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# Chapter A

## Background Topics

### Introduction

The appendices to follow contain a short list of topics extracted from pre-calculus and calculus courses.

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### A.1 Calculus

The selected topics from differential and integral calculus are used in differential equations. The special notation of differential equations is introduced, along with some ideas of Isaac Newton concerning the elementary kinetics formula  $D = RT$ , which has the physical interpretation  $\text{Distance} = \text{Rate} \times \text{Time}$ .

#### Derivative

The calculus derivative  $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$  makes sense provided the indicated limit exists. Implicit in the formula is the *assumption* that  $f$  is defined in an open interval of the form  $|x - x_0| < H$ . Differential equations use this standard notation, plus the **Leibniz notation**

$$\frac{df}{dx} = f'(x).$$

Variable names used in science and engineering often follow this standard:

$$\begin{aligned}y &= \text{dependent variable,} \\x &= \text{independent variable.}\end{aligned}$$

Within certain disciplines, such as kinetics, the variable names change, and the following standard exists:

$$\begin{aligned}x &= \text{displacement, dependent variable,} \\t &= \text{time, independent variable,}\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt} &= \text{velocity} & \frac{d^2x}{dt^2} &= \text{acceleration} \\&= x'(t) & &= x''(t) \\&= \dot{x}(t) & &= \ddot{x}(t) \\&= Dx(t), & &= D^2x(t).\end{aligned}$$

The functional notation  $y(x)$  means  $y$  is a dependent variable which depends on the independent variable  $x$ . For example,  $x(t)$  means displacement  $x$  depends on time  $t$ . In a graphic, it is expected that  $x$  is the vertical axis and  $t$  is the horizontal axis. The dot-notation  $\dot{x}(t)$  and  $\ddot{x}(t)$ , instead of  $x'(t)$  and  $x''(t)$ , is common in literature on statics and dynamics. Operator notation  $Dx$ ,  $D^2x$  appears in differential equations literature and in computer algebra systems, e.g., `maple` and `mathematica`.

## Slope, Rates and Averages

The derivative can be interpreted geometrically as the **slope** of the line tangent to a curve at a point; see Figure 1.

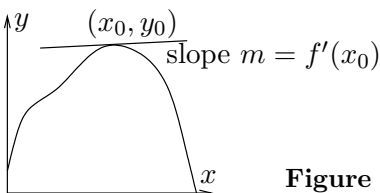


Figure 1. Slope of the tangent line.

The **tangent line** itself can be viewed as the **linearization** of the curve. For example, if the curve is the path of an automobile which at speedometer reading  $v$  instantly skids off the road, then the car follows the tangent line with constant speed  $v$ . Travel along the tangent line is **linear motion** at constant speed.

The line equation tangent to  $y = f(x)$  at  $x = x_0$  is given by the **point-slope form** of a line

$$\begin{aligned}y - y_0 &= m(x - x_0), \\y_0 &= f(x_0), \quad m = f'(x_0).\end{aligned}$$

## A.1 Calculus

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The notation  $y(x)$ , usual in differential equations, conflicts with the notation from geometry. In handwritten and blackboard work it is recommended to change  $x$  and  $y$  to capital letters  $X$  and  $Y$ , then replace  $f$  by  $y$ , as follows:

$$\begin{aligned} Y - y_0 &= m(X - x_0), \\ y_0 &= y(x_0), \quad m = y'(x_0). \end{aligned}$$

Other forms of a straight line in coordinate geometry are the **slope-intercept form**  $y = mx + b$ , the **standard form**  $Ax + By + C = 0$  and the **parametric form**

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \end{cases} \quad -\infty < t < \infty.$$

In the parametric form, the vector  $a\vec{\mathbf{i}} + b\vec{\mathbf{j}}$  is tangent to the line. For example,  $a = 0$  and  $b = 1$  gives a vertical line through  $(x_0, y_0)$ .

Applied sciences interpret the derivative  $f'(x)$  as the **rate of change** of  $y = f(x)$  with respect to  $x$ . Typical interpretations appear below.

$\dot{x}(t) \approx$	change in displacement $x$ for a unit change in $t$
$\frac{dQ}{dt} \approx$	change in charge $Q$ for a unit change in $t$
$\dot{Q}(t) \approx$	change in current $I = \dot{Q}$ for a unit change in $t$
$A'(t) \approx$	expected decrease in the amount $A$ of radioactive material for time interval $[t, t + 1]$

The **average** of  $n$  samples  $y_1, \dots, y_n$  is defined to be

$$\frac{y_1 + y_2 + \cdots + y_n}{n}.$$

The term **simple average** is sometimes used. The **average value**  $\bar{f}$  of a continuous function  $f(x)$  on  $[a, b]$  is defined by

$$\bar{f} = \frac{\int_a^b f(x) dx}{b - a}.$$

This abstract notion has connections with the simple average. The theory of the integral  $\int_a^b f(x) dx$  includes the **rectangular rule** for numerical integration. For step size  $h = (b - a)/n$  and sample values  $y_1 = f(a)$ ,  $y_2 = f(a + h)$ ,  $\dots$ ,  $y_n = f(a + nh - h)$  it gives the approximation formula

$$\int_a^b f(x) dx \approx h(y_1 + y_2 + \cdots + y_n).$$

Multiply this relation by  $1/(b - a)$  and replace the left side by the average value  $\bar{f}$ . Then

$$\bar{f} \approx \frac{y_1 + y_2 + \cdots + y_n}{n},$$

or in words,

The average value  $\bar{f}$  is approximately a simple average of  $n$  samples of  $f$ , taken at equi-spaced points in  $[a, b]$ .

In the language of kinetics,  $f$  is **velocity** and  $\bar{f}$  is the **average velocity** or the **speed**.

The language of kinetics agrees with common public notions of speed. For example, the average of various speedometer reading samples during an automobile trip give a good indication of the average speed of the car on the trip. The average speed  $R = \bar{f}$  is related to the trip time  $T = b - a$  and the trip mileage  $D$  by the classical formula  $D = RT$ , which is taught in elementary school.

The expression for the trip mileage  $D$  in terms of the instantaneous velocity  $f$ ,

$$D = \int_a^b f(x)dx,$$

is due to the creative genius of Isaac Newton. This relation of Newton today appears in texts as the **fundamental theorem of calculus**.

### Fundamental Theorem of Calculus

The foundations of the study of differential equations rests with Newton's discovery of a way to state the relation  $D = RT$  using instantaneous velocities instead of speed averages.

#### Theorem A.1 (Fundamental theorem of calculus)

Let  $G$  be continuous and let  $F$  be continuously differentiable on  $[a, b]$ . Then

$$(a) \quad F(b) - F(a) = \int_a^b F'(x)dx,$$

$$(b) \quad \frac{d}{dx} \int_a^x G(t)dt = G(x).$$

Part (a) of the fundamental theorem is used by calculus students to evaluate integrals. In differential equations, it is applied to find solutions.

Part (b) of the fundamental theorem computes the instantaneous rate of an averaging process. Calculus students use it to check answers to integration problems. In differential equations it is used to verify solutions.

The justification of  $D = RT$  for instantaneous rates  $f(x) = F'(x)$  is contained in part (a): divide both sides by  $b - a$  and interpret the right side as *the average velocity* or *speed* to get the formula  $D/T = R$ .

#### Example A.1 (Leibniz Notation)

Change  $y''(x) + y(x)$  into Leibniz notation.

**Solution:**

## A.1 Calculus

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$$\begin{aligned}y''(x) &= \frac{d}{dx}y'(x) \\ &= \frac{d}{dx} \frac{dy}{dx} \\ &= \frac{d^2y}{dx^2}\end{aligned}$$

Definition of second derivative.

Leibniz notation for the first derivative.

Leibniz notation.

Therefore, the converted expression is  $\frac{d^2y}{dx^2} + y$ .

### Example A.2 (Notation Conversion)

Convert the equation  $\frac{du}{dt} = u + e^t \sin t$  to dot notation.

**Solution:** By convention,  $\frac{du}{dt} = \dot{u}(t)$  and  $u = u(t)$ . Therefore, the converted equation is  $\dot{u}(t) = u(t) + e^t \sin t$ .

### Example A.3 (Slope of the Tangent Line)

Compute the slope  $m$  of the line tangent to  $y = x \sin x$  at  $x = \pi/2$ .

**Solution:**

$$\begin{aligned}m &= y' \\ &= (x \sin x)' \\ &= \sin x + x \cos x\end{aligned}$$

Definition of slope and derivative.

Definition of  $y$ .

Product rule and derivative tables.  
Variable  $x$  to be replaced by  $\pi/2$ .

$$\begin{aligned}&= \sin(\pi/2) + \frac{1}{2}\pi \cos(\pi/2) \\ &= 1\end{aligned}$$

Replacement  $x = \pi/2$ .

Identities  $\cos(\pi/2) = 0$ ,  $\sin(\pi/2) = 1$   
applied.

### Example A.4 (Tangent Line Equation)

Find the tangent line equation at  $x = \pi/2$  for  $y = x \sin x$  in point-slope form and in slope-intercept form.

**Solution:** The point-slope equation in an  $XY$ -system is  $Y - y_0 = m(X - x_0)$ . In this formula,  $x_0 = \pi/2$ ,  $y_0 = x_0 \sin x_0 = \pi/2$ . Example A.3 gives  $m = 1$ . The tangent line equation in point-slope form is  $Y - \pi/2 = (1)(X - \pi/2)$ , which simplifies to the slope-intercept form  $Y = X$ .

### Example A.5 (Line Equations)

Convert the line equation  $y - 2 = 5(x - 3)$  to slope-intercept and parametric forms.

**Solution:** The *slope-intercept* form  $y = 5x - 13$  is found by expansion to an explicit equation for  $y$ . A *parametric* form can be found by setting  $x = t$  and then  $y = 5x - 13 = 5t - 13$ . The vector form is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 5t - 13 \end{pmatrix}, \quad -\infty < t < \infty.$$

**Example A.6 (Decay Law Derivation)**

Derive the decay law  $\frac{dA}{dt} = kA(t)$  from the sentence

*Radioactive material decays at a rate proportional to the amount present.*

**Solution:** The sentence is first dissected into English phrases **1** to **4**.

- |                                       |  |
|---------------------------------------|--|
| <b>1:</b> <i>Radioactive material</i> | The phrase causes the invention of a symbol $A$ for the amount present at time $t$ .                       |
| <b>2:</b> <i>decays at a rate</i>     | It means $A$ undergoes decay. Then $A$ changes. Calculus conventions imply the rate of change is $dA/dt$ . |
| <b>3:</b> <i>proportional to</i>      | Literally, it means <i>equal to a constant multiple of</i> . Let $k$ be the proportionality constant.      |
| <b>4:</b> <i>the amount present</i>   | The amount of radioactive material present is $A(t)$ .   |

The four phrases are translated into mathematical notation as follows.

- |                        |                                     |
|------------------------|-------------------------------------|
| <b>Phrases 1 and 2</b> | Symbol $dA/dt$ .                    |
| <b>Phrase 3</b>        | Equal sign '=' and a constant $k$ . |
| <b>Phrase 4</b>        | Symbol $A(t)$ .                     |

Let  $A(t)$  be the amount present at time  $t$ . The translation is  $\frac{dA}{dt} = kA(t)$ .

**Example A.7 (Average Value)**

Given  $f(x) = xe^x + \sin^2(\pi x)$ , find the average value on  $0 \leq x \leq 2$ .

**Solution:** The value is  $\frac{1}{2}e^2 + 1$ . The details:

$\bar{f} = \frac{1}{2} \int_0^2 f(x) dx$	Definition of average value, page <a href="#">1007</a> .
$= \frac{1}{2} \int_0^2 [xe^x + \sin^2(\pi x)] dx$	Substitute for $f(x)$ .
$= \frac{1}{2} (x-1)e^x \Big _{x=0}^{x=2}$	Integral tables.
$+ \frac{1}{4\pi} (-\cos \pi x \sin \pi x + \pi x) \Big _{x=0}^{x=2}$	
$= \frac{1}{2} e^2 + 1$	Use $\sin(n\pi) = 0$ .

**Example A.8 (Speed)**

Find the speed for a car trip of 2 hours, given the velocity profile

$$\dot{x}(t) = \begin{cases} 1200t & 0 \leq t \leq 0.05, \\ 60 & 0.05 \leq t \leq 2. \end{cases}$$

**Solution:** The speed  $R$  is given by

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$$\begin{aligned} R &= \frac{1}{2} \int_0^2 \dot{x}(t) dt && \text{Average value of } \dot{x}, \text{ page } 1007. \\ &= \frac{1}{2} \left( \int_0^{0.05} 1200t dt + \int_{0.05}^2 60 dt \right) && \text{Use } \int_a^b f = \int_a^c f + \int_c^b f. \\ &= \frac{1}{2} (600(0.05)^2 + 60(2 - 0.05)) && \text{Evaluate integrals.} \\ &= \frac{237}{4}. && \text{About } 59.25 \text{ mph.} \end{aligned}$$

The unrealistic 3-minute acceleration to 60 mph can be replaced by a more realistic 18-second acceleration to give 59.925 mph.

### Example A.9 (Speed Estimation)

Estimate the average speed of a car which accelerates from 0 to 65 miles per hour in 12 seconds.

**Solution:** The purpose of this example is to explain the layman's answer of 65/2 mph. The answer must be justified in the context of calculus.

If the acceleration is constant, then  $\ddot{x}(t) = a = \text{constant}$ . Therefore,  $\dot{x}(t) = at$ , since  $\dot{x}(0) = 0$ . Let  $t_0 = 12/3600$  hours. The average speed  $R$  for time interval  $0 \leq t \leq t_0$  is

$$\begin{aligned} R &= \frac{1}{t_0} \int_0^{t_0} \dot{x}(t) dt && \text{Definition of average speed, page } 1007. \\ &= \frac{a}{t_0} \frac{t_0^2}{2} && \text{Evaluate integral with } \dot{x} = at. \\ &= \frac{65}{2} && \text{Because } 65 = \dot{x}(t_0) = at_0. \end{aligned}$$

It can be argued on physical grounds that no car has constant acceleration, so the answer 65/2 is merely an estimate. The layman's answer can be obtained by averaging the two speeds 0 and 65.

### Example A.10 (Integral Identity)

Verify the integral evaluation  $\int_0^1 xe^x dx = 1$ .

**Solution:**

$$\begin{aligned} I &= \int_0^1 xe^x dx && \text{Integral } I \text{ to be evaluated.} \\ &= \int_0^1 (xe^x - e^x)' dx && \text{Identity } xe^x = (xe^x - e^x)' \text{ derived below.} \\ &= (xe^x - e^x) \Big|_{x=0}^{x=1} && \text{Apply the fundamental theorem of calculus, part (a). See page } 1008. \\ &= 1 && \text{Use } e^0 = 1. \end{aligned}$$

The identity  $xe^x = (xe^x - e^x)'$  applied in the solution above is obtained by experiment, as follows.

$$(xe^x)' = (1)e^x + xe^x \qquad \text{Product rule } (uv)' = u'v + uv'$$



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$$= (e^x)' + xe^x \quad \text{Term } xe^x \text{ isolated on the right.}$$

Solving the last equation for  $xe^x$  gives the identity  $xe^x = (xe^x - e^x)'$ . A more systematic method for finding such identities is *integration by parts*.

### Example A.11 (Integral Answer Check)

Verify the identity

$$\int_0^x t \ln(1+t) dt = \frac{1}{2} (x^2 - 1) \ln(1+x) + \frac{x}{2} - \frac{x^2}{4}.$$

**Solution:** Both sides evaluate to zero at  $x = 0$ , because  $\ln(1) = 0$ . According to the fundamental theorem of calculus, part (b), page 1008, it is sufficient to differentiate the answer on the right and verify that the derivative so obtained matches the integrand on the left. Let RHS denote the right hand side. Then

$$\begin{aligned} \text{RHS}' &= \left( \frac{x^2 - 1}{2} \ln(1+x) + \frac{x}{2} - \frac{x^2}{4} \right)' && \text{The } \textit{Right Hand Side} \text{ of the identity, to be} \\ &= x \ln(1+x) + \frac{x^2 - 1}{2x + 2} + \frac{1}{2} - \frac{x}{2} && \text{Product rule, power rule and the identity} \\ &= x \ln(1+x). && (\ln(u))' = u'/u. \\ & && \text{Simplified derivative of the RHS.} \end{aligned}$$

The derivative of RHS matches the integrand of the left side, which completes the verification.

### Example A.12 (Distance Estimate)

Estimate the distance  $D$  traveled by an automobile in two hours, and its average speed  $R$ , given that for  $t = 20$  to  $t = 120$  the speedometer readings every 20 minutes are 55, 70, 66, 71, 72, 65 miles per hour.

**Solution:** The answers are 133 miles and 66.5 mph. To estimate the values of  $R$  and  $D$ , it will be assumed that the speed was constant during the 20-minute period before the reading. The actual velocity  $\dot{x}(t)$  of the automobile is related to the average velocity  $R$  by the formula

$$R = \frac{1}{120} \int_0^{120} \dot{x}(t) dt.$$

The samples are used to find the average  $R$  as follows.

$$\begin{aligned} R &\approx \frac{55 + 70 + 66 + 71 + 72 + 65}{6} && \text{Used } \bar{f} \approx \frac{y_1 + \cdots + y_n}{n}, \text{ page 1007.} \\ &= \frac{399}{6} && \text{About 66.5 miles per hour.} \end{aligned}$$

Then  $D = RT$  implies  $D \approx \frac{399}{6} \frac{120}{60} = 133$  miles.

## Exercises A.1

**Derivative notation** Convert from the given notation, prime, dot, Leibniz or operator, to the other three forms.

1.  $\frac{du}{dt}$
2.  $\dot{u}(t_0)$
3.  $\ddot{u}(1+t)$
4.  $\frac{dx}{dt} = 1 + x(t)$
5.  $D^2w(x) = 1 + w(x) + x$
6.  $Dy(x) = y^{-2}(x)$
7.  $\ln(w(r)) = \frac{dw}{dr}$
8.  $e^{-y(x)} = y'(x)$
9.  $\dot{y}(t) = 1 + t$
10.  $\dot{x}(t) = e^{-2x(t)}$

**Slope** Compute the slope of the line tangent to the curve at the given point.

11.  $y = x^2 - 3x + 1, \quad x = 0.$
12.  $y = x^5 - x + 2, \quad x = 2.$
13.  $y = \sin x + x, \quad x = \pi/4.$
14.  $y = \cos x - x, \quad x = \pi/4.$
15.  $y = \tan^{-1} x + e^{-x} \ln(1+x), \quad x = 1.$
16.  $y = \sin^{-1} x + e^x \ln(2+x), \quad x = 1.$

**Tangent line equation** Find the tangent line equation in the three possible forms, point-slope, slope-intercept and parametric.

17.  $y = x^3 - x, \quad x = 1.$
18.  $y = x^3 + x + 1, \quad x = 0.$
19.  $y = \sin^{-1}(x), \quad x = 1/2.$
20.  $y = \tan^{-1}(x), \quad x = 1.$

21.  $y = e^{-x}, \quad x = \ln(2).$
22.  $y = \ln(1+x), \quad x = 0.$
23.  $y = \frac{1+x}{1-x}, \quad x = 0.$
24.  $y = \frac{1-x^2}{1+x^2}, \quad x = 0.$

**Rates Model as a rate of change equation.**

25. The expected change in charge  $Q$  is equal to the electromotive force  $\sin(\omega t)$ .
26. The damping force  $F$  is proportional to the instantaneous change in  $x(t)$ .
27. The angular rate of change is proportional to the external force  $\cos(\omega t)$ .
28. The amount in a bank account changes at a rate proportional to the current balance.
29. The expected population change is proportional to the present population  $P$ .
30. The temperature flux and the temperature difference from the surrounding medium are proportional.

**Average value** Find the average value of  $f$  on  $[a, b]$ ,

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

31.  $xe^{-x}, \quad 0 \leq x \leq 1.$
32.  $\frac{1}{2}e^x - \frac{1}{2}e^{-x}, \quad 0 \leq x \leq 2.$
33.  $\ln x, \quad 1 \leq x \leq 3.$
34.  $\sec x, \quad 0 \leq x \leq \pi/4.$
35.  $x^3 - x, \quad 0 \leq x \leq 2.$
36.  $\frac{x-1}{x+1}, \quad 0 \leq x \leq 1.$
37.  $\frac{\sin x}{1+\cos x}, \quad 0 \leq x \leq \pi/4.$
38.  $\sin^3 x \cos x, \quad 0 \leq x \leq \pi.$

39.  $\frac{1}{1+x^2}$  on  $0 \leq x \leq 1/2$ ,  $4/5$  on  $1/2 \leq x \leq 1$ .

40.  $\frac{1}{x}$  on  $1 \leq x \leq 2$ ,  $\frac{5}{8} \frac{x^2}{1+x^2}$  on  $2 \leq x \leq 3$ .

41.  $\tan x$  on  $0 \leq x \leq \pi/4$ , and  $1+(x-\pi/4)$  on  $\pi/4 \leq x \leq \pi/3$ .

42.  $\cot x$  on  $\pi/4 \leq x \leq \pi/2$ , and  $x - \pi/2$  on  $\pi/2 \leq x \leq \pi$ .

**Integral identities** Verify the given integration identity by applying the fundamental theorem of calculus.

43.  $\int_0^1 \frac{1+t}{2+t} dt = 1 + \ln \frac{2}{3}$ .

44.  $\int_0^1 \frac{1+t^2}{2+t} dt = 5 \ln \frac{3}{2} - \frac{3}{2}$ .

45.  $\int_0^\pi t \sin(2t) dt = \frac{\pi-2}{4}$ .

46.  $\int_0^{\pi/2} t \cos(2t) dt = -\frac{1}{2}$ .

47.  $\int_0^1 te^{-t} dt = 1 - \frac{2}{e}$ .

48.  $\int_0^1 t^2 e^{-t} dt = 2 - \frac{5}{e}$ .

49.  $\int_0^x \sin^4(t) \cos(t) dt = \frac{\sin^5(x)}{5}$ .

50.  $\int_0^x \tan(t) dt = -\ln(\cos x)$ .

**Car trip** Estimate the average speed  $R$  and the distance traveled  $D$  on a car trip, given the velocity samples.

51. Every 10 minutes from  $t = 10$  to  $t = 120$  minutes, 51, 62, 55, 53, 60, 67, 61, 67, 55, 70, 71, 66 miles per hour.

52. Every 15 minutes from  $t = 15$  to  $t = 225$  minutes, 90, 92, 110, 112, 120, 113, 109, 90, 95, 97, 60, 90, 100, 105, 103 kilometers per hour.

53. Every 5 minutes from  $t = 5$  to  $t = 75$  minutes, 45, 60, 61, 63, 60, 58, 61, 65, 25, 40, 45, 60, 65, 59, 60 miles per hour.

54. Every 5 minutes from  $t = 5$  to  $t = 100$  minutes, 50, 90, 100, 120, 110, 112, 130, 120, 110, 40, 60, 100, 90, 80, 20, 55, 130, 130, 120, 125 kilometers per hour.

## A.2 Graphics

Engineers and scientists prefer a computer approach to graphing solutions to differential equations  $y' = f(x, y)$ . In special cases, the fastest graph generation is by hand or by calculator. Experience with hand calculations and calculators is useful for judging the accuracy of a computer graphic.

Small *numeric data sets* may be graphed by hand using graph paper or engineering paper. Large data sets are best graphed using a computer spreadsheet program, e.g., Microsoft **Excel**, a computer algebra system, e.g., **maple** or **mathematica**, a numerical laboratory, e.g., **matlab**, **octave** or **scilab**, or a freely available graphing program, e.g., **gnuplot**.

### The Standard Curve Library

Feasibility for hand graphing of equations in *explicit form* is tested using the **standard curve library**, which includes the following equation types.

$y = mx + b$	Equation of a line in slope-intercept form. Includes constant equations. Increases for $m > 0$ , decreases for $m < 0$ .
$y = x^n,$ $y = \frac{1}{x^n}$	Power curves. Even for $n$ even, odd for $n$ odd. Reciprocal powers have asymptote at $x = 0$ . Special cases $y = x^2$ , $y = 1/x$ occur often.
$y = \sin x,$ $y = \tan x$	The sine is $2\pi$ -periodic and the cosine graph is a translation by $\pi/2$ . The tangent is a $\pi$ -periodic curve with asymptotes at odd multiples of $\pi/2$ .
$y = e^x,$ $y = \ln x$	All exponential and logarithmic curves are obtained from these basic graphs.

### Four Transformations

The *standard curve library* is modified for use by allowing four transformations. The first two transformations are **rigid motions**, that is, the shape is unchanged. The last two are not rigid motions.

- (1) Replace  $x$  by  $x - x_0$  and  $y$  by  $y - y_0$ . The effect is to change the origin of coordinates in the graph from  $(0, 0)$  to  $(x_0, y_0)$ .
- (2) Replace  $y$  by  $-y$ . The effect on a paper graphic is to turn the paper over, swapping horizontal edges. Examples are  $y = x^2$  and  $y = -x^2$ .
- (3) Replace  $y$  by  $1/y$ . The effect is to swap the roles of 0 and  $\infty$  in the original graph. Examples are  $y = x^2$  and  $y = 1/x^2$ .
- (4) Replace  $y$  by  $ky$  with  $k > 0$ . The effect is to change the  $y$ -axis scale. Examples are  $y = x^2$  and  $y = 4x^2$  ( $k = 1/4$ ).

## Special Equations

The standard curve library includes some special equations in *implicit* form  $F(x, y) = c$ . Recognized from the subject of **analytic geometry** are the following.

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \quad \textbf{Circle.} \text{ Radius } r \text{ and center } (x_0, y_0).$$

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1 \quad \textbf{Ellipse.} \text{ Semiaxes } a, b \text{ and center } (x_0, y_0).$$

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1 \quad \textbf{Hyperbola.} \text{ Center } (x_0, y_0).$$

## Polynomial Quotients

**Rough graphs** of polynomials and rational functions can be made with curve library methods. The graphs are accurate for the sign of  $y$  and the general shape. The following equation types are suited for use with the library.

$$y = (x - a_1)^{n_1} \cdots (x - a_k)^{n_k} \quad \text{Factored polynomial curves. Roots at } a_1, \dots, a_k.$$

$$y = \frac{(x - a_1)^{n_1} \cdots (x - a_k)^{n_k}}{(x - b_1)^{m_1} \cdots (x - b_\ell)^{m_\ell}} \quad \text{Factored rational curve. The roots are at } a_1, \dots, a_k \text{ and the vertical asymptotes are at } b_1, \dots, b_\ell.$$

The curve graphic at each root looks like a power curve  $Y = X^n$  and at each vertical asymptote it looks like an inverse power curve  $Y = 1/X^n$ , subject to the four transformations on page 1015.

### Example A.13 (Curve Library Graphing Methods)

Apply curve library methods to graph on one set of axes for  $-2 \leq x \leq 2$  the equations

$$y = 2x - 1, \quad y = (x - 1)^2, \quad y = -(x + 1)^4, \quad y = -1/x.$$

**Solution:** The curve library templates for the given graphs are

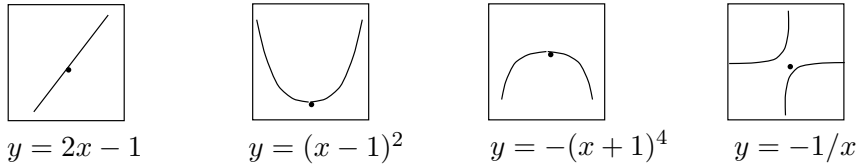
$$Y = X, \quad Y = X^2, \quad Y = -X^4, \quad Y = -1/X.$$

Transformation of the four types described on page 1015 are applied to change the templates into the correct figures. They are:

$Y = X$	Replace $X$ by $2x$ and $Y$ by $y + 1$ to obtain $y = 2x - 1$ .
$Y = X^2$	Replace $X$ by $x - 1$ and $Y$ by $y$ to obtain $y = (x - 1)^2$ .
$Y = -X^4$	Replace $X$ by $x + 1$ and $Y$ by $y$ to get $y = -(x + 1)^4$ .
$Y = -1/X$	Replace $X$ by $x$ and $Y$ by $-y$ to obtain $y = -1/x$ .

## A.2 Graphics

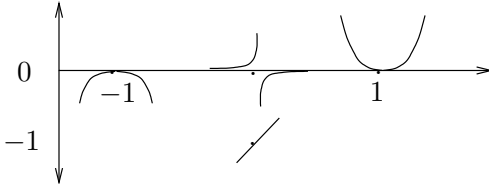
The transformations amount to changing the origin and/or flipping the curve compared to the template graphic from the curve library. The graphics to be assembled onto one set of axes appear in Figure 2.



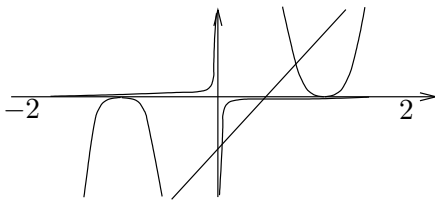
**Figure 2. Transformed template graphics**

**with centers**  $(0, -1)$ ,  $(1, 0)$ ,  $(-1, 0)$ ,  $(0, 0)$ .

The final graphic is completed by assembling the transformed template graphics onto a single set of axes, locating each template onto its center. In the intermediate stage of completion, Figure 3, some portions of the graphic are left incomplete. The final graphic is Figure 4.



**Figure 3. Combined graphic made from the four templates.**



**Figure 4. Combined graphic on  $-2 \leq x \leq 2$  for curves  $y = 2x - 1$ ,  $y = (x - 1)^2$ ,  $y = -(x + 1)^4$  and  $y = -1/x$ .**

### Example A.14 (Factored Polynomial Graphs)

Apply curve library methods to make a rough graph of the factored polynomial

$$y = -3x(x - 1)^2(x - 2)^3(x - 3), \quad 0 \leq x \leq 4.$$

**Solution:** The distinct factors correspond to the template graphics to be used in the assembly of the final graphic:

$$y = c_1x, \quad y = c_2(x - 1)^2, \quad y = c_3(x - 2)^3, \quad y = c_4(x - 3).$$

The constants  $c_1$ ,  $c_2$ ,  $c_3$ ,  $c_4$  are evaluated from the original curve equation  $y = -3x(x - 1)^2(x - 2)^3(x - 3)$  by arguing that, for  $(x, y)$  close to the center of each template graphic, the template and the original should be graphically the same. For example, if  $x$  is close to  $x = 2$ , then

$$\begin{aligned} y &= -3x(x - 1)^2(x - 2)^3(x - 3) \\ &= (-3x(x - 1)^2(x - 3))(x - 2)^3 \\ &\approx [-3x(x - 1)^2(x - 3)|_{x=2}](x - 2)^3 \\ &= 6(x - 2)^3 \end{aligned}$$

Original equation. Analysis near  $x = 2$ .

Isolate all factors not containing the factor  $(x - 2)$ .

Isolated factors are nearly constant close to  $x = 2$ .

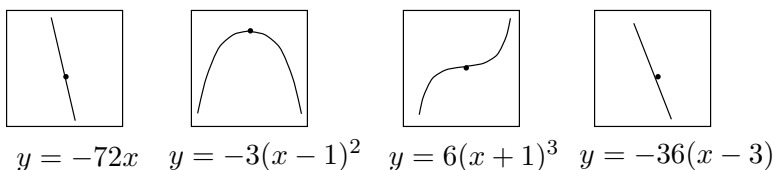
Found template  $y = 6(x - 2)^3$ .

## A.2 Graphics

By this process,  $y = -3x(x - 1)^2(x - 2)^3(x - 3)$  has template equations

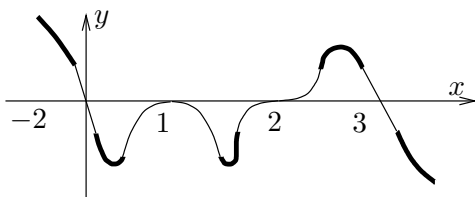
$$y = -72x, \quad y = -3(x - 1)^2, \quad y = 6(x - 2)^3, \quad y = -36(x - 3).$$

As will be seen, the equalities  $c_1 = -72$ ,  $c_2 = -3$ ,  $c_3 = 6$ ,  $c_4 = -36$  are not actually used, *only the signs matter*. Therefore, knowing  $c_1 < 0$ ,  $c_2 < 0$ ,  $c_3 > 0$ ,  $c_4 < 0$  is enough for a rough graphic. This information is rapidly obtained by counting signs of the factors involved. The graphics for the templates, taken from the *standard curve library*, appear in Figure 5.



**Figure 5.** Template graphics for  $y = -3x(x - 1)^2(x - 2)^3(x - 3)$ .

To make the final graphic in Figure 6, the templates are located at their respective centers on one set of axes, then they are connected with a smooth curve (boldface). The connections stay either in the upper or the lower half-plane, because all zeros of  $y$  are accounted for by the template graphics.



**Figure 6.** Final graph for the polynomial  $y = -3x(x - 1)^2(x - 2)^3(x - 3)$ .

The graphic is accurate for the sign of  $y$ . The general shape is correct, but details like maxima, minima and slopes are flawed. Nevertheless, the hand graphic is perhaps more useful than a computer graphic.

Polynomial graphs exit the paper at  $x = \pm\infty$  in the same way as their leading term, which could be called the **horizontal asymptote**. In the present example, the leading term is  $y = -3x^7$ , which is a curve from the standard curve library. This information can be used to detect fundamental graphing errors.

### Example A.15 (Graphing Polynomial Fractions)

Apply curve library methods to make a rough graph of the rational function

$$y = \frac{-3x(x - 1)^2(x - 4)}{(x + 1)^2(x - 2)(x - 3)}.$$

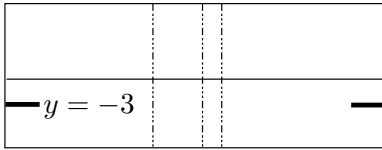
**Solution:** The rational function is a the quotient of two quadrics. By long division of the polynomials it follows that  $y = -3 + r/q$  where the degree of  $r$  is less than the degree of  $q$ . Therefore,  $r/q \approx 0$  at  $\pm\infty$ , and this means  $y = -3$  is the horizontal asymptote.

The effect of these remarks about asymptotes is to declare that the graph exits the paper left and right along the line  $y = -3$ .

The vertical asymptotes  $x = -1$ ,  $x = 2$ ,  $x = 3$  similarly cause the graph to exit the paper top and/or bottom. The curve library method uses this information implicitly.

## A.2 Graphics

A rough graphic can be drawn immediately from this basic information. The portions of the graph that appear in Figure 7 are approximate only, valid for values of  $x$  very close to  $x = -\infty$  or  $x = \infty$ . In Figure 7, vertical asymptotes are shown, even though they already appear in library templates. It is a matter of taste to add vertical asymptotes to this figure. If added, then just portions of the vertical lines near  $|y| = \infty$  are valid.



**Figure 7. Rough graphic near  $x = \pm\infty$  and  $y = \pm\infty$ .**

The remainder of the graphic is obtained from the assembly of curve library templates. The distinct factors of the numerator and denominator of the rational function become the templates:

$$y = c_1x, \quad y = c_2(x - 1)^2, \quad y = c_3(x - 4),$$

$$y = \frac{d_1}{(x + 1)^2}, \quad y = \frac{d_2}{x - 2}, \quad y = \frac{d_3}{x - 3}.$$

Calculation of the constants  $c_1, c_2, c_3, d_1, d_2, d_3$  is unnecessary, only the signs matter for template selection. For example, to compute  $d_1$ :

$$y = \frac{-3x(x - 1)^2(x - 4)}{(x + 1)^2(x - 2)(x - 3)}$$

Given rational function. Analysis near  $x = -1$ .

$$= \left[ \frac{-3x(x - 1)^2(x - 4)}{(x - 2)(x - 3)} \right] \frac{1}{(x + 1)^2}$$

Isolate all factors not containing  $(x + 1)$ .

$$\approx \frac{-3(-1)(-1 - 1)^2(-1 - 4)}{(-1 - 2)(-1 - 3)} \frac{1}{(x + 1)^2}$$

Substitute  $x = -1$  into the isolated factors.

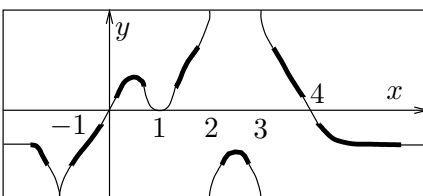
$$= -5 \frac{1}{(x + 1)^2}$$

Template equation found.

The logic of the substitution  $x = -1$  into the *isolated factors* is that they are nearly constant for  $x \approx -1$ . The template equation  $y = -5/(x + 1)^2$  at center  $(-1, 0)$  will be used to plot the final graphic. Just the sign of  $d_1 = -5$  is needed, which can be obtained by counting signs, the actual value 5 being irrelevant for the graphic.

By similar methods, the signs of the constants are found to be  $c_1 > 0, c_2 > 0, c_3 < 0, d_1 < 0, d_2 < 0, d_3 > 0$ . The six templates arise from four different library curves.

The six templates are placed at their centers and joined by a smooth curve (boldface) to produce the final graphic. See Figure 8. The plot needs some explanation. First, nothing is to scale, although the signs are correct for  $y$  and the general shape is valid. The curve goes off the paper left and right at  $|x| = \infty$ , the exit curve being  $y = -3$ . The curve also goes off the paper on the bottom edge at  $x = -1, 2, 3$  and on the top edge at  $x = 2, 3$ , in the manner shown. The maxima and minima of the curve have not been computed, so this information is not to scale either.



**Figure 8. Graphic for  $y = \frac{-3x(x - 1)^2(x - 4)}{(x + 1)^2(x - 2)(x - 3)}$ .**

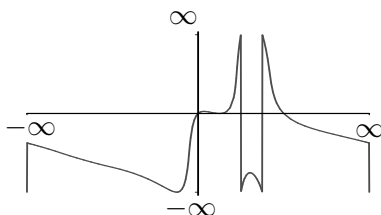


## A.2 Graphics

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Computer algebra systems like `maple` and `mathematica` can produce similar plots, with limitations. Below is the `maple` code for the resulting plot in Figure 9. The unwanted and incorrect vertical lines in the plot are an artifact of the discontinuities. It is especially difficult to see some of the fine features, e.g., the double zero of  $y$  at  $x = 1$  and the horizontal asymptote values.

```
# Maple V 5.1
F:=x-> -3*x*(x-1)^2*(x-4)/((x+1)^2*(x-2)*(x-3));
plot(F(x),x=-infinity..infinity);
```



**Figure 9.** Graphic by `maple` for  $y = \frac{-3x(x-1)^2(x-4)}{(x+1)^2(x-2)(x-3)}$ .

### Example A.16 (Computer Graphing)

Graph the 20 data points generated by the approximation formula

$$y(x + 0.05) \approx y(x) + 0.05(x + y(x)), \quad y(0) = 1,$$

from  $x = 0$  to  $x = 1$  in uniform steps of 0.05, using a computer.

**Solution:** The formula is applied as a **recursion formula**, which details how to generate from a *given table pair*  $x, y$  the *next table pair*  $X, Y$  via the formulas

$$X = x + 0.05, \quad Y = y + 0.05(x + y).$$

Mathematical translation includes elimination of the approximation symbol ( $\approx$ ) and the use of equal signs ( $=$ ) in the final formulas.

The first step is to generate a table of values. Then the table is plotted by a standard method. The process of determining the data pairs can be done by hand as follows.

$$x = 0, \quad y = 1$$

The first data pair arises from  $y(0) = 1$ , which means  $y = 1$  at  $x = 0$ .

$$X = x + 0.05$$

The next  $x$ -value is the old one plus 0.05.

$$Y = y + 0.05(x + y)$$

Approximation formula for the next  $y$ -value.

$$= 1 + 0.05(0 + 1)$$

Use  $x = 0, y = 1$ .

$$= 1.05.$$

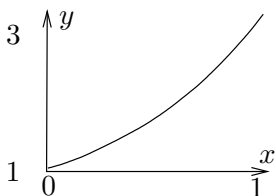
The second data pair is  $X = 0.05, Y = 1.05$ .

The first three pairs of values are verified to be

$$(0.00, 1.000), \quad (0.05, 1.050), \quad (0.10, 1.105).$$

A `maple` plot of the data uses the following code, resulting in Figure 10. Similar mechanisms for plotting data points are available in `matlab`, `mathematica`, `gnuplot` and `scilab`. Libreoffice `CALC` and Microsoft `Excel` can be used for such graphics.

## A.2 Graphics



**Figure 10.** Sample computer graphic for the approximation formula

$$y(x + 0.05) \approx y(x) + 0.05(x + y(x)), \quad y(0) = 1.$$

```
# Maple V, plot data points in list L
L:= [0.00,1.000], [0.05,1.050], [0.10,1.105],
    [0.15,1.165], [0.20,1.231], [0.25,1.303],
    [0.30,1.380], [0.35,1.464], [0.40,1.555],
    [0.45,1.653], [0.50,1.758], [0.55,1.870],
    [0.60,1.992], [0.65,2.121], [0.70,2.260],
    [0.75,2.408], [0.80,2.566], [0.85,2.734],
    [0.90,2.913], [0.95,3.104], [1.00,3.307]:
plot([L]);
```

In computer algebra systems, it is possible to avoid typing the numeric data, because of the formulas  $X = x + 0.05$ ,  $Y = y + 0.05(x + y)$ . To generate the list L in maple, execute the two code groups below.

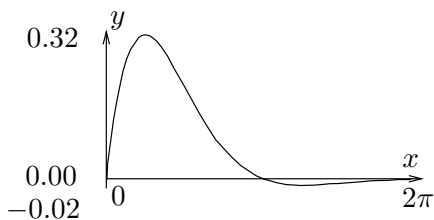
```
# Execute the first group once
X:=0:Y:=1:L:=[X,Y]:
# Execute the second group 20 times
Y:=Y+0.05*(X+Y):X:=X+0.05:L:=L,[X,Y]:
```

### Example A.17 (Computer Plotting)

Graph by computer the explicit equation  $y = e^{-x} \sin(x)$  on  $0 \leq x \leq 2\pi$ .

**Solution:** Plot commands for five plotting systems are given below. The graphic in Figure 11 represents the maple output.

<code>plot(exp(-x)*sin(x),x=0..2*Pi);</code>	maple
<code>Plot[{exp(-x) sin(x)}, {x,0,2 Pi}];</code>	mathematica
<code>plot [0:2*pi] exp(-x)*sin(x)</code>	gnuplot
<code>x=0:0.1:2*PI; y=exp(-x).*sin(x); plot(x,y)</code>	matlab and scilab



**Figure 11.** Computer plot of  $y = e^{-x} \sin(x)$  on  $0 \leq x \leq 2\pi$ .

### Example A.18 (Computer Plotting)

Plot by computer the implicit equation  $x^2 + 2y^2 + xy = 10$ .

**Solution:** Some background will be reviewed, which provides the source of intuition for plotting similar implicit equations. Quadratic forms  $Ax^2 + 2Bxy + Cy^2 = D$  are studied in *analytic geometry*, and it is known how to classify the graphic based upon the sign of  $B^2 - AC$ . A change of variables to eliminate the cross term  $xy$  would result in a hand solution to this example, an ellipse with semiaxes  $a \approx 3.55$  and  $b \approx 2.13$  rotated about  $-22.5$  degrees with major axis along the line  $y = (1 - \sqrt{2})x$ . The exact semiaxis lengths are given by

$$\frac{1}{a^2} = \frac{3 - \sqrt{2}}{20}, \quad \frac{1}{b^2} = \frac{3 + \sqrt{2}}{20}.$$

The graphic in Figure 12 is the result of the `maple` code below. Plots of *implicit equations* require tweaking of the domain and various plot parameters. The feature is not available in some programs, e.g., `gnuplot`.

```
# Maple V
with(plots):
eq:=x^2+2*y^2 + x*y = 10:
opt:=scaling=constrained,grid=[40,40]:
implicitplot(eq,x=-4..4,y=-4..4,opt);
```

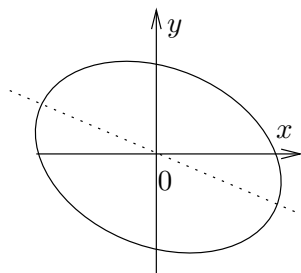


Figure 12. Implicit plot of  $x^2 + 2y^2 + xy = 10$ .

## Exercises A.2 [↗](#)

**Curve library graphics** Apply the curve library method to construct by hand a graphic of the given equations on one set of axes.

- $y = 2x + 1, y = 3(x + 1)^2$
- $y = \frac{-1}{x + 1}, y = -2x - 1$
- $y = \frac{2}{(x + 1)^2}$
- $y = \frac{-1}{(x + 1)^3}$
- $y = x^2, y = (x - 1)^4, y = (x - 2)^6$
- $y = \frac{1}{x + 1}, y = \frac{1}{(x - 1)^2}$

**Factored polynomial graphics** Apply the curve library method to construct by hand a graphic of the given factored polynomial on one set of axes.

- $y = -2x(x - 1)^2$
- $y = 2x(x + 1)^3$
- $y = -(x + 1)^2(x - 1)^3$
- $y = (x + 1)^3(x - 1)^4$
- $y = (x + 1)(x - 1)^3(x + 2)$
- $y = -x^3(1 - x)(1 + x)$
- $y = \pi(x + 1)(x - 1)(x + 2)^2$
- $y = \pi^2(x + 1)(x - 1)(x + 2)^3$

**Factored rational graphics** Apply the curve library method to construct by hand a graphic of the given factored rational function on one set of axes.

15.  $y = \frac{x-1}{x+1}$

16.  $y = \frac{2x+1}{x+2}$

17.  $y = \frac{x(x+1)}{(x+2)(x-2)}$

18.  $y = \frac{x(2x+1)}{(x+2)(x-2)}$

19.  $y = \frac{-x(1-x)}{(x+1)(x-2)}$

20.  $y = \frac{5x(x+1)}{(x-1)(x-2)}$

**Computer plotting of tables** Make a table of values  $x = 0$  to  $x = 1$  in steps of 0.05 for the given approximate equation and plot the table of values. Cite the *recursion* formulas applied to obtain the next table pair from the previous table pair.

21.  $y(x+0.05) \approx y(x) + 0.05(1-y(x)),$   
 $y(0) = 1$

22.  $y(x+0.05) \approx y(x) + 0.05(1+y(x)),$   
 $y(0) = 1$

23.  $y(x+0.05) \approx y(x) + 0.05(x-y(x)),$   
 $y(0) = 0$

24.  $y(x+0.05) \approx y(x) + 0.05(2x+y(x)),$   
 $y(0) = 0$

25.  $y(x+0.05) \approx y(x) + 0.05(\sin x + xy(x)),$   
 $y(0) = 2$

26.  $y(x+0.05) \approx y(x) + 0.05(\sin x - x^2y(x)),$   
 $y(0) = 2$

**Computer plots of explicit equations** Plot by computer the given explicit equation over  $0 \leq x \leq 1$ .

27.  $y = e^{-x} \sin \pi x$

28.  $y = e^{-x} \cos \pi x$

29.  $y = e^{-x} \ln(1+x)$

30.  $y = e^{-x} \ln(1+x^2)$

31.  $y = \sin(\pi x) \sin^2(2\pi x)$

32.  $y = \sin(\pi x) \cos^2(\pi x)$

**Implicit plots** Plot by computer or by hand the given implicit equation.

33.  $x^2 + y^2 + 3xy = 10$

34.  $x^2 + y^2 - 3xy = 10$

35.  $x^2 - (y+1)^2 = 1$

36.  $x^2 - y^2 + xy = 10$

37.  $x(x-1)y = 5$

38.  $xy(1+y^2) = 10$

## A.3 Explicit and Implicit Answers

Important to engineers and scientists are methods by which an existing *answer* can be tested for correctness. Given here are tests for explicit and implicit equations, as applied to the initial value problem

$$(1) \quad \begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

It is possible to test mathematical equations of the form  $y = y(x)$  and  $F(x, y) = 0$ , to see if they represent a solution to the problem (1). Both methods rely upon the expansion of the left side (LHS) and the right side (RHS) of equations. The two sides are compared for equality, either symbolically or else as constants. A proposed *answer* passes the test if the two sides are equal, that is, LHS = RHS.

### Explicit Equations

An *explicit equation*  $y = y(x)$  represents a solution of (1) provided checkpoints (a), (b) hold below.

- (a) The equation  $y' = f(x, y)$  is expanded using  $y = y(x)$  to produce a LHS and a RHS that depend on  $x$ . The expressions LHS and RHS are tested for symbolic equality at each  $x$  in the domain of  $y(x)$ .
- (b) The equation  $y(x_0) = y_0$  has a constant LHS, evaluated using the given expression for  $y(x)$  and the value  $x = x_0$ . The constant RHS is  $y_0$ . The expressions LHS and RHS are tested for numerical equality.

### Implicit Equations

A given *implicit equation*  $F(x, y) = 0$  represents a solution of (1) provided checkpoints (c), (d) hold below.

- (c) Briefly, *implicit differentiation* of  $F(x, y) = 0$  reproduces (1). Technically, the equation  $f(x, y) = -F_x(x, y)/F_y(x, y)$  is expanded using the formulas for  $F$  and its partial derivatives  $F_x$  and  $F_y$ , to produce a LHS and a RHS which are expressions in the two symbols  $x, y$ . The symbolic equality LHS = RHS must hold for all  $(x, y)$  satisfying  $F(x, y) = 0$ .
- (d) Initial condition  $y(x_0) = y_0$  is tested by expansion of the equation  $F(x_0, y_0) = 0$  into LHS and RHS. The constant expressions must be equal, LHS = RHS.

The equation  $F(x, y) = 0$  can be viewed as a **conservation law**, e.g., if  $F$  is energy, then  $F = 0$  says the energy is constant along the path of a particle.

### A.3 Explicit and Implicit Answers

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Implicit differentiation results in the **dynamical equation** for the conservation law. This equation describes the dynamics or change, hence it is expressed in terms of the rate of change  $dy/dx$ .

The formal verification of (c) depends upon the **chain rule for 2 variables**

$$\frac{dF(x, y)}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt}.$$

Technical assumptions which allow  $y$  to be found as a function of  $x$  in the equation  $F(x, y) = 0$  appear in the *implicit function theorem*, page 1041, where the critical assumption  $F_y(x_0, y_0) \neq 0$  is made.

The chain rule is applied to the equation  $F(x, y) = 0$ , setting  $x = t$ ,  $y = y(t)$ , to give

$$F_x(t, y(t))(1) + F_y(t, y(t)) \frac{dy(t)}{dt} = 0.$$

Substitution of  $t = x$  and  $y' = f(x, y)$  into this equation justifies Part (c) of the test.

## Computer Algebra Methods

The ideas outlined above for checking an explicit or implicit equation can be implemented in most computer algebra systems (abbreviation CAS). It suffices to create the two CAS symbols LHS and RHS and then test for equality of LHS and RHS in all the relevant variables.

It sometimes transparent that LHS and RHS are equal, due to automatic CAS simplifications. There are instances where equality is completely opaque, because of insufficient CAS simplifications. To the rescue comes this idea: define ZERO to be the difference of LHS and RHS. The CAS symbol ZERO should reduce to zero, after simplifications are performed. See Examples A.21, A.22, page 1026 for details.

Realistically, engineers and scientists will migrate to CAS verifications, after intuition has been gained from many hand computations. Even in the simplest applications, something can go wrong, so experts advise: *verify the results by hand and by machine, verify it more than once, and check it from different viewpoints.*<sup>1</sup>

### Example A.19 (Verify an Explicit Solution)

Verify the explicit solution  $y = x - 1 + 2e^{-x}$  for  $y' = x - y$ ,  $y(0) = 1$ .

**Solution:** The *initial condition*  $y(0) = 1$  is verified as follows.

$y(0) = (x - 1 + 2e^{-x}) _{x=0}$	Compute the left side of $y(0) = 1$ where $y(x) = x - 1 + 2e^{-x}$ .
$= 0 - 1 + 2e^0$	Evaluate.

---

<sup>1</sup>Picture a person walking in the rain, dripping wet, holding in one hand a closed umbrella.

### A.3 Explicit and Implicit Answers

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= 1.

Therefore, the two sides of  $y(0) = 1$  are equal.

The *differential equation* is verified in a slightly different way, by independent expansion of the left and right sides.

LHS =  $y'$

$$= (x - 1 + 2e^{-x})'$$

$$= 1 - 2e^{-x},$$

RHS =  $x - y$

$$= x - (x - 1 + 2e^{-x})$$

$$= 1 - 2e^{-x}.$$

The left side of  $y' = x - y$  is  $y'$ .

Insert  $y(x) = x - 1 + 2e^{-x}$ .

Apply derivative rules.

The right side of  $y' = x - y$  is  $x - y$ .

Insert  $y(x) = x - 1 + 2e^{-x}$ .

Simplified RHS.

Therefore, LHS=RHS.

#### Example A.20 (Verify an Implicit Solution)

Verify the implicit solution  $3x^2 + y^2 = c$  for the equation  $y' = -3x/y$ .

**Solution:**

$$f(x, y) = -3x/y$$

$$F(x, y) = 3x^2 + y^2$$

$$F_x(x, y) = 6x$$

$$F_y(x, y) = 2y$$

$$Z = F_x(x, y) + F_y(x, y)f(x, y)$$

$$= 6x + 2y(-3x/y)$$

$$= 0.$$

The right side of  $y' = -3x/y$  is called  $f(x, y)$ .

The level curve  $F(x, y) = c$  duplicates the proposed solution  $3x^2 + y^2 = c$ .

Partial derivative in  $x$ .

Partial derivative in  $y$ .

Test the differential equation. Expect  $Z$  to be zero.

Substitute partials and  $f(x, y) = -3x/y$ .

Simplify.

Therefore, implicit differentiation of  $F(x, y) = c$  reproduces the differential equation  $y' = -3x/y$ ; see page 1025.

#### Example A.21 (Verify Explicit Solution by Computer)

Verify the explicit solution  $y = e^{-x}$  for  $y' = -y$ ,  $y(0) = 1$  using a computer algebra system.

**Solution:** The illustration will be for `maple`.

`y:=x->exp(-x):`

The `maple` code for solution  $y = e^{-x}$ .

`LHS:=diff(y(x),x):`

The left side of  $y' = -y$  is  $y'(x)$ .

`RHS:=-y(x):`

The right side of  $y' = -y$  is  $-y(x)$ .

`ZERO:=LHS-RHS;`

The expression `ZERO` depends symbolically on  $x$ .

`Z:=y(0)-1;`

Write `Z` as the difference of the left and right sides of the equation  $y(0) = 1$ .

Evaluation of `ZERO` should give the symbolic answer 0, because  $LHS = RHS$  is equivalent to  $LHS - RHS = 0$ . Evaluation of the constant `Z` should give constant 0. This verifies the differential equation and initial condition by computer algebra methods. In unusual

### A.3 Explicit and Implicit Answers

cases, it may be necessary to force simplifications or to interpret the answers. Simplification is forced by the `maple` command `simplify(ZERO)` while interpretation may be required to conclude that an expression, e.g., `sin(n*Pi)`, evaluates to zero.

Since `maple V 5.1`, there is a special function `odetest`, designed to do the above test. It is valuable because it eliminates errors made by re-typing formulas.

#### Example A.22 (Verify Implicit Solution by Computer)

Verify the implicit solution  $x^2 + y^2 = c$  for the equation  $y' = -x/y$  using a computer algebra system.

**Solution:** The illustration will be for `maple`.

`F:=(x,y)->x*x+y*y-c`

Write  $x^2 + y^2 = c$  as  $F = 0$  where  $F = x^2 + y^2 - c$ .

`f:=(x,y)->-x/y`

The right side of  $y' = -x/y$  is  $f(x, y)$ .

`Fx:=(x,y)->diff(F(x,y),x)`

Partial derivative in  $x$ .

`Fy:=(x,y)->diff(F(x,y),y)`

Partial derivative in  $y$ .

`ZERO:=Fx(x,y)+f(x,y)*Fy(x,y)`

Variable `ZERO` is the left side of  $F_x + F_y y' = 0$  with  $y' = f(x, y)$ .

Evaluation of `ZERO` should give the answer 0. This verifies the implicit solution of the differential equation by computer algebra methods.

In `maple V 5.1`, the function `odetest` will test implicit solutions.

### Exercises A.3

**Verify an Explicit Solution** Apply the methods in Example A.19, page 1025, to verify the given solution of the initial value problem.

1.  $I(t) = I_0 e^{-2t}$ ,  
 $I' + 2I = 0$ ,  $I(0) = I_0$ .

2.  $Q(t) = Q_0 e^{-0.2t}$ ,  
 $Q' = -0.2Q$ ,  $Q(0) = Q_0$ .

3.  $A(t) = 100e^{kt}$ ,  
 $A' = kA$ ,  $A(0) = 100$ .

4.  $P(t) = 1000e^{ht}$ ,  
 $P' = hP$ ,  $P(0) = 1000$ .

5.  $y(x) = -1 + \sqrt{4 + x^2 - 2x}$ ,  
 $y' = \frac{x-1}{y+1}$ ,  $y(0) = 1$ .

6.  $y(x) = -1 + \sqrt{2 + 2e^x - 2x}$ ,  
 $y' = \frac{e^x - 1}{y+1}$ ,  $y(0) = 1$ .

7.  $y(x) = e^{x^2/2}$ ,  
 $y' = xy$ ,  $y(0) = 1$ .

8.  $y(x) = e^{x^3/3}$ ,  
 $y' = x^2 y$ ,  $y(0) = 1$ .

9.  $y(x) = e^{1-\cos(x)}$ ,  
 $y' = \sin(x)y$ ,  $y(0) = 1$ .

10.  $y(x) = e^{\sin(x)}$ ,  
 $y' = \cos(x)y$ ,  $y(0) = 1$ .

**Verify an Implicit Solution** Apply the methods in Example A.20, page 1026, to verify the given implicit solution of the differential equation. If an initial condition is given, then verify it also.

11.  $xy^2 + x^2y + xy = c$ ,  
 $y' = -\frac{y(y+2x+1)}{x(2y+x+1)}$ .

12.  $x^2y^2 + x^3y + xy^2 = c$ ,  
 $y' = -\frac{y(2xy+3x^2+y)}{x(2xy+x^2+2y)}$ .



$$13. \quad x \sin y + \cos(xy) = c, \\ y' = -\frac{-\sin(y) + \sin(xy)y}{x(-\cos(y) + \sin(xy))}.$$

$$14. \quad x^2 \cos(y) + \sin(xy^2) = c, \\ y' = \frac{2x \cos(y) + \cos(xy^2)y^2}{x(x \sin(y) - 2 \cos(xy^2)y)}.$$

$$15. \quad x^2 e^y + e^{x-y} = 1 + e, \\ y' = -\frac{2xe^y + e^{x-y}}{x^2 e^y - e^{x-y}}, \quad y(1) = 0.$$

$$16. \quad x^3 e^{-y} + x e^{2x-y} = 1 + e^2, \\ y' = \frac{3x^2 + e^{2x} + 2xe^{2x}}{x(x^2 + e^{2x})}, \\ y(1) = 0.$$

**Verify an Explicit Solution by Computer** Apply the methods in Example A.21, page 1026, to verify the given solution of the initial value problem.

$$17. \quad y(x) = \sqrt[3]{3x}, \\ y' = 1/y^2, \quad y(1/3) = 1.$$

$$18. \quad y(x) = \sqrt[4]{4x}, \\ y' = 1/y^3, \quad y(1/4) = 1.$$

$$19. \quad y(x) = e^{-x^2/2}, \\ y' = -xy, \quad y(0) = 1.$$

$$20. \quad y(x) = \pi e^{-x^3/3}, \\ y' = -x^2 y, \quad y(0) = \pi.$$

$$21. \quad y(x) = x e^{\cos(x)-1}, \\ y' = (1/x - \sin(x))y, \\ y(2\pi) = 2\pi.$$

$$22. \quad y(x) = \tan x + e^{\sin(x)}, \\ y' = \sec^2 x - \sin x + y \cos(x), \\ y(0) = 1.$$

**Verify Implicit Solution by Computer**

Apply the methods in Example A.22, page 1027, to verify the given implicit solution of the differential equation. If an initial condition is given, then verify it also.

$$23. \quad xy = 2, \quad y' = -y/x, \quad y(2) = 1.$$

$$24. \quad x^2 y = 2, \quad y' = -2y/x, \quad y(1) = 2.$$

$$25. \quad x e^y + y e^x = c, \quad y' = -\frac{e^y + y e^x}{x e^y + e^x}.$$

$$26. \quad x e^{-y} + y e^{-x} = c, \\ y' = \frac{e^{-y} - y^2 e^{-x}}{x e^{-y} - 2 y e^{-x}}.$$

$$27. \quad x \sin y + \cos(xy) = c, \\ y' = \frac{\sin(y) - \sin(xy)y}{x(\sin(xy) - \cos(y))}.$$

$$28. \quad x^2 \cos(y) + \sin(xy^2) = c, \\ y' = \frac{2x \cos(y) + \cos(xy^2)y^2}{x(-x \sin(y) + 2 \cos(xy^2)y)}.$$

## A.4 Numerical and Graphical Answers

Given here are tests for numeric tables and graphics, as applied to the initial value problem

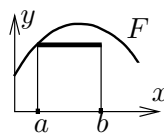
$$(1) \quad \begin{aligned} y' &= f(x, y), \\ y(x_0) &= y_0. \end{aligned}$$

The numerical tests are based upon numerical integration methods from calculus. The ideas lead to the numerical methods of Euler, Heun and Runge-Kutta, which are studied in the text.

### Numerical Integration Approximations

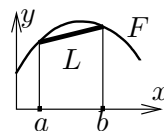
Reproduced here for future reference are calculus topics: the **rectangular rule**, the **trapezoidal rule** and **Simpson's rule** for the numerical approximation of an integral  $\int_a^b F(x)dx$ . The approximations are valid for  $b - a$  small. Larger intervals must be subdivided, then the rule applies to the small subdivisions.

**Rectangular Rule.** The approximation uses Euler's idea of replacing the integrand by a constant. The value of the integral is approximately the area of a rectangle of width  $b - a$  and height  $F(a)$ .



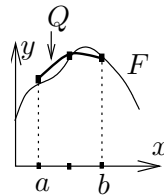
$$(2) \quad \int_a^b F(x)dx \approx (b - a)F(a).$$

**Trapezoidal Rule.** The rule replaces the integrand  $F(x)$  by a linear function  $L(x)$  which connects the planar points  $(a, F(a))$ ,  $(b, F(b))$ . The value of the integral is approximately the area under the curve  $L$ , which is the area of a trapezoid.



$$(3) \quad \int_a^b F(x)dx \approx \frac{b - a}{2} (F(a) + F(b)).$$

**Simpson's Rule.** The rule replaces the integrand  $F(x)$  by a quadratic polynomial  $Q(x)$  which connects the planar points  $(a, F(a))$ ,  $((a + b)/2, F((a + b)/2))$ ,  $(b, F(b))$ . The value of the integral is approximately the area under the quadratic curve  $Q$ .



$$(4) \quad \int_a^b F(x)dx \approx \frac{b - a}{6} (F(a) + 4F((a + b)/2) + F(b)).$$

**Simpson's Polynomial Rule.** If  $Q(x)$  is a linear, quadratic or cubic polynomial, then (proof on page 1036)

$$(5) \quad \int_a^b Q(x)dx = \frac{b-a}{6} (Q(a) + 4Q((a+b)/2) + Q(b)).$$

Integrals of linear, quadratic and cubic polynomials can be evaluated *exactly* using Simpson's polynomial rule (5); see Example A.26, page 1035.

**Remarks on Simpson's Rule.** The right side of (4) is exactly the integral of  $Q(x)$ , which is evaluated by equation (5). The appearance of  $F$  instead of  $Q$  on the right in equation (4) is due to the relations  $Q(a) = F(a)$ ,  $Q((a+b)/2) = F((a+b)/2)$ ,  $Q(b) = F(b)$ , which arise from the requirement that  $Q$  connect three points along curve  $F$ .

The quadratic interpolation polynomial  $Q(x)$  is determined uniquely from the three data points; see page 1037 for a formula for  $Q$  and a derivation. It is interesting that Simpson's rule depends only upon the uniqueness and not upon the actual formula for  $Q$ !

## Graphic and Numeric Table Test

Studied here is a general problem:

Find a test which verifies a given graphic or numeric table, given only the  $xy$ -pairs and  $y' = f(x, y)$ .

The test should work with a hand calculator, a spreadsheet or a computer algebra system. Important to the test is the ability to *spot-check* the graphic or table, testing just one or two data items.

To be presented here are the *Euler*, *trapezoidal* and *Simpson tests*. They detect errors in graphics by pixel criteria; see page 1032 for details. All tests have limitations and flaws. If the data items are far apart, then the approximation is poor and the test fails. Use is limited to *detection of gross errors*.

**Equivalent Integral Equation.** Fundamental to understanding the *tests* is the **equivalent integral equation**

$$(6) \quad y(x) = y(x_0) + \int_{x_0}^x f(t, y(t))dt$$

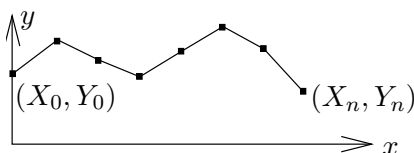
for the first order initial value problem  $y' = f(x, y)$ ,  $y(x_0) = y_0$ . Equation (6) is justified on page 1036.

**Raw Data.** Graphics produced in computer algebra systems or in computer spreadsheets require raw plot data, either implicitly or explicitly supplied. It will be assumed that this data is available as a table of  $xy$ -values, or equivalently, as a list of pairs

$$(X_0, Y_0), (X_1, Y_1), \dots, (X_n, Y_n).$$

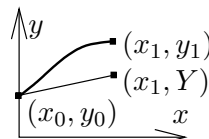
It is necessary in the tests to evaluate  $f(x, y)$  at the points of this list. No other evaluations of  $f$  are used, for the simple tests.

A linear connection (“connect-the-dots”) of the data points is used by many computer programs; many points are required for a smooth result. Typical detail is shown in Figure 13.



**Figure 13. Linear connection of raw data points**  $(X_0, Y_0), (X_1, Y_1), \dots, (X_n, Y_n)$  **in a computer graphic.**

**Euler’s Test.** The test applies to one pair of consecutive points from the raw plot data list. **Euler’s test** is related to **Euler’s numerical method**, which is the oldest and simplest numerical method for first order differential equations.



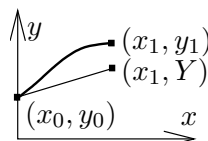
The test is named after Leonhard Euler<sup>2</sup> (1707-1783), Swiss physicist and mathematician. The test is justified on page 1037.

**Step 1.** Let  $(x_0, y_0)$  and  $(x_1, y_1)$  denote consecutive pairs from the raw plot data list  $(X_0, Y_0), (X_1, Y_1), \dots, (X_n, Y_n)$ .

**Step 2.** Compute  $h = x_1 - x_0$  and  $Y = y_0 + hf(x_0, y_0)$ .

**Step 3.** Test equality of  $y_1$  and  $Y$ .

**Trapezoidal Test.** The tests applies to a consecutive pair of points from the raw plot data list. The **trapezoidal test** is related to the **modified Euler numerical method**, or **Heun’s method**. The justification appears on page 1037.



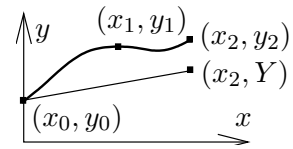
**Step 1.** Let  $(x_0, y_0)$  and  $(x_1, y_1)$  denote consecutive pairs from the raw plot data list  $(X_0, Y_0), (X_1, Y_1), \dots, (X_n, Y_n)$ .

**Step 2.** Compute  $h = x_1 - x_0$  and  $Y = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1))$ .

**Step 3.** Test equality of  $y_1$  and  $Y$ .

<sup>2</sup>His name is pronounced *Oiler*, and not *Yuler*.

**Simpson's Test.** The test is applied to three consecutive pairs from the raw plot data list. Assume uniformly-spaced  $X$ -data. The **Simpson test** is related to the **Runge-Kutta numerical method** for first order differential equations. Justification is on page 1037.



**Step 1.** Let  $(x_0, y_0)$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$  denote three consecutive pairs from the raw plot data list. It is assumed that  $x_1 = (x_0 + x_2)/2$ .

**Step 2.** Let  $Y = y_0 + \frac{x_2 - x_0}{6}(f(x_0, y_0) + 4f(x_1, y_1) + f(x_2, y_2))$ .

**Step 3.** Test equality of  $y_2$  and  $Y$ .

**Pass and Fail.** A given test can **pass** or **fail** according to how the resulting approximation is judged. A graph *passes* the test if the ideal data point  $(x, y)$  and the approximate data point  $(x, Y)$  land on the same pixel, that is, the dots cannot be distinguished in the graphic. Arithmetically, the test is an inequality

$$\frac{|y - Y|}{|d - c|} < \frac{1}{M},$$

where  $M$  is the number of  $y$ -pixels in the graphic on  $c \leq y \leq d$ . Otherwise, the graph *fails*.

Exercises in this text use the **standard graphic**, a  $3\frac{1}{4}$ -inch square graphic at 300 dots per inch, which is about  $1000 \times 1000$  pixels. The same graphic displayed on a video monitor uses considerably fewer pixels.

There are two standard ways to measure the approximations:

**Absolute Error.** The **absolute error** is  $E = |y - Y|$ . The standard graphic of 1000 pixels on  $c \leq y \leq d$  will *pass* the test if  $E < \frac{d-c}{1000}$ .

**Relative Error.** The **relative error** is  $E = |y - Y|/|y|$ . Since  $Y = (1 \pm E)y$ , it measures the percentage error.

Mostly, it is used for  $y$ -ranges  $c \leq y \leq d$  where  $c > 0$  or  $d < 0$  (division by zero is problematic). The standard graphic of 1000 pixels on  $0 < c \leq y \leq d$  will *pass* the test if  $E < \frac{d-c}{1000d}$ .

To distinguish the two measurements, apply the definitions to  $y = 1000$  and  $Y = 1001$ : the absolute error is 1 and the relative error is  $1/1000$ .

**Uniformly-Spaced and Adaptive Data.** In computer workbenches like `matlab` or `scilab`, the  $x$ -values will be uniformly spaced. In other systems, uniform spacing can be arranged, but the default may be **non-uniform data** or **adaptive data**, e.g., `maple`. Graphics systems normally document how to print out the plot data used for the graphic, even if the plot was done by implicit or

## A.4 Numerical and Graphical Answers

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automatic means. To get uniformly spaced data in `maple`, some preparation is required, as the following illustration shows. Uniformly spaced data is required in the Simpson test, page [1032](#).

```
y:=x->20*exp(-3*x):a:=0.0: b:=1.0:
# Adaptive plot saved in variable P
P:=plot(y(x),x=a..b);
# Uniform x-data plot saved in Q. Maple V 5.1
Q:=plot(y(x),x=a..b,adaptive=false,
        sample=[seq(i*h,i=0..(b-a)/h)]);
```

### Example A.23 (Spot Check)

A graphic for the differential equation  $y' = x + y$  has window  $0 \leq x \leq 0.5$ ,  $1 \leq y \leq 2$  and uses  $1000 \times 1000$  pixels. Two adjacent plot data entries are  $(0.180, 1.21443)$  and  $(0.195, 1.23562)$ . Spot-check these entries with Euler's test.

**Solution:** The plot data passes Euler's test, page [1031](#), because the target value 1.23562 is close to the Euler approximation 1.2353464, with less than one pixel difference in the plot. The steps of the justification appear below.

$x_0 = 0.180$ , $y_0 = 1.21443$	The first data point $(0.180, 1.21443)$ .
$x_1 = 0.195$ , $y_1 = 1.23562$	The second data point $(0.195, 1.23562)$ .
$h = x_1 - x_0 = 0.015$	Define the step size.
$Y = y_0 + h(x_0 + y_0)$	Apply Euler's test, page <a href="#">1031</a> .
$= 1.21443$	Substitute $x_0$ , $y_0$ , $h$ .
$+ 0.015(0.18 + 1.21443)$	
$= 1.2353464$ .	Expected to be close to $y_1 = 1.23562$ .

The *absolute error* is  $E = 0.0002736$ , which is less than the cutoff value of  $E^* = (d - c)/1000 = 0.001$ . The data passes Euler's test.

### Example A.24 (Trapezoidal Test)

A graphic for the differential equation  $y' = x + y$  has window  $0 \leq x \leq 0.5$ ,  $1 \leq y \leq 2$  and uses  $1000 \times 1000$  pixels. Find the worst absolute error  $|y_1 - Y|$  made according to the trapezoidal test for the associated plot data below and report pass or fail.

$(0.180, 1.21443)$ ,  $(0.195, 1.23562)$ ,  $(0.210, 1.25736)$ ,  
 $(0.225, 1.27965)$ ,  $(0.240, 1.30250)$ ,  $(0.255, 1.32592)$ .

**Solution:** The cutoff value for the absolute error is  $(d - c)/1000 = 0.001$ . It will be justified below that the worst absolute error according to the Trapezoidal test is 0.0000057. In short, the data passes the test.

The first pair of points in the plot data passes the Trapezoidal test, page [1031](#), because the target value  $y_1 = 1.23562$  is close to the test's value  $Y = 1.2356179$ , with absolute error  $|y_1 - Y| = 0.0000021$ . The steps of the justification appear below.

## A.4 Numerical and Graphical Answers

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$x_0 = 0.180, y_0 = 1.21443$	Initial point (0.180, 1.21443).
$x_1 = 0.195, y_1 = 1.23562$	Next point (0.195, 1.23562).
$h = 0.015$	The value $h = x_1 - x_0$ should be small.
$f(x, y) = x + y$	Right side of the differential equation.
$Y = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1))$	Trapezoidal test, page 1031.
$= y_0 + \frac{h}{2}(x_0 + y_0 + x_1 + y_1)$	Expand functional expressions.
$= 1.21443 + \frac{0.015}{2}(2.82505)$	Expand expressions.
$= 1.2356179.$	Calculator result. The absolute error $E =  y_1 - Y $ is 0.0000021.

This *process* can be carried out on the other four pairs of points, in a similar way, to find the five absolute errors

0.0000021, 0.0000052, 0.000000075, 0.0000036, 0.0000057.

The largest error is 0.0000057.

Details of a `maple` implementation appear below. The errors made with its ten-digit exact arithmetic may differ from those of a calculator.

```
# Execute the first group once.
f:=(x,y)->x+y:
L:=[[.180, 1.21443],[.195, 1.23562],
    [.210, 1.25736],[.225, 1.27965],
    [.240, 1.30250],[.255, 1.32592]]:
n:=1:
# Execute the second group 5 times.
x0:=L[n][1]:y0:=L[n][2]:
x1:=L[n+1][1]:y1:=L[n+1][2]:
Y:=y0+(x1-x0)*0.5*(f(x0,y0)+f(x1,y1)):
n:=n+1: ABSerror:=abs(y1-Y);
```

New to the `maple` code is the *list* `L` of pairs of points. The syntax `L[n]` refers to item  $n$  of the 6 items in the list, a pair. Syntax `L[n][1]` means the first entry of that pair.

### Example A.25 (Simpson Test)

A graphic for the differential equation  $y' = x + y$  has window  $0 \leq x \leq 0.5, 1 \leq y \leq 2$  and uses  $1000 \times 1000$  pixels. Given the data set below, compute the Simpson test prediction for each triple of data points. Report the four absolute errors and judge pass or fail.

(0.180, 1.21443), (0.195, 1.23562), (0.210, 1.25736),  
(0.225, 1.27965), (0.240, 1.30250), (0.255, 1.32592).

## A.4 Numerical and Graphical Answers

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**Solution:** The cutoff value for the absolute error is  $(d - c)/1000 = 0.001$ . It will be justified below that the four absolute errors according to the Simpson test are 0.0000087, 0.0000065, 0.0000023 and 0.0000079. The data passes Simpson's test.

The first three pairs of points in the plot data pass the Simpson test, page 1032, because the target value  $y_1 = 1.25736$  is close to the test's value  $Y = 1.257351350$ , with absolute error  $|y_1 - Y| = 0.0000087$ . The steps of the justification are below.

$x_0 = 0.180, y_0 = 1.21443$	Initial point (0.180, 1.21443).
$x_1 = 0.195, y_1 = 1.23562$	Second point (0.195, 1.23562).
$x_2 = 0.210, y_2 = 1.25736$	Third point (0.210, 1.23562).
$f(x, y) = x + y$	The right side of the differential equation.
$Y = y_0 + \frac{x_2 - x_0}{6} (f(x_0, y_0) + 4f(x_1, y_1) + f(x_2, y_2))$	Simpson test, page 1032.
$= y_0 + 0.005(x_0 + y_0 + 4(x_1 + y_1) + x_2 + y_2)$	Expand functional expressions.
$= 1.21443 + 0.005(8.58427)$	Substitute constants.
$= 1.2573513.$	Absolute error $E =  y_2 - Y  = 0.0000087$ .

This *process* can be carried out in a similar way on the other triples of points:

Second: (0.195, 1.23562), (0.210, 1.25736), (0.225, 1.27965),

Third: (0.210, 1.25736), (0.225, 1.27965), (0.240, 1.30250),

Fourth: (0.225, 1.27965), (0.240, 1.30250), (0.255, 1.32592).

The absolute errors for these last three cases are 0.0000065, 0.0000023 and 0.0000079.

Details of a maple implementation appear below.

```
# Execute the first group once.
f:=(x,y)->x+y:
L:=[[.180, 1.21443],[.195, 1.23562],
    [.210, 1.25736],[.225, 1.27965],
    [.240, 1.30250],[.255, 1.32592]]:
n:=1:
# Execute the second group 4 times.
x0:=L[n][1]:y0:=L[n][2]:
x1:=L[n+1][1]:y1:=L[n+1][2]:
x2:=L[n+2][1]:y2:=L[n+2][2]:
Y:=y0+(x2-x0)*(f(x0,y0)+
    4*f(x1,y1)+f(x2,y2))/6:
n:=n+1: ABSerror:=abs(y2-Y);
```

### Example A.26 (Polynomial Quadrature)

Apply Simpson's polynomial rule (5) to verify  $\int_1^2 (x^3 - 16x^2 + 4)dx = -355/12$ .

**Solution:** The application proceeds as follows:

$$I = \int_1^2 Q(x)dx \qquad \text{Evaluate integral } I \text{ using } Q(x) = x^3 - 16x^2 + 4.$$



## A.4 Numerical and Graphical Answers

---

$$\begin{aligned}
 &= \frac{2-1}{6} (Q(1) + 4Q(3/2) + Q(2)) && \text{Apply Simpson's polynomial rule (5).} \\
 &= \frac{1}{6} (-11 + 4(-229/8) - 52) && \text{Use } Q(x) = x^3 - 16x^2 + 4. \\
 &= -\frac{355}{12}. && \text{Equality verified.}
 \end{aligned}$$

**Integral Equation Justification.** Let  $f(x, y)$  be continuous for  $a < x < b$ ,  $-\infty < y < \infty$ . Assume  $(x_0, y_0)$  is in the domain. It will be justified that the initial value problem  $y' = f(x, y)$ ,  $y(x_0) = y_0$  is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t))dt.$$

The case  $x \geq x_0$  will be considered, the other case  $x \leq x_0$  being similar. *Equivalence* means a solution of the initial value problem is a solution of the integral equation, and conversely.

The integral equation is obtained from the initial value problem as follows: details.

$$\begin{aligned}
 y'(t) &= f(t, y(t)) && \text{The given equation with } x \text{ replaced by } t. \\
 \int_{x_0}^x y'(t)dt &= \int_{x_0}^x f(t, y(t))dt && \text{Integrate both sides on } x_0 \leq t \leq x. \text{ It is assumed that } \\
 &&& \text{ } y, y', f \text{ are continuous, which insures both integrals} \\
 &&& \text{are defined.} \\
 y(x) - y(x_0) &= \int_{x_0}^x f(t, y(t))dt && \text{Apply the fundamental theorem of calculus, page 1008,} \\
 &&& \text{part (a).}
 \end{aligned}$$

Conversely, if the integral equation is assumed, then  $y(x)$  is differentiable by the fundamental theorem of calculus, page 1008, part (b). Differentiate across both sides of the integral equation to obtain  $y' = f(x, y)$ . Finally, substitute  $x = x_0$  into the integral equation to obtain the initial condition  $y(x_0) = y_0$ .

**Simpson's Polynomial Rule Proof.** Let  $Q(x)$  be a linear, quadratic or cubic polynomial. It will be verified that

$$(7) \quad \int_a^b Q(x)dx = \frac{b-a}{6} (Q(a) + 4Q((a+b)/2) + Q(b)).$$

If the formula holds for polynomial  $Q$  and  $c$  is a constant, then the formula also holds for the polynomial  $cQ$ . Similarly, if the formula holds for polynomials  $Q_1$  and  $Q_2$ , then it also holds for  $Q_1 + Q_2$ . Consequently, it suffices to show that the formula is true for the special polynomials  $1, x, x^2$  and  $x^3$ , because then it holds for all combinations  $Q(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ .

Only the special case  $Q(x) = x^3$  will be treated here. The other cases are left to the exercises. The details:

$$\begin{aligned}
 \text{RHS} &= \frac{b-a}{6} \left( Q(a) + 4Q\left(\frac{a+b}{2}\right) + Q(b) \right) && \text{Evaluate the right side of equation} \\
 &&& (7). \\
 &= \frac{b-a}{6} \left( a^3 + \frac{1}{2}(a+b)^3 + b^3 \right) && \text{Substitute } Q(x) = x^3. \\
 &= \frac{b-a}{6} \frac{3}{2} (a^3 + a^2b + ab^2 + b^3) && \text{Expand } (a+b)^3 = a^3 + 3a^2b + 3ab^2 + \\
 &&& b^3 \text{ and simplify.}
 \end{aligned}$$

## A.4 Numerical and Graphical Answers

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$$\begin{aligned} &= \frac{3}{12}(b^4 - a^4) && \text{Multiply and simplify.} \\ \text{LHS} &= \int_a^b Q(x) dx && \text{Evaluate the left hand side (LHS) of} \\ &= \int_a^b x^3 dx && \text{equation (7).} \\ &= (b^4 - a^4)/4 && \text{Substitute } Q(x) = x^3. \\ &= \text{RHS.} && \text{Evaluate.} \\ & && \text{Compare with the RHS.} \end{aligned}$$

**Euler Test Proof:** To justify Euler's test, page 1031, apply the equivalent integral equation (6) and the rectangular rule (2) with  $F(t) = f(t, y(t))$ ,  $a = x_0$  and  $b = x_0 + h = x_1$ . This gives a first approximation

$$(8) \quad y(x_0 + h) \approx y(x_0) + hF(x_0).$$

Then apply approximation  $y(x_0) \approx y_0$  to the right side of (8) to give the approximation  $Y = y_0 + hf(x_0, y_0)$ .

**Trapezoidal Test Proof:** To justify the trapezoidal test, page 1031, begin with the equivalent integral equation (6) and approximate the integral using the trapezoidal rule (3), with  $F(t) = f(t, y(t))$ ,  $a = x_0$  and  $b = x_0 + h = x_1$ . This gives a first approximation

$$(9) \quad y(x_0 + h) \approx y(x_0) + \frac{h}{2}(F(x_0) + F(x_1))$$

Then apply approximations  $y(x_0) \approx y_0$  and  $y(x_1) \approx y_1$  to the right side of (9) to give the approximation  $Y = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1))$ .

**Simpson Test Proof:** To justify the Simpson test, page 1032, begin with the equivalent integral equation (6) and approximate the integral using Simpson's rule (4), with  $F(t) = f(t, y(t))$ ,  $a = x_0$  and  $b = x_2$ . This gives a first approximation

$$(10) \quad y(x_0 + h) \approx y(x_0) + \frac{x_2 - x_0}{6}(F(x_0) + 4F(x_1) + F(x_2))$$

Then apply approximations  $y(x_0) \approx y_0$ ,  $y(x_1) \approx y_1$  and  $y(x_2) \approx y_2$  to the right side of (10) to give the approximation

$$Y = y_0 + \frac{x_2 - x_0}{6}(f(x_0, y_0) + 4f(x_1, y_1) + f(x_2, y_2)).$$

**Quadratic Interpolation Proof:** Given  $a < b$  and the three data points  $(a, Y_0)$ ,  $((a + b)/2, Y_1)$ ,  $(b, Y_2)$ , it will be verified that the quadratic curve  $Q(X)$  which connects the points is given by

$$\begin{aligned} Q(X) &= Y_0 + (4Y_1 - Y_2 - 3Y_0) \frac{X - a}{b - a} \\ &\quad + (2Y_2 + 2Y_0 - 4Y_1) \frac{(X - a)^2}{(b - a)^2}. \end{aligned}$$

The term *quadratic* is meant loosely: it can be a constant or linear function as well. The solution is presented as two lemmas.<sup>3</sup> The first lemma contains the essential ideas. The second simply translates the variables.

---

<sup>3</sup>What's a lemma? It's a helper theorem, used to dissect long proofs into short pieces.

## A.4 Numerical and Graphical Answers

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**Lemma A.1** Given  $y_1$  and  $y_2$ , define  $A = y_2 - y_1$ ,  $B = 2y_1 - y_2$ . Then the quadratic  $y = x(Ax + B)$  fits the data items  $(0, 0)$ ,  $(1, y_1)$ ,  $(2, 2y_2)$ .

**Lemma A.2** Given  $Y_0$ ,  $Y_1$  and  $Y_2$ , define  $y_1 = Y_1 - Y_0$ ,  $y_2 = \frac{1}{2}(Y_2 - Y_0)$ ,  $A = y_2 - y_1$ ,  $B = 2y_1 - y_2$  and  $x = 2(X - a)/(b - a)$ . Then quadratic  $Y(X) = Y_0 + x(Ax + B)$  fits the data items  $(a, Y_0)$ ,  $((a + b)/2, Y_1)$ ,  $(b, Y_2)$ .

To verify the first lemma, the formula  $y = x(Ax + B)$  is tested to go through the given data points  $(0, 0)$ ,  $(1, y_1)$  and  $(2, 2y_2)$ . For example, the last pair is tested by the steps

$$\begin{aligned}y(2) &= 2(2A + B) && \text{Apply } y = x(Ax + B) \text{ with } x = 2. \\ &= 4y_2 - 4y_1 + 4y_1 - 2y_2 && \text{Use } A = y_2 - y_1 \text{ and } B = 2y_1 - y_2. \\ &= 2y_2. && \text{Therefore, the quadratic fits data item } (2, 2y_2).\end{aligned}$$

The other two data items are tested similarly, details omitted here.

To verify the second lemma, observe that it is just a change of variables in the first lemma,  $Y = Y_0 + y$ . The data fit is checked as follows:

$$\begin{aligned}Y(b) &= Y_0 + y(2) && \text{Apply formulas } Y(X) = Y_0 + y(x), y(x) = x(Ax + B) \\ & && \text{with } X = b \text{ and } x = 2. \\ &= Y_0 + 2y_2 && \text{Apply data fit } y(2) = 2y_2. \\ &= Y_2. && \text{The quadratic fits the data item } (b, Y_2).\end{aligned}$$

The other two items are checked similarly, details omitted here. This completes the proof of the two lemmas. The formula for  $Q$  is obtained from the second lemma as  $Q = Y_0 + Bx + Ax^2$  with substitutions for  $A$ ,  $B$  and  $x$  performed to obtain the given equation for  $Q$  in terms of  $Y_0$ ,  $Y_1$ ,  $Y_2$ ,  $a$ ,  $b$  and  $X$ .

## Exercises A.4

**Euler Test: Spot Check** Apply the methods of Example A.23, page 1033, to compute for the given differential equation the absolute error made by Euler's test for the given data. Report *pass* or *fail* for each exercise. Assume absolute error cutoff value  $(d - c)/1000 = 0.001$ .

1.  $y' = 2y + \sin(x)$ ,  
 (0.1, 0.005346),  
 (0.2, 0.022884),  
 (0.3, 0.055148).

2.  $y' = -y + \cos(x)$ ,  
 (0.1, 0.095000),  
 (0.2, 0.180003),  
 (0.3, 0.255019).

3.  $y' = y(1 - y) + 5$ ,  
 (0.400, 1.877093),  
 (0.405, 1.893746),  
 (0.410, 1.910168).

4.  $y' = y(2 - y) + 10$ ,  
 (0.400, 3.547216),  
 (0.405, 3.569489),  
 (0.410, 3.591196).

5.  $y' = 1 + y^2$ ,  
 (0.100, 0.100335),  
 (0.105, 0.105388),  
 (0.110, 0.110446).

6.  $y' = 4 + 4y^2$ ,  
 (0.100, 0.422793),  
 (0.105, 0.446573),  
 (0.110, 0.470781).

**Trapezoidal Test** Apply the methods of Example A.24, page 1033, to compute for the given differential equation the relative error  $E = |y_1 - Y|/|y_1|$  made by the Trapezoidal test for the given data. Report for each exercise *pass* or *fail* and the three error values. Assume the given relative error cutoff value  $E^*$ .

7.  $y' = 2y + \sin(x)$ ,  $E^* = 0.001$ ,  
 (0.1, 0.005346), (0.2, 0.022884),  
 (0.3, 0.055148), (0.4, 0.105129).

8.  $y' = -y + \cos(x)$ ,  $E^* = 0.00009$ ,  
 (0.1, 0.095000), (0.2, 0.180003),  
 (0.3, 0.255019), (0.4, 0.320080).

9.  $y' = y(1 - y) + 5$ ,  $E^* = 0.00024$ ,  
 (0.100, 0.516828),  
 (0.125, 0.647873),  
 (0.150, 0.777953),  
 (0.175, 0.621714).

10.  $y' = y(2 - y) + 10$ ,  $E^* = 0.0013$ ,  
 (0.100, 1.067919),  
 (0.125, 1.341712),  
 (0.150, 1.610877),  
 (0.175, 1.871962).

11.  $y' = \frac{1 - x}{1 + y}$ ,  $E^* = 0.0004$ ,  
 (0.100, 0.090871),  
 (0.125, 0.111024),  
 (0.150, 0.130265),  
 (0.175, 0.148641).

12.  $y' = \frac{1 + x}{1 - y}$ ,  $E^* = 0.00047$ ,  
 (0.100, 0.111181),  
 (0.125, 0.143043),  
 (0.150, 0.176896),  
 (0.175, 0.212996).

**Simpson Test** Apply the ideas in Example A.25, page 1034, to compute for the given differential equation the relative error  $E = |y_2 - Y|/|y_2|$  made by the Simpson test for the given data. Report for each exercise *pass* or *fail* and the three error values. Assume the given relative error cutoff value  $E^*$ .

13.  $y' = 2y + \sin(x)$ ,  $E^* = 0.0008$ ,  
 (0.2, 0.022884), (0.3, 0.055148),  
 (0.4, 0.105129).

14.  $y' = -y + \cos(x)$ ,  $E^* = 0.00044$ ,  
 (0.2, 0.180003), (0.3, 0.255019),  
 (0.4, 0.320080).

15.  $y' = y(1 - y) + 5$ ,  $E^* = 0.000451$ ,  
 (0.2, 1.031950), (0.3, 1.495883),  
 (0.4, 1.877093).

16.  $y' = y(2 - y) + 10$ ,  $E^* = 0.0004$ ,  
 (0.2, 2.121932), (0.3, 2.970036),  
 (0.4, 3.547216).

17.  $y' = \frac{1 - x}{1 + y}$ ,  $E^* = 0.0004$ ,  
 (0.2, 0.166190), (0.3, 0.228821),  
 (0.4, 0.280625).

18.  $y' = \frac{1 + x}{1 - y}$ ,  $E^* = 0.00068$ ,  
 (0.2, 0.251669), (0.3, 0.443224),  
 (0.4, 0.800000).

**Simpson's Rule** The following exercises use formulas and techniques found in the proof on page 1036 and in Example A.26, page 1035.

19. Verify with Simpson's rule (5) for cubic polynomials the equality  $\int_1^2 (x^3 + 16x^2 + 4)dx = 541/12$ .

20. Verify with Simpson's rule (5) for cubic polynomials the equality  $\int_1^2 (x^3 + x + 14)dx = 77/4$ .

21. Let  $f(x)$  satisfy  $f(0) = 1$ ,  $f(1/2) = 6/5$ ,  $f(1) = 3/4$ . Apply Simpson's rule with one division to verify that  $\int_0^1 f(x)dx \approx 131/120$ .

22. Let  $f(x)$  satisfy  $f(0) = -1$ ,  $f(1/2) = 1$ ,  $f(1) = 2$ . Apply Simpson's rule with

one division to verify that  $\int_0^1 f(x)dx \approx 5/6$ .

23. Verify Simpson's equality (5), assuming  $Q(x) = 1$  and  $Q(x) = x$ .

24. Verify Simpson's equality (5), assuming  $Q(x) = x^2$ .

**Quadratic Interpolation** The following exercises use formulas and techniques from the proof on page 1037.

25. Verify directly that the quadratic polynomial  $y = x(7 - 4x)$  goes through the points (0, 0), (1, 3), (2, -2).

26. Verify directly that the quadratic polynomial  $y = x(8 - 5x)$  goes through the points (0, 0), (1, 3), (2, -4).

27. Compute the quadratic interpolation polynomial  $Q(x)$  which goes through the points (0, 1), (0.5, 1.2), (1, 0.75).

28. Compute the quadratic interpolation polynomial  $Q(x)$  which goes through the points (0, -1), (0.5, 1), (1, 2).

29. Verify the remaining cases in Lemma A.1, page 1038.

30. Verify the remaining cases in Lemma A.2, page 1038.

## A.5 Implicit Functions

The subject of **implicit function theory** treats the problem of solving an equation  $F(x, y) = 0$  for  $y$  in terms of  $x$ . In differential equations, it is the theoretical basis for extracting an explicit solution  $y(x)$  from an implicit solution  $F(x, y) = 0$ .

### Theorem A.2 (Implicit Function Theorem)

Let  $F(x, y)$ ,  $F_x(x, y)$ ,  $F_y(x, y)$  be defined and continuous in an open region  $D$  in the plane. Assume  $(x_0, y_0)$  is the center of a disk contained entirely in  $D$  and  $F_y(x_0, y_0) \neq 0$ . Then there is a number  $H > 0$  and a function  $y = y(x)$  such that

- [1]  $y(x_0) = y_0$ ,
- [2]  $y$  is continuous on  $|x - x_0| < H$ ,
- [3]  $(x, y(x))$  is in  $D$  for  $|x - x_0| < H$ ,
- [4]  $F(x, y(x)) = 0$  for  $|x - x_0| < H$ .

Further, if another function  $y = Y(x)$  satisfies [1]–[4] on  $|x - x_0| < H$ , then  $y(x) = Y(x)$  for  $|x - x_0| < H$ .

The proof of Theorem A.2 appears in various references, for example, see Taylor-Mann [?] and Marsden-Tromba [?]. Results of this type are theoretical, that is, devoid of a method for finding the function  $y(x)$ .

## Practical Numerical Methods

Item [4] in Theorem A.2 together with the chain rule  $\frac{d}{dt}F(x(t), y(t)) = F_x x'(t) + F_y y'(t)$  implies that  $y(x)$  satisfies the initial value problem

$$(1) \quad y' = -\frac{F_x(x, y)}{F_y(x, y)}, \quad y(x_0) = y_0.$$

Problem (1) is the basis for practical numerical methods which are used in applications to calculate and graph the implicit solution  $y(x)$  of the equation  $F(x, y) = 0$ . See Example A.27, page 1042.

## Computer Algebra Methods

Computer algebra systems `maple` and `mathematica` have facilities for solving an equation  $F(x, y) = 0$  for  $y$  in terms of  $x$ . Limited support exists for making graphics directly from the implicit equation  $F(x, y) = 0$ . See Example A.28, page 1042.

Work-alike systems such as `matlab`, `octave` and `scilab` can be applied to solve implicit equations, although the work involved is always more tedious. One idea of merit is to model the implicit equation  $F(x, y) = 0$  as several initial value problems, then apply differential equation numerical solution methods to graph the solutions. See Example A.29, page 1043.

## A.5 Implicit Functions

### Example A.27 (Modeling an Implicit Function Problem)

Model the implicit equation  $x^2 + 4y^4 = 4$  at  $x = 0$ ,  $y = 1$  as an initial value problem for a function  $y(x)$  defined near  $x = 0$ .

**Solution:** Let  $F(x, y) = x^2 + 4y^4$ . Then  $x^2 + 4y^4 = 4$  can be written as  $F(x, y) = 4$ . We verify  $F(0, 1) = 4$ . The chain rule  $(d/dt)F(x(t), y(t)) = F_x x'(t) + F_y y'(t)$  with  $x = t$  and  $y = y(t)$  gives from  $F(x, y) = 4$  the equation  $\frac{dy}{dt} = -F_x(t, y(t))/F_y(t, y(t))$ . Compute  $F_x = 2x$  and  $F_y = 16y^3$ . The initial value problem is

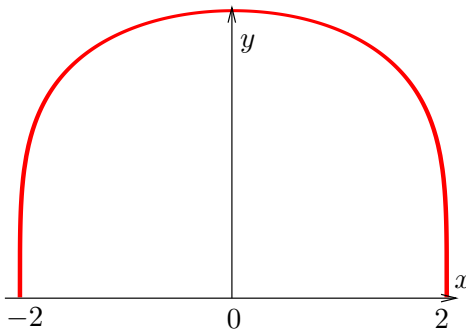
$$\frac{dy}{dt} = -\frac{t}{8y^3}, \quad y(0) = 1.$$

### Example A.28 (Solving $F(x, y) = 0$ Symbolically)

Solve symbolically for  $y$  as a function of  $x$  in the implicit equation  $x^2 + 4y^4 = 4$  at  $x = 0$ ,  $y = 1$  both by hand and by computer.

**Solution:** College algebra methods apply to solve  $x^2 + 4y^4 = 4$  for  $y$  in terms of  $x$ , giving  $y(x) = \sqrt[4]{1 - x^2/4}$ . The graph is defined on  $-2 \leq x \leq 2$ ; see Figure 14. The college algebra details:

$4y^4 = 4 - x^2$	Start with $x^2 + 4y^4 = 4$ and isolate $y$ on the left.
$ y  = \sqrt[4]{1 - x^2/4}$	Divide by 4 and take the fourth root of both sides.
$y = \sqrt[4]{1 - x^2/4}$	Replace $ y $ by $\pm y$ and resolve the sign with $y = 1$ at $x = 0$ .



**Figure 14.** Implicit solution  $y(x)$  of  $x^2 + 4y^4 = 4$  at  $x = 0$ ,  $y = 1$ .

The computer algebra system `maple` partially solves the problem with the command

```
solve(x^2+4*y^4=4,y);
```

Reported are four answers:

$$\begin{aligned} & \frac{1}{2} \sqrt[4]{-4x^2 + 16}, & \frac{1}{2} \sqrt{-1} \sqrt[4]{-4x^2 + 16}, \\ & -\frac{1}{2} \sqrt[4]{-4x^2 + 16}, & -\frac{1}{2} \sqrt{-1} \sqrt[4]{-4x^2 + 16}. \end{aligned}$$

Only one satisfies  $y = 1$  at  $x = 0$ , namely the first. It is typical in computer algebra systems to spend time sorting out the system's answer.

**Remark on Algebraic Complexity.** The more complicated implicit equation  $x^2 + 4y^4 + xy = 4$  does not have a useful or simple college algebra solution. To understand the

## A.5 Implicit Functions

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complications, execute the following `maple` code, which displays several pages of answers involving cube roots of sixth degree polynomials. None of the answers are useful, it being easier to employ the ideas of Example A.27, page 1042.

```
allvalues([solve(x^2+4*y^4+x*y=4,y)]);
```

### Example A.29 (Solving $F(x, y) = 0$ Numerically)

Solve numerically by computer for  $y$  as a function of  $x$  in the implicit equation  $x^2 + 4y^4 = 4$  at  $x = 0$ ,  $y = 1$ . Plot  $y(x)$  on  $0 \leq x \leq 2$ .

**Solution:** It was shown in Example A.27, page 1042, that the problem is equivalent to the differential equation problem

$$\frac{dy}{dx} = -\frac{x}{8y^3}, \quad y(0) = 1.$$

The plot on  $0 \leq x \leq 2$  will look like the right half of Figure 14. The `maple` code:

```
with(DEtools):
de:=diff(y(x),x)=-x/(8*y(x)^3):
DEplot(de,y(x),x=0..2,[[y(0)=1]],arrows=NONE);
```

A more simplistic approach, which is also capable of direct computation of values of  $y(x)$ , is to use the `maple` function `dsolve`.

```
# Maple V 5.1
de:=diff(y(x),x)=-x/(8*y(x)^3):ic:=y(0)=1:
p:=dsolve({de,ic},y(x),numeric);
Y:=x->rhs(p(x)[2]);
plot(Y,0..2);
```



## Exercises A.5

### Modeling an Implicit Function Problem

Apply the ideas in Example A.27, page 1042 to model the given implicit equation as an initial value problem for a function  $y(x)$  defined near  $x = 0$ .

1.  $x^2 + xy^4 + y = 1$ ,  
 $x = 0, y = 1$ .
2.  $x + xy^4 + y = 1$ ,  
 $x = 0, y = 1$ .
3.  $x + y^2 \ln(x + 1) + y = 2$ ,  
 $x = 0, y = 2$ .
4.  $e^x + y^2 \ln(x + 1) + y = 1$ ,  
 $x = 0, y = 2$ .
5.  $\sin x + y^3 \cos x + y^2 = 2$ ,  
 $x = 0, y = 1$ .
6.  $\tan x + y^2 \sec x + y^3 = 2$ ,  
 $x = 0, y = 1$ .
7.  $e^x + y^2 x^2 + xy + 2y = 3$ ,  
 $x = 0, y = 1$ .
8.  $e^{-x} + -y^2 x^2 + xy + 2y = 3$ ,  
 $x = 0, y = 1$ .

### Solve $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ Symbolically

Solve symbolically for  $y$  as a function of  $x$  in the given implicit equation both by hand and by computer. Apply the methods of Example A.28, page 1042.

9.  $x^2 + 5y^4 = 5$ ,  
 $x = 0, y = 1$ .
10.  $x^2 + 5y^2 = 5$ ,  
 $x = 0, y = 1$ .
11.  $x^2 + y^2 + 2y = 3$ ,  
 $x = 0, y = 1$ .

12.  $x^2 + 4y^2 - 2y = 2$ ,  
 $x = 0, y = 1$ .
13.  $\sin x + y^4 = 1$ ,  
 $x = 0, y = 1$ .
14.  $\sin x + y^4 + 2y^2 = 3$ ,  
 $x = 0, y = 1$ .
15.  $-\sin x + \cos y = 1$ ,  
 $x = 0, y = 0$ .
16.  $\sin x + \cos y = 1$ ,  
 $x = 0, y = 0$ .

### Solve $F(x, y) = 0$ Numerically

Solve numerically by computer for  $y$  as a function of  $x$  in the given implicit equation. Plot  $y(x)$  on an interesting interval. See Example A.29, page 1043 for methods.



17.  $x^2 + x + 4 + \cos y = 5$ ,  
 $x = 0, y = 0$ .
18.  $x^2 + x + 6 - \cos(y) = 5$ ,  
 $x = 0, y = 0$ .
19.  $x^2 + y^3 + 2y = 3$ ,  
 $x = 0, y = 1$ .
20.  $x^2 + 4y^3 - 2y = 2$ ,  
 $x = 0, y = 1$ .
21.  $\sin x + y^4 + y = 2$ ,  
 $x = 0, y = 1$ .
22.  $\sin x + y^4 + 2y = 3$ ,  
 $x = 0, y = 1$ .
23.  $-\sin x + y + \cos y = 1$ ,  
 $x = 0, y = 0$ .
24.  $\sin x - y + \cos y = 1$ ,  
 $x = 0, y = 0$ .

# PDF Sources

## Text, Solutions and Corrections

**Author:** Grant B. Gustafson, University of Utah, Salt Lake City 84112.

**Paperback Textbook:** There are 12 chapters on differential equations and linear algebra, book format 7 x 10 inches, 1077 pages. Copies of the textbook are available in two volumes at **Amazon** Kindle Direct Publishing for Amazon's cost of printing and shipping. No author profit. Volume I chapters 1-7, ISBN 9798705491124, 661 pages. Volume II chapters 8-12, ISBN 9798711123651, 479 pages. Both paperbacks have extra pages of backmatter: background topics Chapter A, the whole book index and the bibliography.

**Textbook PDF with Solution Manual:** Packaged as one PDF (13 MB) with hyperlink navigation to displayed equations and theorems. The header in an exercise set has a blue hyperlink  to the same section in the solutions. The header of the exercise section within a solution Appendix has a red hyperlink  to the textbook exercises. Solutions are organized by chapter, e.g., Appendix 5 for Chapter 5. Odd-numbered exercises have a solution. A few even-numbered exercises have hints and answers. Computer code can be mouse-copied directly from the PDF. Free to use or download, no restrictions for educational use.

## Sources at Utah:

<https://math.utah.edu/~gustafso/indexUtahBookGG.html>

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<https://github.com/ggustaf/github.io> and mirror

<https://gitlab.com/ggustaf/answers>.

**Communication:** To contribute a solution or correction, ask a question or request an answer, click the link below, then create a GitHub issue and post. Contributions and corrections are credited, privacy respected.

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