page 977.

9. $2x^2y'' + xy' - y = 0$

Solution: This is a Frobenius equation and also a Cauchy-Euler equation. There is an exact solution:

$y_1 = x$, $y_2 = x^{-1/2}$.

It is not necessary to apply the Frobenius theorem.

```
# Exercise 9, Frobenius solutions
de:=2*x^2*diff(y(x),x,x)+x*diff(y(x),x)-y(x)=0;
dsolve(de,y(x));# Exact solution
Order:=10;dsolve(de,y(x),series);# Series solution
```

■

10. $4x^2y'' + (2x - 7)y' + 6y = 0$

11. $4x^2(x + 1)y'' + x(3x - 1)y' + y = 0$

Solution:
There is a regular singular point at $x = -1$ and also at $x = 0$. Let's focus on $x = 0$ for simplicity. Define $a = 4(x+1)$, $b = 3x-1$, $c = 1$ to form Frobenius equation $ax^2y'' + bxy' + cy = 0$. The associated Cauchy-Euler equation is $4x^2y'' - xy' + y = 0$ and then the indicial equation is $4r(r - 1) - r + 1 = 0$ with larger root $r_1 = 1$ and smaller root $r_2 = \frac{1}{4}$. The problem falls into case **(a)** of the Frobenius theorem page 971: $r_1 \neq r_2$ and $r_1 - r_2$ not an integer. There are two Frobenius series solutions.

$$y_1 = x\left(1 - \frac{3}{7}x + \frac{9\,x^2}{77} - \frac{9\,x^3}{385} + \frac{27\,x^4}{7315} + \cdots\right)$$

$$y_2 = \sqrt[4]{x}\left(1 - \frac{3}{4}x + \frac{9\,x^2}{32} - \frac{9\,x^3}{128} + \frac{27\,x^4}{2048} + \cdots\right)$$

The length details follow Example 12.1 page 971.

```
# Exercise 11, Frobenius solutions, type (a)
de:=4*x^2*diff(y(x),x,x)+x*(3*x-1)*diff(y(x),x)+y(x)=0;
solve(4*r*(r-1)-r+1=0,r);
dsolve(de,y(x));# Exact solution
Order:=10;dsolve(de,y(x),series);# Series solution
```

■

12. $3x^2y'' + xy' - (1+x)y = 0$

Solution: Roots 1, $-1/3$. Case (a) of the Frobenius theorem. Details omitted.

13. $x^2y'' + 3xy' + (1+x)y = 0$

Solution: There is a regular singular point at $x = 0$. Define $a = 1$, $b = 3$, $c = 1+x$ to form Frobenius equation $ax^2y'' + bxy' + cy = 0$. The associated Cauchy-Euler equation is $x^2y'' + 3xy' + y = 0$ and then the indicial equation is $r(r-1) + 3r + 1 = 0$ with equal roots $r_1 = -1$ and $r_2 = -1$. The problem falls into case **(c)** of the Frobenius theorem page 971: $r_1 = r_2$. There are two Frobenius series solutions.

$$
\begin{aligned}
y_1 &= \frac{1}{x}\left(1 - x + \frac{1}{4}\,x^2 - \frac{1}{36}\,x^3 + \frac{x^4}{576} + \cdots\right)\\
y_2 &= \frac{\ln|x|}{x}\left(1 - x + \frac{1}{4}\,x^2 - \frac{1}{36}\,x^3 + \frac{x^4}{576} + \cdots\right)\\
&\quad + \frac{1}{x}\left(2\,x - \frac{3}{4}\,x^2 + \frac{11\,x^3}{108} - \frac{25\,x^4}{3456} + \cdots\right)
\end{aligned}
$$

The details follow Example 12.3 page 977.

```
# Exercise 13, Frobenius solutions, type (c)
de:=x^2*diff(y(x),x,x)+3*x*diff(y(x),x)+(1+x)*y(x)=0;
solve(r*(r-1)+3*r+1=0,r);
dsolve(de,y(x));# Exact solution
Order:=10;dsolve(de,y(x),series);# Series solution
```

■

14. $xy'' + (1-x)y' + 3y = 0$

Solution: Roots $0, 0$. Case (c) of the Frobenius theorem. Details omitted.

15. $x^2y'' + x(x-1)y' + (1-x)y = 0$

Solution:
There is a regular singular point at $x = 0$. Define $A = 1$, $B = -1$, $C = 1$ to form Cauchy-Euler equation $Ax^2y'' + Bxy' + Cy = 0$, which is $x^2y'' - xy' + y = 0$. Then the indicial equation is $r(r-1) - r + 1 = 0$ with equal roots $r_1 = 1$ and $r_2 = 1$. The problem falls into case **(c)** of the Frobenius theorem page 977: $r_1 = r_2$. The exact solution involves the **exponential integral function** $Ei(x)$, not discussed in the textbook, and not discussed here either.

There are two Frobenius series solutions:

$$y_1 = x$$

$$y_2 = x\ln|x|\,1 + x\left(-x + 1/4\,x^2 - 1/18\,x^3 + \frac{x^4}{96} + \cdots\right)$$

The details follow Example 12.3 page 977.

```
# Exercise 15, Frobenius solutions, type (c)
de:=x^2*diff(y(x),x,x)+x*(x-1)*diff(y(x),x)+(1-x)*y(x)=0;
solve(r*(r-1)-r + 1=0,r);
dsolve(de,y(x));# Exact solution
Order:=10;dsolve(de,y(x),series);# Series solution
```

■

16. $xy'' + (2x+3)y' + 4y = 0$

Solution: Roots 0 and -2, which are unequal and differ by an integer. Case (b) of the Frobenius theorem. The details are especially involved but follow case (b) Example 12.2 page 973.

The answers:

$$y_1 = 1 - \frac{4}{3}\,x + x^2 - \frac{8\,x^3}{15} + \frac{2}{9}\,x^4 + \cdots$$

$$y_2 = \frac{1}{x^2}\left(-2 + 4\,x^2 - \frac{16}{3}\,x^3 + 4\,x^4 + \cdots\right)$$

The exact answer: $y = \dfrac{c_1}{x^2} + \dfrac{c_2\,e^{-2\,x}\,(2\,x+1)}{x^2}$

```
# Exercise 16, Frobenius solutions, type (b)
de:=x^2*diff(y(x),x,x)+x*(2*x+3)*diff(y(x),x)+4*x*y(x)=0;
solve(r*(r-1)+3*r + 0=0,r);
dsolve(de,y(x));# Exact solution
Order:=5;dsolve(de,y(x),series);# Series solution
```

■

12.6 Bessel Functions

Values of J_0 and J_1

Use series representations and identities to find an identity for values of the following functions. Use a computer algebra system to compute the answers.

1. $J_0(1)$

Solution:
Identity for J_0.
Let $p = 0$ in the series identities. Because $J_0(-x) = J_0(x)$ (J_0 is even), then only even term are present in the series:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{(n!)^2}$$

Then

$$\begin{aligned} J_0(1) &= \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n}}{(n!)^2} \\ &= 0.7651976866 \text{ by the } \texttt{maple} \text{ code } \textit{infra}. \end{aligned}$$

Most computer systems support GNU C library functions, which includes the Bessel functions with function names $J0, J1, Jn$. A convenient online site is https://www.wolframalpha.com/, which provides a free online calculator for Bessel functions. The Wolfram answer:

$J_0(1) \approx$ 0.76519768655796655144971752610266322090927428975532524186154754 91

```
# Exercise 1, Values of J[0] and J[1]
J[0](1)=evalf(BesselJ(0,1));
sum((-1)^n * (1/2)^(2*n) / (n!)^2,n=0..infinity);
# Reported:  BesselJ(0, 1)
```
■

2. $J_1(1)$

3. $J_0(1/2)$

Solution: Answer: $J_0(1/2) = .9384698072$

```
# Exercise 3, Values of J[0] and J[1]
J[0](1/2)=evalf(BesselJ(0,1/2));
sum((-1)^n * (1/4)^(2*n) / (n!)^2,n=0..infinity);
```
■

4. $J_1(1/2)$

Bessel Function Properties

Prove the following relations by expanding LHS and RHS in series.

5. $J_0'(x) = -J_1(x)$

Solution:

$$\begin{aligned}
\text{LHS} &= J_0'(x) \\
&= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{(n!)^2} \\
&= \sum_{n=1}^{\infty}(2n)(1/2)\frac{(-1)^n(x/2)^{2n-1}}{(n!)^2} && \text{because } \tfrac{d}{dx} \text{ erases the } n=0 \text{ term} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^n(x/2)^{2n-1}}{(n-1)!\,(n!)} && \text{cancel common factors } n \text{ and } 2 \\
\text{RHS} &= -J_1(x) \\
&= -\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{1+2k}}{k!(1+k)!} && \text{by the } J_p \text{ identity page 981.} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}(x/2)^{1+2k}}{k!(1+k)!} && \text{move minus sign inside summation} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^n(x/2)^{2n-1}}{(n-1)!(n)!} && \text{index change } 2n-1 = 1+2k \ (n-1=k)
\end{aligned}$$

Then LHS = RHS, proving the identity. ■

6. $J_1'(x) = J_0(x) - \dfrac{1}{x}J_1(x)$

7. $(x^pJ_p(x))' = x^pJ_{p-1}(x)$,
$p \ge 1$

Solution: Assume $p \ge 1$. The J_p identity:

$$J_p = \sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{p+2n}}{n!(n+p)!}$$

Then the left side of the claimed identity is

$$\begin{aligned}
\text{LHS} &= (x^pJ_p(x))' \\
&= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2p+2n}\,2^p}{n!(n+p)!} && \text{move } x^p \text{ inside summation} \\
&= \sum_{n=0}^{\infty}\frac{(2p+2n)(1/2)(-1)^n(x/2)^{2p+2n-1}\,2^p}{n!(n+p)!} && \tfrac{d}{dx} \text{ term-by-term} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2p+2n-1}\,2^p}{n!(n+p-1)!} && \text{cancel common factors}
\end{aligned}$$

$$\begin{aligned} \text{RHS} &= x^p J_{p=1}(x) \\ &= x^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2p-1+2n}}{n!(n+p-1)!} && \text{by the } J_p \text{ identity} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2p-1+2n}\, 2^p}{n!(n+p-1)!} && \text{move } x^p \text{ inside summation} \end{aligned}$$

Then LHS = RHS, proving the identity. ■

8. $\left(x^{-p}J_p(x)\right)' = -x^{-p}J_{p+1}(x)$,
$p \geq 0$

Bessel Function Recursion Proofs

Add and subtract the expanded equations of the previous exercises.

9. $J_{p+1} = \dfrac{2p}{x} J_p(x) - J_{p-1}(x)$,
$p \geq 1$

Solution:

$\boxed{1}$ Given $(x^p J_p(x))' = x^p J_{p-1}(x)$, $p \geq 1$ from Exercise 7. Expand and divide by x^p:

$J_p' + (p/x)J_p = J_{p-1}$

$\boxed{2}$ Given $\left(x^{-p}J_p(x)\right)' = -x^{-p}J_{p+1}(x)$, $p \geq 0$ from Exercise 8. Expand and divide by x^{-p}:

$-(p/x)J_p + J_p' = -J_{p+1}$

Subtract $\boxed{2}$ from $\boxed{1}$:

$(p/x)J_p + (p/x)J_p = J_{p-1} + J_{p+1}$

$(p/x)J_p + (p/x)J_p - J_{p-1} = J_{p+1}$

which proves the claimed identity. ■

10. $J_{p+1}(x) = -2J_p'(x) + J_{p-1}(x)$,
$p \geq 1$

Recurrence Relations

Use results of the previous exercises.

11. Express J_3 and J_4 in terms of J_0 and J_1.

Solution:

Given $J_{p+1} = \dfrac{2p}{x} J_p(x) - J_{p-1}(x)$ from Exercise 9, insert $p = 1$, $p = 2$, $p = 3$ to get identities

$$\begin{aligned}
J_2 &= \frac{2}{x} J_1(x) - J_0(x) \\
J_3 &= \frac{4}{x} J_2(x) - J_1(x) \\
&= \frac{4}{x}(\frac{2}{x} J_1(x) - J_0(x)) - J_1(x) \\
&= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x) \\
J_4 &= \frac{6}{x} J_3(x) - J_2(x) \\
&= \frac{6}{x}\left(\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right) - J_2(x) \\
&= \frac{6}{x}\left(\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right) - \frac{2}{x} J_1(x) + J_0(x) \\
&= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)
\end{aligned}$$

■

12. Prove by induction that $J_p(x) = c_1(1/x)J_0(x) + c_2(1/x)J_1(x)$ where c_1 and c_2 are polynomials.

Laplace Transform

Assume Laplace identity $\mathcal{L}(J_n(t)) = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}}$ holds for $s \geq 0$. Prove the following results.

13. $\int_0^\infty J_{n+1}(x)dx = \int_0^\infty J_{n-1}(x)dx$

for integers $n > 0$.

Solution: The integrals left and right are obtained from the corresponding Laplace integral $\int_0^\infty f(t)e^{-st}dt$ by setting $f(t) = J_{n+1}(t)$ or $f(t) = J_{n-1}(t)$ and then $s = 0$. In the Laplace identity for J_n, power $n \to n+1$ or $n \to n-1$ is applied to $((0+1)-0)^n$ to give factor 1, then both sides of the proposed identity match.

The identity may also be proved from Exercise 10 by integrating the identity therein. The catch: the additional result $\lim_{x\to\infty} J_p(x) = 0$ is required along with integrability of all functions appearing in Exercise 10. ■

14. $$\int_0^\infty \frac{J_n(x)dx}{x} = \frac{1}{n}$$

for integers $n > 0$

Solution: Hint: Use the details from Exercise 13.

Bessel Function Bounds

Assume L. J. Landau's result $J_p(x) \le c|x|^{-1/3}$ for all x and $p > 0$, where $c = 0.78574687\ldots$ is the best possible constant. Prove the following results.

15. $\lim_{x\to\infty} J_1(x) = 0$

Solution: Limit $x \to \infty$ across Landau's inequality.

16. $\lim_{x\to\infty} J_0'(x) = 0$

12.7 Legendre Polynomials

Equivalent Legendre Equations

Prove the following are equivalent to
$(1-x^2)y''-2xy'+n(n+1)y=0$

1. $((1-x^2)y')' + n(n+1)y = 0$

Solution:
Expand by the calculus product rule

$$\begin{aligned}((1-x^2)y')' &= (1-x^2)'y' + (1-x^2)y'' \\ &= -2xy' + (1-x^2)y''\end{aligned}$$

■

2. Let $x = \cos\theta$, $' = \frac{d}{d\theta}$, then
$\sin\theta y''+\cos\theta y'+n(n+1)\sin\theta y=0$.

Solution:
Use $\frac{dy}{dx} = \frac{dy}{d\theta}\frac{dx}{d\theta} = -y'\sin(\theta)$ and similarly for $\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}$. Details omitted.

Proof of Bonnet's Recursion

3. Define $c_n = \frac{1}{n!2^n}$.
Prove $c_m = 2(m+1)c_{m+1}$.

Solution:
This is Lemma A in the proof of Bonnet's recursion by Rodrigues' formula.

$$\begin{aligned}2(m+1)c_{m+1} &= \frac{2(m+1)}{(m+1)!2^{m+1}} \\ &= \frac{2(m+1)}{2(m+1)\,m!2^m} \\ &= \frac{1}{m!2^m} \\ &= c_m \quad ■\end{aligned}$$

4. Let $D = \frac{d}{dx}$, $u = x^2 - 1$. Verify $D^2u^2 = 12x^2 - 4$ using D and the binomial theorem.

Solution:

$$\begin{aligned}D^2(u^2) &= D^2((x^2-1)^2) \\ &= D^2\textstyle\sum_{r=0}^{2}\binom{2}{r}x^{2r}(-1)^{2-r} \qquad \text{by the binomial theorem} \\ &= \textstyle\sum_{r=0}^{2}\binom{2}{r}D^2(x^{2r})(-1)^{2-r}\end{aligned}$$

$$= \binom{2}{0}(0)(-1)^2 + \binom{2}{1}(2)(-1) + \binom{2}{2}(4)(3)(x^2)(-1)^0$$
$$= 0 + (2)(2)(-1) + (1)(4)(3)(x^2)$$
$$= 12x^2 - 4 \quad \blacksquare$$

5. Prove Bonnet's recursion from the generating function equation

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Solution:
Historically, Bonnet's recursion was discovered by differentiation of the generating function on t to obtain

$$(1) \quad (x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

Then match coefficients of t^n to find Bonnet's recursion. Reference:

https://en.wikipedia.org/wiki/Legendre_polynomials

Series techniques are used to re-write each side of equation (1) as one series indexed on t^k. This step is subject to error. The `maple` code below can check the work.

Coefficient matching gives the following equations:

$P_1 = xP_0$

$2P_2 = 3xP_1 - P_0$

$3P_3 = 5xP_2 - 2P_1$

$4P_4 = 7xP_3 - 3P_2$

The pattern is Bonnet's recursion

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$$

Bonnet's recursion can be proved by other means. The text proves the recursion using Rodrigues' formula for P_n. $\blacksquare$

```
# Exercise 5, Proof of Bonnet's recursion
# Check coefficient matching, first 10 terms
A:=n->(t^2-2*x*t+1)*n*P[n]*t^(n-1);
B:=n->(x-t)*P[n]*t^n;
q := N->-sum(A(n),n=1..N)+sum(B(n),n=0..N);
seq([coeff(q(10),t,j)],j=0..10);
```

6. Prove that $P_n(1) = 1$ directly from Rodrigues' formula.

Solution:
The trick is write $(x^2-1)^n = vw$ where $v = (x-1)^n$, $w = (x+1)^n$. Expand with the Leibnitz rule

$$D^n(vw) = \sum_{r=0}^{n} \binom{n}{r} (D^r v)(D^{n-r} w)$$

Then $D^r v$ at $x = 1$ is zero except for $r = n$, so the expansion at $x = 1$ has a single term. Evaluate the single term to prove $P_n(1) = \frac{1}{n!2^n} D^n((x^2-1)^n)\big|_{x=1} = 1$. ■

Boundary Data at $x = \pm 1$

Use these identities:
(1) $(a+b)^k = \sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r}$
(2) $(uv)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$

7. In Rodrigues' formula, let Let $y = x - 1$ to prove
$P_n(y+1) = \frac{1}{n!2^n} \left(\frac{d}{dy}\right)^n \left(y^2 + 2y\right)^n$

Solution:
Let $u = x^2 - 1$, $D = \frac{d}{dx}$, $c_n = \frac{1}{n!2^n}$. Then $u == (y+1)^2 - 1 = y^2 + y$ and $\frac{d}{dy} = \frac{d}{dx}$. The calculus chain rule then implies

$n!2^n P_n(y+1) = D^n u^n = (\frac{d}{dy})^n (y^2 + y)^n$. ■

8. Verify from identity (1):
$\left(y^2+2y\right)^n = \sum_{r=0}^{n} \binom{n}{r} 2^r y^{2n-r}$

9. Prove $P_n(1) = 1$ from Bonnet's recursion.

Solution:
Proceed by induction. Cases $P_0(1) = 1$ and $P_1(1) = 1$ are proved by identities $P_0(x) = 1$, $P_1(x) = x$. Assume $n > 1$ and induction hypothesis $P_k(1) = 1$ for all $0 \le k \le n$. Then

$(n+1)P_{n+1}(1) = (2n+1)P_n(1) - nP_{n-1}(1) = (2n+1(1) - 2n(1) = 1.$

The induction is complete. ■

10. Assume $P_n(-x) = (-1)^n P_n(x)$ and $P_n'(1) = \dfrac{n(n+1)}{2}$. Prove
$P_n(-1) = (-1)^n$ and
$P_n'(-1) = (-1)^n \dfrac{n(n+1)}{2}$.

Legendre Integrals

Use Legendre properties page 986 .

11. Use $(2n+1)P_n = P'_{n+1} - P'_{n-1}$ to prove $\int_0^1 P_n(x)dx = 0$ for $n > 0$ even.

Solution:

Exercise 9 proves $P_k(1) = 1$ for all $k \geq 0$. For k odd, P_k has only odd powers in its series, therefore $P_k(0) = 0$. The fundamental theorem of calculus gives

$$\begin{aligned}(2n+1)\int_0^1 P_n(x)dx &= \int_0^1 (P'_{n+1}(x) - P'_{n-1}(x))dx \\ &= P_{n+1}(1) - P_n(0) - P_{n-1}(1) + P_{n-1}(0) \\ &= 1 - 1 - 0 + 0 \quad \text{because } n-1 \text{ and } n-1 \text{ are both odd}\end{aligned}$$

when n is even. ■

12. Use Bonnet's recursion to show that $\int_0^1 P_n(x)dx = \frac{P_{n-1}(0)}{n+1}$ for $n > 0$.

12.8 Orthogonality

Legendre series. Establish the following results.

1. Prove using orthogonality that $\int_{-1}^{1} P_n(x)F(x)dx = 0$ for any polynomial $F(x)$ of degree less than n.

Solution: Let F have degree $m < n$. Orthogonality makes $P_0, \ldots, P_{n-1}$ independent, a basis for the vector space of all polynomials of degree less than n. Because $m < n$ then $F(x) = \sum_{k=0}^{n-1} c_k P_k(x)$ holds for some coefficients $\{c_k\}$. Use orthogonality of P_n with $P_0, \ldots, P_{n-1}$:

$$\int_{-1}^{1} P_n(x)F(x)dx = \sum_{k=0}^{n-1} c_k P_n(x)P_k(x)dx = 0$$

■

2. Use identity
$xP_n'(x) - P_{n-1}'(x) = nP_n(x)$
to prove $\int_{-1}^{1} |P_n(x)|^2 dx = \frac{2}{2n+1}$.

Solution: The provided identity is derived from the two basic Legendre identities involving derivatives of P_n, found in section 12.7 page 986. Let $I = \int_{-1}^{1} P_n^2 dx$. Integrate by parts using $u = P_n^2$, $dv = dx$ to find an equation with I on both sides, then solve for I, which depends on P_n and P_n'. Replace factor xP_n' in the expression for I by using the provided identity. Use Exercise 1 to eliminate the term involving P_{n-1}'. Identities $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ are required. Details omitted. ■

3. Let $\left\langle f, g \right\rangle = \int_0^{\pi} f(x)g(x)\sin(x)dx$. Show that the sequence $\{P_n(\cos x)\}$ is orthogonal on $0 \le x \le \pi$ with respect to inner product $\left\langle f, g \right\rangle$.

Solution:
The plan: change variables $x = \cos t$ in inner product $\left\langle f, g \right\rangle$.

$$\begin{aligned}
\left\langle f, g \right\rangle &= \textstyle\int_0^{\pi} f(t)g(t)\sin(t)dt \\
&= \textstyle\int_1^{-1} f(\arccos x)g(\arccos x)(-dx) \\
&= \textstyle\int_{-1}^{1} f(\arccos x)g(\arccos x)dx \\
&= (f(\arccos x), g(\arccos x)) \quad \text{where } (F,G) = \textstyle\int_{-1}^{1} FG dx
\end{aligned}$$

Let $f(t) = P_n(\cos t)$, $g(t) = P_m(\cos t)$. Then $f(\arccos x) = P_n(x)$, $g(\arccos x) = P_m(x)$ and

$$\left\langle f, g \right\rangle = (P_n, P_m)$$

which is zero for $n \neq m$ by orthogonality of the Legendre polynomials. The sequence $\{P_n(\cos x)\}$ is orthogonal. ■

4. Let $F(x) = \sin^3(x) - \sin(x)\cos(x)$. Expand F as a Legendre series $F(x) = \sum_{n=0}^{\infty} c_n P_n(\cos x)$.

Solution: The coefficients are shadow projections using the inner product in Exercise 3. Details omitted.

Chebyshev Series. The **Chebyshev polynomials** are $T_n(x) = \cos(n \arccos(x))$ with inner product $(f,g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{-1/2}dx$.

5. Show that $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$.

Solution:

The identities are proved from trig identities.

$$\begin{aligned}
T_0(x) &= \cos(0 \arccos(x)) \\
&= \cos(0) \\
&= 1 \\
T_1(x) &= \cos((1) \arccos(x)) \\
&= x \\
T_2(x) &= \cos(2 \arccos(x)) \\
&= \cos(2\theta) \quad \text{where } x = \cos\theta \\
&= 2\cos^2\theta - 1 \quad \text{by identity } \cos(2\theta) = 2\cos^2\theta - 1 \\
&= 2x^2 - 1 \quad ■
\end{aligned}$$

6. Show that $T_3(x) = 4x^3 - 3x$.

7. Prove that (f,g) satisfies the abstract properties of an inner product.

Solution: Singularities of the integrand present a problem, because of the division by $\sqrt{1-x^2}$. Let's find another expression for the inner product where f, g are restricted to remove the singularities.

$$\begin{aligned}
(f,g) &= \int_{-1}^{1} f(x)g(x)(1-x^2)^{-1/2}dx \\
&= \int_{\pi}^{0} f(\cos t)g(\cos t)(1-\cos^2 t)^{-1/2}(-\sin t)dt \quad \text{where } x = \cos t \\
&= \int_{0}^{\pi} f(\cos t)g(\cos t)(\sin^2 t)^{-1/2} \sin t\, dt \quad \text{by } \cos^2 t + \sin^2 t = 1 \\
&= \int_{0}^{\pi} F(t)G(t)|\sin t|^{-1} \sin t\, dt \quad \text{where } F(t) = f(\cos t),\ G(t) = g(\cos t) \\
&= \int_{0}^{\pi} F(t)G(t)\, dt \quad \text{because } \sin t > 0 \text{ on } 0 < t < \pi.
\end{aligned}$$

There is an issue at $t = 0$ and $t = \pi$: the singularity is removable by examination of limits. However, the resulting integral is a known inner product on the vector space of continuous functions.

To be proved: the inner product is defined when f, g are continuous. Let $\rho(x) = (1 = x^2)^{-1/2}$ on $-1 < x < 1$ and $\rho(\pm 1) = 0$. Let $|f(x)| \le M_1$, $|g(x)| \le M_2$ on $-1 \le x \le 1$. Then Details:

$$\left|\int_{-1}^{1} fg\rho\,dx\right| \le |\int_{-1}^{1} |f||g|\rho\,dx$$

$$\le |\int_{-1}^{1} M_1 M_2 \rho\,dx$$

$$= \pi M_1 M_2$$

The integral exists for f, g continuous and (f, g) makes sense on the vector space of continuous functions on $[-1, 1]$. Because compositions of continuous functions are continuous then F, G are continuous and (f, g) is an inner product satisfying the abstract properties. ■

8. Show that T_n is a solution of the **Chebyshev equation** $(1 - x^2)y'' - xy' + n^2 y = 0$.

9. Prove that $\{T_n\}$ is orthogonal relative to the weighted inner product (f, g).

Solution:

Let $\rho(x) = (1 - x^2)^{-1/2}$. To be proved:

(1) $\int_{-1}^{1} T_n T_m \rho\,dx = 0$ for $n \ne m$

(2) $\int_{-1}^{1} T_0 T_0 \rho\,dx = \pi$

(3) $\int_{-1}^{1} T_n T_n \rho\,dx = \frac{\pi}{2}$ for $n > 0$

Exercise 7 provides this formula:

$\int_{-1}^{1} fg\rho\,dx = \int_0^{\pi} F(t)G(t)\,dt$ where $F(t) = f(\cos t)$, $G(t) = g(\cos t)$

Let $f = T_n$, $g = T_m$. Then

$$F(t) = T_n(\cos t)$$
$$= \cos(n \arccos(\cos t))$$
$$= \cos(nt). \text{ Then:}$$

$$\int_{-1}^{1} fg\rho\,dx = \int_0^{\pi} \cos(nt)\cos(mt)\,dt$$

Proof of (1)

Orthogonality of the trig functions $\cos nt, \sin nt$ on $[-\pi, \pi]$ implies

$$\int_{-1}^{1} fg\rho\,dx = \int_0^{\pi} \cos(nt)\cos(mt)\,dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,dt \quad \text{because of an even integrand}$$

$$= 0 \quad \text{for } n \neq m$$

Proof of (2)
The integral of 1 over $[0, \pi]$ is π.

Proof of (3)
The problem reduces to the integral over $[0, \pi]$ of $\cos^2(nx)$, which is $\pi/2$ for integers $n > 0$. ■

10. Prove: $T_n(x)$ is an even function for n even and an odd function for n odd.

Hermite Polynomials. Define the Hermite polynomials by $H_0(x) = 1$, $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$.
Define the inner product
$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx$.

11. Verify: $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$.

Solution:
Let $u = e^{-x^2}$, $D = \dfrac{d}{dx}$. Then

$Du = -2xu$ and $H_1 = (-1)\dfrac{1}{u}Du = 2x$.
$D^2u = D(-2xu) = -2u - 2x(-2xu) = (4x^2 - 2)u$. Then

$H_2 = (-1)^2\dfrac{1}{u}D^2u = 4x^2 - 2$

$D^3u = D(D^2u) = D((4x^2-2)u) = 8xu + (4x^2-2)(-2xu) = (-8x^3+12x)u$.
Then

$H_3 = (-1)^3\dfrac{1}{u}(8x - 8x^3 + 4x)u = 8x^3 - 12x$

$D^4u = D((-8x^3 + 12x)u) = (-24x^2 + 12)u + (-8x^3 + 12x)(-2x)u$

$H_4 = (-1)^4\dfrac{1}{u}((-24x^2+12)u+(-8x^3+12x)(-2x)u) = -48x^2+12+16x^4$ ■

```
# Exercise 11, Answer check
seq(simplify(HermiteH(i,x)),i=0..4);
```

12. Prove: $H_n(-x) = (-1)^n H_n(x)$.

13. Prove $H_n'(x) = 2xH_n(x) - H_{n+1}(x)$.
Then use recursion $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ to show $H_n'(x) = 2nH_{n-1}(x)$.

Solution:
Let $u = e^{-x^2}$, $D = \dfrac{d}{dx}$. Then

$$
\begin{aligned}
H_n' &= (-1)^n D(u^{-1}D^n u) \\
&= (-1)^n \left((-1)u^{-2}(-2xu)D^n u + u^{-1}D^{n+1}u\right) \quad \text{because } Du = -2xu \\
&= 2x(-1)^n u^{-1}D^n u + (-1)^n u^{-1}D^{n+1}u \\
&= 2xH_n - H_{n+1}
\end{aligned}
$$

Recursion identity $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ is inserted into the preceding identity to give

$$
\begin{aligned}
H_n'(x) &= 2xH_n(x) - H_{n+1}(x) \\
&= 2xH_n(x) - (\boldsymbol{2xH_n(x) - 2nH_{n-1}(x)}) \\
&= 2nH_(n-1)(x) \quad \blacksquare
\end{aligned}
$$

14. Let $y = H_5 = 32x^5 - 160x^3 + 120x$. Show y satisfies **Hermite's equation** $y'' - 2xy' + 2ny = 0$ with $n = 5$.

Solution: Answer check:

```
# Exercise 14, Answer check
de:=diff(y(x),x,x) -2*x*diff(y(x),x) + 2*n*y(x)=0;
p:=subs(n=5,y(x)=32*x^5-160*x^3+120*x,de);
simplify(p);
```

15. Prove recursion
$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

Solution:

Let $u = e^{-x^2}$, $D = \dfrac{d}{dx}$. Then $H_{n+1} = (-1)^{n+1}u^{-1}D^{n+1}u$

$$
\begin{aligned}
&= (-1)^{n+1}u^{-1}D^n Du \\
&= (-1)^{n+1}u^{-1}D^n(-2xu) \quad \text{because } Du = -2xu \\
&= 2(-1)^n u^{-1}D^n(xu) \\
&= 2(-1)^n u^{-1}\sum_{r=0}^{n}\binom{n}{r} D^r x\, D^{n-r}u \quad \text{by the Leibniz rule} \\
&= 2(-1)^n u^{-1}\left(\binom{n}{0} xD^n u + \binom{n}{1}(1)D^{n-1}u\right) \\
&= 2x(-1)^n u^{-1}u^{-1}D^n u + 2n(-1)^n u^{-1}D^{n-1}u \\
&= 2xH_n + 2nH_{n-1} \quad \blacksquare
\end{aligned}
$$

16. Show that the sequence $\{H_n(x)\}$ is orthogonal with respect to (f, g).

Alternate Laguerre Polynomials. Define the alternate Laguerre polynomials by $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$. Define $(f, g) = \int_0^\infty f(x)g(x)e^{-x}dx$. A **warning**: Laguerre polynomials in the literature are $\frac{1}{n!}L_n$.

17. Prove: $L_1(x) = 1 - x$ and
$L_2(x) = 2 - 4x + x^2$.

Solution:

$$\begin{aligned} L_1 &= e^x(xe^{-x})' \\ &= e^x(e^{-x} - xe^{-x}) = 1 - x \\ L_2 &= e^x(x^2e^{-x})'' \\ &= e^x(2e^{-x} - 4xe^{-x} + x^2e^{-x}) \\ &= 2 - 4x + x^2 \end{aligned}$$

■

18. Prove:
$L_3(x) = 6 - 18x + 9x^2 - x^3$.

19. Prove that (f, g) satisfies the abstract properties for an inner product.

Solution:

Non-negativity: $(f, f) = \int_0^\infty |f(x)|^2 e^{-x} dx \geq 0$

Uniqueness: $(f, f) = \int_0^\infty |f(x)|^2 e^{-x} dx = 0$ implies integrand $f = 0$

Symmetry: $(f, g) = \int_0^\infty f(x)g(x)e^{-x} dx = (g, f)$ because $fg = gf$.

Homogeneity:

$k(f, g) = \int_0^\infty kf(x)g(x)e^{-x} dx$

$(kf, g) = \int_0^\infty (kf(x))g(x)e^{-x} dx$

Therefore $k(f, g) = (kf, g)$.

Additivity:

$$\begin{aligned} (f + g, h) &= \int_0^\infty (f(x) + g(x))h(x)e^{-x} dx \\ &= \int_0^\infty f(x)h(x)e^{-x} dx + \int_0^\infty g(x)h(x)e^{-x} dx \\ &= (f, h) + (g, h) \end{aligned}$$

■

20. Show that L_0, L_1, L_2, L_3 are orthogonal with respect to the inner product (f, g), using direct integration methods.

Solution: By definition, $L_0 = 1$. Use Exercises 17 and 18.

21. Prove:
$L_n(x) = \sum_{k=0}^n \frac{(-1)^k (n!)^2}{(n-k)!(k!)^2} x^k$.

Solution: The method of proof is direct expansion of the formula for L_n using the Leibnitz formula

$$D^n(vw) = \sum_{r=0}^n \binom{n}{r} (D^r v)(D^{n-r} w), \quad D = \frac{d}{dx}$$

Let $v = x^n$ and $w = e^{-x}$. Then

$$D^n(vw) = \sum_{r=0}^{n} \binom{n}{r} (D^r v)(D^{n-r} w)$$
$$= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \left(\frac{n!}{(n-r)!} x^{n-r} \right) ((-1)^{n-r} e^{-x})$$

Change index with $k = n - r$. Then

$$D^n(vw) = \sum_{k=0}^{n} \frac{n!}{(n-k)!(k)!} \left(\frac{n!}{(k)!} x^k \right) ((-1)^k e^{-x})$$
$$= \sum_{k=0}^{n} \frac{(n!)^2}{(n-k)!(k!)^2} (-1)^k x^k e^{-x}$$

Multiply by e^x:

$$e^x D^n(vw) = \sum_{k=0}^{n} \frac{(n!)^2}{(n-k)!(k!)^2} (-1)^k x^k$$

Then $L_n = e^x D^n(vw) = \sum_{k=0}^{n} \frac{(n!)^2}{(n-k)!(k!)^2} (-1)^k x^k$.

■

22. Show that $\{L_n\}$ is an orthogonal sequence with respect to (f, g).

Solution: Hint: Use Laguerre's differential equation and the same integration tricks as for Legendre's equation.

23. Find an expression for a polynomial solution to **Laguerre's equation** $xy'' + (1-x)y' + ny = 0$ using Frobenius theory.

Solution:
The standard form is $x^2y'' + x(1-x)y' + nxy = 0$, where $n \geq 0$ is an integer. The indicial equation is $r(r-1) + (1)r + 0 = 0$ with double root $0, 0$. The equation falls into case (c) of the Frobenius Theorem page 970: there are two series solutions

$$\begin{aligned} y_1 &= \; = x^0 \textstyle\sum_{k=0}^{\infty} c_k x^k, \quad c_0 \neq 0, \\ y_2 &= \; = y_1(x) \ln|x| + \textstyle\sum_{k=1}^{\infty} d_k x^k. \end{aligned}$$

Solution y_2 is not a polynomial, it will not be used.

To be proved: Assume trial solution $y = x^r \sum_{k=0}^{\infty} c_k x^k$ with $r = 0$ the largest root of the indicial equation. It will be shown that the recursion relation is

$$c_{k+1} = \frac{k-n}{(k+1)^2} c_k$$

with solution

$$c_k = \frac{(-1)^k (n!)^2}{(k!)^2 (n-k)!}, \quad \text{for choice } c_0 = n!$$

Find a formula for c_k. The method substitutes the trial series y into the differential equation and then resolve the coefficients. The details:

$$\begin{aligned}
x^2 y'' &= x^2 \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} \\
&= \sum_{j=0}^{\infty} j(j-1) c_j x^j \\
x(1-x)y' &= x(1-x) \sum_{k=1}^{\infty} k c_k x^{k-1} \\
&= \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=1}^{\infty} k c_k x^{k+1} \\
&= \sum_{j=1}^{\infty} j c_j x^j - \sum_{j=2}^{\infty} (j-1) c_{j-1} x^j \\
&= \sum_{j=1}^{\infty} (j c_j - (j-1) c_{j-1}) x^j \qquad \boxed{\mathbf{1}} \\
nxy &= x \sum_{k=0}^{\infty} n c_k x^k \\
&= \sum_{k=0}^{\infty} n c_k x^{k+1} \\
&= \sum_{j=1}^{\infty} n c_{j-1} x^j
\end{aligned}$$

An indexing trick was used at step $\boxed{\mathbf{1}}$:

$$\sum_{j=2}^{\infty} (j-1) c_{j-1} x^j = \sum_{j=1}^{\infty} (j-1) c_{j-1} x^j$$

The trick works because the $j = 1$ term is zero.

Substitute the trial solution into $x^2 y'' + x(1-x)y' + nxy = 0$:

$$\sum_{j=0}^{\infty} j(j-1) c_j x^j + \sum_{j=1}^{\infty} (j c_j - (j-1) c_{j-1}) x^j + \sum_{j=1}^{\infty} n c_{j-1} x^j = 0$$

$$\sum_{j=1}^{\infty} \left(j(j-1) c_j + j c_j - (j-1) c_{j-1} + n c_{j-1} = 0 \right) x^k = 0$$

The recursion is obtained by setting all coefficients on the left to zero, then simplify:

$j(j-1)c_j + jc_j - (j-1)c_{j-1} + nc_{j-1} = 0$

$j^2 c_j + (n-j+1)c_{j-1} = 0, \quad k \geq 2$

Solve for c_j to find the recursion:

$c_j = -\dfrac{n-j+1}{j^2} c_{j-1}, \quad j \geq 1$, with c_0 given

Replace $j = k+1$:

$c_{k+1} = \dfrac{k-n}{(k+1)^2} c_k, \quad k \geq 0$, with c_0 given

Solve the recursion.

Apply the first order recursion formulas page 957:

$x_k = c_k$, $a_k = \frac{k-n}{(k+1)^2}$, $b_k = 0$

$$\begin{aligned} c_{k+1} &= x_0 \textstyle\prod_{r=0}^{k} a_k \\ &= c_0 \textstyle\prod_{r=0}^{k} \frac{r-n}{(r+1)^2} \qquad \text{for } k \geq 0 \text{ and } k < n \\ &= c_0 \dfrac{(-1)^k (n!)}{(k!)^2 (n-k)!} \qquad \text{for } 0 \leq k < n. \end{aligned}$$

Choose $c_0 = n!$, then a polynomial solution to Laguerre's equation of order n is given by

$y(x) = \sum_{k=0}^{n} \dfrac{(-1)^k (n!)^2}{(k!)^2 (n-k)!} x^k$

The alternate Laguerre polynomials are labeled $L_0, L_1, \ldots$, the first few given by equations in Exercise 25.

```
# Exercise 23, Solve by Frobenius theory
# Laguerre's equation answer check
de:=x^2*diff(y(x),x,x)+x*(1-x)*diff(y(x),x)+N*x*y(x);
c0:=n!:C:=(k,n)->(n!)*product((r-n)/(r+1)^2,r=0..k);
Y:=(x,n)->n!+sum(C(k,n)*x^(k+1),k=0..n);
N:=3;Y(x,N);
# Check de solution.
Q:=simplify(expand(subs(y(x)=Y(x,N),de)));
# High to low coefficients
koeffs:=seq(C(N-1-i,N),i=0..N-1),N!;
B:=(k,n)->(n!)*(-1)^k*(n!)/(k!)^2/(n-k)!;
Z:=(x,n)->sum(B(k,n)*x^(k),k=0..n);
Z(x,N);
```

■

24. Show that $y = e^x \frac{d^n}{dx^n}(x^n e^{-x})$ satisfies **Laguerre's equation**: $xy'' + (1 - x)y' + ny = 0$.

25. Verify by computer the **Laguerre** formulas

$$\begin{aligned} L_0(x)&=1\\ L_1(x)&=-x+1\\ L_2(x)&=x^2-4x+2\\ L_3(x)&=-x^3+9x^2-18x+6 \end{aligned}$$

Solution:
```
# Exercise 25, Compute Laguerre polynomials
altLaguerreL:=(n,x)->factorial(n)*LaguerreL(n,x);
for k from 0 to 4 do
simplify(altLaguerreL(k,x)) od;
```
■

26. Find to 6 digits by computer the roots of $L_4(x)$.

Solution: The roots are used in Gauss-Laguerre Quadrature. Reference: https://mathworld.wolfram.com/Laguerre-GaussQuadrature.html

27. Prove: Up to a constant, L_n is the only polynomial solution of $xy'' + (1 - x)y' + ny = 0$, $n \ge 0$ an integer.

Solution:
The Frobenius method in Exercise 23 produces two independent solutions y_1, y_2:

$$\begin{aligned} y_1 &= \; = x^0 \sum_{k=0}^{\infty} c_k x^k, \quad c_0 \neq 0,\\ y_2 &= \; = y_1(x)\ln|x| + \sum_{k=1}^{\infty} d_k x^k. \end{aligned}$$

Let y be another polynomial solution of Laguerre's equation. Then $y = d_1 y_1 + d_2 y_2$ for some constants d_1, d_2. Because $y(0)$ is finite (y is a polynomial) then $d_2 = 0$. Therefore, y is a constant multiple of the Frobenius polynomial solution y_1, which can be selected to equal L_n. ■

28. Assume standard Laguerre polynomials $\{\mathcal{L}_n\}$ satisfy recurrence
$$(n+1)\mathcal{L}_{n+1}(x)=(2n+1-x)\mathcal{L}_n(x) - n\mathcal{L}_{n-1}(x).$$
Prove: The alternate Laguerre polynomials $\{L_n\}$ satisfy recurrence
$$L_{n+1}(x)=(2n+1-x)L_n(x) - n^2 L_{n-1}(x).$$

Solution: Use $\mathcal{L}_n = (n!)\,L_n$.

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