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Chapter 8

Laplace Transform

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The Laplace transform solves differential equations. Besides being a different and efficient alternative to variation of parameters and undetermined coefficients, Laplace's method is especially advantageous for a forcing term that is piecewise–defined, periodic or impulsive.

The Laplace method. It has humble beginnings as an extension of *the method of quadrature* to higher order differential equations and systems. The method is based upon ordinary calculus integrals:

Multiply the differential equation by the Laplace integrator $dx = e^{-st}dt$ and integrate across the equation from $t = 0$ to $t = \infty$. Isolate left the Laplace integral $\int_{t=0}^{t=\infty} y(t)e^{-st}dt$. Look up the answer $y(t)$ in a Laplace integral table.

Definition 8.1 (Laplace Integral)
The **Laplace integral** or the **direct Laplace transform** of a function $f(t)$ defined for $0 \le t < \infty$ is the answer to the Newton calculus integration problem $\int_0^\infty f(t)e^{-st}dt$. Special notation replaces the integral notation in literature:

$$\mathcal{L}(f(t)) \quad \text{replaces} \quad \int_0^\infty f(t)\, e^{-st}\, dt.$$

Decoding and Encoding. The $\mathcal{L}$–notation recognizes that integration always proceeds over $t = 0$ to $t = \infty$ and that the integral always has a fixed *integrator* $e^{-st}dt$ instead of the expected dx. These minor differences distinguish *Laplace integrals* from the *ordinary integrals* found on the inside covers of calculus texts.

When reading mathematical text, replace symbol $\mathcal{L}$ by these words: **Laplace of**. Notation $\mathcal{L}(f(t))$ decodes into calculus by replacing $\mathcal{L}$ by $\int_0^\infty$, then append the Laplace integrator $e^{-st}dt$. For instance, notation $\mathcal{L}(t^2)$ decodes to $\int_0^\infty (t^2)\, e^{-s\,t}\, dt$. To encode $\int_0^\infty (\sin t)\, e^{-s\,t}\, dt$ to $\mathcal{L}(\sin t)$, replace $\int_0^\infty$ by $\mathcal{L}$, then erase Laplace integrator $e^{-st}dt$.

History. The first application of the Laplace method might have been in the 1910 work of H. Bateman [**?**], who transformed Rutherford's radioactive decay equation $\frac{d}{dt}A(t) = -hA(t)$ by setting $a(x) = \int_0^\infty e^{-xt}A(t)dt$, thereby obtaining an equation in variable x (Laplace theory uses s instead of x). The first example presented here will parallel Bateman's 1910 exposition, in which he derived several properties of the Laplace integral as well as isolating what is today called *Laplace's method.* He used Lerch's 1903 theorem published in *Acta Mathematica*. The name *Laplace Transform* dates back to Euler 1763 and Spitzer 1878, which nowadays refers to the linear map $f \to \mathcal{L}(f(t)) \equiv \int_0^\infty e^{-st}f(t)dt$.

8.1   Laplace Method Introduction

The foundation of Laplace theory is **Lerch's 1903 cancellation law**

(1)
$$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt \qquad \text{implies} \qquad y(t) = f(t),$$
$$\text{or}$$
$$\mathcal{L}(y(t) = \mathcal{L}(f(t)) \qquad \text{implies} \qquad y(t) = f(t).$$

In differential equation applications, $y(t)$ is the unknown appearing in the equation while $f(t)$ is an explicit expression extracted or computed from Laplace integral tables. See page 596.

An Illustration. Laplace's method will be applied to solve the initial value problem[1]
$$\frac{dy}{dt} = -1, \quad y(0) = 0.$$
No background in Laplace theory is assumed here, only a calculus background is used. Calculus verifies the answer $y(t) = -t$.

The Plan. The method obtains an equation $\mathcal{L}(y(t)) = \mathcal{L}(-t)$, then Lerch's cancellation law implies that the $\mathcal{L}$-symbols cancel, which gives the differential equation solution $y(t) = -t$.

The Laplace method is advertised as a *generalization of the method of quadrature* to higher order differential equations and systems of differential equations. In addition to quadrature, the method uses *table lookup*: solution $y(t)$ is found from a special integral table.

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[1]Laplace theory uses t instead of x. Prime notation y' means $\frac{dy}{dt}$.

Laplace Integral

The integral $\int_0^\infty g(t)e^{-st}dt$ is called the **Laplace integral** of the function $g(t)$. It is defined by $\lim_{N\to\infty} \int_0^N g(t)e^{-st}dt$ and depends on variable s. The ideas will be illustrated for $g(t) = 1$, $g(t) = t$ and $g(t) = t^2$, producing the integral formulas in Table 1, *infra*.

$\boxed{1}$ $\int_0^\infty (1)e^{-st}dt = -(1/s)e^{-st}\big|_{t=0}^{t=\infty}$ Laplace integral of $g(t) = 1$.

              $= 1/s$ Assumed $s > 0$.

$\boxed{2}$ $\int_0^\infty (t)e^{-st}dt = \int_0^\infty -\frac{d}{ds}(e^{-st})dt$ Laplace integral of $g(t) = t$.

              $= -\frac{d}{ds}\int_0^\infty (1)e^{-st}dt$ $\int \frac{d}{ds}F(t,s)dt = \frac{d}{ds}\int F(t,s)dt$.

              $= -\frac{d}{ds}(1/s)$ By $\boxed{1}$.

              $= 1/s^2$ Differentiate.

$\boxed{3}$ $\int_0^\infty (t^2)e^{-st}dt = \int_0^\infty -\frac{d}{ds}(te^{-st})dt$ Laplace integral of $g(t) = t^2$.

              $= -\frac{d}{ds}\int_0^\infty (t)e^{-st}dt$

              $= -\frac{d}{ds}(1/s^2)$ By $\boxed{2}$.

              $= 2/s^3$

Table 1.   **The Laplace Integral $\int_0^\infty g(t)e^{-st}dt$ for $g(t) = 1$, t and t^2.**

$$\int_0^\infty (1)e^{-st}\,dt = \frac{1}{s}, \qquad \int_0^\infty (t)e^{-st}\,dt = \frac{1}{s^2}, \qquad \int_0^\infty (t^2)e^{-st}\,dt = \frac{2}{s^3}.$$

$$\text{In summary,} \quad \mathcal{L}(t^n) = \frac{n!}{s^{1+n}}$$

Illustration Details for $y' = -1$, $y(0) = 0$

The **Laplace method** will be applied to find the solution $y(t) = -t$ of the problem

$$y' = -1, \quad y(0) = 0.$$

Laplace's method in Table 2 is entirely different from variation of parameters or undetermined coefficients. The method uses only basic calculus and college algebra. In the second Table 3, a succinct version of the first Table 2 is given, using $\mathcal{L}$-notation. The briefer exposition is a model for Laplace Method details as found in references.

Table 2. Laplace Method Details for Illustration $y' = -1$, $y(0) = 0$.

$y'(t)e^{-st}dt = -e^{-st}dt$	Multiply $y' = -1$ by $e^{-st}dt$.
$\int_0^\infty y'(t)e^{-st}dt = \int_0^\infty -e^{-st}dt$	Integrate $t = 0$ to $t = \infty$.
$\int_0^\infty y'(t)e^{-st}dt = -1/s$	Use Table 1 forwards.
$s\int_0^\infty y(t)e^{-st}dt - y(0) = -1/s$	Integrate by parts on the left.
$\int_0^\infty y(t)e^{-st}dt = -1/s^2$	Use $y(0) = 0$ and divide.
$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty (-t)e^{-st}dt$	Use Table 1 backwards.
$y(t) = -t$	Apply Lerch's cancellation law. Solution found.

Table 3. Laplace Method $\mathcal{L}$-notation
Details for $y' = -1$, $y(0) = 0$ translated from Table 2.

$\mathcal{L}(y'(t)) = \mathcal{L}(-1)$	Apply $\mathcal{L}$ across $y' = -1$, or multiply $y' = -1$ by $e^{-st}dt$, integrate $t = 0$ to $t = \infty$.
$\mathcal{L}(y'(t)) = -1/s$	Use Table 1 forwards.
$s\mathcal{L}(y(t)) - y(0) = -1/s$	Integrate by parts on the left.
$\mathcal{L}(y(t)) = -1/s^2$	Use $y(0) = 0$ and divide.
$\mathcal{L}(y(t)) = \mathcal{L}(-t)$	Apply Table 1 backwards.
$y(t) = -t$	Invoke Lerch's cancellation law.

In Lerch's law, the formal rule of erasing the integral signs is valid *provided* the integrals are equal for large s and certain conditions hold on y and f — see Theorem 8.2. The illustration in Table 2 shows that Laplace theory requires an in-depth study of a **special integral table**, a table which is a true extension of the usual table found on the inside covers of calculus books; see Table 1 and section 8.2, Table 4 page 601.

The $\mathcal{L}$-notation for the direct Laplace transform produces briefer details, as witnessed by the translation of Table 2 into Table 3. It is advised to move from Laplace integral notation to the $\mathcal{L}$–notation as soon as possible, in order to highlight goalposts in the method.

Some Transform Rules

The formal properties of calculus integrals plus the integration by parts formula used in Tables 2 and 3 leads to these **rules** for the Laplace transform:

$\mathcal{L}(f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$	The integral of a sum is the sum of the integrals.

$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$	Constants c pass through the integral sign.
$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0)$	The t-derivative rule, or integration by parts. See Theorem 8.3.
$\mathcal{L}(y(t)) = \mathcal{L}(f(t))$ implies $y(t) = f(t)$	Lerch's cancellation law. See Theorem 8.2.

The four rules above appear in Bateman's 1910 publication [**?**]. The first two rules are referenced as **linearity of the Laplace transform**, which allow manipulation of the symbol $\mathcal{L}$ with rules known from calculus and matrix algebra. Laplace symbol $\mathcal{L}$ manipulates like matrix multiply.

Existence of the Transform

The Laplace integral $\int_0^\infty e^{-st} f(t)\, dt$ is known to exist in the sense of the improper integral definition[2]

$$\int_0^\infty g(t)dt = \lim_{N\to\infty} \int_0^N g(t)dt$$

provided $f(t)$ belongs to a class of functions known in the literature as functions of **exponential order**. For this class of functions the relation

$$(2) \qquad \lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$$

is required to hold for some real number α, or equivalently, for some constants M and α,

$$(3) \qquad |f(t)| \le Me^{\alpha t}.$$

In addition, $f(t)$ is required to be **piecewise continuous** on each finite subinterval of $0 \le t < \infty$, a term defined as follows.

Definition 8.2 (Piecewise Continuous)
A function $f(t)$ is **piecewise continuous** on a finite interval $[a, b]$ provided there exists a partition $a = t_0 < \cdots < t_n = b$ of the interval $[a, b]$ and functions f_1, f_2, ..., f_n continuous on $(-\infty, \infty)$ such that for t not a partition point

$$(4) \qquad f(t) = \begin{cases} f_1(t) & t_0 < t < t_1, \\ \vdots & \vdots \\ f_n(t) & t_{n-1} < t < t_n. \end{cases}$$

The values of f at partition points are undecided by equation (4). In particular, equation (4) implies that $f(t)$ has one-sided limits at each point of $a < t < b$ and appropriate one-sided limits at the endpoints. Therefore, f has at worst a **jump discontinuity** at each partition point.

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[2]An advanced calculus background is assumed for the Laplace transform existence proof. Applications of Laplace theory require only a calculus background.

Theorem 8.1 (Existence of $\mathcal{L}(f)$)
Let $f(t)$ be piecewise continuous on every finite interval in $t \geq 0$ and satisfy $|f(t)| \leq Me^{\alpha t}$ for some constants M and α. Then:

1. Laplace integral $\mathcal{L}(f(t))$ exists for $s > \alpha$.

2. **Laplace is zero at** $s = \infty$: $\lim_{s \to \infty} \mathcal{L}(f(t)) = 0$.[3]

Proof on page 598.

Theorem 8.2 (Lerch 1903)
If $f_1(t)$ and $f_2(t)$ are continuous, of exponential order and for all $s > s_0$

$$\int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty f_2(t)e^{-st}dt,$$

then $f_1(t) = f_2(t)$ for $t \geq 0$.[4]

Proofs in French: Lerch (1903) [?] and English: Widder [?]. See also [?].

Theorem 8.3 (Parts Rule or t-Derivative Rule)
Let $f(t)$ be continuous and of exponential order. Let $f'(t)$ be piecewise continuous and of exponential order. Then $\mathcal{L}(f'(t))$ exists for all large s and $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$.

Proof on page 639.

Theorem 8.4 (Euler Solution Atoms have Laplace Integrals)
Let $f(t)$ be $t^n e^{at}$ or the real or imaginary part of $t^n e^{at+ibt}$ where a, b are real, $b > 0$ and $n \geq 0$ is an integer. Briefly, f is an **Euler solution atom**. Then f is of exponential order and $\mathcal{L}(f(t))$ exists. Further, if $g(t)$ is a linear combination of Euler atoms, then $\mathcal{L}(g(t))$ exists.

Proof on page 598.

Remark. Because solutions to undetermined coefficient problems are a linear combination of Euler solution atoms, then Laplace's method applies to all such differential equations. This is the class of all constant-coefficient higher order linear differential equations, and all systems of differential equations with constant coefficients, having a forcing term which is a linear combination of Euler solution atoms.

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[3]Literature might write $F(s)$ for $\mathcal{L}(f(f))$ and $\lim_{s \to \infty} F(s) = 0$

[4]The result extends to piecewise continuous functions provided the conclusion is weakened to: *at points where both f_1, f_2 are continuous, $f_1(t) = f_2(t)$*. Reference: *CRC Concise Encyclopedia of Mathematics* by Weisstein.

Examples and Methods

Example 8.1 (Laplace Method)
Solve the initial value problem $y' = 5 - 2t$, $y(0) = 1$ by the Laplace method to obtain $y(t) = 1 + 5t - t^2$.

Solution: Laplace's method is outlined in Tables 2 and 3. The $\mathcal{L}$-notation of Table 3 will be used to find the solution $y(t) = 1 + 5t - t^2$.

$$\mathcal{L}(y'(t)) = \mathcal{L}(5 - 2t) \qquad \text{Apply } \mathcal{L} \text{ across } y' = 5 - 2t.$$
$$= 5\mathcal{L}(1) - 2\mathcal{L}(t) \qquad \text{Linearity of the transform.}$$
$$= \frac{5}{s} - \frac{2}{s^2} \qquad \text{Use Table 1 forwards.}$$
$$s\mathcal{L}(y(t)) - y(0) = \frac{5}{s} - \frac{2}{s^2} \qquad \text{Apply the parts rule, Theorem 8.3.}$$
$$\mathcal{L}(y(t)) = \frac{1}{s} + \frac{5}{s^2} - \frac{2}{s^3} \qquad \text{Use } y(0) = 1 \text{ and divide.}$$
$$\mathcal{L}(y(t)) = \mathcal{L}(1) + 5\mathcal{L}(t) - \mathcal{L}(t^2) \qquad \text{Apply Table 1 backwards.}$$
$$= \mathcal{L}(1 + 5t - t^2) \qquad \text{Linearity of the transform.}$$
$$y(t) = 1 + 5t - t^2 \qquad \text{Use Lerch's cancellation law.}$$

Example 8.2 (Laplace Method)
Solve by Laplace's method the initial value problem $y'' = 10$, $y(0) = y'(0) = 0$ to obtain $y(t) = 5t^2$.

Solution: The $\mathcal{L}$-notation of Table 3 will be used to find the solution $y(t) = 5t^2$.

$$\mathcal{L}(y''(t)) = \mathcal{L}(10) \qquad \text{Apply } \mathcal{L} \text{ across } y'' = 10.$$
$$s\mathcal{L}(y'(t)) - y'(0) = \mathcal{L}(10) \qquad \text{Apply the parts rule to } y', \text{ that is, replace } f \text{ by } y'$$
$$\text{in Theorem 8.3.}$$
$$s[s\mathcal{L}(y(t)) - y(0)] - y'(0) = \mathcal{L}(10) \qquad \text{Repeat the parts rule, on } y.$$
$$s^2\mathcal{L}(y(t)) = \mathcal{L}(10) \qquad \text{Use } y(0) = y'(0) = 0.$$
$$\mathcal{L}(y(t)) = \frac{10}{s^3} \qquad \text{Use Table 1 forwards. Then divide.}$$
$$\mathcal{L}(y(t)) = \mathcal{L}(5t^2) \qquad \text{Apply Table 1 backwards.}$$
$$y(t) = 5t^2 \qquad \text{Invoke Lerch's cancellation law.}$$

Example 8.3 (Exponential Order)
Show that $f(t) = e^t \cos t + t$ is of exponential order.

Solution: The proof must show that $f(t)$ is piecewise continuous on every interval $[a, b]$ and then find an $\alpha > 0$ such that $\lim_{t \to \infty} f(t)/e^{\alpha t} = 0$.

The given $f(t)$ is continuous on $(-\infty, \infty)$. Given interval $[a, b]$, define $t_0 = a, t_1 = b$ and $f_1(t) = f(t)$. Then (4) holds. Definition 8.2 implies f is piecewise continuous.

From L'Hospital's rule in calculus, $\lim_{t \to \infty} p(t)/e^{\alpha t} = 0$ for any polynomial p and any $\alpha > 0$. Choose $\alpha = 2$, then

$$\lim_{t \to \infty} \frac{f(t)}{e^{2t}} = \lim_{t \to \infty} \frac{\cos t}{e^t} + \lim_{t \to \infty} \frac{t}{e^{2t}} = 0.$$

Proofs and Technical Details

Proof of Theorem 8.1, Existence Laplace Integral:
Details 1. It has to be shown that the Laplace integral of f is finite for $s > \alpha$. Advanced calculus implies that it is sufficient to show that the integrand is absolutely bounded above by an integrable function $g(t)$. Take $g(t) = Me^{-(s-\alpha)t}$. Then $g(t) \geq 0$. Furthermore, g is integrable, because

$$\int_0^\infty g(t)dt = \frac{M}{s - \alpha}.$$

Inequality $|f(t)| \leq Me^{\alpha t}$ implies the absolute value of the Laplace transform integrand $f(t)e^{-st}$ is estimated by

$$\left| f(t)e^{-st} \right| \leq Me^{\alpha t}e^{-st} = g(t).$$

Details 2. The limit statement $\lim_{s\to\infty} \mathcal{L}(f(t)) = 0$ follows from $|\mathcal{L}(f(t))| \leq \int_0^\infty g(t)dt = \frac{M}{s - \alpha}$, because the right side of this inequality has limit zero at $s = \infty$. ∎

Proof of Theorem 8.4, Euler Atoms:
Function $f(t) = t^n e^{at}$ is everywhere continuous. By calculus, $\ln|x| \leq 2x$ for $x \geq 1$. Define $c = 2|n| + |a|$. Then $|f(t)| = e^{n \ln|t| + at} \leq e^{ct}$ for $t \geq 1$, which proves f is of exponential order. Similarly, $f(t) = \mathcal{R}e(t^n e^{at+ibt})$ is everywhere continuous and $|f(t)| \leq |t^n e^{at+ibt}| = |t^n e^{at}| \leq e^{ct}$. Details for $f(t) = \mathcal{I}m(t^n e^{at+ibt})$ are similar. Then f is of exponential order in all three cases. The Laplace integral exists by Theorem 8.1 page 596. ∎

Exercises 8.1 ⬧

Laplace method
Solve the given initial value problem using Laplace's method.

1. $y' = -2$, $y(0) = 0$.

2. $y' = 1$, $y(0) = 0$.

3. $y' = -t$, $y(0) = 0$.

4. $y' = t$, $y(0) = 0$.

5. $y' = 1 - t$, $y(0) = 0$.

6. $y' = 1 + t$, $y(0) = 0$.

7. $y' = 3 - 2t$, $y(0) = 0$.

8. $y' = 3 + 2t$, $y(0) = 0$.

9. $y'' = -2$, $y(0) = y'(0) = 0$.

10. $y'' = 1$, $y(0) = y'(0) = 0$.

11. $y'' = 1 - t$, $y(0) = y'(0) = 0$.

12. $y'' = 1 + t$, $y(0) = y'(0) = 0$.

13. $y'' = 3 - 2t$, $y(0) = y'(0) = 0$.

14. $y'' = 3 + 2t$, $y(0) = y'(0) = 0$.

Exponential order
Show that $f(t)$ is of exponential order, by finding a constant $\alpha \geq 0$ in each case such that $\lim_{t\to\infty} \dfrac{f(t)}{e^{\alpha t}} = 0$.

15. $f(t) = 1 + t$

16. $f(t) = e^t \sin(t)$

17. $f(t) = \sum_{n=0}^N c_n t^n$, for any choice of the constants $c_0, \ldots, c_N$.

18. $f(t) = \sum_{n=1}^N c_n \sin(nt)$, for any choice of the constants $c_1, \ldots, c_N$.

Existence of transforms
Let $f(t) = te^{t^2} \sin(e^{t^2})$. Establish these results.

19. The function $f(t)$ is not of exponential order.

20. The Laplace integral of $f(t)$, $\int_0^\infty f(t)e^{-st}dt$, converges for all $s > 0$.

Jump Magnitude

For f piecewise continuous, define the **jump** at t by

$$J(t) = \lim_{h \to 0+} f(t+h) - \lim_{h \to 0+} f(t-h).$$

Compute $J(t)$ for the following f.

21. $f(t) = 1$ for $t \geq 0$, else $f(t) = 0$

22. $f(t) = 1$ for $t \geq 1/2$, else $f(t) = 0$

23. $f(t) = t/|t|$ for $t \neq 0$, $f(0) = 0$

24. $f(t) = \sin t/|\sin t|$ for $t \neq n\pi$, $f(n\pi) = (-1)^n$

Taylor series

The series relation $\mathcal{L}(\sum_{n=0}^\infty c_n t^n) = \sum_{n=0}^\infty c_n \mathcal{L}(t^n)$ often holds, in which case the result $\mathcal{L}(t^n) = n!s^{-1-n}$ can be employed to find a series representation of the Laplace transform. Use this idea on the following to find a series formula for $\mathcal{L}(f(t))$.

25. $f(t) = e^{2t} = \sum_{n=0}^\infty (2t)^n/n!$

26. $f(t) = e^{-t} = \sum_{n=0}^\infty (-t)^n/n!$

Transfer of Radiance

The differential equation $\frac{d}{dr}N + \alpha N = N^*$ models laser beam radiance (absorption and scattering out of the beam) in a medium like water, where r is the distance from the source.

27. Solve $\frac{d}{dr}N + 2N = 1, N(0) = 20$ by Laplace's method.
Ans: $N(r) = \frac{1}{2} + \frac{39}{2}\,e^{-2\,r}$.
Hint: Obtain $\mathcal{L}(N(t)) = \frac{1+20\,s}{s(s+2)} = \frac{1}{2s} + \frac{39}{2(s+2)}$ using $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ from the Forward Table page 601.

28. Solve $\frac{d}{dr}N + 2N = 1 - e^{-r}, N(0) = 25$ by any method.
Ans: $N(r) = \frac{1}{2} - e^{-r} + \frac{51}{2}\,e^{-2\,r}$.
Hint: A particular solution is $N_p = \frac{1}{2} - e^{-r}$. Superposition applies. See also Example 8.11 page 609.

Piecewise-Defined Functions

29. Define a piecewise continuous function $f(t)$ on $[-1, 1]$ that agrees with $\frac{\sin(t)}{|t|}$ except at $t = 0$. Suggestion: use Taylor expansion $\sin(t) = t - t^3/6 + \cdots$ to define continuous functions f_1, f_2 on $-\infty < t < \infty$.

30. Explain in detail why $1/t$ is not piecewise continuous on $[-1, 1]$. ∎

31. Find $\mathcal{L}(f(t))$, given
$$f(t) = \begin{cases} 1 & 1 \leq t < 2, \\ 0 & \text{otherwise.} \end{cases}$$

32. Find $\mathcal{L}(\mathbf{pulse}(t, a, b))$, given
$$\mathbf{pulse}(t, a, b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise.} \end{cases}$$

33. Define
$$f(t) = \begin{cases} 1 & 1 \leq t < 2, \\ 2 & 3 \leq t < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Find the weights c_1, c_2 such that
$$f(t) = c_1\,\mathbf{pulse}(t, 1, 2) + c_2\,\mathbf{pulse}(t, 3, 4).$$

34. Let
$$f(t) = \cos(t)\,\mathbf{pulse}(t, 0, \pi) + (\sin(t) - 1)\,\mathbf{pulse}(t, \pi, 2\pi)$$
Write f as a piecewise-defined function and graph it.

Piecewise Continuous Definition

Let $g(t)$ be zero for $t < 0$ and have on $t \geq 0$ at most finitely many points of discontinuity, at which finite right and left hand limits exist.

This definition is an alternative way to define *piecewise continuous*, crafted for Laplace theory.

35. Let t_1, t_2 be consecutive points of discontinuity of g. Define a function $g_1(t)$ continuous on $-\infty < t < \infty$ such that $g(t) = g_1(t)$ on $t_1 \leq t \leq t_2$.

The whole real line is the required domain of g_1, which must be defined using g itself and right and left hand limit values of g.

36. Let t_1, t_2, t_3 be consecutive points of discontinuity of g. Invent functions $g_1(t)$, $g_2(t)$ continuous on $-\infty < t < \infty$ such that $g(t) = g_1(t)$ on $t_1 \leq t \leq t_2$ and $g(t) = g_2(t)$ on $t_2 \leq t \leq t_3$.

37. Define g_1, g_2 as in Exercise 36 above. Compute the **jump** at $t = t_2$, $J(t_2) = g(t_2 + 0) - g(t_2 - 0)$, in terms of g_1, g_2.

38. Using the preceding steps, prove that g is piecewise continuous according to the definition given in the text.

8.2   Laplace Integral Table

The objective in developing Laplace integral Table 4 and Table 6 is to keep the table size small. Table manipulation rules in Table 5 page 601 effectively increase the table size manyfold, making it possible to solve typical differential equations from electrical and mechanical models. The combination of Laplace tables plus the table manipulation rules is called the **Laplace transform calculus**.

Table 4 is considered to be a table of minimum size. Table 6 adds a number of special-use entries.

Derivations are postponed to page 650. The theory of the generalized factorial function, the **gamma function** $\Gamma(x)$, is on page 603. The **Dirac impulse** $\delta(t)$ is defined in Section 8.6 page 644.

Table 4.   Minimal Forward Laplace Integral Table with $\mathcal{L}$-notation

$\int_0^\infty (t^n)e^{-st}\,dt = \dfrac{n!}{s^{1+n}}$	$\mathcal{L}(t^n) = \dfrac{n!}{s^{1+n}}$
$\int_0^\infty (e^{at})e^{-st}\,dt = \dfrac{1}{s-a}$	$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$
$\int_0^\infty (\cos bt)e^{-st}\,dt = \dfrac{s}{s^2+b^2}$	$\mathcal{L}(\cos bt) = \dfrac{s}{s^2+b^2}$
$\int_0^\infty (\sin bt)e^{-st}\,dt = \dfrac{b}{s^2+b^2}$	$\mathcal{L}(\sin bt) = \dfrac{b}{s^2+b^2}$

Table 5.   Minimal Forward and Backward Laplace Integral Tables

Forward Table		Backward Table	
$\mathcal{L}(t^n)$	$= \dfrac{n!}{s^{1+n}}$	$\dfrac{1}{s^{1+n}}$	$= \mathcal{L}\left(\dfrac{t^n}{n!}\right)$
$\mathcal{L}(e^{at})$	$= \dfrac{1}{s-a}$	$\dfrac{1}{s-a}$	$= \mathcal{L}\left(e^{at}\right)$
$\mathcal{L}(\cos bt)$	$= \dfrac{s}{s^2+b^2}$	$\dfrac{s}{s^2+b^2}$	$= \mathcal{L}(\cos bt)$
$\mathcal{L}(\sin bt)$	$= \dfrac{b}{s^2+b^2}$	$\dfrac{1}{s^2+b^2}$	$= \mathcal{L}\left(\dfrac{\sin bt}{b}\right)$

On first reading of LaPlace theory, learn Table 5 and back-burner the other tables. To fully understand Table 6 *below* requires hours of Laplace use.

Table 6.   Extended Laplace Integral Table

<div align="center">

Forward Table

</div>

$\mathcal{L}(t^n)$	$=$	$\dfrac{n!}{s^{1+n}}$
$\mathcal{L}(e^{at})$	$=$	$\dfrac{1}{s-a}$
$\mathcal{L}(\cos bt)$	$=$	$\dfrac{s}{s^2+b^2}$
$\mathcal{L}(\sin bt)$	$=$	$\dfrac{b}{s^2+b^2}$

Conventions and Shortcuts

Zero Assumed on $t < 0$. Laplace theory assumes a given $f(t)$ is zero for $t < 0$. Therefore, a given $f(t)$ in Laplace calculations can be formally replaced by $f(t)u(t)$, where u is the unit step defined by $u(t) = 1$ for $t \geq 0$, $u(t) = 0$ for $t < 0$.

Exponential Order. Unless specifically assumed otherwise, any $f(t)$ in Laplace theory is assumed to have exponential order so that $\mathcal{L}(f(t))$ exists. Exceptions: Function $f(t) = t^\alpha$ is not of exponential order for $\alpha < 0$, but $\mathcal{L}(f(t))$ exists; Dirac impulse $\delta(t)$ is in the extended table, but it is not of exponential order, because $\delta(t)$ is not a *function*.

Unit Step and Ramp. Table entry $f(t) = 1$ is called the **unit step** and entry $f(t) = t$ is called the **unit ramp**. Entry $f(t) = 1$ is equivalent to $u(t)$, whose graph shape resembles a staircase step. Entry $f(t) = t$ is equivalent to $tu(t)$, whose graph shape resembles a wheelchair ramp.

Step and Ramp Inputs. Digital design might refer to $y''(t) + y(t) = u(t)$ as an *oscillator with step input*. Similarly, $y''(t) + y(t) = tu(t)$ is an *oscillator with unit ramp input*.

Trigonometric Shortcut. Even function $f(t) = \cos(bt)$ $\mathcal{L}$-transforms to an odd fraction $F(s) = \frac{s}{s^2+b^2}$. Similarly, odd function $f(t) = \sin(bt)$ transforms to even fraction $\frac{b}{s^2+b^2}$.

> Trig table entries $\cos, \sin$ change even-odd under $\mathcal{L}$-transformation.

Gamma Function

In mathematical physics, the **Gamma function** or the **generalized factorial function** is given by the identity

$$(1) \qquad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \quad x > 0.$$

This function is tabulated and available in computer languages such as `Fortran`, `C`, `C++` and `C#`. It is also available in computer algebra systems and numerical laboratories, such as `maple`, `matlab`, `mathematica`.

Fundamental Properties of $\Gamma(x)$

The generalized factorial function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ has the following fundamental properties.

    1 $\Gamma(1) = 1$

    2 $\Gamma(1 + x) = x\Gamma(x)$ for $x > 0$.

    3 $\Gamma(1 + n) = n!$ for integers $n \geq 1$.

Details for relations **1**, **2** and **3**: Start with $\int_0^\infty e^{-t} dt = 1$, which gives $\Gamma(1) = 1$, hence **1**. Use this identity and successively relation **2** to obtain relation **3**. To prove identity **2**, integration by parts is applied, as follows:

$$
\begin{aligned}
\Gamma(1 + x) &= \int_0^\infty e^{-t} t^x dt && \text{Definition.}\\
&= -t^x e^{-t}\big|_{t=0}^{t=\infty} + \int_0^\infty e^{-t} x t^{x-1} dt && \text{Use } u = t^x,\ dv = e^{-t} dt.\\
&= x \int_0^\infty e^{-t} t^{x-1} dt && \text{Boundary terms are zero}\\
& && \text{for } x > 0.\\
&= x\Gamma(x).
\end{aligned}
$$

Examples and Methods

Example 8.4 (Forward Table)
Let $f(t) = t(t - 5) - \sin 2t + e^{3t}$. Compute $\mathcal{L}(f(t))$ using the forward Laplace table and transform linearity properties.

Solution:

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(t^2 - 5t - \sin 2t + e^{3t}) && \text{Expand } t(t - 5).\\
&= \mathcal{L}(t^2) - 5\mathcal{L}(t) - \mathcal{L}(\sin 2t) + \mathcal{L}(e^{3t}) && \text{Linearity applied.}\\
&= \frac{2}{s^3} - \frac{5}{s^2} - \frac{2}{s^2 + 4} + \frac{1}{s - 3} && \text{Forward Table.}
\end{aligned}
$$

Example 8.5 (Backward Table)
Use the backward Laplace table plus transform linearity properties to solve for $f(t)$ in the equation
$$\mathcal{L}(f(t)) = \frac{s}{s^2 + 16} + \frac{2}{s - 3} + \frac{s + 1}{s^3}.$$

Solution:

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \frac{s}{s^2 + 16} + 2\frac{1}{s - 3} + \frac{1}{s^2} + \frac{1}{2}\frac{2}{s^3} && \text{Convert to table entries.} \\
&= \mathcal{L}(\cos 4t) + 2\mathcal{L}(e^{3t}) + \mathcal{L}(t) + \tfrac{1}{2}\mathcal{L}(t^2) && \text{Backward Laplace table.} \\
&= \mathcal{L}(\cos 4t + 2e^{3t} + t + \tfrac{1}{2}t^2) && \text{Linearity applied.} \\
f(t) &= \cos 4t + 2e^{3t} + t + \tfrac{1}{2}t^2 && \text{Lerch's cancellation law.}
\end{aligned}
$$

Example 8.6 (Unit Step and Pulses)
Find $\mathcal{L}(f(t))$ in Figure 1.
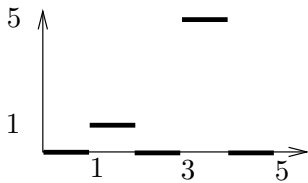
Figure 1. **A piecewise defined function $f(t)$ on $0 \leq t < \infty$: $f(t) = 0$ except for $1 \leq t < 2$ and $3 \leq t < 4$.**

Solution: A **pulse** on $[a, b]$ is defined by

$$\textbf{pulse}(t, a, b) = u(t - a) - u(t - b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise.} \end{cases}$$

The formula for $f(t)$:

$$
\begin{aligned}
f(t) &= \begin{cases} 1 & 1 \leq t < 2, \\ 5 & 3 \leq t < 4, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & 1 \leq t < 2, \\ 0 & \text{otherwise} \end{cases} + 5\begin{cases} 1 & 3 \leq t < 4, \\ 0 & \text{otherwise} \end{cases} \\
&= f_1(t) + 5f_2(t), \quad \text{where} \\
& \quad f_1(t) = u(t - 1) - u(t - 2) = \textbf{pulse}(t, 1, 2), \\
& \quad f_2(t) = u(t - 3) - u(t - 4) = \textbf{pulse}(t, 3, 4).
\end{aligned}
$$

The extended Laplace table gives

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(f_1(t)) + 5\mathcal{L}(f_2(t)) && \text{Linearity.} \\
&= \mathcal{L}(u(t - 1)) - \mathcal{L}(u(t - 2)) + 5\mathcal{L}(f_2(t)) && \text{Substitute for } f_1. \\
&= \frac{e^{-s} - e^{-2s}}{s} + 5\mathcal{L}(f_2(t)) && \text{Extended table used.} \\
&= \frac{e^{-s} - e^{-2s} + 5e^{-3s} - 5e^{-4s}}{s} && \text{Similarly for } f_2.
\end{aligned}
$$

Example 8.7 (Dirac Impulse)

A machine shop tool that repeatedly hammers a die is modeled by a Dirac impulse model $f(t) = \sum_{n=1}^{N} \delta(t - n)$. Verify the formula $\mathcal{L}(f(t)) = \sum_{n=1}^{N} e^{-ns}$.

Solution:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}\left(\sum_{n=1}^{N} \delta(t - n)\right) \\
&= \sum_{n=1}^{N} \mathcal{L}(\delta(t - n)) && \text{Linearity.} \\
&= \sum_{n=1}^{N} e^{-ns} && \text{Extended Laplace table.}
\end{aligned}$$

Example 8.8 (Square wave)

A periodic camshaft force $f(t)$ applied to a mechanical system has the idealized graph shown in Figure 2. Verify formulas $f(t) = 1 + \textbf{sqw}(t)$ and $\mathcal{L}(f(t)) = \frac{1}{s}(1 + \tanh(s/2))$.
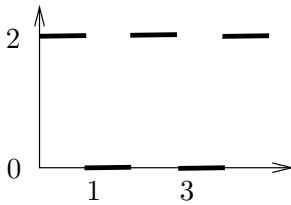
Figure 2. A periodic force $f(t)$ applied to a mechanical system.

Solution:

$$\begin{aligned}
1 + \textbf{sqw}(t) &= \begin{cases} 1 + 1 & 2n \leq t < 2n + 1, \ n = 0, 1, \ldots, \\ 1 - 1 & 2n + 1 \leq t < 2n + 2, \ n = 0, 1, \ldots, \end{cases} \\
&= \begin{cases} 2 & 2n \leq t < 2n + 1, \ n = 0, 1, \ldots, \\ 0 & \text{otherwise,} \end{cases} \\
&= f(t).
\end{aligned}$$

By the extended Laplace table, $\mathcal{L}(f(t)) = \mathcal{L}(1) + \mathcal{L}(\textbf{sqw}(t)) = \dfrac{1}{s} + \dfrac{\tanh(s/2)}{s}$.

Example 8.9 (Sawtooth wave)

Express the P-periodic sawtooth wave represented in Figure 3 as $f(t) = ct/P - c\,\textbf{floor}(t/P)$ and obtain the formula

$$\mathcal{L}(f(t)) = \frac{c}{Ps^2} - \frac{ce^{-Ps}}{s - se^{-Ps}}.$$
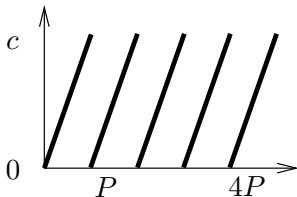
Figure 3. A P-periodic sawtooth wave $f(t)$ of height $c > 0$.

Solution: The representation originates from geometry, because the periodic function f can be viewed as derived from ct/P by subtracting the correct constant from each of intervals $[P, 2P]$, $[2P, 3P]$, etc.

The technique used to verify the identity is to define $g(t) = ct/P - c\,\textbf{floor}(t/P)$ and then show that g is P-periodic and $f(t) = g(t)$ on $0 \le t < P$. Two P-periodic functions equal on the base interval $0 \le t < P$ have to be identical, hence the representation follows.

Periodicity: Let's show $g(u + P) - g(u) = 0$ for all u. Used below is the identity $\textbf{floor}(1 + x) = 1 + \textbf{floor}(x)$. Details: Let $x = u/P$, then

$$\begin{aligned}
g(u + P) - g(u) &= c\frac{u+P}{P} - c\,\textbf{floor}\left(\frac{u+P}{P}\right) - g(u) \\
&= cx + c - c\,\textbf{floor}(1 + x) - cx + c\,\textbf{floor}(x) \\
&= 0.
\end{aligned}$$

Base interval equality: On $0 \le t < P$, define $x = t/P$ so that $0 \le x < 1$. Then $\textbf{floor}(x) = 0$ and $f(t) = ct/P = cx$. Compute $g(t) = ct/P - c\,\textbf{floor}(t/P) = cx - c\,\textbf{floor}(x) = cx = f(t)$.
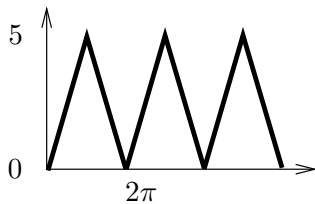
Laplace Calculation:

$$\mathcal{L}(f(t)) = \frac{c}{P}\mathcal{L}(t) - c\mathcal{L}(\textbf{floor}(t/P)) \qquad\qquad \text{Linearity.}$$

$$= \frac{c}{Ps^2} - \frac{ce^{-Ps}}{s - se^{-Ps}} \qquad\qquad \text{Basic and extended table applied.}$$

Example 8.10 (Triangular wave)
Express the triangular wave f of Figure 4 in terms of the square wave **sqw** and obtain

$$\mathcal{L}(f(t)) = \frac{5}{\pi s^2}\tanh(\pi s/2).$$

Figure 4. A 2π-periodic triangular wave $f(t)$ of height 5.

Solution: The representation of f in terms of **sqw** is $f(t) = 5\int_0^{t/\pi} \textbf{sqw}(x)dx$.

Details: A 2-periodic triangular wave of height 1 is obtained by integrating the square wave of period 2. A wave of height c and period 2 is given by $c\,\textbf{trw}(t) = c\int_0^t \textbf{sqw}(x)dx$. Then $f(t) = c\,\textbf{trw}(2t/P) = c\int_0^{2t/P} \textbf{sqw}(x)dx$ where $c = 5$ and $P = 2\pi$.

Laplace calculation: Use the extended Laplace table as follows.

$$\mathcal{L}(f(t)) = \frac{5}{\pi}\mathcal{L}(\pi\,\textbf{trw}(t/\pi)) = \frac{5}{\pi s^2}\tanh(\pi s/2).$$

Exercises 8.2 ↗

Laplace Transform Forward Table

Using the basic Laplace table and linearity properties of the transform, compute $\mathcal{L}(f(t))$. Do not use the direct Laplace transform!

1. $\mathcal{L}(2t)$

2. $\mathcal{L}(4t)$

3. $\mathcal{L}(1 + 2t + t^2)$

4. $\mathcal{L}(t^2 - 3t + 10)$

5. $\mathcal{L}(\sin 2t)$

6. $\mathcal{L}(\cos 2t)$

7. $\mathcal{L}(e^{2t})$

8. $\mathcal{L}(e^{-2t})$

9. $\mathcal{L}(t + \sin 2t)$

10. $\mathcal{L}(t - \cos 2t)$

11. $\mathcal{L}(t + e^{2t})$

12. $\mathcal{L}(t - 3e^{-2t})$

13. $\mathcal{L}((t + 1)^2)$

14. $\mathcal{L}((t + 2)^2)$

15. $\mathcal{L}(t(t + 1))$

16. $\mathcal{L}((t + 1)(t + 2))$

17. $\mathcal{L}(\sum_{n=0}^{10} t^n / n!)$

18. $\mathcal{L}(\sum_{n=0}^{10} t^{n+1} / n!)$

19. $\mathcal{L}(\sum_{n=1}^{10} \sin nt)$

20. $\mathcal{L}(\sum_{n=0}^{10} \cos nt)$

Laplace Backward Table

Solve the given equation for the function $f(t)$. Use the basic table and linearity properties of the Laplace transform.

21. $\mathcal{L}(f(t)) = s^{-2}$

22. $\mathcal{L}(f(t)) = 4s^{-2}$

23. $\mathcal{L}(f(t)) = 1/s + 2/s^2 + 3/s^3$

24. $\mathcal{L}(f(t)) = 1/s^3 + 1/s$

25. $\mathcal{L}(f(t)) = 2/(s^2 + 4)$

26. $\mathcal{L}(f(t)) = s/(s^2 + 4)$

27. $\mathcal{L}(f(t)) = 1/(s - 3)$

28. $\mathcal{L}(f(t)) = 1/(s + 3)$

29. $\mathcal{L}(f(t)) = 1/s + s/(s^2 + 4)$

30. $\mathcal{L}(f(t)) = 2/s - 2/(s^2 + 4)$

31. $\mathcal{L}(f(t)) = 1/s + 1/(s - 3)$

32. $\mathcal{L}(f(t)) = 1/s - 3/(s - 2)$

33. $\mathcal{L}(f(t)) = (2 + s)^2 / s^3$

34. $\mathcal{L}(f(t)) = (s + 1)/s^2$

35. $\mathcal{L}(f(t)) = s(1/s^2 + 2/s^3)$

36. $\mathcal{L}(f(t)) = (s + 1)(s - 1)/s^3$

37. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{1+n}$

38. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{2+n}$

39. $\mathcal{L}(f(t)) = \sum_{n=1}^{10} \dfrac{n}{s^2 + n^2}$

40. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} \dfrac{s}{s^2 + n^2}$

Laplace Table Extension

Compute the indicated Laplace integral using the extended Laplace table, page 602.

41. $\mathcal{L}(u(t - 2) + 2u(t))$

42. $\mathcal{L}(u(t - 3) + 4u(t))$

43. $\mathcal{L}(u(t - \pi)(u(t) + u(t - 1)))$

44. $\mathcal{L}(u(t - 2\pi) + 3u(t - 1)u(t - 2))$

45. $\mathcal{L}(\delta(t - 2))$

46. $\mathcal{L}(5\delta(t - \pi))$

47. $\mathcal{L}(\delta(t - 1) + 2\delta(t - 2))$

48. $\mathcal{L}(\delta(t - 2)(5 + u(t - 1)))$

60. $\mathcal{L}(f(t)) = 5e^{-3s}$

49. $\mathcal{L}(\mathbf{floor}(3t))$

50. $\mathcal{L}(\mathbf{floor}(2t))$

61. $\mathcal{L}(f(t)) = \dfrac{e^{-s/3}}{s(1 - e^{-s/3})}$

51. $\mathcal{L}(5\,\mathbf{sqw}(3t))$

52. $\mathcal{L}(3\,\mathbf{sqw}(t/4))$

62. $\mathcal{L}(f(t)) = \dfrac{e-2s}{s(1 - e^{-2s})}$

53. $\mathcal{L}(4\,\mathbf{trw}(2t))$

63. $\mathcal{L}(f(t)) = \dfrac{4\tanh(s)}{s}$

54. $\mathcal{L}(5\,\mathbf{trw}(t/2))$

55. $\mathcal{L}(t + t^{-3/2} + t^{-1/2})$

64. $\mathcal{L}(f(t)) = \dfrac{5\tanh(3s)}{2s}$

56. $\mathcal{L}(t^3 + t^{-3/2} + 2t^{-1/2})$

65. $\mathcal{L}(f(t)) = \dfrac{4\tanh(s)}{3s^2}$

Inverse Laplace, Extended Table
Solve the given equation for $f(t)$, using the extended Laplace integral table.

66. $\mathcal{L}(f(t)) = \dfrac{5\tanh(2s)}{11s^2}$

57. $\mathcal{L}(f(t)) = e^{-s}/s$

67. $\mathcal{L}(f(t)) = \dfrac{1}{\sqrt{s}}$

58. $\mathcal{L}(f(t)) = 5e^{-2s}/s$

59. $\mathcal{L}(f(t)) = e^{-2s}$

68. $\mathcal{L}(f(t)) = \dfrac{1}{\sqrt{s^3}}$

8.3 Laplace Transform Rules

In Table 7, the basic table manipulation rules are summarized. Full statements and proofs of the rules appear in section 8.5, page 637.

The rules are applied here to several key examples using the 8 rules. Partial fraction expansions including Heaviside's coverup method will be delayed to the section on Heaviside's Method page 8.4.

Table 7. Laplace transform Rules

$\mathcal{L}(f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$ $\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$	Linearity. The Laplace of a sum is the sum of the Laplaces. Constants move through the $\mathcal{L}$-symbol.
$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0)$	The t-derivative or parts rule. Derivatives $\mathcal{L}(y')$ are replaced in transformed equations.
$\mathcal{L}\left(\int_0^t g(x)dx\right) = \frac{1}{s}\mathcal{L}(g(t))$	Forward t-integral rule.
$\frac{1}{s}\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t g(x)dx\right)$	Backward t-integral rule.
$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t))$	Forward s-differentiation rule. Each erased t-factor inserts $-\frac{d}{ds}$ in front of $\mathcal{L}$.
$\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}((-t)f(t))$	Backward s-differentiation rule.
$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))\vert_{s\to(s-a)}$	Forward First Shifting rule.
$\mathcal{L}(f(t))\vert_{s\to(s-a)} = \mathcal{L}(e^{at}f(t))$	Backward First Shifting rule. Multiplying f by e^{at} replaces s by $s-a$.
$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}(g(t+a)),$	Forward Second Shifting rule.
$e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a))$	Backward Second Shifting rule.
$\mathcal{L}(f(t)) = \dfrac{\int_0^P f(t)e^{-st}dt}{1 - e^{-Ps}}$	Rule for P-periodic functions. Assumed: $f(t+P) = f(t)$.
$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}((f*g)(t))$	Convolution rule. Define $(f*g)(t) = \int_0^t f(x)g(t-x)dx$.

Examples and Methods

Example 8.11 (Rutherford Decay)
Solve the radioactive chain decay problem $x' + 2x = -e^{-2t}$, $x(0) = 10$ by Laplace's method.

Solution: The solution is $x = e^{-2t} - te^{-2t}$. The details:

$\mathcal{L}(x' + 2x) = \mathcal{L}(-e^{-2t})$	$\boxed{1}$ Apply $\mathcal{L}$ across the equation.
$\mathcal{L}(x') + 2\mathcal{L}(x) = -\mathcal{L}(e^{-2t})$	Linearity of $\mathcal{L}$.
$s\mathcal{L}(x) - x(0) + 2\mathcal{L}(x) = -\mathcal{L}(e^{-2t})$	Parts rule.
$(s+2)\mathcal{L}(x) = 10 - \mathcal{L}(e^{-2t})$	Use $x(0) = 10$. Collect left on $\mathcal{L}(x)$.
$(s+2)\mathcal{L}(x) = 10 - \frac{1}{s+2}$	Forward Laplace table: $\mathcal{L}(e^{-2t}) = \frac{1}{s+2}$.

$\mathcal{L}(x) = \frac{10}{s+2} - \frac{1}{(s+2)^2}$ Divide to isolate $\mathcal{L}(x)$ left.

$\mathcal{L}(x) = \frac{10}{s+2} - \frac{1}{s^2}\big|_{s\to s+2}$ $\boxed{2}$ First shifting rule preparation.

$\mathcal{L}(x) = \mathcal{L}(e^{-2t}) - \mathcal{L}(t)\big|_{s\to s+2}$ Backward Laplace table:
$\frac{1}{s-a} = \mathcal{L}(e^{at})$, $\frac{1}{s^2} = \mathcal{L}(t)$.

$\mathcal{L}(x) = \mathcal{L}(e^{-2t}) - \mathcal{L}(te^{-2t})$ $\boxed{3}$ Backwards first shifting theorem.

$\mathcal{L}(x) = \mathcal{L}(e^{-2t} - te^{-2t})$ Linearity.

$x = e^{-2t} - te^{-2t}$ Lerch's theorem: cancel $\mathcal{L}$ on each side.

Laplace's method: Multiply across by $e^{-st}dt$, then integrate across $t = 0$ to $t = \infty$. It is the same as applying $\mathcal{L}$ across the equation.

The **details** used algebraic steps and Laplace rules to obtain $\mathcal{L}(x(t))$ on the left and $\mathcal{L}(\text{some } t\text{-expression})$ on the right. In the last step Lerch's theorem applies to cancel $\mathcal{L}$ on each side, which isolates the solution $x(t) = \text{some } t\text{-expression}$.

$\boxed{1}$: Think of $\mathcal{L}$ as a matrix and Laplace's method as matrix multiply.

$\boxed{2}$: Fraction $\frac{1}{(s+2)^2}$ is $\frac{1}{w^2}$ using substitution $w = s + 2$. Mentally replace w by s and search the Backward Table for a matching entry. The intuition comes from u-substitution in calculus, but because u is the unit step function in Laplace theory, symbol **w is used instead of u** in substitution examples.

$\boxed{3}$: The backwards first shifting theorem in words: Remove $\big|_{s\to s-a}$ by inserting exponential e^{at} inside the scope of $\mathcal{L}$.

Example 8.12 (Harmonic oscillator)
Solve the initial value problem $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$ by Laplace's method.

Solution: The solution is $x(t) = \sin t$. The details:

$\mathcal{L}(x'') + \mathcal{L}(x) = \mathcal{L}(0)$ Apply $\mathcal{L}$ across the equation.

$s\mathcal{L}(x') - x'(0) + \mathcal{L}(x) = 0$ The t-derivative or parts rule.

$s[s\mathcal{L}(x) - x(0)] - x'(0) + \mathcal{L}(x) = 0$ Again the parts rule.

$(s^2 + 1)\mathcal{L}(x) = 1$ Use $x(0) = 0$, $x'(0) = 1$.

$\mathcal{L}(x) = \dfrac{1}{s^2 + 1}$ Divide to isolate $\mathcal{L}(x(t))$ left.

$= \mathcal{L}(\sin t)$ Forward Laplace table.

$x(t) = \sin t$ Lerch's cancellation law.

Example 8.13 (Forward Table First Shifting Rule)

Show the steps for the identity $\mathcal{L}(t^2 e^{-3t}) = \dfrac{2}{(s + 3)^3}$.

Solution:

$\mathcal{L}(t^2 e^{-3t}) = \mathcal{L}(t^2)\big|_{s\to s-(-3)}$ First shifting rule.

$= \left(\dfrac{2}{s^{2+1}}\right)\Big|_{s\to s-(-3)}$ Forward Laplace table.

$$= \frac{2}{(s+3)^3} \qquad \text{Identity verified.}$$

Example 8.14 (Backward Table First Shifting Rule I)

Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+7}{s^2+4s+8}$.

Solution: The answer is $f(t) = e^{-2t}(\cos 2t + \frac{5}{2}\sin 2t)$. The details:

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \frac{s+7}{(s+2)^2+4} & \text{Complete the square.}\\[2mm]
&= \frac{w+5}{w^2+4} & \text{Replace } s+2 \text{ by } w.\\[2mm]
&= \frac{w}{w^2+4} + \frac{5}{2}\frac{2}{w^2+4} & \text{Split into table entries.}\\[2mm]
&= \frac{s}{s^2+4} + \frac{5}{2}\frac{2}{s^2+4}\Big|_{s\to w=s+2} & \text{Shifting rule preparation.}\\[2mm]
&= \mathcal{L}\left(\cos 2t + \tfrac{5}{2}\sin 2t\right)\big|_{s\to w=s+2} & \text{Basic Laplace table.}\\[2mm]
&= \mathcal{L}(e^{-2t}(\cos 2t + \tfrac{5}{2}\sin 2t)) & \text{First shifting rule.}\\[2mm]
f(t) &= e^{-2t}(\cos 2t + \tfrac{5}{2}\sin 2t) & \text{Lerch's cancellation law.}
\end{aligned}
$$

Example 8.15 (Backward Table First Shifting Rule II)

Solve the equation $\mathcal{L}(f(t)) = \dfrac{s+2}{2^2+2s+2}$ for $f(t)$.

Solution: The answer is $f(t) = e^{-t}\cos t + e^{-t}\sin t$. The details:

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \frac{s+2}{s^2+2s+2} & \text{Signal for this method: the denominator has complex roots.}\\[2mm]
&= \frac{s+2}{(s+1)^2+1} & \text{Complete the square, denominator.}\\[2mm]
&= \frac{w+1}{w^2+1} & \text{Substitute } w \text{ for } s+1.\\[2mm]
&= \frac{w}{w^2+1} + \frac{1}{w^2+1} & \text{Split into Laplace table entries.}\\[2mm]
&= (\mathcal{L}(\cos t) + \mathcal{L}(\sin t))\big|_{s\to w=s+1} & \text{Basic Laplace table.}\\[2mm]
&= \mathcal{L}(e^{-t}\cos t) + \mathcal{L}(e^{-t}\sin t) & \text{First shifting rule.}\\[2mm]
f(t) &= e^{-t}\cos t + e^{-t}\sin t & \text{Lerch's cancellation law.}
\end{aligned}
$$

Example 8.16 (Damped oscillator)
Solve by Laplace's method the initial value problem $x'' + 2x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$.

Solution: The solution is $x(t) = e^{-t} \cos t$. The details:

$\mathcal{L}(x'') + 2\mathcal{L}(x') + 2\mathcal{L}(x) = \mathcal{L}(0)$ Apply $\mathcal{L}$ across the equation.

$s\mathcal{L}(x') - x'(0) + 2\mathcal{L}(x') + 2\mathcal{L}(x) = 0$ The t-derivative rule on x'.

$s[s\mathcal{L}(x) - x(0)] - x'(0)$
$+2[\mathcal{L}(x) - x(0)] + 2\mathcal{L}(x) = 0$ The t-derivative rule on x.

$(s^2 + 2s + 2)\mathcal{L}(x) = 1 + s$ Use $x(0) = 1$, $x'(0) = -1$.

$\mathcal{L}(x) = \dfrac{s+1}{s^2 + 2s + 2}$ Divide to isolate $\mathcal{L}(x)$.

$\quad = \dfrac{s+1}{(s+1)^2 + 1}$ Complete the square.

$\quad = \dfrac{w}{w^2 + 1}\Big|_{w = s+1}$ Replace $s + 1$ by w.

$\quad = \mathcal{L}(\cos t)|_{s \to w = s+1}$ Backward table: $\frac{s}{s^2+1} = \mathcal{L}(\cos t)$.

$\quad = \mathcal{L}(e^{-t} \cos t)$ First shifting rule.

$x(t) = e^{-t} \cos t$ Lerch's cancellation law.

Example 8.17 (Forward Table s-Differentiation)

Show the steps for the identity $\mathcal{L}(t^2 \, e^{5t}) = \dfrac{2}{(s-5)^3}$.

Solution:

$\mathcal{L}(t^2 e^{5t}) = \left(-\dfrac{d}{ds}\right)\left(-\dfrac{d}{ds}\right)\mathcal{L}(e^{5t})$ Apply s-differentiation.

$\quad = (-1)^2 \dfrac{d}{ds}\dfrac{d}{ds}\left(\dfrac{1}{s-5}\right)$ Basic Laplace table.

$\quad = \dfrac{d}{ds}\left(\dfrac{-1}{(s-5)^2}\right)$ Calculus power rule $(u^n)' = nu^{n-1}u'$.

$\quad = \dfrac{2}{(s-5)^3}$ Identity verified.

Example 8.18 (Backward Table s-Differentiation)

Solve the equation $\mathcal{L}(f(t)) = \dfrac{2s}{(s^2 + 1)^2}$ for $f(t)$.

Solution: The solution is $f(t) = t \sin t$. The details:

$\mathcal{L}(f(t)) = \dfrac{2s}{(s^2 + 1)^2}$

$\quad = -\dfrac{d}{ds}\left(\dfrac{1}{s^2 + 1}\right)$ Calculus power rule $(u^n)' = nu^{n-1}u'$.

$\quad = -\dfrac{d}{ds}\left(\mathcal{L}(\sin t)\right)$ Basic Laplace table.

$$= \mathcal{L}(t \sin t) \qquad \text{Apply the } s\text{-differentiation rule.}$$
$$f(t) = t \sin t \qquad \text{Lerch's cancellation law.}$$

Example 8.19 (Forward Table Second shifting rule)

Show the steps for the identity $\mathcal{L}(\sin(t)\, u(t - \pi)) = \dfrac{-e^{-\pi s}}{s^2 + 1}$, where $u(t)$ is the **unit step function**: $u(t) = 1$ for $t \geq 0$, $u(t) = 0$ otherwise.

Solution: The second shifting rule is applied as follows, where LHS and RHS abbreviate the left and right hand side.

$$\text{LHS} = \mathcal{L}(\sin t\, u(t - \pi)) \qquad \text{Left side of the identity.}$$
$$= \mathcal{L}(g(t) u(t - a)) \qquad \text{Choose } g(t) = \sin t, \ a = \pi.$$
$$= e^{-as} \mathcal{L}(g(t + a)) \qquad \text{Second form, second shifting theorem.}$$
$$= e^{-\pi s} \mathcal{L}(\sin(t + \pi)) \qquad \boxed{1}\ \text{Substitute } a = \pi, \ g(t) = \sin(t).$$
$$= e^{-\pi s} \mathcal{L}(-\sin t) \qquad \text{Trig rules } \sin(a + b) = \sin a \cos b + \sin b \cos a \text{ and } \sin \pi = 0, \ \cos \pi = -1.$$
$$= e^{-\pi s} \frac{-1}{s^2 + 1} \qquad \text{Forward Laplace table.}$$
$$= \text{RHS} \qquad \text{Identity verified.}$$

$\boxed{1}$: Easy for some readers, difficult for others. How did we change symbol $g(t + a)$ into $\sin(t + \pi)$? For $g(t) = \sin t$, the replacement process $g \to \sin$ and $a \to \pi$ can be written as $g(t + a) = g(x)|_{x=t+a} = \sin(x)|_{x=t+\pi} = \sin(t + \pi)$.

Example 8.20 (Backward Table Second Shifting Rule)

Solve the equation $\mathcal{L}(f(t)) = e^{-3s} \dfrac{s + 1}{s^2 + 2s + 2}$ for $f(t)$.

Solution: The answer is $f(t) = e^{3-t} \cos(t - 3)$ for $t \geq 3$, $f(t) = 0$ otherwise. The details:

$$\mathcal{L}(f(t)) = e^{-3s} \frac{s + 1}{(s + 1)^2 + 1} \qquad \text{Complete the square.}$$
$$= e^{-3w+3} \frac{w}{w^2 + 1} \qquad \text{Let } w = s + 1, \text{ like a calculus } u\text{-substitution.}$$
$$= e^{-3w+3} \left(\mathcal{L}(\cos t) \right)|_{s \to w} \qquad \text{Backward table: } \frac{s}{s^2+1} = \mathcal{L}(\cos t)$$
$$= e^3 \left(e^{-3s} \mathcal{L}(\cos t) \right)\big|_{s \to w} \qquad \text{Regroup factor } e^{-3w}.$$
$$= e^3 \left(\mathcal{L}(\cos(t - 3)u(t - 3)) \right)|_{s \to w = s+1} \qquad \text{Second shifting rule, 1st form.}$$
$$= e^3 \mathcal{L}(\cos(t - 3)u(t - 3)e^{-t}) \qquad \text{First shifting rule.}$$
$$f(t) = e^{3-t} \cos(t - 3)u(t - 3) \qquad \text{Lerch's cancellation law.}$$

Example 8.21 (Trigonometric formulas)

Show the steps used to obtain these Laplace identities:

(a) $\mathcal{L}(t\cos at) = \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$

(c) $\mathcal{L}(t^2\cos at) = \dfrac{2(s^3 - 3sa^2)}{(s^2 + a^2)^3}$

(b) $\mathcal{L}(t\sin at) = \dfrac{2sa}{(s^2 + a^2)^2}$

(d) $\mathcal{L}(t^2\sin at) = \dfrac{6s^2 a - a^3}{(s^2 + a^2)^3}$

Solution: The details for **(a)**:

$$\mathcal{L}(t\cos at) = -(d/ds)\mathcal{L}(\cos at) \qquad \text{Use } s\text{-differentiation.}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right) \qquad \text{Basic Laplace table.}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2} \qquad \text{Calculus quotient rule.}$$

The details for **(c)**:

$$\mathcal{L}(t^2\cos at) = -(d/ds)\mathcal{L}((-t)\cos at) \qquad \text{Use } s\text{-differentiation.}$$

$$= \frac{d}{ds}\left(-\frac{s^2 - a^2}{(s^2 + a^2)^2}\right) \qquad \text{Result of } \textbf{(a)}.$$

$$= \frac{2s^3 - 6sa^2)}{(s^2 + a^2)^3} \qquad \text{Calculus quotient rule.}$$

The similar details for **(b)** and **(d)** are left as exercises.

Example 8.22 (Exponential Formulas)

Show the steps used to obtain these Laplace identities:

(a) $\mathcal{L}(e^{at}\cos bt) = \dfrac{s - a}{(s - a)^2 + b^2}$

(c) $\mathcal{L}(te^{at}\cos bt) = \dfrac{(s - a)^2 - b^2}{((s - a)^2 + b^2)^2}$

(b) $\mathcal{L}(e^{at}\sin bt) = \dfrac{b}{(s - a)^2 + b^2}$

(d) $\mathcal{L}(te^{at}\sin bt) = \dfrac{2b(s - a)}{((s - a)^2 + b^2)^2}$

Solution: Details for **(a)**:

$$\mathcal{L}(e^{at}\cos bt) = \mathcal{L}(\cos bt)|_{s\to s-a} \qquad \text{First shifting rule.}$$

$$= \left(\frac{s}{s^2 + b^2}\right)\Big|_{s\to s-a} \qquad \text{Basic Laplace table.}$$

$$= \frac{s - a}{(s - a)^2 + b^2} \qquad \text{Verified } \textbf{(a)}.$$

Details for **(c)**:

$$\mathcal{L}(te^{at}\cos bt) = \mathcal{L}(t\cos bt)|_{s\to s-a} \qquad \text{First shifting rule.}$$

$$= \left(-\frac{d}{ds}\mathcal{L}(\cos bt)\right)\Big|_{s\to s-a} \qquad \text{Apply } s\text{-differentiation.}$$

$$= \left(-\frac{d}{ds}\left(\frac{s}{s^2 + b^2}\right)\right)\Big|_{s\to s-a} \qquad \text{Basic Laplace table.}$$

$$= \left. \left(\frac{s^2 - b^2}{(s^2 + b^2)^2} \right) \right|_{s \to s - a} \qquad \text{Calculus quotient rule.}$$

$$= \frac{(s - a)^2 - b^2}{((s - a)^2 + b^2)^2} \qquad \text{Verified \textbf{(c)}.}$$

Left as exercises are **(b)** and **(d)**.

Example 8.23 (Hyperbolic Functions)

Establish these Laplace transform facts about $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$.

(a) $\mathcal{L}(\cosh at) = \dfrac{s}{s^2 - a^2}$ $\qquad$ **(c)** $\mathcal{L}(t \cosh at) = \dfrac{s^2 + a^2}{(s^2 - a^2)^2}$

(b) $\mathcal{L}(\sinh at) = \dfrac{a}{s^2 - a^2}$ $\qquad$ **(d)** $\mathcal{L}(t \sinh at) = \dfrac{2as}{(s^2 - a^2)^2}$

Solution: The details for **(a)**:

$$\mathcal{L}(\cosh at) = \tfrac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) \qquad \text{Definition plus linearity of } \mathcal{L}.$$

$$= \frac{1}{2} \left(\frac{1}{s - a} + \frac{1}{s + a} \right) \qquad \text{Basic Laplace table.}$$

$$= \frac{s}{s^2 - a^2} \qquad \text{Identity \textbf{(a)} verified.}$$

The details for **(d)**:

$$\mathcal{L}(t \sinh at) = -\frac{d}{ds} \left(\frac{a}{s^2 - a^2} \right) \qquad \text{Apply the } s\text{-differentiation rule.}$$

$$= \frac{a(2s)}{(s^2 - a^2)^2} \qquad \text{Calculus power rule; \textbf{(d)} verified.}$$

Left as exercises are **(b)** and **(c)**.

Example 8.24 (Rectified sine wave)

Compute the Laplace transform of the rectified sine wave $f(t) = |\sin \omega t|$ and show that it can be expressed in the form

$$\mathcal{L}(|\sin \omega t|) = \frac{\omega \, \coth \left(\frac{\pi s}{2\omega} \right)}{s^2 + \omega^2}.$$

Solution: The periodic function formula will be applied with period $P = 2\pi/\omega$. The calculation reduces to the evaluation of $J = \int_0^P f(t) e^{-st} dt$. Because $\sin \omega t \leq 0$ on $\pi/\omega \leq t \leq 2\pi/\omega$, integral J can be written as $J = J_1 + J_2$, where

$$J_1 = \int_0^{\pi/\omega} \sin \omega t \, e^{-st} dt, \quad J_2 = \int_{\pi/\omega}^{2\pi/\omega} -\sin \omega t \, e^{-st} dt.$$

Integral tables give the result

$$\int \sin \omega t \, e^{-st} \, dt = -\frac{\omega e^{-st} \cos(\omega t)}{s^2 + \omega^2} - \frac{s e^{-st} \sin(\omega t)}{s^2 + \omega^2}.$$

Then

$$J_1 = \frac{\omega(e^{-\pi*s/\omega} + 1)}{s^2 + \omega^2}, \quad J_2 = \frac{\omega(e^{-2\pi s/\omega} + e^{-\pi s/\omega})}{s^2 + \omega^2},$$

$$J = \frac{\omega(e^{-\pi s/\omega} + 1)^2}{s^2 + \omega^2}.$$

The remaining challenge is to write the answer for $\mathcal{L}(f(t))$ in terms of $\coth(u) = \frac{\cosh(u)}{\sinh(u)}$ where $\cosh(u) = \frac{1}{2}e^u + \frac{1}{2}e^{-u}$ and $\sinh(u) = \frac{1}{2}e^u - \frac{1}{2}e^{-u}$. The details:

$$\mathcal{L}(f(t)) = \frac{J}{1 - e^{-Ps}} \qquad\qquad \text{Periodic function formula.}$$

$$= \frac{J}{(1 - e^{-Ps/2})(1 + e^{-Ps/2})} \qquad \begin{array}{l}\text{Apply } 1 - x^2 = (1-x)(1+x)\\ \text{where } x = e^{-Ps/2}.\end{array}$$

$$= \frac{\omega(1 + e^{-Ps/2})}{(1 - e^{-Ps/2})(s^2 + \omega^2)} \qquad \text{Cancel factor } 1 + e^{-Ps/2}.$$

$$= \frac{e^{Ps/4} + e^{-Ps/4}}{e^{Ps/4} - e^{-Ps/4}} \frac{\omega}{s^2 + \omega^2} \qquad \text{Factor out } e^{-Ps/4}, \text{ then cancel.}$$

$$= \frac{2\cosh(Ps/4)}{2\sinh(Ps/4)} \frac{\omega}{s^2 + \omega^2} \qquad \text{Apply cosh, sinh identities.}$$

$$= \frac{\omega\coth(Ps/4)}{s^2 + \omega^2} \qquad\qquad \text{Use } \frac{\cosh u}{\sinh u} = \coth u.$$

$$= \frac{\omega\coth\left(\frac{\pi s}{2\omega}\right)}{s^2 + \omega^2} \qquad\qquad \text{Identity verified.}$$

Example 8.25 (Half–wave Rectification)

Determine the Laplace transform of the half–wave rectification $g(t)$ of $\sin\omega t$, in which the negative cycles of $\sin\omega t$ have been replaced by zero to define $g(t)$. Show in particular that

$$\mathcal{L}(g(t)) = \frac{1}{2}\frac{\omega}{s^2 + \omega^2}\left(1 + \coth\left(\frac{\pi s}{2\omega}\right)\right)$$

Solution: The half–wave rectification of $\sin\omega t$ is $g(t) = (\sin\omega t + |\sin\omega t|)/2$. The Forward Table plus the result of Example 8.24 gives

$$\mathcal{L}(2g(t)) = \mathcal{L}(\sin\omega t) + \mathcal{L}(|\sin\omega t|)$$

$$= \frac{\omega}{s^2 + \omega^2} + \frac{\omega\cosh(\pi s/(2\omega))}{s^2 + \omega^2}$$

$$= \frac{\omega}{s^2 + \omega^2}(1 + \cosh(\pi s/(2\omega))$$

Dividing by 2 produces the identity.

Exercises 8.3 ☑

First Order Linear DE

Display the Laplace method details which verify the supplied answer.

The first two exercises use forward and back- *ward Laplace tables plus the first shifting theo-* *rems. The others require a calculus background* *in partial fractions.*

1. $x' + x = e^{-t}$, $x(0) = 1$;
$x(t) = (1+t)e^{-t}$.

2. $x' + 2x = -e^{-2t}$, $x(0) = 1$;
$x(t) = (1-t)e^{-2t}$.

3. $x' + x = 1$, $x(0) = 1$; $x(t) = 1$.

4. $x' + 4x = 4$, $x(0) = 1$; $x(t) = 1$.

5. $x' + x = t$, $x(0) = -1$; $x(t) = t - 1$.

6. $x' + x = t$, $x(0) = 1$;
$x(t) = t - 1 + 2e^{-t}$.

Second Order Linear DE
Display the Laplace method details which verify the supplied answer.
The first 4 exercises require only forward and backward Laplace tables and the first shifting theorems. The others require methods in partial fractions beyond a calculus background.

7. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 1$;
$x(t) = \cos t + \sin t$.

8. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 2$;
$x(t) = \cos t + 2\sin t$.

9. $x'' + 2x' + x = 0$, $x(0) = 0$, $x'(0) = 1$;
$x(t) = te^{-t}$.

10. $x'' + 2x' + x = 0$, $x(0) = 1$, $x'(0) = -1$;
$x(t) = e^{-t}$.

11. $x'' + 3x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$;
$x(t) = e^{-t}$.

12. $x'' + 3x' + 2x = 0$, $x(0) = 1$, $x'(0) = -2$;
$x(t) = e^{-2t}$.

13. $x'' + 3x' = 0$, $x(0) = 5$, $x'(0) = 0$;
$x(t) = 5$.

14. $x'' + 3x' = 0$, $x(0) = 1$, $x'(0) = -3$;
$x(t) = e^{-3t}$.

15. $x'' + x = 1$, $x(0) = 1$, $x'(0) = 0$;
$x(t) = 1$.

16. $x'' = 2$, $x(0) = 0$, $x'(0) = 0$; $x(t) = t^2$.

Forward Integral Rule
The rule is $\mathcal{L}\left(\int_0^t g(r)dr\right) = \frac{1}{s}\mathcal{L}(g(t))$

17. Relate this rule to the convolution rule with $f(t) = 1$.

18. Compute $\mathcal{L}\left(\int_0^t \sin(r)dr\right)$.

19. Compute $\mathcal{L}\left(\int_0^t (r+1)^3\, dr\right)$.

20. Compute $\mathcal{L}\left(\int_0^t \textbf{sqw}(r)dr\right)$, where **sqw** is the square wave of period 2. Use the Extended Laplace Table.

Backward Integral Rule
Apply rule $\frac{1}{s}\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t g(r)dr\right)$ and Lerch's theorem to solve for $f(t)$.

21. $\mathcal{L}(f(t)) = \frac{1}{s(s^2+1)}$

22. $\mathcal{L}(f(t)) = \frac{1}{s}\frac{s+1}{s^2+1}$

23. $\mathcal{L}(f(t)) = \frac{1}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$

24. $\mathcal{L}(f(t)) = \frac{1}{s}\frac{e^{-s}}{s}$
Hint: $\mathcal{L}(u(t-a)) = \frac{1}{s}e^{-as}$.

The s–Integral Rule
Identity $\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty \mathcal{L}(f(t))\,ds$
requires piecewise continuous $f(t)$ of exponential order with $\lim_{t\to 0+}\frac{f(t)}{t} = L$.

25. Prove the identity.

26. Compute $\mathcal{L}\left(\frac{\sin(t)}{t}\right)$.

Forward First Shifting Rule
Apply $\mathcal{L}(f(t)e^{at}) = \mathcal{L}(f(t))|_{s\to s-a}$ to find the Laplace transform.

27. $\mathcal{L}(te^t)$

28. $\mathcal{L}(te^t + e^{2t})$

29. $\mathcal{L}(\sin(t)e^t)$

30. $\mathcal{L}(\sin(2t)e^{2t} + \cos(t)e^t)$

31. $\mathcal{L}(t\cosh(2t))$ using identity
$\cosh(w) = \frac{1}{2}e^w + \frac{1}{2}e^{-w}$.

32. $\mathcal{L}((t+1)^3\, e^t)$

Backward First Shifting Rule
Apply $\mathcal{L}(f(t))|_{s\to s-a} = \mathcal{L}(f(t)e^{at})$ and Lerch's theorem to solve for $f(t)$.

33. Explain for $\mathcal{L}(t^2)\big|_{s \to s-4}$ the rule
Erase a shift $|_{s \to s-a}$ by inserting e^{at} inside the scope of $\mathcal{L}$.

34. $\mathcal{L}(f(t)) = \frac{s}{s^2+1}\Big|_{s \to s-1}$

35. $\mathcal{L}(f(t)) = \frac{s-1}{(s-1)^2+4}$

36. $\mathcal{L}(f(t)) = \frac{8}{(s+1)^2+4}$

37. $\mathcal{L}(f(t)) = \frac{s+1}{s^2+2s+5}$

38. $\mathcal{L}(f(t)) = \frac{4}{s^2+8s+17}$

39. $\mathcal{L}(f(t)) = \frac{2}{(s+1)^2}$

40. $\mathcal{L}(f(t)) = \frac{1}{(s+2)^{101}}$

Forward s-Differentiation
Apply $\mathcal{L}((-t)f(t)) = \frac{d}{ds}\mathcal{L}(f(t))$ to find the Laplace transform.

41. Explain for $\mathcal{L}((-t)\cos(t))$ the rule
Multiplying by $(-t)$ differentiates the Laplace transform..

42. $\mathcal{L}((-t)\sin(2t))$

43. $\mathcal{L}((-t)\sinh(2t))$, using identity $\sinh(w) = \frac{1}{2}e^w - \frac{1}{2}e^{-w}$.

44. $\mathcal{L}(te^t \sin(2t) + te^{2t}\cos(t))$

Backward s-Differentiation
Apply $\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}((-t)f(t))$ and Lerch's theorem to solve for $f(t)$.

45. Explain for $\frac{d}{ds}\mathcal{L}(\cos(t))$ the rule
Erase $\frac{d}{ds}$ by inserting factor $(-t)$ inside the scope of $\mathcal{L}$.

46. $\mathcal{L}(f(t)) = \frac{d}{ds}\frac{s}{s^2+4}$

47. $\mathcal{L}(f(t)) = \frac{d^2}{ds^2}\frac{1}{(s+1)^5}$

48. $\mathcal{L}(f(t)) = \frac{d^3}{ds^3}\frac{s+1}{s^2+2s+5}$

Unit Step and Pulse
Define
$$\mathbf{pulse}(t,a,b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{else,} \end{cases}$$
which is a tool for encoding and decoding piecewise-defined functions.

49. Prove the identity
$$\mathbf{pulse}(t,a,b) = u(t-a) - u(t-b),$$
where u is the **unit step**.

50. Prove the Laplace formula
$$\mathcal{L}(\mathbf{pulse}(t,a,b)) = \frac{e^{-at} - e^{-bt}}{s}$$

51. Verify that $f(t)$ defined by
$$\begin{cases} 2 & 1 \le t < 2, \\ 0 & \text{else} \end{cases} + \begin{cases} 3 & 3 \le t < 4, \\ 0 & \text{else} \end{cases}$$
encodes to representation
$2\,\mathbf{pulse}(t,1,2) + 3\,\mathbf{pulse}(t,3,4)$.

52. Decode $f(t)$ into a piecewise–defined function and graph it by hand, no computer, given $f(t)$ is
$e^t\,\mathbf{pulse}(t,1,3) + e^{-t}\,\mathbf{pulse}(t,4,6)$

53. Decode $f(t)$ into a piecewise–defined function and graph it, no computer, given $f(t)$ is the sum
$\sum_{n=1}^3 |\sin(n\pi t)|\,\mathbf{pulse}(t,2n,2n+1)$

54. Encode as a combination of pulses
$$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ -2 & 3 \le t < 4, \\ 1 & 5 \le t < 6, \\ 0 & \text{else,} \end{cases}$$
showing all encoding details.　　Ans:
$f(t) = \mathbf{pulse}(t,1,2) - 2\,\mathbf{pulse}(t,3,4)$
$+ \mathbf{pulse}(t,5,6)$.

Alternate Second Shifting Rule
$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(w)|_{w=t+a}\right)$. No Laplace here. The focus is on function notation and finding $g(t+a) = g(w)|_{w=t+a}$, which means *substitute $w = t+a$ into the $g(w)$–formula.*

55. Let $g(t) = te^{-t}$. Verify identity $g(w)|_{w=t+2} = e^{-2}(te^{-t} + 2e^{-t})$.

56. Let $g(t) = t^3$. Verify identity $g(w)|_{w=t+2} = 8 + 12t + 6t^2 + t^3$.

57. Typical polynomial $g(w) = 1 + 2w^2 + 3w^4$ upon substitution $w = t+a$ requires expansions for $(t+a)^2$ and $(t+a)^4$. Pascal's Triangle can be useful. Find the answer for $g(t+a) = g(w)|_{w=t+a}$.

8.3 Laplace Transform Rules

58. Polynomial $1+2w^2+3w^4$ upon substitution $w = t - b$ is a Taylor polynomial expansion
$f(t) = \sum_{n=0}^{4} \frac{f^{(n)}(b)}{n!} (t - b)^n$.
Find the Maclaurin expansion
$f(t) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} t^n$.

Forward Second Shifting Rule
$\mathcal{L}(g(t)u(t - a)) = e^{-as}\mathcal{L}(g(t + a))$
Find $\mathcal{L}(f(t))$, where u is the unit step.

59. $f(t) = u(t - \pi)$

60. $f(t) = e^t\, u(t - 1)$

61. $f(t) = t^3 u(t - \pi)$

62. $f(t) = e^t\, \mathbf{pulse}(t, 1, 2)$, where
$\mathbf{pulse}(t, a, b) = u(t - a) - u(t - b)$.

63. $f(t) = te^t u(t - 2)$

64. $f(t) = t\sin(t)u(t - \pi)$

Backward Second Shifting Rule
$e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t - a)u(t - a))$
Find $f(t)$ using the rule and Lerch's theorem, giving a piecewise–defined display and a unit step or pulse formula.

65. $\mathcal{L}(f(t)) = \frac{1}{s}e^{-3s}$
Ans: $f(t) = u(t - 3) = \begin{cases} 1 & t \geq 3, \\ 0 & \text{else,} \end{cases}$

66. $\mathcal{L}(f(t)) = \frac{1}{s^2}e^{3-3s}$

67. $\mathcal{L}(f(t)) = \frac{4}{s^2 + 8s + 17}e^{-2s}$

68. $\mathcal{L}(f(t)) = \frac{4 + s}{s^2 + 8s + 17}e^{-3s}$

69. $\mathcal{L}(f(t)) = \left(\frac{1}{s^2} + \frac{2}{s^3}\right)e^{-2s}$

70. $\mathcal{L}(f(t)) = \frac{1}{(s - 4)^2}e^{-2s}$

Trigonometric Formulas
Supply the details in Example 8.21.

71. $\mathcal{L}(t\sin at) = \frac{2as}{(s^2 + a^2)^2}$

72. $\mathcal{L}(t^2 \sin at) = \frac{6s^2 a - a^3}{(s^2 + a^2)^3}$

Exponential Formulas
Supply the details in Example 8.22.

73. $\mathcal{L}(e^{at}\sin bt) = \frac{b}{(s - a)^2 + b^2}$

74. $\mathcal{L}(te^{at}\sin bt) = \frac{2b(s - a)}{((s - a)^2 + b^2)^2}$

Hyperbolic Functions
Supply the details in Example 8.23.

75. $\mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}$

76. $\mathcal{L}(t\cosh at) = \frac{s^2 + a^2}{(s^2 - a^2)^2}$

Waves
Use Laplace ideas from Examples 8.24 and 8.25. Each $f(t)$ can be expressed as a **pulse train**, which is an expression $\sum_{n=1}^{\infty} f_n(t)\,\mathbf{pulse}(t, a_i, b_i)$ to which the second shifting theorem applies.

77. Find $\mathcal{L}(f(t))$ for the square wave
$f(t) = \sum_{n=0}^{\infty}(-1)^n\,\mathbf{pulse}(t, n, n + 1)$

78. Define pulse train
$f(t) = \sum_{n=0}^{\infty} f_n(t)\,\mathbf{pulse}(t, n, n + 1)$,
$f_{2n}(t) = t - 2n, f_{2n+1}(t) = 2 - t + 2n.$
Show that $f(t + 2) = f(t)$ and
$f(t) = \begin{cases} t & 0 \leq t < 1, \\ 2 - t & 1 \leq t \leq 2. \end{cases}$

79. Find $\mathcal{L}(f(t))$ for
$f(t) = \begin{cases} |\sin(2t)| & 0 \leq t \leq \pi, \\ 0 & \pi \leq t \leq 2\pi, \end{cases}$
and $f(t + r\pi) = f(t)$.

80. Find $\mathcal{L}(f(t))$ for
$f(t) = \begin{cases} 1 & 0 \leq t \leq \pi, \\ |\sin(t)| & \pi \leq t \leq 2\pi, \end{cases}$
and $f(t + 2\pi) = f(t)$.

81. Given $f(t) = \frac{1}{2}(|\sin t| + \sin t)$, called the **Half–wave rectification** of the sine wave, derive
$\mathcal{L}(f(t)) = \frac{1}{(s^2+1)(1-e^{-\pi s})}$

82. Solve for 2–periodic function $f(t)$:
$\mathcal{L}(f(t)) = \frac{1}{s}\tanh\left(\frac{s}{2}\right).$
Use the Extended Laplace Integral Table.

8.4　Heaviside's Partial Fraction Method

This clever algebraic shortcut is used to solve an equation like

$$\mathcal{L}(f(t)) = \frac{2s}{(s+1)(s^2+1)}$$

for the time domain function $f(t) = -e^{-t} + \cos t + \sin t$. The details in Heaviside's method involve a sequence of easy-to-learn college algebra steps. The practical method was popularized by English electrical engineer Oliver Heaviside (1850–1925).

More precisely, **Heaviside's method** starts with a polynomial quotient

(1)
$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m}$$

and computes an expression $f(t)$ such that

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \mathcal{L}(f(t)) \equiv \int_0^\infty f(t) e^{-st} dt.$$

Symbols $a_0, \ldots, a_n, b_0, \ldots, b_m$ are **real** constants. Heaviside's method assumes **limit zero at** $s = \infty$ for polynomial quotient (1). [5]

Partial Fraction Theory

It is a college algebra theorem that a rational function (1) can be expressed as the sum of **partial fractions**.

Definition 8.3 (Partial Fraction)
A partial fraction is a polynomial fraction with a constant in the numerator and a polynomial denominator having exactly one root, i.e.,

(2)
$$\textbf{partial fraction} = \frac{C}{(s-s_0)^k}.$$

The numerator C in (2) is a real or complex constant. The denominator has exactly one root $s = s_0$, real or complex. We expect power $(s-s_0)^k$ to be a **divisor of the denominator** in fraction (1).

Real Root Case. If s_0 in (2) is a real number, then C is *real*.

Complex Root Case. If $s_0 = a + ib$ in (2), then $(s - \overline{s_0})^k$ also divides the denominator in (1), where $\overline{s_0} = a - ib$ is the complex conjugate of s_0. The corresponding partial fractions used in the expansion turn out to be complex

---

[5]Otherwise, fraction (1) equals by long division a polynomial plus a remainder. Heaviside's method applies to the remainder.

conjugates of one another, which can be paired and re-written as a fraction with numerator $Q(s)$ a *real* polynomial.

$$(3) \qquad \frac{C}{(s - s_0)^k} + \frac{\overline{C}}{(s - \overline{s_0})^k} = \frac{Q(s)}{((s - a)^2 + b^2)^k}.$$

To illustrate, if $C = u + iv$, then

$$\frac{C}{(s - 2i)^2} + \frac{\overline{C}}{(s + 2i)^2} = \frac{(C + \overline{C})s^2 + 4i(\overline{C} - C)s - 4(C + \overline{C})}{(s^2 + 4)^2}$$

$$= \frac{2us^2 + 8vs - 8u}{(s^2 + 4)^2}.$$

The numerator $2us^2 + 8vs - 8u$ can be expanded by the college algebra division algorithm as $Q(s) = A_1(s^2 + 4) + A_2s + A_3$, with real coefficients A_1, A_2, A_3. Then the fraction can be written as

$$\frac{Q(s)}{(s^2 + 4)^2} = \frac{A_1}{s^2 + 4} + \frac{A_2s + A_3}{(s^2 + 4)^2}.$$

Similarly, numerator $2us^3 - 12vs^2 - 24us + 16v$ expands as $A_1(s^2 + 4)^2 + A_2(s^2 + 4) + A_3s + A_4$ in the following example:

$$\frac{u + iv}{(s - 2i)^3} + \frac{u - iv}{(s + 2i)^3} = \frac{2us^3 - 12vs^2 - 24us + 16v}{(s^2 + 4)^3}$$

$$= \frac{A_1}{s^2 + 4} + \frac{A_2}{(s^2 + 4)^2} + \frac{A_3s + A_4}{(s^2 + 4)^3}$$

for some *real coefficients* A_1, A_2, A_3, A_4.

This discussion generalizes to all powers $k > 1$. Partial fractions with denominator $(s - s_0)^k$ and $(s - \overline{s_0})^k$ with $s_0 = $ a complex number are paired and the division algorithm is employed as in the examples to replace the pair of terms by a sum of terms of the form

$$\frac{\text{linear polynomial in } s}{((s - a)^2 + b^2)^j}, \quad 1 \le j \le k.$$

The numerator has the form $c_1 + c_2s$ with *real* coefficients c_1, c_2. This **real partial fraction form** is preferred over the sum of complex fractions, because integral tables and Laplace tables typically contain only formulas with *real coefficients*. See Example 8.26, page 629.

Simple Roots

Assume that (1) has *real coefficients* and the denominator of the fraction (1) has **distinct real roots** s_1, ..., s_N and **distinct complex roots** $\alpha_1 \pm i\beta_1$, ...,

$\alpha_M \pm i\beta_M$. The partial fraction expansion of (1) is a sum given in terms of *real* constants A_p, B_q, C_q by

(4)
$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \sum_{p=1}^{N} \frac{A_p}{s - s_p} + \sum_{q=1}^{M} \frac{B_q + C_q(s - \alpha_q)}{(s - \alpha_q)^2 + \beta_q^2}.$$

Multiple Roots

Assume (1) has *real coefficients* and the denominator of the fraction (1) has possibly *multiple roots*. Let N_p be the multiplicity of real root s_p and let M_q be the multiplicity of complex root $\alpha_q + i\beta_q$ ($\beta_q > 0$), $1 \le p \le N$, $1 \le q \le M$. The partial fraction expansion of (1) is given in terms of *real* constants $A_{p,k}$, $B_{q,k}$, $C_{q,k}$ by

(5)
$$\sum_{p=1}^{N} \sum_{1 \le k \le N_p} \frac{A_{p,k}}{(s - s_p)^k} + \sum_{q=1}^{M} \sum_{1 \le k \le M_q} \frac{B_{q,k} + C_{q,k}(s - \alpha_q)}{((s - \alpha_q)^2 + \beta_q^2)^k}.$$

Summary

A polynomial quotient p/q with limit zero at infinity has a unique expansion into partial fractions. A partial fraction is either a constant divided by a divisor of q having exactly one real root, or else a linear function divided by a real divisor of q, having exactly one complex conjugate pair of roots.

Sampling Method

Consider the expansion in partial fractions

(6)
$$\frac{s - 1}{s(s + 1)^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{(s + 1)^2} + \frac{Ds + E}{s^2 + 1}.$$

The five undetermined real constants A through E are found by **clearing the fractions**, that is, multiply (6) by the denominator on the left to obtain the polynomial equation

(7)
$$\begin{aligned} s - 1 = \ & A(s + 1)^2(s^2 + 1) + Bs(s + 1)(s^2 + 1) \\ & + Cs(s^2 + 1) + (Ds + E)s(s + 1)^2. \end{aligned}$$

Next, five different **samples** of s are substituted into (7) to obtain equations for the five unknowns A through E.[6] Always use the **roots of the denominator**

---

[6] The values chosen for s are called **samples**, that is, they are cleverly chosen values. The number of s-values sampled equals the number of symbols A, B, ... to be determined, which in turn equals the degree of the denominator.

to start: $s = 0$, $s = -1$, $s = i$, $s = -i$ are the roots of $s(s+1)^2(s^2+1) = 0$. Each complex root results in two equations, by taking real and imaginary parts. The complex conjugate root $s = -i$ is not used, because it duplicates equations already obtained from $s = i$. The three roots $s = 0$, $s = -1$, $s = i$ give only four equations, so we **invent another sample** $s = 1$ to get the last equation:

$$
\begin{array}{rcll}
-1 &=& A & (s = 0) \\
-2 &=& -2C & (s = -1) \\
i - 1 &=& (Di + E)i(i+1)^2 & (s = i) \\
0 &=& 8A + 4B + 2C + 4(D + E) & (s = 1)
\end{array}
$$

(8)

Because D and E are real, the complex equation $(s = i)$ becomes two equations, as follows.

$i - 1 = (Di + E)i(i^2 + 2i + 1)$ Expand power $(i+1)^2$.

$i - 1 = -2Di - 2E$ Simplify using $i^2 = -1$.

$1 = -2D$ Equate imaginary parts.

$-1 = -2E$ Equate real parts.
             Root i created 2 equations!

The 5×5 system of linear algebraic equations is solved for answers $A = -1$, $B = 3/2$, $C = 1$, $D = -1/2$, $E = 1/2$.

Method of Atoms

Consider the expansion in partial fractions

(9)
$$
\frac{2s - 2}{s(s+1)^2(s^2+1)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{(s+1)^2} + \frac{ds + e}{s^2 + 1}.
$$

Clearing the fractions in (9) gives the polynomial equation

(10)
$$
\begin{aligned}
2s - 2 &= a(s+1)^2(s^2+1) + bs(s+1)(s^2+1) \\
&\quad + cs(s^2+1) + (ds+e)s(s+1)^2.
\end{aligned}
$$

The **method of atoms** expands all polynomial products and collects on powers of s. Functions 1, s, s^2, ... are by *definition* called **Euler solution atoms**, hence the terminology. The coefficients of the powers are matched to give 5 equations in the five unknowns a through e. Some details:

(11)
$$
\begin{aligned}
2s - 2 &= (a + b + d)\, s^4 + (2a + b + c + 2d + e)\, s^3 \\
&\quad + (2a + b + d + 2e)\, s^2 + (2a + b + c + e)\, s + a
\end{aligned}
$$

Matching powers of s implies the 5 equations

$$
a + b + d = 0, \; 2a + b + c + 2d + e = 0, \; 2a + b + d + 2e = 0,
$$
$$
2a + b + c + e = 2, \; a = -2.
$$

Solving, the unique solution is $a = -2$, $b = 3$, $c = 2$, $d = -1$, $e = 1$.

Heaviside's Coverup Method

Assume distinct roots in the denominator of fraction (1). Extensions to multiple-root cases can be made; see page 625.

To illustrate Oliver Heaviside's 1890 ideas, consider the problem details

$$
(12) \qquad \frac{2s+1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}
$$

$$
= \mathcal{L}(A) + \mathcal{L}(Be^t) + \mathcal{L}(Ce^{-t})
$$

$$
= \mathcal{L}(A + Be^t + Ce^{-t})
$$

The first line in (12) uses college algebra partial fractions. The second and third lines use the basic Laplace table and linearity of $\mathcal{L}$. Missing here are the values of constants A, B, C. Heaviside's ideas provide an efficient method to evaluate $A = -1, B = \frac{3}{2}, C = -\frac{1}{2}$. Then $\mathcal{L}(y) = \frac{2s+1}{s(s-1)(s+1)} = \mathcal{L}(-1 + \frac{3}{2}e^t - \frac{1}{2}e^{-t})$ implies $y = -1 + \frac{3}{2}e^t - \frac{1}{2}e^{-t}$.

Mysterious Details

Oliver Heaviside proposed to find $A = -1, B = \frac{3}{2}, C = -\frac{1}{2}$ in (12) by a **cover-up method**. The method is completely mental, no writing at all. We explain in detail how Heaviside found $C = -\frac{1}{2}$.

Heaviside starts with the identity

$$
(13) \qquad \frac{2s+1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}.
$$

The cover–up method finds C by **mentally** clearing the fraction $\frac{\mathbf{C}}{\mathbf{s+1}}$, that is, multiply (13) by the denominator $\boxed{\mathbf{s+1}}$ of the partial fraction $\frac{\mathbf{C}}{\mathbf{s+1}}$ to obtain the *partially-cleared fraction relation*

$$
\frac{(2s+1)\boxed{(s+1)}}{s(s-1)\boxed{(s+1)}} = \frac{A\boxed{(s+1)}}{s} + \frac{B\boxed{(s+1)}}{s-1} + \frac{C\boxed{(s+1)}}{\boxed{(s+1)}}.
$$

Set $\boxed{(s+1)} = 0$ in the display. Cancellations left and right plus annihilation of two terms on the right give the answer for C:

$$
\left.\frac{2s+1}{s(s-1)}\right|_{\boxed{s+1}=0} = 0 + 0 + C.
$$

Heaviside's cryptic *instructions* are to cover–up the matching factors $(s+1)$ on the left and right in (13) with box $\boxed{(s+1)}$ (Heaviside used his fingertips), then

evaluate on the left at the *root s* which causes the box contents to be zero. The other terms on the right are replaced by zero. Heaviside would find $C = -\frac{1}{2}$ by placing his fingers over the factors $(s + 1)$ left and right in (13), the boxes $\boxed{(s + 1)}$ below being his finger tips:

$$\left.\frac{2s + 1}{s(s - 1)\boxed{(s + 1)}}\right|_{\boxed{s+1}=0} = \frac{C}{\boxed{(s + 1)}}.$$

The factor $(s + 1)$ in (13) is by no means special: the same procedure applies to find A and B. The method works for denominators with simple roots, that is, no repeated roots are allowed. Heaviside's method in words:[7]

> To determine C in partial fraction $\frac{C}{s-s_0}$, multiply the relation by $(s - s_0)$, to partially clear the fraction. Substitute root s of equation $s - s_0 = 0$ into the partially cleared relation.

Extension to Multiple Roots

Heaviside's method can be extended to the case of repeated roots. The basic idea is to *factor–out the repeats.* To illustrate, consider the partial fraction expansion details

$$R = \frac{1}{(s + 1)^2(s + 2)}$$
A sample rational function having repeated roots.

$$= \frac{1}{s + 1}\left(\frac{1}{(s + 1)(s + 2)}\right)$$
Factor–out the repeats.

$$= \frac{1}{s + 1}\left(\frac{1}{s + 1} + \frac{-1}{s + 2}\right)$$
Apply the cover–up method to the simple root fraction.

$$= \frac{1}{(s + 1)^2} + \frac{-1}{(s + 1)(s + 2)}$$
Multiply. Observe that $\frac{1}{(s+1)^2}$ is a partial fraction!

$$= \frac{1}{(s + 1)^2} + \frac{-1}{s + 1} + \frac{1}{s + 2}$$
Apply the cover–up method to the last fraction on the right.

Term $\frac{1}{(s+1)^2}$ has constant numerator and denominator with only one root. It is already a partial fraction.[8] Therefore the work centers on expansion of quotients in which the denominator has two or more roots.

---

[7] Root $s = s_0$ is called a **pole** and the answer C is called a **residue**. See page 627.

[8] Re–read the definition of partial fraction page 620.

Special Methods

Heaviside's method has a useful extension for the case of roots of multiplicity two. To illustrate, consider these details:

$$R = \frac{1}{(s+1)^2(s+2)}$$ $\boxed{1}$ A fraction with multiple roots.

$$= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$ $\boxed{2}$ See equation (5), page 622.

$$= \frac{A}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}$$ $\boxed{3}$ Find B and C by Heaviside's cover–up method.

$$= \frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}$$ $\boxed{4}$ Details below.

Details $\boxed{4}$. Multiply the equation $\boxed{1} = \boxed{2}$ by $s+1$ to partially clear fractions, the same step as the cover-up method:

$$\frac{1}{(s+1)(s+2)} = A + \frac{B}{s+1} + \frac{C(s+1)}{s+2}.$$

Don't substitute s from $s+1 = 0$, because it gives infinity for the second term. Instead, set $s = \infty$ to get the equation $0 = A + C$. Because $C = 1$ from $\boxed{3}$, then $A = -1$.

The illustration works for one root of multiplicity two, because $s = \infty$ will resolve the coefficient not found by the cover–up method.

In general, if the denominator in (1) has a root s_0 of multiplicity k, then the partial fraction expansion contains terms

$$\frac{C_1}{s - s_0} + \frac{C_2}{(s - s_0)^2} + \cdots + \frac{C_k}{(s - s_0)^k}.$$

Heaviside's cover–up method directly finds C_k, but not C_1 to C_{k-1}.

Cover-up Method and Complex Numbers

Consider the partial fraction expansion

$$\frac{10}{(s+1)(s^2+9)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9}.$$

The symbols A, B, C are real. The value of A can be found directly by the cover-up method, giving $A = 1$. To find B and C, multiply the fraction expansion by $s^2 + 9$, in order to partially clear fractions, then formally set $s^2 + 9 = 0$ to obtain the two equations

$$\frac{10}{s+1} = Bs + C, \quad s^2 + 9 = 0.$$

The method applies the identical idea used for one real root. By clearing fractions in the first, the equations become

$$10 = Bs^2 + Cs + Bs + C, \quad s^2 + 9 = 0.$$

Substitute $s^2 = -9$ into the first equation to give the linear equation

$$10 = (-9B + C) + (B + C)s.$$

Because this linear equation has two complex roots $s = \pm 3i$, then real constants B, C satisfy the 2×2 system

$$
\begin{array}{rcrcr}
-9B & + & C & = & 10, \\
B & + & C & = & 0.
\end{array}
$$

Solving gives $B = -1$, $C = 1$.

The same method applies especially to fractions with 3-term denominators, like $s^2 + s + 1$. The only change made in the details is the replacement $s^2 \to -s - 1$. By repeated application of $s^2 = -s - 1$, the first equation can be distilled into one linear equation in s with two roots. As before, a 2×2 system results.

Residues, Poles and Oliver Heaviside

The language of **residues** and **poles** invaded engineering literature years ago, blamed in part on engineers who studied the foundations of complex variables. The terminology formalizes the naming of partial fraction theory constants and roots that appear in Oliver Heaviside's *cover-up method*, detailed above, which is an electrical engineering partial fraction shortcut that dates back to the year 1890.

Residues and poles do not provide any new mathematical tools for solving partial fraction problems. The service of residues and poles is to provide a new language for discussing the answers, a language that appears in current engineering and science literature. If you know how to compute coefficients in partial fractions using Heaviside's shortcut, then you already know how to find residues and poles.

A Key Example. Heaviside's shortcut finds the coefficients $c_1 = \frac{1}{2}, c_2 = -5, c_3 = \frac{5}{2}$ in the expansion

$$\frac{5 - 2(s+2)(s+3)}{(s+1)(s+2)(s+3)} = \frac{c_1}{s+1} + \frac{c_2}{s+2} + \frac{c_3}{s+3}$$

by clearing the fractions one at a time, each clearing followed by substitution of the corresponding root found in the denominator.

For instance, to clear the fraction for c_2 requires multiplication by $(s+2)$, to give the intermediate step (Heaviside did it mentally, writing nothing)

$$\frac{5 - 2(s+2)(s+3)}{(s+1)(s+3)} = \frac{c_1(s+2)}{s+1} + \frac{c_2}{1} + \frac{c_3(s+2)}{s+3}.$$

Root $s = -2$ of $s + 2 = 0$ is substituted above to give $c_2 = -5$.

Table 8. Working Definition of Pole and Residue

A **pole** is the same as a root of the denominator in a quotient $\dfrac{p(x)}{q(x)}$.

A **residue** is the same as a coefficient in the partial fraction expansion of the quotient $\dfrac{p(x)}{q(x)}$ (precise details below).

In the *key example*, the **residue** at **pole** $s = -2$ (the pole is the root of $s+2 = 0$) is **defined** by the equation

$$\lim_{s \to -2} (s + 2) \frac{5 - (2(s + 2)(s + 3)}{(s + 1)(s + 2)(s + 3)}.$$

To evaluate the limit, cancel the common factor $(s + 2)$ and substitute $s = -2$. Oliver Heaviside would be surprised by the unnecessary limit.

Definition 8.4 (Poles and Residues)
A function $f(z)$ of complex variable z has a **pole** at $z = z_0$ provided there is an integer $n \geq 0$ such that $g(z) = (z - z_0)^n f(z)$ can be written as a power series

$$g(z) = g_0 + g_1(z - z_0) + g_2(z - z_0)^2 + \cdots$$

convergent in a disk $|z - z_0| < R$ and $g_0 \neq 0$ (which means $g(z_0) \neq 0$).

The **order of the pole** is the integer n. The **residue** is g_0.

If $f(z)$ has a pole $z = z_0$ of order n, then the **residue** g_0 at the pole can be computed from the limit formula

$$g_0 = \lim_{z \to z_0} (z - z_0)^n f(z).$$

In terms of series expansion, a pole of order n means that

$$f(z) = \frac{g_0}{(z - z_0)^n} + \cdots + g_n + g_{n+1}(z - z_0) + g_{n+2}(z - z_0)^2 + \cdots,$$

which is called a **Laurent Series**.

Table 9. Pole, Residue and Applications

A real pole defines the **damping coefficient** in a transient.

A complex pole on the imaginary axis describes **frequency**.

Residues are **mode shape** information.

Examples and Methods

Example 8.26 (Partial Fractions I)

Show the details of the partial fraction expansion

$$\frac{s^3 + 2s^2 + 2s + 5}{(s-1)(s^2+4)(s^2+2s+2)} = \frac{2/5}{s-1} + \frac{1/2}{s^2+4} - \frac{1}{10}\frac{7+4s}{s^2+2s+2}.$$

Solution:
Background. The problem originates as equality $\boxed{5} = \boxed{6}$ in the sequence of Example 8.28, page 632, which solves for $x(t)$ using the method of partial fractions:

$$\boxed{5} \qquad \mathcal{L}(x) = \frac{s^3 + 2s^2 + 2s + 5}{(s-1)(s^2+4)(s^2+2s+2)}$$

$$\boxed{6} \qquad = \frac{2/5}{s-1} + \frac{1/2}{s^2+4} - \frac{1}{10}\frac{7+4s}{s^2+2s+2}$$

College algebra detail. College algebra partial fractions theory says that there exist real constants A, B, C, D, E satisfying the identity

$$\frac{s^3 + 2s^2 + 2s + 5}{(s-1)(s^2+4)(s^2+2s+2)} = \frac{A}{s-1} + \frac{B+Cs}{s^2+4} + \frac{D+Es}{s^2+2s+2}.$$

As explained on page 621, the complex conjugate roots $\pm 2i$ and $-1\pm i$ are not represented as terms $c/(s-s_0)$, but in the combined real form seen in the above display, which is suited for use with Laplace tables.

The **sampling method** applies to find the constants. In this method, the fractions are cleared to obtain the polynomial relation

$$\begin{aligned}
s^3 + 2s^2 + 2s + 5 \;=\;& A(s^2+4)(s^2+2s+2) \\
&+(B+Cs)(s-1)(s^2+2s+2) \\
&+(D+Es)(s-1)(s^2+4).
\end{aligned}$$

The roots of the denominator $(s-1)(s^2+4)(s^2+2s+2)$ to be inserted into the previous equation are $s=1$, $s=2i$, $s=-1+i$. The conjugate roots $s=-2i$ and $s=-1-i$ are not used. Each complex root generates two equations, by equating real and imaginary parts, therefore there will be 5 equations in 5 unknowns. Substitution of $s=1$, $s=2i$, $s=-1+i$ gives three equations

$$\begin{aligned}
s = 1 & & 10 &= 25A, \\
s = 2i & & -4i-3 &= (B+2iC)(2i-1)(-4+4i+2), \\
s = -1+i & & 5 &= (D-E+Ei)(-2+i)(2-2(-1+i)).
\end{aligned}$$

Writing each expanded complex equation in terms of its real and imaginary parts, explained in detail below, gives 5 equations

$$\begin{aligned}
s = 1 & & 2 &= 5A, \\
s = 2i & & -3 &= -6B & + & \;16C, \\
s = 2i & & -4 &= -8B & - & \;12C, \\
s = -1+i & & 5 &= -6D & - & \;2E, \\
s = -1+i & & 0 &= 8D & - & \;14E.
\end{aligned}$$

The equations are solved to give $A = 2/5$, $B = 1/2$, $C = 0$, $D = -7/10$, $E = -2/5$ (details for B, C below).

Complex equation to two real equations. It is an algebraic mystery how exactly the complex equation

$$-4i - 3 = (B + 2iC)(2i - 1)(-4 + 4i + 2)$$

gets converted into two real equations. The process is explained here.

First, the complex equation is expanded, as though it is a polynomial in variable i, to give the steps

$$
\begin{aligned}
-4i - 3 &= (B + 2iC)(2i - 1)(-2 + 4i) \\
&= (B + 2iC)(-4i + 2 + 8i^2 - 4i) && \text{Expand.} \\
&= (B + 2iC)(-6 - 8i) && \text{Use } i^2 = -1. \\
&= -6B - 12iC - 8Bi + 16C && \text{Expand, use } i^2 = -1. \\
&= (-6B + 16C) + (-8B - 12C)i && \text{Convert to form } x + yi.
\end{aligned}
$$

Next, the two sides are compared. Because B and C are real, then the real part of the right side is $(-6B + 16C)$ and the imaginary part of the right side is $(-8B - 12C)$. Equating matching parts on each side gives the equations

$$
\begin{aligned}
-6B + 16C &= -3, \\
-8B - 12C &= -4,
\end{aligned}
$$

which is a 2×2 linear system for the unknowns B, C.

Solving the 2×2 system. Such a system with a unique solution can be solved by Cramer's rule, matrix inversion or elimination. The answer: $B = 1/2$, $C = 0$.

The easiest method turns out to be elimination. Multiply the first equation by 4 and the second equation by 3, then subtract to obtain $C = 0$. Then the first equation is $-6B + 0 = -3$, implying $B = 1/2$.

Example 8.27 (Partial Fractions II)

Verify the partial fraction expansion

$$\boxed{1} \quad \frac{s^5 + 8\,s^4 + 23\,s^3 + 31\,s^2 + 24\,s + 9}{(s+1)^2\,(s^2 + s + 1)^2} = \frac{4}{s+1} + \frac{5 - 3s}{s^2 + s + 1}.$$

Solution:
Basic partial fraction theory implies that there are unique real constants a, b, c, d, e, f satisfying the equation

$$
\begin{aligned}
\frac{s^5 + 8\,s^4 + 23\,s^3 + 31\,s^2 + 24\,s + 9}{(s+1)^2\,(s^2 + s + 1)^2} &= \frac{a}{s+1} + \frac{b}{(s+1)^2} \\
&+ \frac{c + ds}{s^2 + s + 1} + \frac{e + f\,s}{(s^2 + s + 1)^2}
\end{aligned}
$$

Sanity checks apply when constructing the expansion. First, the number of real constants is always the degree of the denominator, which is 6 in this example. This caused the invention of 6 symbols a, b, c, d, e, f. Only *real* polynomials appear in the fraction

denominators on the right. The following checkpoints are done mentally: *we never in a partial fraction problem write out such details*:

$$
\begin{array}{ll}
s + 1 & \text{divides } (s+1)^2 \left(s^2 + s + 1\right)^2 \\
(s+1)^2 & \text{divides } (s+1)^2 \left(s^2 + s + 1\right)^2 \\
s^2 + s + 1 & \text{divides } (s+1)^2 \left(s^2 + s + 1\right)^2 \\
(s^2 + s + 1)^2 & \text{divides } (s+1)^2 \left(s^2 + s + 1\right)^2
\end{array}
$$

The **sampling** method applies to clear fractions and replace the fractional equation by the polynomial relation

$$
\begin{aligned}
s^5 + 8\,s^4 + 23\,s^3 + 31\,s^2 + 24\,s + 9 \;=\; & a(s+1)(s^2 + s + 1)^2 \\
& + b(s^2 + s + 1)^2 \\
& + (c + ds)(s^2 + s + 1)(s+1)^2 \\
& + (e + f\,s)(s+1)^2
\end{aligned}
$$

However, the prognosis for the resultant algebra is grim: only three of the six required equations can be obtained by substitution of the roots ($s = -1$, $s = -1/2 + i\sqrt{3}/2$) of the denominator. The sampling idea is abandoned, because of the complexity of the 6×6 system of linear equations required to solve for the six constants a through f.

Instead, the fraction R on the left of $\boxed{1}$ is written with repeated factors extracted, as follows:

$$
R = \frac{1}{(s+1)(s^2 + s + 1)} \left(\frac{p(s)}{(s+1)(s^2 + s + 1)} \right),
$$
$$
p(s) = s^5 + 8\,s^4 + 23\,s^3 + 31\,s^2 + 24\,s + 9.
$$

Long division gives the formulas

$$
\frac{p(s)}{(s+1)(s^2 + s + 1)} = s^2 + 6s + 9,
$$
$$
R = \frac{p(s)}{(s+1)^2(s^2 + s + 1)^2} = \frac{(s+3)^2}{(s+1)(s^2 + s + 1)}.
$$

The simplified form of R has a partial fraction expansion

$$
\frac{(s+3)^2}{(s+1)(s^2 + s + 1)} = \frac{a}{s+1} + \frac{b + cs}{s^2 + s + 1}
$$

where a, b, c are real constants. Reuse of earlier symbols a, b, c, d, e, f has occurred, similar to always using symbol x in a quadratic equation. Progress: the dimension of the algebra problem went from 6×6 to 3×3.

Heaviside's cover-up method gives $a = 4$. Applying Heaviside's method again to the quadratic factor implies the pair of equations

$$
\frac{(s+3)^2}{s+1} = b + cs, \quad s^2 + s + 1 = 0.
$$

These equations can be solved for $b = 5$, $c = -3$. The details assume that s is a root of $s^2 + s + 1 = 0$, then

$$
\frac{(s+3)^2}{s+1} = b + cs \qquad\qquad \text{The first equation.}
$$

$$\frac{s^2 + 6s + 9}{s + 1} = b + cs \qquad\qquad \text{Expand.}$$

$$\frac{-s - 1 + 6s + 9}{s + 1} = b + cs \qquad\qquad \text{Use } s^2 + s + 1 = 0.$$

$$5s + 8 = (s + 1)(b + cs) \qquad\qquad \text{Clear fractions.}$$

$$5s + 8 = bs + cs + b + cs^2 \qquad\qquad \text{Expand again.}$$

$$5s + 8 = bs + cs + b - cs - c \qquad\qquad \text{Use } s^2 + s + 1 = 0.$$

Conclusion $5 = b$ and $8 = b - c$ follows because the last equation is linear but has two complex roots. Solve: $b = 5$, $c = -3$.

Finally: $a = 4, b = 5, c = -3$, which verifies $\boxed{1}$.

Example 8.28 (Third Order Initial Value Problem)
Solve the third order initial value problem

$$x''' - x'' + 4x' - 4x = 5e^{-t}\sin t,$$
$$x(0) = 0, \quad x'(0) = x''(0) = 1.$$

Solution:
The answer is
$$x(t) = \frac{2}{5}e^t + \frac{1}{4}\sin 2t - \frac{3}{10}e^{-t}\sin t - \frac{2}{5}e^{-t}\cos t.$$

Method. Apply $\mathcal{L}$ to the differential equation. In steps $\boxed{1}$ to $\boxed{3}$ the Laplace integral of $x(t)$ is isolated, by applying linearity of $\mathcal{L}$, integration by parts $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ and the basic Laplace table.

$$\mathcal{L}(x''') - \mathcal{L}(x'') + 4\mathcal{L}(x') - 4\mathcal{L}(x) = 5\mathcal{L}(e^{-t}\sin t) \qquad\qquad \boxed{1}$$

$$(s^3\mathcal{L}(x) - s - 1) - (s^2\mathcal{L}(x) - 1) + 4(s\mathcal{L}(x)) - 4\mathcal{L}(x) = \frac{5}{(s+1)^2 + 1} \qquad \boxed{2}$$

$$(s^3 - s^2 + 4s - 4)\mathcal{L}(x) = 5\frac{1}{(s+1)^2 + 1} + s \qquad\qquad \boxed{3}$$

Steps $\boxed{5}$ and $\boxed{6}$ use the college algebra theory of partial fractions, the details of which appear in Example 8.26, page 629. Steps $\boxed{7}$ and $\boxed{8}$ write the partial fraction expansion in terms of Laplace table entries. Step $\boxed{9}$ converts the s-expressions, which are basic Laplace table entries, into Laplace integral expressions. Algebraically, we replace s-expressions by expressions in symbols $\mathcal{L}$ and t.

$$\mathcal{L}(x) = \frac{\frac{5}{(s+1)^2+1} + s}{s^3 - s^2 + 4s - 4} \qquad\qquad \boxed{4}$$

$$= \frac{s^3 + 2s^2 + 2s + 5}{(s - 1)(s^2 + 4)(s^2 + 2s + 2)} \qquad\qquad \boxed{5}$$

$$= \frac{2/5}{s - 1} + \frac{1/2}{s^2 + 4} - 1/10\,\frac{7 + 4\,s}{s^2 + 2\,s + 2} \qquad\qquad \boxed{6}$$

$$= \frac{2/5}{s - 1} + \frac{1/2}{s^2 + 4} - 1/10\,\frac{3 + 4(s + 1)}{(s + 1)^2 + 1} \qquad\qquad \boxed{7}$$

$$= \frac{2/5}{s - 1} + \frac{1/2}{s^2 + 4} - \frac{3/10}{(s + 1)^2 + 1} - \frac{(2/5)(s + 1)}{(s + 1)^2 + 1} \quad \boxed{8}$$

$$= \mathcal{L}\left(\frac{2}{5}e^t + \frac{1}{4}\sin 2t - \frac{3}{10}e^{-t}\sin t - \frac{2}{5}e^{-t}\cos t\right) \quad \boxed{9}$$

The last step $\boxed{10}$ applies Lerch's cancellation theorem to $\mathcal{L}(x(t)) = \boxed{9}$.

$$x(t) = \frac{2}{5}e^t + \frac{1}{4}\sin 2t - \frac{3}{10}e^{-t}\sin t - \frac{2}{5}e^{-t}\cos t \qquad \boxed{10}$$

Example 8.29 (Second Order System)

Solve for $x(t)$ and $y(t)$ in the 2nd order system of linear differential equations

$$\begin{aligned}2x'' - x' + 9x - y'' - y' - 3y = 0, & \quad x(0) = x'(0) = 1, \\ 2'' + x' + 7x - y'' + y' - 5y = 0, & \quad y(0) = y'(0) = 0.\end{aligned}$$

Solution: The answer is

$$x(t) = \frac{1}{3}e^t + \frac{2}{3}\cos(2\,t) + \frac{1}{3}\sin(2\,t),$$

$$y(t) = \frac{2}{3}e^t - \frac{2}{3}\cos(2\,t) - \frac{1}{3}\sin(2\,t).$$

Transform. The intent of steps $\boxed{1}$ and $\boxed{2}$ is to transform the initial value problem into two equations in two unknowns. Used repeatedly in $\boxed{1}$ is integration by parts $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$. No Laplace tables were used. In $\boxed{2}$ the substitutions $x_1 = \mathcal{L}(x)$, $x_2 = \mathcal{L}(y)$ are made to produce two equations in the two unknowns x_1, x_2.

$$\begin{aligned}(2s^2 - s + 9)\mathcal{L}(x) + (-s^2 - s - 3)\mathcal{L}(y) = 1 + 2s, \\ (2s^2 + s + 7)\mathcal{L}(x) + (-s^2 + s - 5)\mathcal{L}(y) = 3 + 2s,\end{aligned} \qquad \boxed{1}$$

$$\begin{aligned}(2s^2 - s + 9)x_1 &+& (-s^2 - s - 3)x_2 &=& 1 + 2s, \\ (2s^2 + s + 7)x_1 &+& (-s^2 + s - 5)x_2 &=& 3 + 2s.\end{aligned} \qquad \boxed{2}$$

Step $\boxed{3}$ uses Cramer's rule. Equations $\boxed{2}$ are of the form $ax_1 + bx_2 = e$, $cx_1 + dx_2 = f$. Cramer's rule expresses answers x_1, x_2 by determinant fractions

$$x_1 = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

The variable names x_1, x_2 stand for the Laplace integrals of the unknowns $x(t)$, $y(t)$, respectively. The answers, following a tedious calculation:

$$\begin{cases} x_1 = \dfrac{s^2 + 2/3}{s^3 - s^2 + 4\,s - 4}, \\[2mm] x_2 = \dfrac{10/3}{s^3 - s^2 + 4\,s - 4}. \end{cases} \qquad \boxed{3}$$

Step $\boxed{4}$ writes each fraction resulting from Cramer's rule as a partial fraction expansion suited for backward Laplace table look-up (details after $\boxed{6}$). Step $\boxed{5}$ does the table look-up and prepares for step $\boxed{6}$ to apply Lerch's cancellation law, in order to display the answers $x(t)$, $y(t)$.

$$\begin{cases} x_1 = \dfrac{1/3}{s-1} + \dfrac{2}{3}\dfrac{s}{s^2+4} + \dfrac{1}{3}\dfrac{2}{s^2+4}, \\[2mm] x_2 = \dfrac{2/3}{s-1} - \dfrac{2}{3}\dfrac{s}{s^2+4} - \dfrac{1}{3}\dfrac{2}{s^2+4}. \end{cases}$$

<div style="text-align:right">□ 4</div>

$$\begin{cases} \mathcal{L}(x(t)) = \mathcal{L}\left(\dfrac{1}{3}e^t + \dfrac{2}{3}\cos(2t) + \dfrac{1}{3}\sin(2t)\right), \\[2mm] \mathcal{L}(y(t)) = \mathcal{L}\left(\dfrac{2}{3}e^t - \dfrac{2}{3}\cos(2t) - \dfrac{1}{3}\sin(2t)\right). \end{cases}$$

<div style="text-align:right">□ 5</div>

$$\begin{cases} x(t) = \dfrac{1}{3}e^t + \dfrac{2}{3}\cos(2t) + \dfrac{1}{3}\sin(2t), \\[2mm] y(t) = \dfrac{2}{3}e^t - \dfrac{2}{3}\cos(2t) - \dfrac{1}{3}\sin(2t). \end{cases}$$

<div style="text-align:right">□ 6</div>

Partial fraction details. Shown below is how to obtain the expansion

$$\frac{s^2 + 2/3}{s^3 - s^2 + 4s - 4} = \frac{1/3}{s-1} + \frac{2}{3}\frac{s}{s^2+4} + \frac{1}{3}\frac{2}{s^2+4}.$$

The denominator $s^3 - s^2 + 4s - 4$ factors into $s - 1$ times $s^2 + 4$. Partial fraction theory implies that there is an expansion with *real coefficients* A, B, C of the form

$$\frac{s^2 + 2/3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs + C}{s^2+4}.$$

Let's verify $A = 1/3$, $B = 2/3$, $C = 2/3$. Clear the fractions to obtain the polynomial equation

$$s^2 + 2/3 = A(s^2 + 4) + (Bs + C)(s - 1).$$

Instead of using $s = 1$ and $s = 2i$, which are roots of the denominator, invent samples $s = 1$, $s = 0$, $s = -1$ to get a *real* 3×3 system for A, B, C:

$$\begin{array}{lll} s = 1: & 1 + 2/3 & = & A(1+4) + 0, \\ s = 0: & 0 + 2/3 & = & A(4) + C(-1), \\ s = -1: & 1 + 2/3 & = & A(1+4) + (-B+C)(-2). \end{array}$$

Write this system as an augmented matrix G with variables A, B, C assigned to the first three columns of G:

$$G = \left(\begin{array}{ccc|c} 5 & 0 & 0 & 5/3 \\ 4 & 0 & -1 & 2/3 \\ 5 & 2 & -2 & 5/3 \end{array}\right)$$

Using computer assist, calculate

$$\mathbf{rref}(G) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 2/3 \end{array}\right)$$

Then A, B, C are the last column entries of $\mathbf{rref}(G)$, which verifies the partial fraction expansion.

Heaviside cover-up detail. It is possible to rapidly check that $A = 1/3$ using the cover-up method. Less obvious is that the cover-up method also applies to the fraction with complex roots.

The idea is to multiply the fraction decomposition by s^2+4 to partially clear the fractions and then set $s^2+4=0$. This process formally sets s equal to one of the two roots $s=\pm 2i$. Complex numbers are avoided entirely by solving for B, C in the pair of equations

$$\frac{s^2+2/3}{s-1}=A(0)+(Bs+C),\quad s^2+4=0.$$

Because $s^2=-4$, the first equality is simplified to $\dfrac{-4+2/3}{s-1}=Bs+C$. Swap sides of the equation, then cross-multiply to obtain $Bs^2+Cs-Bs-C=-10/3$ and then use $s^2=-4$ again to simplify to $(-B+C)s+(-4B-C)=-10/3$. Because this linear equation in variable s has two solutions, then $-B+C=0$ and $-4B-C=-10/3$. Solve this 2×2 system by elimination to obtain $B=C=2/3$.

The algebraic method: First, find two equations in symbols s, B, C. Next, symbol s is eliminated to give two equations in symbols B, C. Finally, the 2×2 system for B, C is solved.

Exercises 8.4 ↗

Partial Fraction Mistakes

1. How many real constants appear in the partial fraction expansion of the fraction $\dfrac{s+1}{s^2(s+2)(s+3)^2}$?

2. How many real constants appear in the partial fraction expansion of $\dfrac{s+1}{s^2(s^2+4)(s^2+2s+5)^2}$?

3. Guido expanded $\dfrac{s+1}{s(s+2)(s+3)^2}$
to get $\dfrac{a}{s}+\dfrac{b}{s+2}+\dfrac{c}{(s+3)^2}$.
What is the mistake?

4. Helena made this expansion:
$$\frac{s+1}{s(s+2)}=\frac{a}{s}+\frac{b}{s+2}+\frac{c}{s+3}$$
The expansion is correct! Explain how you know that $c=0$ without computing anything.
This example explains why fractions on the right have denominators dividing the denominator on the left.

5. Marco made an expansion:
$$\frac{s+1}{s(s^2+4)}=\frac{a}{s}+\frac{b}{s+2}+\frac{c}{s-2}$$
Explain why it is a mistake.
This example explains why sanity checks have more than one item to check.

6. Violeta made an expansion
$$\frac{s+2}{s(s-2)(s+2)}=\frac{a}{s}+\frac{b}{s-2}+\frac{c}{s+2}$$
Explain why $c=0$ without computing anything.
This example explains why common factors of numerator and denominator should be removed.

7. Find the mistake in expansion
$$\frac{(s+2)^2}{s(s-2)}=\frac{a}{s}+\frac{b}{s-2}$$
This example explains why the degree of the numerator and denominator are checkpoints.

8. Is there a mistake here?
$$\frac{(s+2)^2}{s^2(s-2)}=\frac{a}{s}+\frac{b}{s^2}+\frac{c}{s-2}$$

Sampling Method
Apply the sampling method (a *failsafe method*) to verify the given equation.

9. $\dfrac{s}{s^2-1}=\dfrac{1/2}{s-1}+\dfrac{1/2}{s+1}$

10. $\dfrac{s}{s^4-1}=\dfrac{1/4}{s-1}+\dfrac{1/4}{s+1}+\dfrac{-s/2}{s^2+1}$

Method of Atoms
Apply the method of atoms to verify the given equation.

11. $\dfrac{2s}{s^2-1}=\dfrac{1}{s-1}+\dfrac{1}{s+1}$

12. $\dfrac{4s}{s^4-1} = \dfrac{1}{s-1} + \dfrac{1}{s+1} + \dfrac{-2s}{s^2+1}$

Heaviside's 1890 Shortcut

Apply Heaviside's shortcut to verify the given equation.

13. $\dfrac{2s}{s^2-4} = \dfrac{1}{s-2} + \dfrac{1}{s+2}$

14. $\dfrac{s+4}{s^3+4s} = \dfrac{1}{s} + \dfrac{-s+1}{s^2+4}$

Residues and Poles

Compute the residue for the given pole.

15. Residue at $s=2$ for $\dfrac{2s}{s^2-4}$.

16. Residue at $s=0$ for $\dfrac{s+4}{s^3+16s}$.

Scalar Differential Equations

The **transfer function** of $x'' + x = f(t)$ is $H(s) = \frac{1}{s^2+1}$. A common definition is $H(s) = \mathcal{L}(f(t))$ divided by $\mathcal{L}(x(t))$, assuming $x(0) = x'(0) = 0$.

17. Verify for $x'' + x = e^{-t}$ with $x(0) = 0$, $x'(0) = 0$ that $\mathcal{L}(x) = \frac{1}{s+1}\frac{1}{s^2+1}$. Then compute $H(s)$.

18. Explain the transfer function equation
$$H(s) = \frac{1}{\text{characteristic equation}}.$$

19. Solve $\mathcal{L}(x(t)) = \frac{1}{s+1}\frac{1}{s^2+1}$ by Heaviside cover–up for output $x(t) = \frac{1}{2}(e^{-t} - \cos t + \sin t)$.

20. Given $x'' + x = te^{-t}$, $x(0) = x'(0) = 0$, show all steps to find $\mathcal{L}(x(t)) = \frac{1}{(s+1)^2}\frac{1}{s^2+1}$.

First Order System

Using Example 8.29 as a guide, solve the system for $x_1(t)$ by Laplace's method.

21.
$$\begin{cases} x_1' = x_2, \\ x_2' = 4x_1 + 12e^{-t}, \\ x_1(0) = x_2(0) = 0. \end{cases}$$
Ans: $x_1(t) = e^{2t} + 3e^{-2t} - 4e^{-t}$.

22.
$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' = 4x_1 - 4x_2 + x_3 + 10e^{-t}, \\ x_1(0) = x_2(0) = x_3(0) = 0. \end{cases}$$
Ans: $x_1(t) = e^t - e^{-t} - \sin(2t)$.

Second Order System

Using Example 8.29 as a guide, compute $x(t), y(t)$.

23. $\mathcal{L}(x(t)) = \frac{3s^2+2}{(s-1)(s^2+4)}$,
$\mathcal{L}(y(t)) = \frac{10}{(s-1)(s^2+4)}$.
Ans: $x = 2\cos(2t) + \sin(2t) + e^t$,
$y = -2\cos(2t) - \sin(2t) + 2e^t$

24. $\mathcal{L}(x(t)) = \frac{2s^2+4}{(s+1)(s^2+1)}$,
$\mathcal{L}(y(t)) = \frac{2}{(s+1)(s^2+1)}$.
Ans: $x = -\cos(t) + \sin(t) + 3e^{-t}$,
$y = -\cos(t) + \sin(t) + e^{-t}$.

8.5 Transform Properties

Collected here are the major theorems for the manipulation of Laplace transform tables along with their derivations. Those who study in isolation are advised to dwell on the details of proof and re-read the examples of preceding sections. No exercises are appropriate and none are supplied.

Theorem 8.5 (Linearity)
The Laplace transform has these inherited integral properties:

$$\begin{aligned}
\textbf{(a)} \quad & \mathcal{L}(f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t)), \\
\textbf{(b)} \quad & \mathcal{L}(cf(t)) = c\mathcal{L}(f(t)).
\end{aligned}$$

Theorem 8.6 (The t-Derivative Rule or Parts Rule)
Let $y(t)$ be continuous, of exponential order and let $y'(t)$ be piecewise continuous on $t \geq 0$. Then $\mathcal{L}(y'(t))$ exists and

$$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0+).$$

Theorem 8.7 (The t-Integral Rule)
Let $g(t)$ be of exponential order and continuous for $t \geq 0$. Then

$$\mathcal{L}\left(\int_0^t g(x)\, dx\right) = \frac{1}{s}\mathcal{L}(g(t))$$

or equivalently

$$\mathcal{L}(g(t)) = s\mathcal{L}\left(\int_0^t g(x)\, dx\right)$$

Theorem 8.8 (The s-Differentiation Rule)
Let $f(t)$ be of exponential order. Then

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t)).$$

Theorem 8.9 (First Shifting Rule)
Let $f(t)$ be of exponential order and $-\infty < a < \infty$. Then

$$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))|_{s\to(s-a)}.$$

Theorem 8.10 (Second Shifting Rule)
Let $f(t)$ and $g(t)$ be of exponential order and assume $a \geq 0$. Then

$$\begin{aligned}
\textbf{(a)} \quad & \mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f(t)), \\
\textbf{(b)} \quad & \mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a)).
\end{aligned}$$

Theorem 8.11 (Periodic Function Rule)
Let $f(t)$ be of exponential order and satisfy $f(t + P) = f(t)$. Then

$$\mathcal{L}(f(t)) = \frac{\int_0^P f(t)e^{-st}dt}{1 - e^{-Ps}}.$$

Theorem 8.12 (Convolution Rule)
Let $f(t)$ and $g(t)$ be of exponential order. Then

$$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(x)g(t-x)dx\right).$$

Theorem 8.13 (Laplace at Infinity is Zero)
Let $f(t)$ be of piecewise continuous and of exponential order. Then

$$\lim_{s\to\infty} \mathcal{L}(f(t)) = 0.$$

Theorem 8.14 (Initial and Final Value Rules)
Let $f(t)$ and $f'(t)$ be functions of exponential order. Then, when all indicated limits exist,

$$\begin{aligned}
\mathbf{1}. \quad f(0+) &= \lim_{t\to 0+} f(t) = \lim_{s\to\infty} s\mathcal{L}(f(t)), \\
\mathbf{2}. \quad f(\infty) &= \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\mathcal{L}(f(t)).
\end{aligned}$$

Initial and Final Value Pitfalls

In Theorem 8.14, impulses and higher order singularities at $t = 0$ are disallowed, because hypotheses require $s\mathcal{L}(f(t))$ to be bounded.

Examples $f(t) = \sin t$ and $f(t) = e^t$ don't satisfy hypotheses for **2** because $f(\infty)$ is undefined, but **1** applies for both examples.

A **pole**, defined precisely on page 628, is a root of the denominator in a fraction $F(s) = \mathcal{L}(f(t))$. The location of the poles influences the possibility of using Theorem 8.14:

> If there are poles in the right s-plane, then $f(t)$ will contain exponentially growing terms, which implies $f(\infty)$ does not exist.
>
> If there are pairs of complex conjugate poles on the imaginary axis, then $f(t)$ will contain sinusoids and $f(\infty)$ is not defined.
>
> Poles in the left s-plane contribute exponentially decaying terms to $f(t)$ which do not affect the final value.
>
> Signal $f(t)$ has possibly a constant final value, the steady state of the signal, only when there are poles at the origin of the s-plane.

Proofs and Details

Proof of Theorem 8.5 (Linearity):

$$
\begin{aligned}
\text{LHS} &= \mathcal{L}(f(t) + g(t)) && \text{Left side of the identity in (a).}\\
&= \int_0^\infty (f(t) + g(t))e^{-st}dt && \text{Direct transform.}\\
&= \int_0^\infty f(t)e^{-st}dt + \int_0^\infty g(t)e^{-st}dt && \text{Calculus integral rule.}\\
&= \mathcal{L}(f(t)) + \mathcal{L}(g(t)) && \text{Equals RHS; identity (a) verified.}\\
\text{LHS} &= \mathcal{L}(cf(t)) && \text{Left side of the identity in (b).}\\
&= \int_0^\infty cf(t)e^{-st}dt && \text{Direct transform.}\\
&= c\int_0^\infty f(t)e^{-st}dt && \text{Calculus integral rule.}\\
&= c\mathcal{L}(f(t)) && \text{Equals RHS; identity (b) verified.}
\end{aligned}
$$

Proof of Theorem 8.6 (t-Derivative or parts rule): Already $\mathcal{L}(f(t))$ exists, because f is of exponential order and continuous. On an interval $[a, b]$ where f' is continuous, integration by parts using $u = e^{-st}$, $dv = f'(t)dt$ gives

$$
\begin{aligned}
\int_a^b f'(t)e^{-st}dt &= f(t)e^{-st}\big|_{t=a}^{t=b} - \int_a^b f(t)(-s)e^{-st}dt\\
&= -f(a)e^{-sa} + f(b)e^{-sb} + s\int_a^b f(t)e^{-st}dt.
\end{aligned}
$$

On any interval $[0, N]$, there are finitely many intervals $[a, b]$ on each of which f' is continuous. Add the above equality across these finitely many intervals $[a, b]$. The boundary values on adjacent intervals match and the integrals add to give

$$
\int_0^N f'(t)e^{-st}dt = -f(0+)e^0 + f(N)e^{-sN} + s\int_0^N f(t)e^{-st}dt.
$$

Take the limit across this equality as $N \to \infty$. Then the right side has limit $-f(0) + s\mathcal{L}(f(t))$, because of the existence of $\mathcal{L}(f(t))$ and $\lim_{t\to\infty} f(t)e^{-st} = 0$ for large s. Therefore, the left side has a limit, and by definition $\mathcal{L}(f'(t))$ exists and $\mathcal{L}(f'(t)) = -f(0) + s\mathcal{L}(f(t))$.

Proof of Theorem 8.7 (t-Integral rule): Let $f(t) = \int_0^t g(x)dx$. Then f is of exponential order and continuous. The details:

$$
\begin{aligned}
\mathcal{L}(\textstyle\int_0^t g(x)dx) &= \mathcal{L}(f(t)) && \text{By definition.}\\
&= \frac{1}{s}\mathcal{L}(f'(t)) && \text{Because } f(0) = 0 \text{ implies } \mathcal{L}(f'(t)) = s\mathcal{L}(f(t)).\\
&= \frac{1}{s}\mathcal{L}(g(t)) && \text{Because } f' = g \text{ by the Fundamental theorem of calculus.}
\end{aligned}
$$

Proof of Theorem 8.8 (s-Differentiation): Let's prove $\mathcal{L}((-t)f(t)) = (d/ds)\mathcal{L}(f(t))$, an equivalent relation. If f is of exponential order, then so is $(-t)f(t)$, therefore $\mathcal{L}((-t)f(t))$ exists. It remains to show the s-derivative exists and satisfies the given equality.

The proof below is based in part upon the calculus inequality

$$
(1) \qquad\qquad \left|e^{-x} + x - 1\right| \le x^2, \quad x \ge 0.
$$

The inequality is obtained from two applications of the *mean value theorem* $g(b) - g(a) = g'(\overline{x})(b - a)$, which gives $e^{-x} + x - 1 = x\overline{x}e^{-x_1}$ with $0 \leq x_1 \leq \overline{x} \leq x$.

In addition, the existence of $\mathcal{L}(t^2|f(t)|)$ is used to define $s_0 > 0$ such that $\mathcal{L}(t^2|f(t)|) \leq 1$ for $s > s_0$. This follows from the transform existence theorem for functions of exponential order, where it is shown that the transform has limit zero at $s = \infty$. See also the proof of Theorem 8.13.

Consider $h \neq 0$ and the Newton quotient $Q(s, h) = (F(s + h) - F(s))/h$ for the s-derivative of the Laplace integral. We have to show that

$$\lim_{h \to 0} |Q(s, h) - \mathcal{L}((-t)f(t))| = 0.$$

This will be accomplished by proving for $s > s_0$ and $s + h > s_0$ the inequality

$$|Q(s, h) - \mathcal{L}((-t)f(t))| \leq |h|.$$

For $h \neq 0$,

$$Q(s, h) - \mathcal{L}((-t)f(t)) = \int_0^\infty f(t) \frac{e^{-st-ht} - e^{-st} + the^{-st}}{h} \, dt.$$

Assume $h > 0$. Due to the exponential rule $e^{A+B} = e^A e^B$, the quotient in the integrand simplifies to give

$$Q(s, h) - \mathcal{L}((-t)f(t)) = \int_0^\infty f(t)e^{-st} \left(\frac{e^{-ht} + th - 1}{h} \right) dt.$$

Inequality (1) applies with $x = ht \geq 0$, giving

$$|Q(s, h) - \mathcal{L}((-t)f(t))| \leq |h| \int_0^\infty t^2|f(t)|e^{-st} dt.$$

The right side is $|h|\mathcal{L}(t^2|f(t)|)$, which for $s > s_0$ is bounded by $|h|$, completing the proof for $h > 0$. If $h < 0$, then a similar calculation is made to obtain

$$|Q(s, h) - \mathcal{L}((-t)f(t))| \leq |h| \int_0^\infty t^2|f(t)|e^{-st-ht} dt.$$

The right side is $|h|\mathcal{L}(t^2|f(t)|)$ evaluated at $s + h$ instead of s. If $s + h > s_0$, then the right side is bounded by $|h|$, completing the proof for $h < 0$.

Proof of Theorem 8.9 (First Shifting Rule): The left side LHS of the equality can be written because of the exponential rule $e^A e^B = e^{A+B}$ as

$$\text{LHS} = \int_0^\infty f(t)e^{-(s-a)t} dt.$$

This integral is $\mathcal{L}(f(t))$ with s replaced by $s - a$, which is precisely the meaning of the right side RHS of the equality. Therefore, LHS = RHS.

Proof of Theorem 8.10 (Second Shifting Rule): The details for (a) are

$$\begin{aligned}
\text{LHS} &= \mathcal{L}(H(t - a)f(t - a)) \\
&= \int_0^\infty H(t - a)f(t - a)e^{-st} dt \qquad \text{Direct transform.}
\end{aligned}$$

$$= \int_a^\infty H(t-a)f(t-a)e^{-st}dt \qquad \text{Because } a \geq 0 \text{ and } H(x) = 0 \text{ for } x < 0.$$

$$= \int_0^\infty H(x)f(x)e^{-s(x+a)}dx \qquad \text{Change variables } x = t - a, \ dx = dt.$$

$$= e^{-sa}\int_0^\infty f(x)e^{-sx}dx \qquad \text{Use } H(x) = 1 \text{ for } x \geq 0.$$

$$= e^{-sa}\mathcal{L}(f(t)) \qquad \text{Direct transform.}$$

$$= \text{RHS} \qquad \text{Identity (a) verified.}$$

In the details for (b), let $f(t) = g(t+a)$, then

$$\text{LHS} = \mathcal{L}(H(t-a)g(t))$$

$$= \mathcal{L}(H(t-a)f(t-a)) \qquad \text{Use } f(t-a) = g(t-a+a) = g(t).$$

$$= e^{-sa}\mathcal{L}(f(t)) \qquad \text{Apply (a).}$$

$$= e^{-sa}\mathcal{L}(g(t+a)) \qquad \text{Because } f(t) = g(t+a).$$

$$= \text{RHS} \qquad \text{Identity (b) verified.}$$

Proof of Theorem 8.11 (Periodic Function Rule):

$$\text{LHS} = \mathcal{L}(f(t))$$

$$= \int_0^\infty f(t)e^{-st}dt \qquad \text{Direct transform.}$$

$$= \sum_{n=0}^\infty \int_{nP}^{nP+P} f(t)e^{-st}dt \qquad \text{Additivity of the integral.}$$

$$= \sum_{n=0}^\infty \int_0^P f(x+nP)e^{-sx-nPs}dx \qquad \text{Change variables } t = x + nP.$$

$$= \sum_{n=0}^\infty e^{-nPs}\int_0^P f(x)e^{-sx}dx \qquad \text{Because } f(x) \text{ is } P\text{--periodic and } e^A e^B = e^{A+B}.$$

$$= \int_0^P f(x)e^{-sx}dx \sum_{n=0}^\infty r^n \qquad \text{The summation has a common factor. Define } r = e^{-Ps}.$$

$$= \int_0^P f(x)e^{-sx}dx \frac{1}{1-r} \qquad \text{Sum the geometric series.}$$

$$= \frac{\int_0^P f(x)e^{-sx}dx}{1-e^{-Ps}} \qquad \text{Substitute } r = e^{-Ps}.$$

$$= \text{RHS} \qquad \text{Periodic function identity verified.}$$

Left unmentioned here is the convergence of the infinite series on line 3 of the proof, which follows from f of exponential order.

Proof of Theorem 8.12 (Convolution rule): The details use Fubini's integration interchange theorem for a planar unbounded region, and therefore this proof involves advanced calculus methods that may be outside the background of the reader. Modern calculus texts contain a less general version of Fubini's theorem for finite regions, usually referenced as *iterated integrals*. The unbounded planar region is written in two ways:

$$\mathbf{D} = \{(r,t) : t \leq r < \infty, \ 0 \leq t < \infty\},$$

$$\mathcal{D} = \{(r,t) : 0 \leq r < \infty, \ 0 \leq r \leq t\}.$$

Readers should pause here and verify that $\mathbf{D} = \mathcal{D}$.

The change of variable $r = x+t$, $dr = dx$ is applied for fixed $t \geq 0$ to obtain the identity

$$(2) \qquad \begin{aligned} e^{-st}\int_0^\infty g(x)e^{-sx}dx &= \int_0^\infty g(x)e^{-sx-st}dx \\ &= \int_t^\infty g(r-t)e^{-rs}dr. \end{aligned}$$

The left side of the convolution identity is expanded as follows:

$$\mathsf{LHS} = \mathcal{L}(f(t))\mathcal{L}(g(t))$$

$$= \int_0^\infty f(t)e^{-st}dt \int_0^\infty g(x)e^{-sx}dx \qquad \text{Direct transform.}$$

$$= \int_0^\infty f(t)\int_t^\infty g(r-t)e^{-rs}drdt \qquad \text{Apply identity (2).}$$

$$= \int_{\mathbf{D}} f(t)g(r-t)e^{-rs}drdt \qquad \text{Fubini's theorem applied.}$$

$$= \int_{\mathcal{D}} f(t)g(r-t)e^{-rs}drdt \qquad \text{Descriptions } \mathbf{D} \text{ and } \mathcal{D} \text{ are the same.}$$

$$= \int_0^\infty \int_0^r f(t)g(r-t)dte^{-rs}dr \qquad \text{Fubini's theorem applied.}$$

Then

$$\mathsf{RHS} = \mathcal{L}\left(\int_0^t f(u)g(t-u)du\right)$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)due^{-st}dt \qquad \text{Direct transform.}$$

$$= \int_0^\infty \int_0^r f(u)g(r-u)due^{-sr}dr \qquad \text{Change variable names } r \leftrightarrow t.$$

$$= \int_0^\infty \int_0^r f(t)g(r-t)dt\,e^{-sr}dr \qquad \text{Change variable names } u \leftrightarrow t.$$

$$= \mathsf{LHS} \qquad \text{Convolution identity verified.}$$

Proof of Theorem 8.13 (Laplace at Infinity is Zero): Assumed is an inequality $|f(t)| \leq Me^{kt}$ for some constants $M \geq 0$ and k. Then

$$\left|\int_0^N f(t)e^{-st}dt\right| \leq \int_0^N |f(t)|e^{-st}dt \leq M\int_0^N e^{-(s-k)t}dt.$$

The integral on the right is estimated for $s > k$ and all $N \geq 0$ as follows:

$$\int_0^N e^{-(s-k)t}dt = \frac{1 - e^{-(s-k)N}}{s-k} \leq \frac{1}{s-k}.$$

Limiting as $N \to \infty$ across the chained inequalities implies the fundamental estimate $|\mathcal{L}(f(t))| \leq \frac{M}{s-k}$, $s > k$, therefore $\lim_{s\to\infty}\mathcal{L}(f(t)) = 0$.

Proof of Theorem 8.14 (Initial and Final Values):
$\boxed{1}$: Write $\mathcal{L}(f'(t))$ in two ways: (1) $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0+)$ using the parts formula, and (2) $\mathcal{L}(f'(t)) = \int_0^\infty f'(t)e^{-st}dt$, using the direct Laplace definition.

A high-powered calculus theorem is needed to tell us that the integral on the right in (2) has limit as $s \to 0+$ equal to $\int_0^\infty f'(t)(1)dt = f(t)|_{t=0}^{t=\infty} = f(\infty) - f(0+)$. The needed result is *Lebesgue's Bounded Convergence Theorem*, which says that under certain conditions (met by the assumed hypotheses here) $\lim_{n\to\infty}\int_0^\infty f_n(t)dt = \int_0^\infty (\lim_{n\to\infty} f_n(t))\,dt$. We take $f_n(t) = f'(t)e^{-s_n t}$ where $\{s_n\}$ is any sequence of positive numbers with limit zero.

Assembling the two ways to write $\mathcal{L}(f'(t))$ implies $\lim_{s\to 0+} s\mathcal{L}(f(f)) - f(0) = f(\infty) - f(0+)$. Cancel $f(0+)$ from each side of this identity. Then $\lim_{s\to 0+} s\mathcal{L}(f(t)) = f(\infty)$.

$\boxed{2}$: Theorem 8.13 implies $\mathcal{L}(f'(t))$ has limit zero as $s \to \infty$. Limit as $s \to \infty$ across the parts formula $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0+)$ to obtain the limit $\lim_{s\to\infty} s\mathcal{L}(f(t)) - f(0+) = 0$, which is the claimed identity.

Exercises 8.5 $\nearrow$

There are no exercises for this section. The content is exclusively statements of theorems and proofs, for the following theorems.

Linearity

The t-Derivative Rule or Parts Rule

The t-Integral Rule

The s-Differentiation Rule

First Shifting Rule

Second Shifting Rule

Periodic Function Rule

Convolution Rule

Initial and Final Value Rules

8.6 Heaviside Step and Dirac Impulse

Heaviside Function

The **unit step function** $\mathbf{u}(t)$ is distinguished from the more precise clone called the **Heaviside function** $H(t)$, which is undefined at $t = 0$. The definitions:

$$\mathbf{u}(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 1 & \text{for} \quad t = 0, \\ 0 & \text{for} \quad t < 0, \end{cases} \qquad H(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ \text{undefined} & \text{for} \quad t = 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

Functions $1, \mathbf{u}(t), H(t)$ agree for $t > 0$ because all functions in Laplace theory are assumed zero for $t < 0$.

An often–used formula involving the unit step function is the **characteristic function** of the interval $a \le t < b$, or **unit pulse**:

$$(1) \qquad \mathbf{pulse}(t, a, b) = \mathbf{u}(t - a) - \mathbf{u}(t - b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise.} \end{cases}$$

To illustrate, a square wave $\mathbf{sqw}(t) = (-1)^{\mathbf{floor}(t)}$ can be written in the series form $\sum_{n=0}^{\infty} (-1)^n \, \mathbf{pulse}(t, n, n+1)$ as a **pulse train**.[9]

Trouble at $t = 0$. Computer algebra systems like `maple` distinguish between the piecewise-defined unit step function and the Heaviside function. The Heaviside function $H(t)$ is left undefined at $t = 0$, whereas the unit step is defined everywhere. This seemingly minor distinction makes more sense when taking formal derivatives. On the domain $t \ne 0$ of H, the ordinary calculus derivative dH/dt is defined and equals zero. In contrast, $\mathbf{u}(t)$ on its domain $-\infty < t < \infty$ fails to have a derivative at one point: $t = 0$.

Fundamental Theorem of Calculus. Calculus rule $\int_a^b f'(t) dt = f(b) - f(a)$ fails for $f = H$, due to integrand $f'(t) = dH/dt = 0$. Riemann and Stieltjes filled the gap in the theory by providing a new definition of integral and corresponding theory of integration, these days called **Riemann–Stieltjes Integration**. In their theory, integral $\int_a^b \frac{dH}{dt} dt$ makes sense and $\int_a^b \frac{dH}{dt} dt = H(b) - H(a)$. Riemann–Stieltjes theory will be used to explain a contribution of Paul Dirac (1902-1984) to Laplace theory called the **Dirac impulse**, denoted δ in the literature.

Dirac Impulse

Following the 1932 work of Paul A. M. Dirac the definition should be

$$\delta(t) = \frac{d}{dt} \mathbf{u}(t).$$

---

[9] A square wave resembles a train of boxcars. ⊓⊔⊓⊔

One year after Dirac introduced impulse δ, he received the Nobel Prize in physics for his quantum theory work. Laurent Schwartz later justified mathematically the use of δ.

A precise mathematical definition of the Dirac impulse is $\delta(t) = d\,\mathbf{u}(t)$, where $\mathbf{u}(t)$ is the unit step and $d\,\mathbf{u}(t)/dt$ has meaning under the integral sign in a Riemann-Stieltjes integral. This definition restrains $d\,\mathbf{u}(t)$ to have meaning only under an integral sign. It is in this sense that the Dirac impulse δ is defined.[10]

Dirac Impulse in Applications. What is the meaning of the differential equation

$$x'' + 16x = 5\delta(t - t_0)?$$

The equation $x'' + 16x = f(t)$ represents an undamped spring-mass system having Hooke's constant 16, subject to external force $f(t)$. In a mechanical context, the Dirac impulse term $5\delta(t - t_0)$ is an *idealization* of a hammer-hit at time $t = t_0 > 0$ with impulse 5. The hammer-hit **injects energy** into the system almost instantaneously.

Forcing term $f(t)$ in $x'' + 16x = f(t)$ can be formally written as a Riemann-Stieltjes integrator $5\,d\,\mathbf{u}(t - t_0)$ where $\mathbf{u}$ is the unit step function: $\mathbf{u}(t) = 1$ on $t \geq 0$, else $\mathbf{u}(t) = 0$.

The Dirac impulse or *derivative of the unit step*, nonsensical as it may appear, is realized in applications via the two–sided or central difference quotient $\dfrac{\mathbf{u}(t + h) - \mathbf{u}(t - h)}{2h} \approx d\,\mathbf{u}(t)$. Given t_0, let $a = t_0 - h$, $b = t_0 + h$ for $h > 0$ very small. A *simplistic approximation* for ideal impulse $5\delta(t - t_0)$ is given by the central difference approximation

$$\frac{5}{2h} \begin{cases} 1 & a \leq t < b \\ 0 & \text{else} \end{cases} = 5\frac{\mathbf{u}(t - a) - \mathbf{u}(t - b)}{b - a}$$
$$= \frac{5}{b - a}\,\mathbf{pulse}(t, a, b)$$

The *impulse*[11] of the actual force f is therefore approximated by

$$\int_{-\infty}^{\infty} f(t)\,dt \approx 5\int_a^b \frac{1}{b - a}\,\mathbf{pulse}(t, a, b)\,dt = 5,$$

due to the integrand being $1/(b - a)$ on $a \leq t < b$ and otherwise 0.

---

[10]The definition of the Dirac Impulse by Laurent Schwartz uses Lebesgue integration theory. In differential equations applications, the Riemann–Stieltjes definition $\delta(t) = d\,\mathbf{u}(t)$ suffices, with an unremarkable quantity of exceptions. The presentation here requires a calculus background but no Lebesgue theory background.

[11]Momentum is defined to be mass times velocity. If the force f is given by Newton's law as $f(t) = \frac{d}{dt}(mv(t))$ and $v(t)$ is velocity, then $\int_a^b f(t)dt = mv(b) - mv(a)$ is the net momentum or impulse on $[a, b]$.

Modeling Impulses

One argument for the Dirac impulse idealization is that an infinity of choices exist for modeling an impulse. There are in addition to the central difference quotient two other popular difference quotients, the forward quotient $(\mathbf{u}(t+h) - \mathbf{u}(t))/h$ and the backward quotient $(\mathbf{u}(t) - \mathbf{u}(t-h))/h$ ($h > 0$ assumed). In reality, h is unknown in any application, and the impulsive force of a hammer hit is hardly constant, as is supposed by this naive modeling.

The modeling logic often applied for the Dirac impulse is that the external force $f(t)$ will be used in the model in a limited manner, in which only the momentum $p = mv$ is important. More precisely, only the change in momentum or impulse is important, $\int_a^b f(t)dt = \Delta p = mv(b) - mv(a)$.

The precise force $f(t)$ is replaced during the modeling by a simplistic piecewise–defined force that has exactly the same impulse Δp. The replacement is justified by arguing that if only the impulse is important, and not the actual details of the force, then both models should give similar results. Most of the intuition for this modeling magic comes from investigation of two models: (1) $f(t)$ is piecewise–defined and depends on h, (2) $f(t)$ is an idealized Dirac impulse. The result of the investigation is that answers from (1) converge as $h \to 0$ to the single idealized answer from (2).

Impulses in Differential Equations. In Laplace theory, there is a natural encounter with Dirac's ideas, because $\mathcal{L}(f(t))$ routinely appears on the right of the equation after transformation. Let $a = t_0 - h, b = t_0 + h$ for small $h > 0$. Then $2h = b - a$. Assume $t_0 > 0$ and $t_0 - h > 0$ for the purpose of illustration. If the input $f(t)$ is a simplistic impulsive force of impulse c, then representation $f(t) = \frac{c}{b-a} \mathbf{pulse}(t, a, b)$ permits a direct computation of the impulse:

$$\text{impulse} = \int_{-\infty}^{\infty} f(t)dt = \int_a^b \frac{c}{b-a} \mathbf{pulse}(t, a, b)dt = c.$$

The Laplace integral $\mathcal{L}(f(t))$ evaluates as follows:

$$\mathcal{L}(f(t)) = \int_0^{\infty} f(t)e^{-st}dt$$
$$= \int_a^b \frac{c}{b-a} \mathbf{pulse}(t, a, b)e^{-st}dt$$
$$= \int_a^b \frac{c}{b-a}(1)e^{-st}dt$$
$$= \frac{c}{b-a} \left(\frac{e^{-sa} - e^{-sb}}{-s} \right) \qquad \boxed{1}$$
$$= c \frac{e^{-sh} - e^{sh}}{2sh} e^{-s t_0} \qquad \boxed{2}$$
$$\approx ce^{-s t_0} \qquad \boxed{3}$$

$\boxed{1}$: Factor the constant $\frac{c}{b-a}$ outside the integral, use the definition of pulse, then integrate the exponential.
$\boxed{2}$: Replace $a = t_0 - h, b = t_0 + h$, simplify, then collect denominator factors s

and $2h = b - a$.

$\boxed{3}$: As $h \to 0$, factor $\frac{e^{sh} - e^{-sh}}{2sh}$ converges to 1 because of L'Hôspital's rule applied to $\frac{e^x - e^{-x}}{2x}$ at $x = 0$.

Mathematical Flaw. The immediate naive modeling conclusion is that the simplistic impulsive force f should be replaced by an equivalent one f^* such that

$$\mathcal{L}(f^*(t)) = c\, e^{-s\, t_0}.$$

Unfortunately, *there is no such function f^**!

The apparent mathematical flaw in this idea was resolved by the work of Laurent Schwartz on **distributions**. In short, there is a solid foundation for introducing f^*, but unfortunately the mathematics involved is not elementary nor especially accessible to those readers whose background is just calculus.[12] The theory of distributions provides a resolution of the mathematical flaw:

$$\mathcal{L}(c\,\delta(t - t_0)) = c\, e^{-s\, t_0}.$$

It is mistake to write $f^*(t) = c\,\delta(t - t_0)$ and call f^* *a function*, because it is not. Expression f^* makes sense only under an integral sign.

Function or Operator? The work of physics Nobel prize winner Paul Dirac (1902–1984) proceeded for about 15 years before the mathematical community developed a sound mathematical theory for his impulsive force representations. A systematic theory was developed in 1936 by the Soviet mathematician S. Sobolev. The French mathematician Laurent Schwartz further developed the theory in 1945. He observed that the idealization δ is not a function but an operator or *linear functional*, in particular, δ maps or *associates* to each function $\phi(t)$ its value at $t = 0$, in short, $\delta(\phi) = \phi(0)$. This fact was observed early on by Dirac and others, during the replacement of simplistic forces by δ.

Laplace Theory and the Dirac Impulse. When Laplace theory manipulates the Dirac impulse δ, it does so by obeying the *under the integral sign rule*. The good news is that answers can be calculated formally, as though f^* was a function. What are we to do when applying the formal rules? We think of $\delta(t - t_0)$ as a simplistic impulse given on an interval $[a, b]$ that shrinks to t_0. A simplistic impulse *is a function*! Laplace theory provides a transition from simplistic impulse modeling to *idealized* Dirac impulse.

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[12] Practising engineers and scientists might be able to ignore the vast literature on distributions, citing the example of physicist Paul Dirac, who succeeded in applying impulsive force ideas without the distribution theory developed by S. Sobolev and L. Schwartz. Those who wish to read current literature on partial differential equations have no such luxury, because the work on distributions has forever changed the required background for reading new literature.

Properties of the Dirac Impulse

Theorem 8.15 (Fundamental Identities for Dirac δ)

Let $\mathbf{u}(t)$ denote the unit step function. Define $\delta(t) = d\,\mathbf{u}(t)$ as a Riemann–Stieltjes integrator. Let $g(t)$ be piecewise continuous and $a \geq 0$. Then

(1) $\displaystyle\int_{-\infty}^{\infty} \delta(t)dt = 1,$ meaning $\int_{-\infty}^{\infty} d\,\mathbf{u}(t) = 1$

(2) $\displaystyle\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a+),$ meaning $\int_{-\infty}^{\infty} g(t)d\,\mathbf{u}(t-a) = g(a+)$

(3) $\mathcal{L}(\delta(t-a)) = e^{-s\,a},$ meaning $\int_{0}^{\infty} e^{-st}\,d\,\mathbf{u}(t-a) = e^{-s\,a}.$

Proof:

Symbol $g(a+)$ means $\lim_{h \to 0+} g(a+h)$, the right–hand limit at $t = a$.

Property **(1)** follows from property **(2)** by choosing $g(t) = \mathbf{u}(t)$ and $a = 0$.

Property **(3)** follows from property **(2)** by choosing $g(t) = \mathbf{u}(t)$.

Details **(2)**: The *definition* of the Dirac impulse is a formal one, in which every occurrence of $\delta(t-a)dt$ under an integrand is replaced by $d\,\mathbf{u}(t-a)$. The differential symbol $d\,\mathbf{u}(t-a)$ is taken in the sense of the Riemann-Stieltjes integral, which is defined in Rudin [**?**] for monotonic integrators $\alpha(x)$ as

$$\int_{a}^{b} f(x)d\alpha(x) = \lim_{N \to \infty} \sum_{n=1}^{N} f(x_n)(\alpha(x_n) - \alpha(x_{n-1})).$$

Required in the definition: $x_0 = a$, $x_N = b$ and $x_0 < x_1 < \cdots < x_N$ forms a partition of $[a,b]$ whose mesh $\max\{|x_j - x_{j-1}| : 1 \leq j \leq N\}$ approaches zero as $N \to \infty$. Used exclusively here is nondecreasing integrator $\alpha = \mathbf{u}$, the unit step.

Steps below verify that the left and right sides in **(2)** are equal.

LHS $= \mathcal{L}(g(t)\delta(t-a))$	Left side of **(2)**.
$= \int_0^\infty g(t)e^{-st}\delta(t-a)dt$	Laplace integral, $a \geq 0$ assumed.
$= \int_0^\infty g(t)e^{-st}d\,\mathbf{u}(t-a)$	Replace $\delta(t-a)dt$ by $d\,\mathbf{u}(t-a)$.
$= \lim_{M\to\infty} \int_0^M g(t)e^{-st}d\,\mathbf{u}(t-a)$	Definition of improper integral.
$= g(a)e^{-sa}$	Explained below.
$=$ RHS	Property **(2)** verified.

To explain the last step, apply the definition of the Riemann-Stieltjes integral to $\alpha = \mathbf{u}$ with given partition $0 = t_0 < t_1 < \cdots < t_N = M$ of $[0, M]$, M a large positive number. It is assumed that the mesh approaches zero as $N \to \infty$. Then

$$\int_0^M g(t)e^{-st}d\,\mathbf{u}(t-a) = \lim_{N\to\infty} \sum_{n=0}^{N-1} g(t_n)e^{-st_n}\left(\mathbf{u}(t_n - a) - \mathbf{u}(t_{n-1} - a)\right)$$

Given point a satisfying $0 \leq a < M$, then this point has to lie in exactly one interval: $t_{n-1} \leq a < t_n$. Then $\mathbf{u}(t_n - a) - \mathbf{u}(t_{n-1} - a) = 1$ and for all other intervals this factor is zero. The sum reduces to a single term $g(t_n)e^{-s\,t_n}$. This term limits to $g(a+)\,e^{-sa}$ as $N \to \infty$, because t_n limits to a from the right. ■

Exercises 8.6 ⎘

Unit Step and Heaviside

1. The unit step $\mathbf{u}(t)$ is defined on the whole real line. Is it piecewise continuous on the whole line?

2. Is there a continuous function on the real line that agrees with the Heaviside function except at $t = 0$?

3. The piecewise continuous function $\mathbf{pulse}(t, a, b)$ is defined everywhere. Redefine $\mathbf{pulse}(t, a, b)$ using $H(t)$ instead of $\mathbf{u}(t)$.

4. Write $f(t) = \mathbf{floor}(t)\,\mathbf{u}(t)$ as a sum of terms, each of which has the form $g(t)\,\mathbf{pulse}(t, a, b)$.

Dirac Impulse

5. Verify $\int_{-\infty}^{\infty} \frac{\mathbf{pulse}(t,a,b)}{b-a}\, dt = 1$.

6. Verify by direct integration that $f(t) = 10\,\mathbf{pulse}(t, -0.001, 0.001)$ represents a simple impulse of 10 at $t = 0$ of duration 0.002. Graph it without using technology.

7. Find $\mathcal{L}(\delta(t - 1) + \delta(t - 2))$.

8. Find $\mathcal{L}(10\,\delta(t - 1) - 5\,\delta(t - 2))$.

9. Solve for $f(t)$ in terms of δ:
$\mathcal{L}(f(t)) = 10e^{-s}$

10. Solve for $f(t)$ in terms of δ:
$\mathcal{L}(f(t)) = 10e^{-s} + \frac{s}{s^2+1}\, e^{-2s}$

11. Find $\mathcal{L}\left(\sum_{n=1}^{10}(1 + n)\delta(t - n)\right)$.

12. A sequence of camshaft impulses happening periodically in a finite time interval have transform $\mathcal{L}(f(t)) = \sum_{i=1}^{N} e^{-c_i s}$. Find the idealized impulse train f.

Riemann–Stieltjes Integral

Evaluate the integrals either directly from the definition or else by using Theorem 8.15.

13. $\int_0^2 d\,\mathbf{u}(t - 1)$

14. $\int_0^{\infty} d\,\mathbf{u}(t - 2)$

15. $\int_0^2 \tanh(t^2 + 1)\, d\,\mathbf{u}(t - 1)$

16. $\int_0^{\infty} \frac{t}{1+t^2}\, d\,\mathbf{u}(t - 2)$

8.7   Laplace Table Derivations

Verified here are two Laplace tables: the minimal Laplace Table 7.2-4 and its extension Table 7.2-6. This section is for reading, designed to enrich lectures and to aid those who study in isolation. Due to density of proof details, there are no exercises.

Derivation of Laplace integral formulas in Table 7.2-4, page 601.

● **Proof of $\mathcal{L}(t^n) = n!/s^{1+n}$:**

The first step is to evaluate $\mathcal{L}(t^n)$ for $n = 0$.

$$\begin{aligned}\mathcal{L}(1) &= \int_0^\infty (1)e^{-st}dt && \text{Laplace integral of } f(t) = 1. \\ &= -(1/s)e^{-st}\big|_{t=0}^{t=\infty} && \text{Evaluate the integral.} \\ &= 1/s && \text{Assumed } s > 0 \text{ to evaluate } \lim_{t\to\infty} e^{-st}. \end{aligned}$$

The value of $\mathcal{L}(t^n)$ for $n = 1$ can be obtained by s-differentiation of the relation $\mathcal{L}(1) = 1/s$, as follows.

$$\begin{aligned}\frac{d}{ds}\mathcal{L}(1) &= \frac{d}{ds}\int_0^\infty (1)e^{-st}dt && \text{Laplace integral for } f(t) = 1. \\ &= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right)dt && \text{Used } \frac{d}{ds}\int_a^b F dt = \int_a^b \frac{dF}{ds}dt. \\ &= \int_0^\infty (-t)e^{-st}dt && \text{Calculus rule } (e^u)' = u'e^u. \\ &= -\mathcal{L}(t) && \text{Definition of } \mathcal{L}(t). \end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}(t) &= -\frac{d}{ds}\mathcal{L}(1) && \text{Rewrite last display.} \\ &= -\frac{d}{ds}(1/s) && \text{Use } \mathcal{L}(1) = 1/s. \\ &= 1/s^2 && \text{Differentiate.} \end{aligned}$$

This idea can be repeated to give $\mathcal{L}(t^2) = -\frac{d}{ds}\mathcal{L}(t)$ and hence $\mathcal{L}(t^2) = 2/s^3$. The pattern is $\mathcal{L}(t^n) = -\frac{d}{ds}\mathcal{L}(t^{n-1})$ which gives $\mathcal{L}(t^n) = n!/s^{1+n}$.

● **Proof of $\mathcal{L}(e^{at}) = 1/(s - a)$:**

The result follows from $\mathcal{L}(1) = 1/s$, as follows.

$$\begin{aligned}\mathcal{L}(e^{at}) &= \int_0^\infty e^{at}e^{-st}dt && \text{Direct Laplace transform.} \\ &= \int_0^\infty e^{-(s-a)t}dt && \text{Use } e^A e^B = e^{A+B}. \\ &= \int_0^\infty e^{-St}dt && \text{Substitute } S = s - a. \\ &= 1/S && \text{Apply } \mathcal{L}(1) = 1/s. \\ &= 1/(s - a) && \text{Back-substitute } S = s - a. \end{aligned}$$

● **Proof of $\mathcal{L}(\cos bt) = s/(s^s + b^2)$ and $\mathcal{L}(\sin bt) = b/(s^s + b^2)$:**

Use will be made of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, usually first introduced in trigonometry. In this formula, θ is a real number in radians and $i = \sqrt{-1}$ is the complex unit.

$$e^{ibt}e^{-st} = (\cos bt)e^{-st} + i(\sin bt)e^{-st}$$

Substitute $\theta = bt$ into Euler's formula and multiply by e^{-st}.

$$\int_0^\infty e^{-ibt}e^{-st}dt = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$$

Integrate $t = 0$ to $t = \infty$. Then use properties of integrals.

$$\frac{1}{s - ib} = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$$

Evaluate the left hand side using $\mathcal{L}(e^{at}) = 1/(s - a)$, $a = ib$.

$$\frac{1}{s - ib} = \mathcal{L}(\cos bt) + i\mathcal{L}(\sin bt)$$

Direct Laplace transform definition.

$$\frac{s + ib}{s^2 + b^2} = \mathcal{L}(\cos bt) + i\mathcal{L}(\sin bt)$$

Use complex rule $1/z = \overline{z}/|z|^2$, $z = A + iB$, $\overline{z} = A - iB$, $|z| = \sqrt{A^2 + B^2}$.

$$\frac{s}{s^2 + b^2} = \mathcal{L}(\cos bt)$$

Extract the real part.

$$\frac{b}{s^2 + b^2} = \mathcal{L}(\sin bt)$$

Extract the imaginary part.

Derivation of Laplace integral formulas in Table 7.2-6, page 602.

● **Proof of the Heaviside formula $\mathcal{L}(\mathbf{u}(t - a)) = e^{-as}/s$.**

$$\begin{aligned}
\mathcal{L}(\mathbf{u}(t - a)) &= \int_0^\infty \mathbf{u}(t - a)e^{-st}dt && \text{Direct Laplace transform. Assume } a \geq 0. \\
&= \int_a^\infty (1)e^{-st}dt && \text{Because } \mathbf{u}(t - a) = 0 \text{ for } 0 \leq t < a. \\
&= \int_0^\infty (1)e^{-s(x+a)}dx && \text{Change variables } t = x + a. \\
&= e^{-as}\int_0^\infty (1)e^{-sx}dx && \text{Constant } e^{-as} \text{ moves outside integral.} \\
&= e^{-as}(1/s) && \text{Apply } \mathcal{L}(1) = 1/s.
\end{aligned}$$

● **Proof of the Dirac impulse formula $\mathcal{L}(\delta(t - a)) = e^{-as}$.**

The *definition* of the Dirac impulse is a formal one, in which every occurrence of $\delta(t-a)dt$ under an integrand is replaced by $d\,\mathbf{u}(t - a)$. The differential symbol $d\,\mathbf{u}(t - a)$ is taken in the sense of the Riemann-Stieltjes integral. This integral is defined in Rudin [?] for monotonic integrators $\alpha(x)$ as the limit

$$\int_a^b f(x)d\alpha(x) = \lim_{N\to\infty} \sum_{n=1}^N f(x_n)(\alpha(x_n) - \alpha(x_{n-1}))$$

where $x_0 = a$, $x_N = b$ and $x_0 < x_1 < \cdots < x_N$ forms a partition of $[a, b]$ whose mesh approaches zero as $N \to \infty$. Instance $\alpha(x) = x$ duplicates the theory of the Riemann integral in calculus.

The steps in computing the Laplace integral of the Dirac impulse appear below. Admittedly, the proof requires advanced calculus skills and a certain level of mathematical maturity. The reward is a fuller understanding of the Dirac symbol $\delta(x)$. More details and further properties of the Dirac impulse can be found in Section 8.6, page 648.

$$\begin{aligned}
\mathcal{L}(\delta(t - a)) &= \int_0^\infty e^{-st}\delta(t - a)dt && \text{Laplace integral, } a \geq 0 \text{ assumed.} \\
&= \int_0^\infty e^{-st}d\,\mathbf{u}(t - a) && \text{Replace } \delta(t - a)dt \text{ by } d\,\mathbf{u}(t - a). \\
&= \lim_{M\to\infty} \int_0^M e^{-st}d\,\mathbf{u}(t - a) && \text{Definition of improper integral.} \\
&= e^{-sa} && \text{Explained below.}
\end{aligned}$$

To explain the last step, apply the definition of the Riemann-Stieltjes integral:

$$\int_0^M e^{-st}d\,\mathbf{u}(t-a) = \lim_{N\to\infty} \sum_{n=0}^{N-1} e^{-st_n}(\mathbf{u}(t_n-a)-\mathbf{u}(t_{n-1}-a))$$

where $0 = t_0 < t_1 < \cdots < t_N = M$ is a partition of $[0, M]$ whose mesh $\max_{1\le n\le N}(t_n - t_{n-1})$ approaches zero as $N \to \infty$. Given a partition, then point a in $0 \le a < M$ lies in exactly one interval: $t_{n-1} \le a < t_n$. By the definition of unit step, $\mathbf{u}(t_n-a)-\mathbf{u}(t_{n-1}-a) = 1$, while for any other interval this factor is zero. Therefore, the sum reduces to a single term e^{-st_n}. This term approaches e^{-sa} as $N \to \infty$, because t_n must approach a from the right.

● **Proof of** $\mathcal{L}(\mathbf{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$:

The library function **floor** present in computer languages C and Fortran is defined by **floor**(x) = greatest whole integer $\le x$, e.g., **floor**$(5.2) = 5$ and **floor**$(-1.9) = -2$. The computation of the Laplace integral of **floor**(t) requires ideas from infinite series, as follows.

$$F(s) = \int_0^\infty \mathbf{floor}(t)e^{-st}dt \qquad \text{Laplace integral definition.}$$
$$= \sum_{n=0}^\infty \int_n^{n+1}(n)e^{-st}dt \qquad \text{On } n \le t < n+1, \mathbf{floor}(t) = n.$$
$$= \sum_{n=0}^\infty \frac{n}{s}(e^{-ns} - e^{-ns-s}) \qquad \text{Evaluate each integral.}$$
$$= \frac{1-e^{-s}}{s}\sum_{n=0}^\infty ne^{-sn} \qquad \text{Common factor removed.}$$
$$= \frac{x(1-x)}{s}\sum_{n=0}^\infty nx^{n-1} \qquad \text{Define } x = e^{-s}.$$
$$= \frac{x(1-x)}{s}\frac{d}{dx}\sum_{n=0}^\infty x^n \qquad \text{Term-by-term differentiation.}$$
$$= \frac{x(1-x)}{s}\frac{d}{dx}\frac{1}{1-x} \qquad \text{Geometric series sum.}$$
$$= \frac{x}{s(1-x)} \qquad \text{Compute the derivative, simplify.}$$
$$= \frac{e^{-s}}{s(1-e^{-s})} \qquad \text{Substitute } x = e^{-s}.$$

To evaluate the Laplace integral of **floor**(t/a), a change of variables is made.

$$\mathcal{L}(\mathbf{floor}(t/a)) = \int_0^\infty \mathbf{floor}(t/a)e^{-st}dt \qquad \text{Laplace integral definition.}$$
$$= a \int_0^\infty \mathbf{floor}(r)e^{-asr}dr \qquad \text{Change variables } t = ar.$$
$$= aF(as) \qquad \text{Apply the formula for } F(s).$$
$$= \frac{e^{-as}}{s(1-e^{-as})} \qquad \text{Simplify.}$$

● **Proof of** $\mathcal{L}(\mathbf{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$:

The square wave defined by $\mathbf{sqw}(x) = (-1)^{\mathbf{floor}(x)}$ is periodic of period 2 and piecewise-defined. Let $\mathcal{P} = \int_0^2 \mathbf{sqw}(t)e^{-st}dt$.

$$\mathcal{P} = \int_0^1 \mathbf{sqw}(t)e^{-st}dt + \int_1^2 \mathbf{sqw}(t)e^{-st}dt \quad \text{Apply } \int_a^b = \int_a^c + \int_c^b.$$

$$= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt \quad \text{Use } \mathbf{sqw}(x) = 1 \text{ on } 0 \leq x < 1 \text{ and } \mathbf{sqw}(x) = -1 \text{ on } 1 \leq x < 2.$$

$$= \frac{1}{s}(1 - e^{-s}) + \frac{1}{s}(e^{-2s} - e^{-s}) \quad \text{Evaluate each integral.}$$

$$= \frac{1}{s}(1 - e^{-s})^2 \quad \text{Collect terms.}$$

An intermediate step is to compute the Laplace integral of $\mathbf{sqw}(t)$:

$$\mathcal{L}(\mathbf{sqw}(t)) = \frac{\int_0^2 \mathbf{sqw}(t)e^{-st}dt}{1 - e^{-2s}} \quad \text{Periodic function formula, page 638.}$$

$$= \frac{1}{s}(1 - e^{-s})^2 \frac{1}{1 - e^{-2s}}. \quad \text{Use the computation of } \mathcal{P} \text{ above.}$$

$$= \frac{1}{s}\frac{1 - e^{-s}}{1 + e^{-s}}. \quad \text{Factor } 1 - e^{-2s} = (1 - e^{-s})(1 + e^{-s}).$$

$$= \frac{1}{s}\frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}. \quad \text{Multiply the fraction by } e^{s/2}/e^{s/2}.$$

$$= \frac{1}{s}\frac{\sinh(s/2)}{\cosh(s/2)}. \quad \text{Use } \sinh u = (e^u - e^{-u})/2, \cosh u = (e^u + e^{-u})/2.$$

$$= \frac{1}{s}\tanh(s/2). \quad \text{Use } \tanh u = \sinh u/\cosh u.$$

To complete the computation of $\mathcal{L}(\mathbf{sqw}(t/a))$, a change of variables is made:

$$\mathcal{L}(\mathbf{sqw}(t/a)) = \int_0^\infty \mathbf{sqw}(t/a)e^{-st}dt \quad \text{Direct transform.}$$

$$= \int_0^\infty \mathbf{sqw}(r)e^{-asr}(a)dr \quad \text{Change variables } r = t/a.$$

$$= \frac{a}{as}\tanh(as/2) \quad \text{See } \mathcal{L}(\mathbf{sqw}(t)) \text{ above.}$$

$$= \frac{1}{s}\tanh(as/2)$$

● **Proof of** $\mathcal{L}(a\,\mathbf{trw}(t/a)) = \dfrac{1}{s^2}\tanh(as/2)$:

The triangular wave is defined by $\mathbf{trw}(t) = \int_0^t \mathbf{sqw}(x)dx$.

$$\mathcal{L}(a\,\mathbf{trw}(t/a)) = \frac{1}{s}(f(0) + \mathcal{L}(f'(t)) \quad \text{Let } f(t) = a\,\mathbf{trw}(t/a). \text{ Use } \mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0), \text{ page 596.}$$

$$= \frac{1}{s}\mathcal{L}(\mathbf{sqw}(t/a)) \quad \text{Use } f(0) = 0, (a\int_0^{t/a}\mathbf{sqw}(x)dx)' = \mathbf{sqw}(t/a).$$

$$= \frac{1}{s^2}\tanh(as/2) \quad \text{Table entry for } \mathbf{sqw}.$$

● **Proof of** $\mathcal{L}(t^\alpha) = \dfrac{\Gamma(1 + \alpha)}{s^{1+\alpha}}$:

$$\mathcal{L}(t^\alpha) = \int_0^\infty t^\alpha e^{-st}dt \quad \text{Direct Laplace transform.}$$

$$= \int_0^\infty (u/s)^\alpha e^{-u}du/s \quad \text{Change variables } u = st, \ du = sdt.$$

$$= \frac{1}{s^{1+\alpha}} \int_0^\infty u^\alpha e^{-u} du$$

$$= \frac{1}{s^{1+\alpha}} \Gamma(1+\alpha).$$

Where $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$, by definition.

The *generalized factorial function* $\Gamma(x)$ is defined for $x > 0$ and it agrees with the classical factorial $n! = (1)(2)\cdots(n)$ in case $x = n+1$ is an integer. In literature, $\alpha!$ means $\Gamma(1+\alpha)$. For more details about the Gamma function, see Abramowitz and Stegun [?], or `maple` documentation.

- **Proof of $\mathcal{L}(t^{-1/2}) = \sqrt{\dfrac{\pi}{s}}$:**

$$\mathcal{L}(t^{-1/2}) = \frac{\Gamma(1 + (-1/2))}{s^{1-1/2}}$$

Apply the previous formula.

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

Use $\Gamma(1/2) = \sqrt{\pi}$.

8.8   Modeling

Laplace Modeling in Engineering

A *differential equation model* in variable t can be subjected to the Laplace transform, which produces an *algebraic model* in transform variable s.

The possibility of equivalence of models

$$mx''(t) + cx'(t) + kx(t) = 0 \quad \text{and} \quad \frac{1}{ms^2 + cs + k},$$

can be understood because of the one-to-one correspondence of the physical parameters m, c, k. Lerch's theorem provides a theoretical foundation which says that the differential equation model in the t-domain and the algebraic model in the s-domain are equivalent, that is, the solution of one model gives the solution to the other model.

$$\boxed{\begin{aligned} &mx'' + cx' + kx = f(t), \\ &x(0) = x'(0) = 0 \end{aligned}} \qquad \longleftrightarrow \qquad \boxed{X(s) = \frac{F(s)}{ms^2 + cs + k}}$$

Figure 5.   Differential Equation and Laplace Model Equivalence

In mechanical, electrical and computer engineering it is commonplace to deal *only* with the Laplace algebraic model, and to back-burner discussions of the differential equation model in the time domain.

Modeling conversations are often exclusively in terms of transforms. Differential equations are *rarely* mentioned! Terminology for such modeling is necessarily

specialized, which gives rise to new contextual meanings to the terms *input* and *output*. For example, an *RLC*-circuit could be discussed with *input*

$$F(s) = \frac{\omega}{s^2 + \omega^2},$$

and the listener must know that this expression is the Laplace transform of the time input $f(t) = \sin \omega t$. The audience would then know that the *RLC*-circuit is driven by a sinusoidal input of natural frequency ω and amplitude one. The *output* could be the Laplace transform

$$X(s) = \frac{1}{s+1} + \frac{d_1 + d_2 \omega}{s^2 + \omega^2}.$$

Lerch's equivalence provides extra information, deemed momentarily useless, that $X(s)$ is the Laplace transform of the time output $x(t) = e^{-t} + d_1 \cos \omega t + d_2 \sin \omega t$.

It is important to know for modeling that fraction $\frac{1}{s+1}$ is the Laplace transform of the transient part of the output, while fraction $\frac{d_1 + d_2 \omega}{s^2 + \omega^2}$ is the Laplace transform of the steady state output.

DC Gain

Background. Gain may be voltage gain (OP-amp, V/V), power gain (RF-amp, W/W) or sensor gain (light, e.g., 5 μV per photon). **Steady state gain** and **DC-gain** are synonyms for the same number. Laplace theory can compute steady-state values for a differential equation.

Signal applications might seek the signal $x(t)$ as the output of an **underdamped** model with switch at $t = 0$:

$$x'' + 2\zeta\omega x' + \omega^2 x = G_{\mathbf{DC}}\,\omega^2\,\mathbf{u}(t).$$

The three parameters $\zeta, \omega, G_{\mathbf{DC}}$ are known respectively as the **damping ratio**, **frequency** and **DC-gain**. Symbol $\mathbf{u}(t)$ is the unit step function. Under-damped for this equation means $\zeta > 1$, the case for complex roots of the characteristic equation. The Euler atoms for the homogeneous problem are exponential decay factors times sines and cosines.

On time interval $0 \le t < \infty$ the unit step $\mathbf{u}(t)$ is replaced by 1:

$$x'' + 2\zeta\omega x' + \omega^2 x = G_{\mathbf{DC}}\,\omega^2$$

Superposition implies $x = x_h + x_p$ with equilibrium solution $x_p(t) = G_{\mathbf{DC}}$ and homogeneous general solution $x_h(t) = c_1 e^{-at} \cos(bt) + c_2 e^{-at} \sin(bt)$, symbols defined by $a = \zeta\omega$ and $b = \omega\sqrt{\zeta^2 - 1}$. Because of the exponential decay of $x_h(t)$, the constant solution $x_p(t) = G_{\mathbf{DC}}$ is the signal steady state. This is why a simulator in a lab given a constant input k has DC-gain equal to the steady state of the output signal divided by k.

The DC-gain can be found mathematically from fractions in the s-domain instead of from time domain formulas. One possibility is to use the *final value rule* page 638 to find the steady state gain (DC-gain) $G_{\mathbf{DC}}$. Assume the input is the unit step function $\mathbf{u}(t)$. Then the steady state value is $\lim_{t \to \infty} x(t) = x(\infty) = \lim_{s \to 0} s\mathcal{L}(x(t))$. The s-domain product $s\mathcal{L}(x(t)) = \dfrac{\mathcal{L}(x(t))}{\mathcal{L}(\mathbf{u}(t))}$ equals the Laplace of the output divided by the Laplace of the input. Evaluation of this quotient at $s = 0$ is the system's steady state gain (DC-gain) $G_{\mathbf{DC}}$.

The **transfer function** $H(s)$ is the Laplace of the output $x(t)$ with input $\mathbf{u}(t)$ and zero initial data. For this special output $x(t)$ the equation $\mathcal{L}(x(t)) = H(s)\mathcal{L}(\mathbf{u}(t)) = \dfrac{H(s)}{s}$ holds. Therefore, the steady state gain (DC-gain) equals $H(0)$.

Illustration. An underdamped system whose transfer function is the fraction $H(s) = \dfrac{2}{s^2 + 2s + 2}$ has DC-gain $H(0) = 1$.

Engineering Inverse Problems

Linear time-invariant systems are used as building blocks to construct complex systems, in which the output of one system is the input of the next system. The systems are modeled by constant-coefficient linear differential equations. Practical applications endeavor to find a mathematical model to represent the block, technically an inverse problem.

What is the Inverse Problem? The terminology *inverse problem* applied to $x'' + px' + qx = f(t)$ means: given input $f(t)$ and output $x(t)$ as numerical data, recover the values of p and q. Imagine the experimental data is in a graph of signal $x(t)$ viewed on an oscilloscope. The graph data is imported into a numerical workbench in order to find the *system parameters* p, q in the predicted mathematical model

$$x'' + px' + qx = f(t), \quad \text{transfer function} = \frac{1}{s^2 + ps + q}.$$

Oscilloscope Experiments. It may help to think of the block as a physical device, like part of a battery charging circuit on a mobile phone. Initial states for a block are $x(0) = 0$, $x'(0) = 1$ or $x(0) = x'(0) = 0$. Possible input signals are zero input $f(t) = 0$, a step input $f(t) = k\,\mathbf{u}(t)$ or an impulse input $f(t) = \delta(t)$. Zero input means no battery. A step input can be considered a toggle which switches in a k–volt battery at $t = 0$. Impulse input $\delta(t)$ is practically a simplistic impulse $\frac{1}{h}(\mathbf{u}(t) - \mathbf{u}(t - h))$ with $h > 0$ very small; a function generator would work. A special BNC cable carries output signal $x(t)$ from the block to the oscilloscope for display. Oscilloscopes can save $x(t)$ data in text format for import into computer software.

Graphical Recognition of Block Types. A nontrivial aspect of an inverse problem is examination of graphical output in order to predict the block type.

The process can be more art than science. Multiple graphics might be produced for accurate prediction of the model type. For simplicity, there are two possibilities:

Non–oscillatory $x'' + px' + qx = 0$, called *over–damped* in applications. It means characteristic equation $r^2 + pr + q = 0$ has two real distinct roots $-a, -b$. Assumed below is $a < b$.

Oscillatory $x'' + px' + qx = 0$, called *under–damped* in applications. It means characteristic equation $r^2 + pr + q = 0$ has complex conjugate roots $r = -a \pm bi$ with $b > 0$.

Skipped in the analysis above is the *critically–damped* case in which the characteristic equation has a double root. This case is technically non–oscillatory and physically indistinguishable from the over–damped case. In spring–mass systems, coefficient p is the damping constant, imagined as a tuning parameter adjusted by a set screw, for which the critically–damped value for p separates the two physically observable classifications **oscillatory** (small $p > 0$) and **non–oscillatory** (large $p > 0$).

The two observable cases are graphed in Figures 6 and 7. The distinction: the first curve touches the t-axis just once, while the second curve touches the t-axis infinitely often. In oscilloscope output, oscillations may be damped severely, looking non–oscillatory like Figure 6. Nonlinear blocks may have output completely different from Figures 6, 7. The choice of model is then art instead of science.
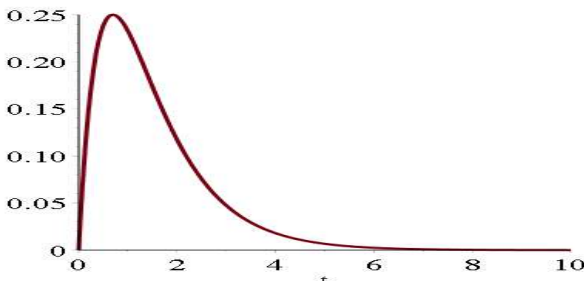
Figure 6. Block Output, Over-Damped. Non–oscillatory output $x(t)$ for $x'' + 3x' + 2x = 0$, $x(0) = 0$, $x'(0) = 1$.
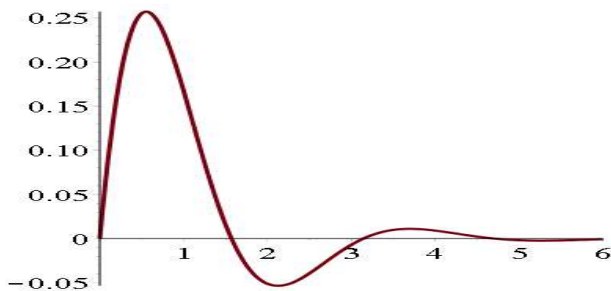
Figure 7. Block Output, Under-Damped Oscillatory output $x(t)$ for $x'' + 2x' + 5x = 0$, $x(0) = 0$, $x'(0) = 1$.

Second Order Models

The models can be represented by

$$x''(t) + 2\zeta\omega x'(t) + \omega^2 x(t) = 0,$$

where ζ is the damping ratio and ω is the undamped natural frequency. Cases $\zeta < 1$, $\zeta = 1$ and $\zeta > 1$ are named **over-damped**, **critically-damped** and **under-damped**, respectively.

The over–damped and under–damped cases specialize respectively to

$$x'' + (a + b)x' + abx = 0, \quad a < b,$$
$$x'' + 2ax' + (a^2 + b^2) x = 0, \quad b > 0.$$

Because ζ, ω can be found from a, b, then the inverse problem seeks values for a, b instead of the damping ratio and undamped frequency.

Theorem 8.16 (Solution Formulas for Second Order Over–Damped)
The differential equation is $x'' + (a + b)x' + abx = f(t)$. Assume $a < b$. Formulas are for $t > 0$.

(1) Zero Input $f(t) = 0$, $x(0) = 0$, $x'(0) = 1$:
$$x(t) = \frac{1}{b - a} \left(e^{-at} - e^{-bt} \right)$$
(2) Step Input $f(t) = k\,\mathbf{u}(t)$, $x(0) = 0$, $x'(0) = 0$:
$$x(t) = \frac{k}{ab} + \frac{k}{a^2 - ab} e^{-at} + \frac{k}{b^2 - ab} e^{-bt}$$
(3) Dirac Input $f(t) = k\delta(t)$, $x(0) = 0$, $x'(0) = 0$:
$$x(t) = \frac{1}{2} \frac{1}{b - a} \left(e^{-at} - e^{-bt} \right)$$

Details for Theorem 8.16: Paper and pencil solutions use Laplace theory. The formulas can be obtained from a CAS like `maple` or `mathematica`, which use Laplace theory to solve the equation. The `maple` code:

```
deOD:=diff(z(t),t,t)+(a+b)*diff(z(t),t)+a*b*z(t);
ic1:=z(0)=0,D(z)(0)=1;ic2:=z(0)=0,D(z)(0)=0;
dsolve({deOD=0,ic1},z(t));dsolve({deOD=k,ic2},z(t));
dsolve({deOD=Dirac(t),ic2},z(t));convert(%,piecewise);
```

Theorem 8.17 (Solution Formulas for Second Order Under–Damped)
The equation is $x''(t) + 2ax'(t) + (a^2 + b^2) x(t) = f(t)$. Assume $b > 0$. Formulas are for $t > 0$.

(1) Zero Input $f(t) = 0$, $x(0) = 0$, $x'(0) = 1$:
$$x(t) = e^{-at} \frac{\sin(b\,t)}{b}$$
(2) Step Input $f(t) = k\,\mathbf{u}(t)$, $x(0) = 0$, $x'(0) = 0$:

$$x(t) = \frac{k - k\,e^{-a\,t}(\frac{a}{b}\sin(b\,t) + \cos(b\,t))}{a^2 + b^2}$$

(3) Dirac Input $f(t) = \delta(t)$, $x(0) = 0$, $x'(0) = 0$:

$$x(t) = \frac{e^{-a\,t}}{2}\frac{\sin(b\,t)}{b}$$

Details for Theorem 8.17: Paper and pencil solutions use Laplace theory. The `maple` code:

```
deUD:=diff(z(t),t,t)+2*a*diff(z(t),t)+(a^2+b^2)*z(t);
ic1:=z(0)=0,D(z)(0)=1;ic2:=z(0)=0,D(z)(0)=0;
dsolve({deUD=0,ic1},z(t));dsolve({deUD=k,ic2},z(t));
dsolve({deUD=Dirac(t),ic2},z(t));convert(%,piecewise);
```

System Parameters for Over-Damped Problems

It is assumed that a signal $x(t)$ is known via a graphic with numerical data available. The graphic must pass the following visual test:

> The curve starts at $t = 0$, $x = 0$ and increases. The region of increase may end at a maximum and after decrease to limit zero, or else the region of increase is a half–line and $x(t)$ limits to a nonzero steady–state value.

Expected is a rich sample of the plot data, because computations use the numeric data, not the graphic. The initial data and the input are assumed to satisfy one of the following three cases.

Zero Input: $x(0) = 0$, $x'(0) = 1$, $f(t) = 0$
Step Input: $x(0) = 0$, $x'(0) = 0$, $f(t) = k\,\mathbf{u}(t)$
Impulse Input: $x(0) = 0$, $x'(0) = 0$, $f(t) = \delta(t)$

A graphic that passes the visual test predicts the model

$$x'' + (a + b)x' + abx = f(t), \text{ Transfer Function} = \frac{1}{(s + a)(s + b)}, \ a < b.$$

The plan is to compute numerical values for a, b from the graphical data. The product of the computation is a mathematical model for the block represented by the graphic.

Example 8.30 (System Parameters: Over-Damped with Zero Input)

Oscilloscope data created Figure 8 from block initial state $x(0) = 0$, $x'(0) = 1$ and zero input. Explain why the graphic predicts over–damped model

$$x''(t) + (a + b)x'(t) + abx(t) = f(t), \quad \text{Transfer Function} = \frac{1}{(s + a)(s + b)}.$$

Then verify system parameters $a = 1$, $b = 2$ from graphical numeric data.
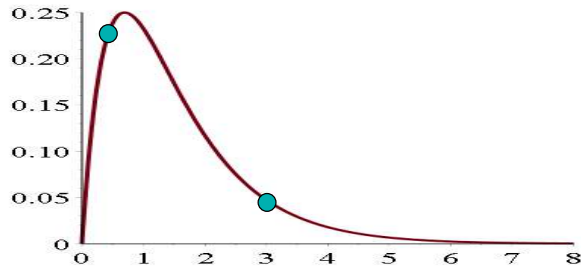
Figure 8. Oscilloscope output $x(t)$ **for Example 8.30.** The cyan dots are located at $(0.5, 0.2387)$ and $(3, 0.0473)$.

Solution: The curve in Figure 8 passes through $t = 0$, $x = 0$, then increases to a maximum and after decreases to zero. The curve fails to have infinity many crossings of the x-axis, therefore the system model is non–oscillatory over–damped.

We have only numeric output data for $x(t)$ and not the differential equation itself, so a, b are unknown. We discuss how to find $a = 1$ and $b = 2$ directly from the numerical data used to plot Figure 8.

Choose two points on the curve, one on the increasing section and one on the decreasing section. For example, the cyan dots in the figure, $t = 0.5, x = 0.2387$ and $t = 3.0, x = 0.0473$. Define $F(t, a, b) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)$, which is symbolic solution **(1)** in Theorem 8.16. Use a CAS like `maple` or `mathematica`, or a workbench like `matlab` to solve for a, b in the equations $F(0.5, a, b) = 0.2387$, $F(3.0, a, b) = 0.0473$. The answer is $b = 1.998164793$, $a = 1.000799323$. Due to $F(t, a, b) = F(t, b, a)$, there are two answers, but only one answer with requirement $a < b$. The `maple` code:

```
F:=(t,a,b)->(exp(-a*t)-exp(-b*t))/(b-a);
fsolve({F(0.5,a,b)=0.2387,F(3.0,a,b)=0.0473},{a,b});
```

Numerical computations like this might be done with algebra and shortcuts, like finding the smaller root from $x(t) \approx e^{-at}/(b - a)$ for large t.

Example 8.31 (System Parameters: Over–Damped with Step Input)

Oscilloscope data created Figure 9 from block state $x(0) = 0$, $x'(0) = 0$ and unit step input. Explain why the graphic predicts over–damped model

$$x''(t) + (a + b)x'(t) + abx(t) = f(t), \quad \text{Transfer Function} = \frac{1}{(s + a)(s + b)}.$$

Then verify system parameters $a = 1$, $b = 2$ from graphical numeric data.
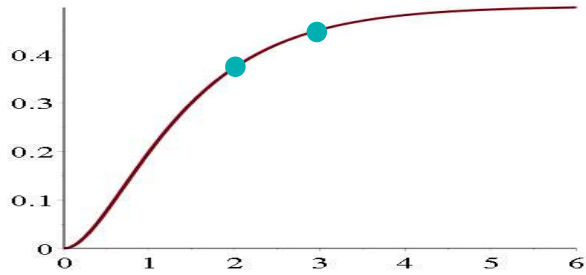
Figure 9. Oscilloscope output $x(t)$ for Example 8.31. The cyan dots are located at $(2, 0.374)$ and $(3, 0.451)$. The steady–state is $y_0 = 1/2$.

Solution:

The response curve has only one t-axis crossing, which classifies it nonoscillatory. The block type prediction is based upon seeing a response curve that starts at $(0, 0)$ and increases to nonzero steady–state, which is $y_0 = \frac{1}{2}$ in this example.

Choose two data points on the graphic, the cyan dots in the figure: $t = 2, y = 0.374$ and $t = 3, y = 0.451$. Let $F(t, a, b) = \frac{1}{ab} + \frac{1}{a^2 - ab} e^{-at} + \frac{1}{b^2 - ab} e^{-bt}$, which is symbolic solution **(2)** in Theorem 8.16. Solve the equations $F(2, a, b) = 0.374$, $F(3, a, b) = 0.451$ in a CAS or numerical workbench to get $b = 1.972417640$, $a = 1.017736613$. Because $F(t, a, b) = F(t, b, a)$, switching a, b values gives another solution. Only one of these meets requirement $a < b$. The `maple` code:
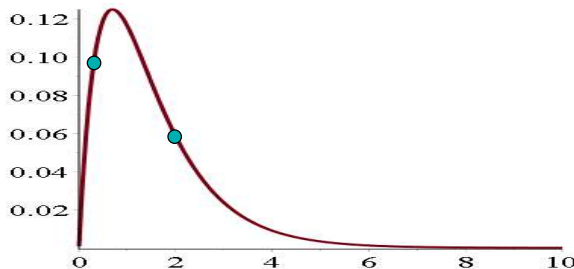
```
F:=(t,a,b)->1/(a*b)+exp(-a*t)/(a^2-a*b)+exp(-b*t)/(b^2-a*b);
fsolve({F(2,a,b)=0.374,F(3,a,b)=0.451},{a,b});
```

Example 8.32 (System Parameters: Over–Damped Impulse Input)

Oscilloscope data created Figure 10 from block state $x(0) = 0$, $x'(0) = 0$ and Dirac input. Explain why the graphic predicts over–damped model

$$x''(t) + (a + b)x'(t) + abx(t) = f(t), \quad \text{Transfer Function} = \frac{1}{(s + a)(s + b)}.$$

Then verify system parameters $a = 1$, $b = 2$ from graphical numeric data.

Figure 10. Oscilloscope output $x(t)$ for Example 8.32. The cyan dots are located at $(0.3, 0.0960)$ and $(2, 0.0585)$.

Solution: The graphic increases from $t = 0$, $x = 0$ to a maximum and after decreases to zero. The response curve has only one t-axis crossing, which classifies it non-oscillatory, hence over–damped.

Let $F(t, a, b) = \frac{1}{2} \frac{1}{b-a} \left(e^{-at} - e^{-bt} \right)$, which is symbolic solution **(3)** in Theorem 8.16 for $t > 0$. Choose two points on the experimental curve, for example the cyan dots in the figure $t = 0.3, x = 0.0960$ and $t = 2, x = 0.0585$. Solve for a, b in the two equations $F(0.3, a, b) = 0.0960$ and $F(2, a, b) = 0.0585$. The answer is $a = 2.000235495$, $b = 1.000004659$. The `maple` code:

```
F:=(t,a,b)->(1/2)*(exp(-a*t)-exp(-b*t))/(b-a);
fsolve({F(0.3,a,b)=0.0960,F(2,a,b)=0.0585},{a,b});
```

System Parameters for Under-Damped Problems

It is assumed that a signal $x(t)$ is known via numerical data for a graphic passing the following visual test:

> The curve starts at $t = 0$, $x = 0$ and has at least two local maxima on $t > 0$.

The initial data and the input are assumed to satisfy one of the following three cases.:

Zero Input: $x(0) = 0$, $x'(0) = 1$, $f(t) = 0$
Step Input: $x(0) = 0$, $x'(0) = 0$, $f(t) = k\,\mathbf{u}(t)$
Impulse Input: $x(0) = 0$, $x'(0) = 0$, $f(t) = \delta(t)$

A graphic that passes the above test predicts the model

$$x'' + 2ax' + \left(a^2 + b^2 \right) x = f(t), \quad \text{Transfer Function} = \frac{1}{(s + a)(s + b)}, \; b > 0.$$

The method computes numerical values for a, b from the graphical data. The computation finds a mathematical model for the block represented by the graphic.

Theorem 8.18 (Parameters for an Under–Damped Model)
Let $x(t)$ be the response curve for $x'' + 2ax' + \left(a^2 + b^2 \right) x = f(t)$ having a maximum at $t = t_1$, $x = x_1$ and next maximum at $t = t_2$, $x = x_2$.

If the input is $f(t) = 0$ or $f(t) = \delta(t)$, then the system parameters are

$$a = \frac{1}{t_2 - t_1} \ln \left| \frac{x_1}{x_2} \right| \quad \text{and} \quad b = \frac{2\pi}{t_2 - t_1}.$$

If the input is $f(t) = k\,\mathbf{u}(t)$ with steady–state solution y_0 different from x_1 and x_2, then the formulas are

$$a = \frac{1}{t_2 - t_1} \ln \left| \frac{x_1 - y_0}{x_2 - y_0} \right| \quad \text{and} \quad b = \frac{2\pi}{t_2 - t_1}.$$

Proof of Theorem 8.18, Underdamped Parameters:
For zero input or impulse input, the equation is $x'' + 2ax' + \left(a^2 + b^2 \right) x = 0$ for $t > 0$ with general solution $x = c_1 e^{-at} \cos(bt) + c_2 e^{-at} \sin(bt)$. Convert to the form $x = Ae^{-at} \cos(bt - \phi)$. For definiteness, assume $x_1 > 0$ and $x_2 > 0$.

Verify $b = \frac{2\pi}{t_2-t_1}$. At max $t = t_1$ the cosine factor must be a max, which is 1, so $bt_1 - \phi = n\pi$. At the next maximum $t = t_2$ the cosine factor must also be 1, so $bt_2 - \phi = n\pi + m\pi$. Because the maxima are consecutive, then $m = 2$ (the period of the cosine is 2π). Subtract the two equations to obtain $2\pi = (bt_2 - \phi) - (bt_1 - \phi) = b(t_2 - t_1)$ and solve for $b = \frac{2\pi}{t_2-t_1}$.

Verify $a = \frac{1}{t_2-t_1}\ln\left|\frac{x_1}{x_2}\right|$, $x_1 = Ae^{-at_1}\cos(bt_1 - \phi)$. Let $x_2 = Ae^{-at_2}\cos(bt_2 - \phi)$. Because $\cos(bt_1 - \phi) = \cos(bt_2 - \phi) = 1$, as argued above, then x_1 divided by x_2 gives $\frac{x_1}{x_2} = e^{at_2-at_1}$. Take the logarithm across this equality and use $\ln(e^u) = u$, then $\ln\left|\frac{x_1}{x_2}\right| = \ln(e^{at_2-at_1}) = a(t_2 - t_1)$. Solve for $a = \frac{1}{t_2-t_1}\ln\left|\frac{x_1}{x_2}\right|$.

**Case** $f(t) = k\,\mathbf{u}(t)$. The equation is $x'' + 2ax' + \left(a^2 + b^2\right)x = k$ with steady–state solution $y_0 = k/(a^2 + b^2)$ and homogeneous solution $x_h = c_1 e^{-at}\cos(bt) + c_2 e^{-at}\sin(bt)$. The change of variables $y(t) = x(t) - y_0$ changes $x'' + 2ax' + \left(a^2 + b^2\right)x = k$ into $y'' + 2ay' + \left(a^2 + b^2\right)y = 0$. Because $x'(t) = y'(t)$, the curves $x(t)$ and $y(t)$ have the same critical points. Further, solution $y(t)$ has consecutive local maxima at $t = t_1$, $y = x(t_1) - y_0 = x_1 - y_0$ and $t = t_2$, $y = x(t_2) - y_0 = x_2 - y_0$. The zero input case applies to compute the parameters $a = \frac{1}{t_2-t_1}\ln\left|\frac{x_1-y_0}{x_2-y_0}\right|$, $b = \frac{2\pi}{t_2-t_1}$.[13] ∎

Example 8.33 (Parameters: Under–Damped, Zero or Impulse Input)

Oscilloscope data created Figure 11 from block state $x(0) = 0$, $x'(0) = 1$ and zero input or block state $x(0) = 0$, $x'(0) = 0$ and impulse input. Explain why the graphic predicts under–damped model

$$x''(t) + 2ax'(t) + \left(a^2 + b^2\right)x(t) = f(t), \text{ Transfer Function} = \frac{1}{(s + a)^2 + b^2}.$$

Then verify system parameters $a = 1$, $b = 2$ from graphical numeric data.
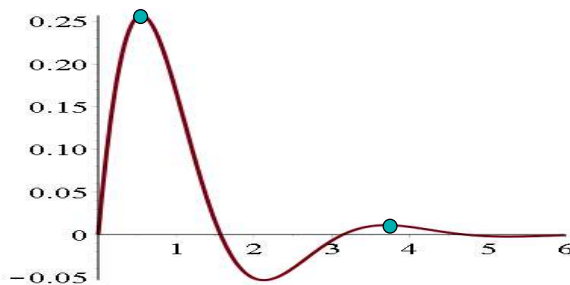
Figure 11. Oscilloscope output $x(t)$ for Example 8.33. The cyan dots are located at $(0.554, 0.257)$ and $(3.695, 0.011)$.

Solution: The curve in Figure 11 passes through $t = 0$, $x = 0$, then increases to a maximum and after decreases to a minimum. The curve crosses the t-axis twice, therefore it will have infinity many crossings of the t-axis: the system model is oscillatory under–damped.

---

[13]Oscilloscopes can display signal $y(t)$ directly, which simplifies external data processing to the zero input case. See Example 8.35.

Only numeric data for $x(t)$ is available and not the differential equation itself, so parameters a, b are unknown. Approximations to $a = 1$ and $b = 2$ are found directly from the numerical data used to plot Figure 11.

Choose two points on the curve, one at the first maximum and one at the very next maximum. These are the cyan dots in the figure, $t_1 = 0.554$, $x_1 = 0.257$ and $t_2 = 3.695$, $x_2 = 0.011$. Apply Theorem 8.18 to obtain $a = \dfrac{\ln(x_1/x_2)}{t_2 - t_1} = 1.003241265$ and $b = \dfrac{2\pi}{t_2 - t_1} = 2.000377366$.

Answer Check. Insert the parameters $a = 1$, $b = 2$ into the predicted model, then plot the symbolic response $x(t)$ for zero input (or $\delta(t)$ for impulse input). If the graphic matches Figure 11, then the computed parameters were likely correct.

Example 8.34 (System Parameters: Under–Damped Step Input)

Oscilloscope data created Figure 12 from block state $x(0) = 0$, $x'(0) = 0$ and unit step input. Explain why the graphic predicts under–damped model

$$x''(t) + 2ax'(t) + \left(a^2 + b^2\right) x(t) = f(t), \text{ Transfer Function} = \frac{1}{(s+a)^2 + b^2}.$$

Then verify system parameters $a = 1$, $b = 4$ from graphical numeric data.
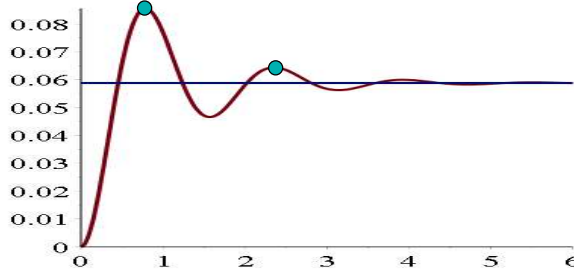
Figure 12. **Oscilloscope output $x(t)$ for Example 8.34.** The cyan dots are located at $(0.7854, 0.0856)$, $(2.3562, 0.0644)$. The steady–state is $y_0 = 1/17$.

Solution: The curve in Figure 12 passes through $t = 0$, $x = 0$, then increases to a local maximum and after decreases to a local minimum. The steady–state is $y_0 = 1/17$. The curve crosses the steady–state $y_0 = 1/17$ twice, therefore it will have infinity many crossings of $y_0 = 1/17$: the system model is oscillatory under–damped.

Choose two points on the curve, one at the first local maximum and one at the very next local maximum. These are the cyan dots in the figure, $t_1 = 0.7854$, $x_1 = 0.0856$, $t_2 = 2.3562$, $x_2 = 0.0644$. Let $y_0 = 1/17$. Apply Theorem 8.18 to obtain $a = \frac{1}{t_2-t_1} \ln \left| \frac{x_1-y_0}{x_2-y_0} \right| = 0.9988333798$ and $b = \frac{2\pi}{t_2-t_1} = 3.999990646$.

Answer Check. Insert the parameters $a = 1$, $b = 4$ into the predicted model, then plot the symbolic response $x(t)$ for $x(0) = x'(0) = 0$ and unit step input. If the graphic matches Figure 12, then the computed parameters are probably correct.

Example 8.35 (Special Case: Under–Damped Step Input)

Oscilloscope data created Figure 13 from block state $x(0) = 0$, $x'(0) = 1$ and unit step input. Explain why the graphic is missing a nonzero steady–state and the predicted model is

$$x''(t) + 2ax'(t) + \left(a^2 + b^2\right) x(t) = f(t), \text{ Transfer Function } = \frac{1}{(s + a)^2 + b^2}.$$

Discuss which formula to use when verifying $a = 1$, $b = 4$.
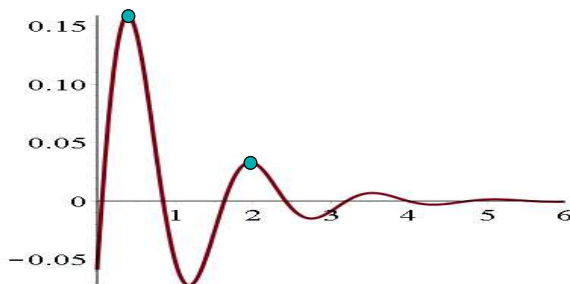
Figure 13. Oscilloscope output $x(t)$ for Example 8.35. The cyan dots are located at $(0.3927, 0.1580)$ and $(1.9635, 0.0330)$. The steady–state is $x = 0$.

Solution: The oscilloscope hardware made a change of variables $y = x - 1/17$ prior to display of the y–data. This changed the steady state from $y_0 = 1/17$ to $y_0 = 0$. The apparent oscillation of the graphic about $x = 0$ predicts the under–damped case: see the proof of Theorem 8.18 page 662.

The figure looks like the zero input case, because y satisfies the homogeneous equation $y'' + 2ay' + (a^2 + b^2)y = 0$ with steady–state zero and initial state $y(0) = -1/17$, $y'(0) = 1$. The single important distinction is that the graph fails to pass through $(0, 0)$.

Apply Theorem 8.18 with $y_0 = 0$ to obtain $a = \dfrac{1}{t_2 - t_1} \ln \left| \dfrac{x_1}{x_2} \right| = 1.0006$ and $b = \dfrac{2\pi}{t_2 - t_1} = 3.999991$.

Exercises 8.8 ☑

Oscillatory and Non–oscillatory

Assume $x'' + px' + qx = 0$ with p, q nonnegative.

Parameter p is imagined as a set screw adjustment on a screen door dashpot, larger p meaning more damping effect.

Parameter q is the Hooke's constant for the spring restoring force.

1. Let $q = 100$, $p = 99$. Verify that the equation is over–damped in two ways:
(1) Graph $x(t)$;

(2) Justify that $r^2 + pr + q = 0$ has real negative roots.

2. Let $q = 100$. The case which is called *critically–damped* happens at exactly one value $p = p^*$ between 0 and 99. Compute p^* numerically. Graph $x(t)$ using $q = 100$, $p = p^*$, $x(0) = 0$, $x'(0) = 1$.

3. Let $q = 100$. Verify that $p = 0$ produces the **harmonic oscillator** $x'' + \omega^2 x = 0$, $\omega = 10$.

Small set screw changes from $p = 0$ to $p > 0$ are still oscillatory. Under–damped means weak dashpot reaction.

4. Let $q = 100$, $p = 2$. Justify oscillatory under–damped from the graph of $x(t)$ and also by solving $r^2 + pr + q = 0$.

Simplistic Dirac Impulse

Define $g(t) = 7 e^{-153800\,t}\,\mathbf{u}(t)$ and
$f(t, a) = \frac{1}{a}\left(u(t) - u(t - a)\right)$, $a > 0$.
The impulse of force h is $\int_{-\infty}^{\infty} h(t)\, dt$.

5. Compute the impulse for $f(t, a)$.
Ans: 1.

6. Plot $f(t, a)$ for $a = 0.1$, 0.001, 0.0001.

7. Calculate the impulse for $g(t)$.
Ans: About 46 times 10^{-6}.

8. Try to find an **RC** discharge circuit with 10 volt *emf* and output $g(t)$.

Circuit response $g(t)$ simulates Dirac impulsive force $\frac{45.5}{1000000}\delta(t)$.

Parameters: Over–Damped

Find $a, b, \omega = \sqrt{ab}, \zeta = \frac{a+b}{2\omega}$ given the plot and two dots on the graph.

9. Step input Figure 9, dots
$(1, 0.1998)$, $(4, 0.4819)$.
Ans: $a = 1.0000$, $b = 1.9997$, $\omega = 1.4141$, $\zeta = 1.0607$.

10. Impulse input Figure 10, dots
$(0.5, 0.1193)$, $(2, 0.0585)$.
Ans: $a = 0.9991$, $b = 2.0021$, $\omega = 1.4143$, $\zeta = 1.0610$.

Parameters: Under–Damped

Find $a, b, \omega = \sqrt{a^2 + b^2}, \zeta = \frac{a}{\omega}$ given the plot and two dots on the graph.

11. Zero input like Figure 11, but consecutive maxima at $(2.5107, 0.0257)$, $(4.6051, 0.0032)$.
Ans: Approximately $a = 1$, $b = 3$.

12. Step input like Figure 13, but steady–state $y_0 = 1/26$ and consecutive maxima at $(0.6283, 0.0205)$, $(1.8850, 0.0058)$.
Ans: Approximately $a = 1$, $b = 5$.

PDF Sources

Text, Solutions and Corrections

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Paperback Textbook: There are 12 chapters on differential equations and linear algebra, book format 7 x 10 inches, 1077 pages. Copies of the textbook are available in two volumes at **Amazon** Kindle Direct Publishing for Amazon's cost of printing and shipping. No author profit. Volume I chapters 1-7, ISBN 9798705491124, 661 pages. Volume II chapters 8-12, ISBN 9798711123651, 479 pages. Both paperbacks have extra pages of backmatter: background topics Chapter A, the whole book index and the bibliography.

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