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Chapter 7

Topics in Linear Differential Equations

Contents

Developed here is the theory for higher order linear constant-coefficient differential equations. Besides a basic formula for the solution of such equations, extensions are developed for the topics of variation of parameters and undetermined coefficients.

Enrichment topics include the Cauchy-Euler differential equation, the Cauchy kernel for second order linear differential equations, and a library of special methods for undetermined coefficients methods, the latter having prerequisites of only basic calculus and college algebra. Developed within the library methods is a verification of the method of undetermined coefficients, via Kümmer's change of variable.

7.1 Higher Order Homogeneous

Presented here is a solution method for higher order linear differential equations with real constant coefficients

$$y^n + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0. \tag{1}$$

This topic was covered earlier, therefore the central purpose of this section is the collection of additional exercises. The only new topics have to do with factorization of polynomials and differential operators. The first subject has to do with efficiency, a shortcut to speed up the process of solving a constant-coefficient linear homogeneous differential equation.

How to Solve Higher Order Equations

The **Characteristic Equation** of (1) is the polynomial equation

$$r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0. \tag{2}$$

The left side of (2) is called the **Characteristic Polynomial**. We assume the coefficients are real numbers.

For a real root $r = a$ of the characteristic equation, symbol k equals its **Algebraic Multiplicity**. Then k is the maximum power such that $(r - a)^k$ divides the characteristic polynomial.

The same symbol k is used for the **algebraic multiplicity** of a complex root $r = a + ib$. Complex roots always come in pairs, $a \pm ib$, because the coefficients of the characteristic polynomial are *real*. This means k is the maximum power such that $((r - a)^2 + b^2)^k$ divides the characteristic polynomial.

Constructing the General Solution

The general solution y of (1) is constructed as a linear combination of n Euler atoms. The list of n Euler atoms is found from the roots of the characteristic equation, by iterating on Step I and Step II below.

Step I: Real Roots

Each multiplicity k real root $r = a$ of the characteristic equation produces a group of k Euler atoms

$$e^{rx},\ xe^{rx},\ \ldots,\ x^{k-1}e^{rx}$$

which are solutions of (1). Append the group to the list of Euler atoms for equation (1).

Step II: Complex Root pairs

Each multiplicity k pair of complex roots $z = a + ib$ and $\overline{z} = a - ib$ of the characteristic equation produces two groups of k distinct Euler atoms

group 1: $e^{ax}\cos bx,\ xe^{ax}\cos bx,\ \ldots,\ x^{k-1}e^{ax}\cos bx,$
group 2: $e^{ax}\sin bx,\ xe^{ax}\sin bx,\ \ldots,\ x^{k-1}e^{ax}\sin bx,$

which are solutions to the differential equation. Append the two groups to the list of Euler atoms for equation (1).

Exponential Solutions and Euler's Theorem

Characteristic equation (2) is formally obtained from the differential equation by replacing $y^{(k)}$ by r^k. This device for remembering how to form the characteristic equation is attributed to **Euler**, because of the following fact.

Theorem 7.1 (Euler's Exponential Substitution)
Let w be a real or complex number. The function $y(x) = e^{wx}$ is a solution of (1) if and only if $r = w$ is a root of the characteristic equation (2).

Steps I and II above are justified from Euler's basic result:

Theorem 7.2 (Euler's Multiplicity Theorem)
Function $y(x) = x^p e^{wx}$ is a solution of (1) if and only if $(r - w)^{p+1}$ divides the characteristic polynomial.

An Illustration of the Higher Order Method

Consider the problem of solving a constant coefficient linear differential equation (1) of order 11 having factored characteristic equation

$$(r-2)^3(r+1)^2(r^2+4)^2(r^2+4r+5) = 0.$$

To be applied is the solution method for higher order equations. Then **Step I** loops on the two linear factors $r-2$ and $r+1$, while **Step 2** loops on the two real quadratic factors r^2+4 and r^2+4r+5.

Hand solutions can be organized by a tabular method for generating the general solution y. The key element is that rows are distinct factors of the characteristic polynomial. This feature insures that each row contains distinct atoms not duplicated in another row.

Factor	Roots	Multiplicity	Atom Groups
$(r-2)^3$	$r = 2, 2, 2$	3	$e^{2x}, xe^{2x}, x^2e^{2x}$
$(r+1)^2$	$r = 1, 1$	2	e^x, xe^x
$(r^2+4)^2$	$r = \pm 2i, \pm 2i$	2	$\cos 2x, x\cos 2x$ $\sin 2x, x\sin 2x$
$(r+2)^2+1$	$r = -2 \pm i$	1	$e^{2x}\cos x$ $e^{2x}\sin x$

The equation has order $n = 11$. Symbols $c_1, \ldots, c_n$ will represent arbitrary constants in the general solution y. A real root of multiplicity k will consume k of these symbols, while a complex conjugate pair of roots of multiplicity k consumes $2k$ symbols. The number of terms added in Step I equals the multiplicity of the root, or twice that in Step II, the case of complex roots. The symbols are used in order, as the general solution is constructed, as follows.

Root(s)	Count	Solution Terms Added
$r = 2, 2, 2$	3	$(c_1 + c_2 x + c_3 x^2)e^{2x}$
$r = -1, -1$	2	$(c_4 + c_5 x)e^{-x}$
$r = \pm 2i, \pm 2i$	4	$(c_6 + c_7 x)\cos 2x + (c_8 + c_9 x)\sin 2x$
$r = -2 \pm i$	2	$c_{10}e^{-2x}\cos x + c_{11}e^{-2x}\sin x$

Then the general solution is

$$\begin{aligned} y \quad = \quad & (c_1 + c_2 x + c_3 x^2)e^{2x} \\ & +(c_4 + c_5 x)e^{-x} \\ & +(c_6 + c_7 x)\cos 2x + (c_8 + c_9 x)\sin 2x \\ & +c_{10}e^{-2x}\cos x + c_{11}e^{-2x}\sin x. \end{aligned}$$

Computer Algebra System Solution

The system `maple` can symbolically solve a higher order equation. Below, `@` is the function composition operator, `@@` is the repeated composition operator and `D` is the differentiation operator. The coding writes the factors of

$$(r-2)^3(r+1)^2(r^2+4)^2(r^2+4r+5)$$

as differential operators $(D-2)^3$, $(D+1)^2$, $(D^2+4)^2$, D^2+4D+5. Then the differential equation is the composition of the component factors. See the next section for details about differential operators.

```
id:=x->x;
F1:=(D-2*id) @@ 3;
F2:=(D+id) @@ 2;
F3:=(D@D+4*id) @@ 2;
F4:=D@D+4*D+5*id;
de:=(F1@F2@F3@F4)(y)(x)=0:
dsolve({de},y(x));
```

Exercises 7.1

Higher Order Factored

Solve the higher order equation with the given characteristic equation. Display the roots according to multiplicity and list the corresponding solution atoms.

1. $(r-1)(r+2)(r-3)^2 = 0$

2. $(r-1)^2(r+2)(r+3) = 0$

3. $(r-1)^3(r+2)^2r^4 = 0$

4. $(r-1)^2(r+2)^3r^5 = 0$

5. $r^2(r-1)^2(r^2+4r+6) = 0$

6. $r^3(r-1)(r^2+4r+6)^2 = 0$

7. $(r-1)(r+2)(r^2+1)^2 = 0$

8. $(r-1)^2(r+2)(r^2+1) = 0$

9. $(r-1)^3(r+2)^2(r^2+4) = 0$

10. $(r-1)^4(r+2)(r^2+4)^2 = 0$

Higher Order Unfactored

Completely factor the given characteristic equation, then the roots according to multiplicity and the solution atoms.

11. $(r-1)(r^2-1)^2(r^2+1)^3 = 0$

12. $(r+1)^2(r^2-1)^2(r^2+1)^2 = 0$

13. $(r+2)^2(r^2-4)^2(r^2+16)^2 = 0$

14. $(r+2)^3(r^2-4)^4(r^2+5)^2 = 0$

15. $(r^3-1)^2(r-1)^2(r^2-1) = 0$

16. $(r^3-8)^2(r-2)^2(r^2-4) = 0$

17. $(r^2-4)^3(r^4-16)^2 = 0$

18. $(r^2+8)(r^4-64)^2 = 0$

19. $(r^2-r+1)(r^3+1)^2 = 0$

20. $(r^2+r+1)^2(r^3-1) = 0$

Higher Order Equations

The exercises study properties of Euler atoms and nth order linear differential equations.

21. (Euler's Theorem)
Explain why the derivatives of atom x^3e^x satisfy a higher order equation with characteristic equation $(r-1)^4 = 0$.

22. (Euler's Theorem)
Explain why the derivatives of atom $x^3\sin x$ satisfy a higher order equation with characteristic equation $(r^2+1)^4 = 0$.

23. (Kümmer's Change of Variable)
Consider a fourth order equation with characteristic equation $(r+a)^4 = 0$ and general solution y. Define $y = ue^{-ax}$. Find the differential equation for u and solve it. Then solve the original differential equation.

24. (Kümmer's Change of Variable)
A polynomial $u = c_0 + c_1x + c_2x^2$ satisfies $u''' = 0$. Define $y = ue^{ax}$. Prove that y satisfies a third order equation and determine its characteristic equation.

25. (Ziebur's Derivative Lemma)
Let y be a solution of a higher order constant-coefficient linear equation. Prove that the derivatives of y satisfy the same differential equation.

26. (Ziebur's Lemma: atoms)
Let $y = x^3e^x$ be a solution of a higher order constant-coefficient linear equation. Prove that Euler atoms e^x, xe^x, x^2e^x are solutions of the same differential equation.

27. (Ziebur's Atom Lemma)
Let y be an Euler atom solution of a higher order constant-coefficient linear equation. Prove that the Euler atoms extracted from the expressions $y, y', y'', \ldots$ are solutions of the same differential equation.

28. (Differential Operators)
Let y be a solution of a differential equation with characteristic equation $(r-1)^3(r+2)^6(r^2+4)^5 = 0$. Explain why y''' is a solution of a differential equation with characteristic equation $(r-1)^3(r+2)^6(r^2+4)^5 r^3 = 0$.

29. (Higher Order Algorithm)
Let atom $x^2 \cos x$ appear in the general solution of a linear higher order equation. Find the pair of complex conjugate roots that constructed this atom, and the multiplicity k. Report the $2k$ atoms which must also appear in the general solution.

30. (Higher Order Algorithm)
Let Euler atom $xe^x \cos 2x$ appear in the general solution of a linear higher order equation. Find the pair of complex conjugate roots that constructed this atom and estimate the multiplicity k. Report the $2k$ atoms which are expected to appear in the general solution.

31. (Higher Order Algorithm)
Let a higher order equation have characteristic equation $(r-9)^3(r-5)^2(r^2+4)^5 = 0$. Explain precisely using existence-uniqueness theorems why the general solution is a sum of constants times Euler atoms.

32. (Higher Order Algorithm)
Explain why any higher order linear homogeneous constant-coefficient differential equation has general solution a sum of constants times Euler atoms.

7.2 Differential Operators

A polynomial in the symbol $D = d/dx$ is called a **Differential Operator** and the formal manipulation of these expressions is called an **Operational Calculus**.

The meaning of an expression such as $D^2 + 3D + 2$ is through linearity, $[D^2 + 3D+2]y$ meaning $D^2y+3Dy+2y$, and each term has the corresponding meaning

$$Dy = y'(x), \quad D^2y = y''(x), \quad \cdots$$

Products of the expressions are defined through composition. For example, $(D+1)(D+2)y$ means $(D+1)(y'+2y)$, which in turn is defined to be $(y'+2y)'+(y'+2y)$. This example suggests that expansion of such factored products is identical to expansion of polynomial $(x + 1)(x + 2)$ into $x^2 + 3x + 2$.

Theorem 7.3 (Commuting Operators)
Let $P = p_0 + \cdots + p_nD^n$ and $Q = q_0 + \cdots + q_mD^m$ be two differential operators with constant coefficients. Define $R = r_0 + \cdots + r_kD^k$ to be the polynomial product expansion of P and Q. Then for every infinitely differentiable function $y(x)$,

$$P(Qy) = Q(Py) = Ry.$$

In short, P and Q commute and their product in either order is the formal expanded polynomial product.

Proof: Define $p_i = 0$ for $i > n$ and $q_j = 0$ for $j > m$, so that P and Q can be written as infinite series. The Cauchy product theorem from series implies that $r_\ell = p_0q_\ell+\cdots+p_\ell q_0$. By definitions, and the Cauchy product theorem,

$$\begin{aligned}
P(Qy) &= \textstyle\sum_{i=0}^\infty p_iD^i(Qy) \\
&= \textstyle\sum_{i=0}^\infty p_iD^i\left(\sum_{j=0}^\infty q_jy^{(j)}\right) \\
&= \textstyle\sum_{i=0}^\infty p_i\left(\sum_{j=0}^\infty q_jy^{(i+j)}\right) \\
&= \textstyle\sum_{i=0}^\infty \sum_{j=0}^\infty p_iq_jy^{(i+j)} \\
&= \textstyle\sum_{\ell=0}^\infty \sum_{j=0}^\ell p_{\ell-j}q_jy^{(\ell)} \\
&= \textstyle\sum_{\ell=0}^\infty r_\ell y^{(\ell)} \\
&= Ry
\end{aligned}$$

Because the series product in reverse order gives the identical answer, the proof is complete.

Factorization

The fundamental theorem of algebra implies that the characteristic equation of a real nth order linear constant-coefficient differential equation has exactly n roots, counted according to multiplicity. Some number of the roots are real and the remaining roots appear in complex conjugate pairs. This implies that every characteristic equation has a **factored form**

$$(r - a_1)^{k_1} \cdots (r - a_q)^{k_q} Q_1(r)^{m_1} \cdots Q_p(r)^{m_p} = 0$$

where $a_1, \ldots, a_q$ are the **distinct real roots** of the characteristic equation of algebraic multiplicities $k_1, \ldots, k_q$, respectively. Factors $Q_1(r), \ldots, Q_p(r)$ are the distinct real quadratic factors of the form $(r - z)(r - \overline{z})$. Symbol z exhausts the **distinct complex roots** $z = a + ib$ with $b > 0$, having corresponding algebraic multiplicities $m_1, \ldots, m_p$. The quadratic $(r - z)(r - \overline{z})$ is normally written $(r - a)^2 + b^2$.

General Solution

An nth order linear homogeneous differential equation with real constant coefficients can be written in D-operator notation via the distinct real linear and quadratic factors of the characteristic equation as

$$\left((D - a_1)^{k_1} \cdots (D - a_q)^{k_q} Q_1(D)^{m_1} \cdots Q_p(D)^{m_p}\right) y = 0.$$

For $Q = (r - a)^2 + b^2$, symbol $Q(D) = (D - a)^2 + b^2$.

Picard's theorem on existence-uniqueness fixes the possible number of independent solutions at exactly n, the order of the differential equation. Each factor, real or quadratic, generates a certain number of distinct Euler solution atoms, the union of which counts to exactly n independent atoms, forming a solution basis for the differential equation.

Specifically, the general solution of

$$(D - a)^{k+1} y = 0$$

is a polynomial $u = c_0 + c_1 x + \cdots + c_k x^k$ with $k + 1$ terms times e^{ax}. This fact is proved by Kümmer's change of variable $y = e^{ax} u$, which finds an equivalent equation $D^{k+1} u = 0$, solvable by quadrature. Details in the exercises.

The general solution of

$$((D - a)^2 + b^2)^{k+1} y = 0$$

is a real polynomial $u_1 = a_0 + \cdots + a_k x^k$ with $k + 1$ terms times $e^{ax}\cos(bx)$ plus a real polynomial $u_2 = b_0 + \cdots + b_k x^k$ with $k + 1$ terms times $e^{ax}\sin(bx)$.

Technical details: Kümmer's change of variable $y = e^{ax} u$ transforms to the equation $(D^2 + b^2)^{k+1} u = 0$. Because $D^2 + b^2 = (D - ib)(D + ib)$, the work done in the preceding paragraph applies, resulting in solutions that are polynomials with $k+1$ terms times e^{ibx} and e^{-ibx}. Taking real and imaginary parts of these solutions give the real solutions $u_1 \cos(bx)$, $u_2 \sin(bx)$. Transforming back multiplies these answers by e^{ax}.

Exercises 7.2

Operator Arithmetic

Compute the operator and solve the corresponding differential equation.

1. $D(D+1)+D$

2. $D(D+1)+D(D+2)$

3. $D(D+1)^2$

4. $D(D^2+1)^2$

5. $D^2(D^2+4)^2$

6. $(D-1)((D-1)^2+1)^2$

Operator Properties.

7. **(Operator Composition)** Multiply $P = D^2 + D$ and $Q = 2D + 3$ to get $R = 2D^3 + 5D^2 + 3D$. Then compute $P(Qy)$ and $Q(Py)$ for $y(x)$ 3-times differentiable, and show both equal Ry.

8. **(Kernels)**
The operators $(D-1)^2(D+2)$ and $(D-1)(D+2)^2$ share common factors. Find the Euler solution atoms shared by the corresponding differential equations.

9. **(Operator Multiply)**
Let differential equation $(D^2 + 2D + 1)y = 0$ be formally differentiated four times. Find its operator and solve the equation. What does this have to do with operator multiply?

10. **(Non-homogeneous Equation)** The differential equation $(D^5 + 4D^3)y = 0$ can be viewed as $(D^2 + 4)u = 0$ and $u = D^3y$. On the other hand, y is a linear combination of the atoms generated from the characteristic equation $r^3(r^2 + 4) = 0$. Use these facts to find a particular solution of the non-homogeneous equation $y''' = 3\cos 2x$.

Kümmer's Change of Variable

Kümmer's change of variable $y = ue^{ax}$ changes a y-differential equation into a u-differential equation. It can be used as a basis for solving homogeneous nth order linear constant coefficient differential equations.

11. Supply details: $y = ue^{ax}$ changes $y'' = 0$ into $u'' + 2au' + a^2u = 0$.

12. Supply details: $y = ue^{ax}$ changes $(D^2 + 4D)y = 0$ into $((D + a)^2 + 4(D + a))u = 0$.

13. Supply details: $y = ue^{ax}$ changes the differential equation $D^n y = 0$ into $(D + a)^n u = 0$.

14. Kümmer's substitution $y = ue^{ax}$ changes the differential equation $(D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = 0$ into $(F^n + a_{n-1}F^{n-1} + \cdots + a_0)u = 0$, where $F = D + a$. Write the proof.

7.3 Higher Order Non-Homogeneous

Continued here is the study of higher order linear differential equations with real constant coefficients.

The **homogeneous equation** is

$$y^n + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0. \tag{1}$$

The variation of parameters formula and the method of undetermined coefficients are discussed for the associated **non-homogeneous equation**

$$y^n + a_{n-1}y^{(n-1)} + \cdots + a_0 y = f(x). \tag{2}$$

Variation of Parameters Formula

The Picard-Lindelöf theorem implies that on $(-\infty, \infty)$ there a unique solution of the initial value problem

$$\begin{aligned} &y^n + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0, \\ &y(0) = \cdots = y^{(n-2)}(0) = 0, \quad y^{(n-1)}(0) = 1. \end{aligned} \tag{3}$$

The unique solution is called **Cauchy's kernel**, written $\mathcal{K}(x)$.

To illustrate, Cauchy's kernel $\mathcal{K}(x)$ for $y''' - y'' = 0$ is obtained from its general solution $y = c_1 + c_2 x + c_3 e^x$ by computing the values of the constants c_1, c_2, c_3 from initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, giving $\mathcal{K}(x) = e^x - x - 1$.

Theorem 7.4 (Higher Order Variation of Parameters)
Let $y^n + a_{n-1}y^{(n-1)} + \cdots + a_0 y = f(x)$ have constant coefficients $a_0, \ldots, a_{n-1}$ and continuous forcing term $f(x)$. Denote by $\mathcal{K}(x)$ Cauchy's kernel for the homogeneous differential equation. Then a particular solution is given by the **Variation of Parameters Formula**

$$y_p(x) = \int_0^x \mathcal{K}(x-u)f(u)du. \tag{4}$$

This solution has zero initial conditions $y(0) = \cdots = y^{(n-1)}(0) = 0$.

Proof: Define $y(x) = \int_0^x \mathcal{K}(x-u)f(u)du$. Compute by the 2-variable chain rule applied to $F(x,y) = \int_0^x \mathcal{K}(y-u)f(u)du$ the formulas

$$\begin{aligned} y(x) &= F(x,x) \\ &= \textstyle\int_0^x \mathcal{K}(x-u)f(u)du, \\ y'(x) &= F_x(x,x,) + F_y(x,x) \\ &= \mathcal{K}(x-x)f(x) + \textstyle\int_0^x \mathcal{K}'(x-u)f(u)du \\ &= 0 + \textstyle\int_0^x \mathcal{K}'(x-u)f(u)du. \end{aligned}$$

The process can be continued to obtain for $0 \le p < n-1$ the general relation

$$y^{(p)}(x) = \int_0^x \mathcal{K}^{(p)} f(u)du.$$

The relation justifies the initial conditions $y(0) = \cdots = y^{(n-1)}(0) = 0$, because each integral is zero at $x = 0$. Take $p = n - 1$ and differentiate once again to give

$$y^{(n)}(x) = \mathcal{K}^{(n-1)}(x - x)f(x) + \int_0^x \mathcal{K}^{(n)} f(u)du.$$

Because $\mathcal{K}^{(n-1)}(0) = 1$, this relation implies

$$y^{(n)} + \sum_{p=0}^{n-1} a_p y^{(p)} = f(x) + \int_0^x \left(\mathcal{K}^{(n)}(x-u) + \sum_{p=0}^{n-1} a_p \mathcal{K}^{(p)}(x-u)\right) f(u)du.$$

The sum under the integrand on the right is zero, because Cauchy's kernel satisfies the homogeneous differential equation. This proves $y(x)$ satisfies the nonhomogeneous differential equation. ■

Undetermined Coefficients Method

The method applies to higher order nonhomogeneous linear differential equations with real constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = f(x). \tag{5}$$

It finds a particular solution y_p of (5) *without* the integration steps present in variation of parameters. The theory was already presented earlier, for the special case of second order differential equations. The contribution of this section is a higher order example and more exercises. The term **Euler atom** is an abbreviation for the phrase *Euler solution atom of a constant-coefficient linear homogeneous differential equation*. A **base atom** is one of e^{ax}, $e^{ax}\cos bx$, $e^{ax}\sin bx$ where symbols a and b are real constants with $b > 0$. Euler atoms are x^n times a base atom $n = 0, 1, 2, 3, \ldots$.

Requirements and limitations:

1. The coefficients on the left side of (5) are constant.
2. The function $f(x)$ is a sum of constants times atoms.

Method of Undetermined Coefficients

Step 1. Define the list of k atoms in a trial solution using Rule I and Rule II [details below]. Multiply these atoms by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define **trial solution** y.

Step 2. Substitute y into the differential equation.

Step 3. Match coefficients of Euler atoms left and right to write out linear algebraic equations for unknowns $d_1, d_2, \ldots, d_k$. Solve the equations.

Step 4. The trial solution y with evaluated coefficients $d_1, d_2, \ldots, d_k$ becomes the particular solution y_p.

Undetermined Coefficients Rule I

Assume $f(x)$ in the equation $y^{(n)} + \cdots + a_0 y = f(x)$ is a sum of constants times Euler atoms. For each atom A appearing in $f(x)$, extract all distinct atoms that appear in A, A', A'', ..., then collect all these computed atoms into a list of k distinct Euler atoms.

If the list **contains a solution of the homogeneous differential equation**, then Rile I FAILS. Otherwise, multiply the k atoms by undetermined coefficients d_1, ..., d_k to form trial solution

$$y = d_1(\text{atom 1}) + d_2(\text{atom 2}) + \cdots + d_k(\text{atom k}).$$

Undetermined Coefficients Rule II

Assume Rule I constructed a list of k Euler atoms but **FAILED**. The particular solution y_p is still a sum of constants times k atoms. Rule II changes some or all of the k atoms, by repeated multiplication by x.

The k-atom list is subdivided into groups with the same base atom, called **group 1**, **group 2**, and so on. Test each group for a solution of the homogeneous differential equation. If found, then multiply each atom in the group by factor x. Repeat until **no group contains a solution of the homogeneous differential equation**. The final set of k Euler atoms is used to construct trial solution

$$y = d_1(\text{atom 1}) + d_2(\text{atom 2}) + \cdots + d_k(\text{atom k}).$$

A Common Difficulty

An able and earnest student working on undetermined coefficients writes:

> *I substituted trial solution y into the differential equation, but then I couldn't solve the equations. What's wrong?*

Trial solution substitution can result in a missing variable d_p on the left. It happens *exactly* when the trial solution contains a term $d_p A$, where A is an Euler solution atom of the homogeneous equation.

To illustrate, suppose $y = d_1 x + d_2 x^2$ is substituted into left side of the differential equation $y''' - y'' = x + x^2$ to get

$$\begin{array}{lcl} d_1[(x)''' - (x)''] & + & d_2[(x^2)''' - (x^2)''] & = & x + x^2, \\ d_1[0] & + & d_2[-2] & = & x + x^2. \end{array}$$

Then d_1 vanishes from the left side, because $(x)''' - (x)''$ evaluates to zero! Equation $(x)''' - (x)'' = 0$ means function $y(x) = x$ is a solution of the homogeneous differential equation for $y''' - y'' = f(x)$. Then d_1 is a **free variable** in the linear algebra problem. The other coefficient d_2 is determined to be zero. The nonsense equation $0 = x + x^2$ tells us **we chose the wrong trial solution**.

What caused the missing variable? Function $y = x$ was a solution of the homogeneous differential equation for $y''' - y'' = x + x^2$.

To prevent the error, test the trial solution before substitution:

Search the Euler atom list for trial solution y for a solution of the homogeneous equation – there shouldn't be any!

The *test* should be used before embarking upon the time–consuming task of writing the linear algebraic equations and solving them.

Illustration: nth Order Undetermined Coefficients

Let's solve

$$y''' - y'' = xe^x + 2x + 1 + 3\sin x$$

Answer:

$$y_p(x) = -\frac{3}{2}x^2 - \frac{1}{3}x^3 - 2xe^x + \frac{1}{2}x^2e^x + \frac{3}{2}\cos x + \frac{3}{2}\sin x.$$

Solution:

Check Applicability. The right side $f(x) = xe^x + 2x + 1 + 3\sin x$ is a sum of terms constructed from Euler atoms xe^x, x, 1, $\sin x$. The left side has constant coefficients. Therefore, the method of undetermined coefficients applies to find a particular solution y_p.

Homogeneous solution. The equation $y''' - y'' = 0$ has general solution y_h equal to a linear combination of Euler atoms 1, x, e^x.

Rule I. The Euler atoms found in $f(x)$ are subjected to repeated differentiation. The six distinct atoms so found are 1, x, e^x, xe^x, $\cos x$, $\sin x$ (drop coefficients to identify new atoms). Three of these are solutions of the homogeneous equation: **Rule I FAILS**.

Rule II. Divide the list of six atoms 1, x, e^x, xe^x, $\cos x$, $\sin x$ into four groups with identical base atom:

Group	Euler Atoms	Base Atom
group 1 :	$1, x$	1
group 2 :	e^x, xe^x	e^x
group 3 :	$\cos x$	$\cos x$
group 4 :	$\sin x$	$\sin x$

Group 1 contains a solution of the homogeneous equation $y''' - y'' = 0$. Rule II says to multiply **group 1** by x. Rule II is repeated, because the new group x, x^2 still contains a solution of the homogeneous equation. The process stops with new group x^2, x^3. **Group 2** contains solution e^x of the homogeneous equation. Rule II says to multiply group 2 by x. The new group xe^x, x^2e^x contains no solution of the homogeneous differential equation $y'' - y = 0$ .The last two groups are unchanged, because neither contains a solution of the homogeneous equation. Then

Group	Atoms	Action
New group 1 :	x^2, x^3	multiplied by x twice
New group 2 :	xe^x, x^2e^x	multiplied once by x
group 3 :	$\cos x$	unchanged
group 4 :	$\sin x$	unchanged

The final groups have been found. The shortest trial solution is

$$\begin{aligned} y &= \text{linear combination of atoms in the new groups} \\ &= d_1x^2 + d_2x^3 + d_3xe^x + d_4x^2e^x + d_5\cos x + d_6\sin x. \end{aligned}$$

Equations for d_1 to d_6. Substitution of trial solution y into $y''' - y''$ requires formulas for y', y'', y''':

$$\begin{aligned} y' &= 2\,d_1x + 3\,d_2x^2 + d_3e^xx + d_3e^x + 2\,d_4xe^x + d_4x^2e^x \\ &\quad - d_5\sin(x) + d_6\cos(x), \\ y'' &= 2\,d_1 + 6\,d_2x + d_3e^xx + 2\,d_3e^x + 2\,d_4e^x + 4\,d_4xe^x + d_4x^2e^x \\ &\quad - d_5\cos(x) - d_6\sin(x), \\ y''' &= 6\,d_2 + d_3e^xx + 3\,d_3e^x + 6\,d_4e^x + 6\,d_4xe^x + d_4x^2e^x \\ &\quad + d_5\sin(x) - d_6\cos(x) \end{aligned}$$

Then

$$\begin{aligned} f(x) &= y''' - y'' && \text{Given equation.} \\ &= 6d_2 - 2d_1 - 6d_2x + (d_3+4d_4)e^x + 2d_4xe^x && \text{Substitute, then} \\ &\quad + (d_5-d_6)\cos(x) + (d_5+d_6)\sin(x) && \text{collect on atoms.} \end{aligned}$$

Because $f(x) \equiv 1 + 2x + xe^x + 3\sin x$, then two linear combinations of the same set of six Euler atoms are equal:

$$\begin{aligned} 1 + 2x + xe^x + 3\sin x &= (6d_2 - 2d_1)(1) + (-6d_2)x \\ &\quad + (d_3+4d_4)e^x + (2d_4)xe^x \\ &\quad + (d_5-d_6)\cos(x) + (d_5+d_6)\sin(x). \end{aligned}$$

Coefficients of Euler atoms on the left and right must match, by independence of atoms. Write out the equations for matching coefficients:

$$\begin{aligned} -2d_1 + 6d_2 &= 1, \\ -6d_2 &= 2, \\ d_3 + 4d_4 &= 0, \\ 2d_4 &= 1, \\ d_5 - d_6 &= 0, \\ d_5 + d_6 &= 3. \end{aligned}$$

Solve. The first four equations can be solved by back-substitution to give $d_2 = -1/3$, $d_1 = -3/2$, $d_4 = 1/2$, $d_3 = -2$. The last two equations are solved by elimination or Cramer's rule to give $d_5 = 3/2$, $d_6 = 3/2$.

Report y_p. The corrected trial solution y with evaluated coefficients d_1 to d_6 becomes the particular solution

$$y_p(x) = -\frac{3}{2}x^2 - \frac{1}{3}x^3 - 2xe^x + \frac{1}{2}x^2e^x + \frac{3}{2}\cos x + \frac{3}{2}\sin x.$$

Exercises 7.3

Variation of Parameters

Solve the higher order equation given by its characteristic equation and right side $f(x)$. Display the Cauchy kernel $\mathcal{K}(x)$ and a particular solution $y_p(x)$ with fewest terms. Use a computer algebra system to evaluate integrals, if possible.

1. $(r-1)(r+2)(r-3)^2 = 0$,
$f(x) = e^x$

2. $(r-1)^2(r+2)(r+3) = 0$,
$f(x) = e^x$

3. $(r-1)^3(r+2)^2r^4 = 0$,
$f(x) = x + e^{-2x}$

4. $(r-1)^2(r+2)^3r^5 = 0$,
$f(x) = x + e^{-2x}$

5. $r^2(r-1)^2(r^2+4r+6) = 0$,
$f(x) = x + e^x$

6. $r^3(r-1)(r^2+4r+6)^2 = 0$,
$f(x) = x^2 + e^x$

7. $(r-1)(r+2)(r^2+1)^2 = 0$,
$f(x) = \cos x + e^{-2x}$

8. $(r-1)^2(r+2)(r^2+1) = 0$,
$f(x) = \sin x + e^{-2x}$

9. $(r-1)^3(r+2)^2(r^2+4) = 0$,
$f(x) = \cos 2x + e^x$

10. $(r-1)^4(r+2)(r^2+4)^2 = 0$,
$f(x) = \sin 2x + e^x$

Undetermined Coefficient Method

A higher order equation is given by its characteristic equation and right side $f(x)$. Display (a) a trial solution, (b) a system of equations for the undetermined coefficients, and (c) a particular solution $y_p(x)$ with fewest terms. Use a computer algebra system to solve for undetermined coefficients, if possible.

11. $(r-1)(r+2)(r-3)^2 = 0$,
$f(x) = e^x$

12. $(r-1)^2(r+2)(r+3) = 0$,
$f(x) = e^x$

13. $(r-1)^3(r+2)^2r^4 = 0$,
$f(x) = x + e^{-2x}$

14. $(r-1)^2(r+2)^3r^5 = 0$,
$f(x) = x + e^{-2x}$

15. $r^2(r-1)^2(r^2+4r+6) = 0$,
$f(x) = x + e^x$

16. $r^3(r-1)(r^2+4r+6)^2 = 0$,
$f(x) = x^2 + e^x$

17. $(r-1)(r+2)(r^2+1)^2 = 0$,
$f(x) = \cos x + e^{-2x}$

18. $(r-1)^2(r+2)(r^2+1) = 0$,
$f(x) = \sin x + e^{-2x}$

19. $(r-1)^3(r+2)^2(r^2+4) = 0$,
$f(x) = \cos 2x + e^x$

20. $(r-1)^4(r+2)(r^2+4)^2 = 0$,
$f(x) = \sin 2x + e^x$

7.4 Cauchy-Euler Equation

The differential equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_0 y = 0$$

is called the **Cauchy-Euler** differential equation of order n. The symbols a_i, $i = 0, \ldots, n$ are constants and $a_n \neq 0$.

The Cauchy-Euler equation is important in the theory of linear differential equations because it has direct application to **Fourier's method** in the study of partial differential equations. In particular, the second order Cauchy-Euler equation

$$ax^2 y'' + bxy' + cy = 0$$

accounts for the bulk of such applications in applied literature.

A second argument for studying the Cauchy-Euler equation is theoretical: it is a single example of a differential equation with non-constant coefficients that has a known closed-form solution. This fact is due to a change of variables $(x, y) \longrightarrow (t, z)$ given by equations

$$x = e^t, \quad z(t) = y(x),$$

which changes the Cauchy-Euler equation into a constant-coefficient differential equation. Since the constant-coefficient equations have closed-form solutions, so also do the Cauchy-Euler equations.

Theorem 7.5 (Cauchy-Euler Equation)
The change of variables $x = e^t$, $z(t) = y(e^t)$ transforms the Cauchy-Euler equation

$$ax^2 y'' + bxy' + cy = 0$$

into its equivalent constant-coefficient equation

$$a\frac{d}{dt}\left(\frac{d}{dt} - 1\right) z + b\frac{d}{dt} z + cz = 0.$$

The result is memorized by the general differentiation formula

$$x^k y^{(k)}(x) = \frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - k + 1\right) z(t). \tag{1}$$

Proof: The equivalence is obtained from the formulas

$$y(x) = z(t), \quad xy'(x) = \frac{d}{dt} z(t), \quad x^2 y''(x) = \frac{d}{dt}\left(\frac{d}{dt} - 1\right) z(t)$$

by direct replacement of terms in $ax^2 y'' + bxy' + cy = 0$. It remains to establish the general identity (1), from which the replacements arise.

The method of proof is mathematical induction. The induction step uses the chain rule of calculus, which says that for $y = y(x)$ and $x = x(t)$,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

The identity (1) reduces to $y(x) = z(t)$ for $k = 0$. Assume it holds for a certain integer k; we prove it holds for $k+1$, completing the induction.

Let us invoke the induction hypothesis LHS = RHS in (1) to write

$$\begin{aligned}
\frac{d}{dt}\mathsf{RHS} &= \frac{d}{dt}\mathsf{LHS} && \text{Reverse sides.}\\
&= \frac{dx}{dt}\frac{d}{dx}\mathsf{LHS} && \text{Apply the chain rule.}\\
&= e^t\frac{d}{dx}\mathsf{LHS} && \text{Use } x = e^t,\ dx/dt = e^t.\\
&= x\frac{d}{dx}\mathsf{LHS} && \text{Use } e^t = x.\\
&= x\left(x^k y^{(k)}(x)\right)' && \text{Expand with } ' = d/dx.\\
&= x\left(kx^{k-1}y^{(k)}(x) + x^k y^{(k+1)}(x)\right) && \text{Apply the product rule.}\\
&= k\,\mathsf{LHS} + x^{k+1}y^{(k+1)}(x) && \text{Use } x^k y^{(k)}(x) = \mathsf{LHS}.\\
&= k\,\mathsf{RHS} + x^{k+1}y^{(k+1)}(x) && \text{Use hypothesis LHS = RHS.}
\end{aligned}$$

Solve the resulting equation for $x^{k+1}y^{(k+1)}$. The result completes the induction. The details, which prove that (1) holds with k replaced by $k+1$:

$$\begin{aligned}
x^{k+1}y^{(k+1)} &= \frac{d}{dt}\mathsf{RHS} - k\,\mathsf{RHS}\\
&= \left(\frac{d}{dt} - k\right)\mathsf{RHS}\\
&= \left(\frac{d}{dt} - k\right)\frac{d}{dt}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - k + 1\right)z(t)\\
&= \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - k\right)z(t)
\end{aligned}$$

Example 7.1 (How to Solve a Cauchy-Euler Equation)
Show the solution details for the equation

$$2x^2y'' + 4xy' + 3y = 0,$$

verifying general solution

$$y(x) = c_1x^{-1/2}\cos\left(\frac{\sqrt{5}}{2}\ln|x|\right) + c_2e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}\ln|x|\right).$$

Solution: The characteristic equation $2r(r-1) + 4r + 3 = 0$ can be obtained as follows:

$2x^2y'' + 4xy' + 3y = 0$	Given differential equation.
$2x^2r(r-1)x^{r-2} + 4xrx^{r-1} + 3x^r = 0$	Use **Euler's substitution** $y = x^r$.
$2r(r-1) + 4r + 3 = 0$	Cancel x^r. **Characteristic equation** found.
$2r^2 + 2r + 3 = 0$	Standard quadratic equation.
$r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$	Quadratic formula complex roots.

Cauchy-Euler Substitution. The second step is to use $y(x) = z(t)$ and $x = e^t$ to transform the differential equation. By Theorem 7.5,

$$2(d/dt)^2z + 2(d/dt)z + 3z = 0,$$

a constant-coefficient equation. Because the roots of the characteristic equation $2r^2 + 2r + 3 = 0$ are $r = -1/2 \pm \sqrt{5}i/2$, then the Euler solution atoms are

$$e^{-t/2}\cos\left(\frac{\sqrt{5}}{2}t\right), \quad e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}t\right).$$

Back-substitute $x = e^t$ and $t = \ln|x|$ in this equation to obtain two independent solutions of $2x^2y'' + 4xy' + 3y = 0$:

$$x^{-1/2}\cos\left(\frac{\sqrt{5}}{2}\ln|x|\right), \quad e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}\ln|x|\right).$$

Substitution Details. Because $x = e^t$, the factor $e^{-t/2}$ is re-written as $(e^t)^{-1/2} = x^{-1/2}$. Because $t = \ln|x|$, the trigonometric factors are back-substituted like this: $\cos\left(\frac{\sqrt{5}}{2}t\right) = \cos\left(\frac{\sqrt{5}}{2}\ln|x|\right)$.

General Solution. The final answer is the set of all linear combinations of the two preceding independent solutions.

Exercises 7.4

Cauchy-Euler Equation

Find solutions y_1, y_2 of the given homogeneous differential equation which are independent by the Wronskian test, page **??**.

1. $x^2y'' + y = 0$

2. $x^2y'' + 4y = 0$

3. $x^2y'' + 2xy' + y = 0$

4. $x^2y'' + 8xy' + 4y = 0$

Variation of Parameters

Find a solution y_p using a variation of parameters formula.

5. $x^2y'' = x$

6. $x^3y'' = e^x$

7. $y'' + 9y = \sec 3x$

8. $y'' + 9y = \csc 3x$

7.5 Variation of Parameters Revisited

The independent functions y_1 and y_2 in the general solution $y_h = c_1 y_1 + c_2 y_2$ of a homogeneous linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

are used to define **Cauchy's kernel**[1]

$$\mathcal{K}(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}. \tag{1}$$

The denominator is the *Wronskian* $W(t)$ of y_1, y_2. Define

$$C_1(t) = \frac{-y_2(t)}{W(t)}, \quad C_2(t) = \frac{y_1(t)}{W(t)}. \tag{2}$$

Then Cauchy's kernel $\mathcal{K}$ has these properties (proved on page 571):

$$\begin{array}{ll}
\mathcal{K}(x,t) = C_1(t)y_1(x) + C_2(t)y_2(x), & \mathcal{K}(x,x) = 0, \\
\mathcal{K}_x(x,t) = C_1(t)y_1'(x) + C_2(t)y_2'(x), & \mathcal{K}_x(x,x) = 1, \\
\mathcal{K}_{xx}(x,t) = C_1(t)y_1''(x) + C_2(t)y_2''(x), & a\mathcal{K}_{xx} + b\mathcal{K}_x + c\mathcal{K} = 0.
\end{array}$$

Theorem 7.6 (Cauchy Kernel Shortcut)
Let a, b, c be constants and let U be the unique solution of $aU'' + bU' + cU = 0$, $U(0) = 0$, $U'(0) = 1$. Then Cauchy's kernel is $\mathcal{K}(x,t) = U(x-t)$.

Proof on page 572.

Theorem 7.7 (Variation of Parameters Formula: Cauchy's Kernel)
Let a, b, c, f be continuous near $x = x_0$ and $a(x) \neq 0$. Let $\mathcal{K}$ be Cauchy's kernel for $ay'' + by' + cy = 0$. Then the non-homogeneous initial value problem

$$ay'' + by' + cy = f, \quad y(x_0) = y'(x_0) = 0$$

has solution

$$y_p(x) = \int_{x_0}^{x} \frac{\mathcal{K}(x,t)f(t)}{a(t)} dt.$$

Proof on page 572.

Specific initial conditions $y(x_0) = y'(x_0) = 0$ imply that y_p can be determined in a laboratory with just one experimental setup. The integral form of y_p shows that it depends *linearly* on the input $f(x)$.

Example 7.2 (Cauchy Kernel)
Verify that the equation $2y'' - y' - y = 0$ has Cauchy kernel $\mathcal{K}(x,t) = \frac{2}{3}(e^{x-t} - e^{-(x-t)/2})$.

[1]Pronunciation *ko–she*.

Solution: The two independent solutions y_1, y_2 are calculated from Theorem **??**, which uses the characteristic equation $2r^2 - r - 1 = 0$. The roots are $-1/2$ and 1. The general solution is $y = c_1 e^{-x/2} + c_2 e^x$. Therefore, $y_1 = e^{-x/2}$ and $y_2 = e^x$.

The Cauchy kernel is the quotient

$$\begin{aligned}
\mathcal{K}(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} && \text{Definition page 569.} \\
&= \frac{e^{-t/2}e^x - e^{-x/2}e^t}{e^{-t/2}e^t + 0.5e^{-t/2}e^t} && \text{Substitute } y_1 = e^{-x/2},\ y_2 = e^x. \\
&= \frac{2}{3}(e^{-t}e^x - e^{-x/2}e^{t/2}) && \text{Simplify.} \\
&= \frac{2}{3}(e^{x-t} - e^{(t-x)/2}) && \text{Final answer.}
\end{aligned}$$

An alternative method to determine the Cauchy kernel is to apply the shortcut Theorem 7.6. We will apply it to check the answer. Solution U must be $U(x) = Ay_1(x) + By_2(x)$ for some constants A, B, determined by the conditions $U(0) = 0$, $U'(0) = 1$. The resulting equations for A, B are $A+B = 0$, $-A/2+B = 1$. Solving gives $-A = B = 2/3$ and then $U(x) = \frac{2}{3}(e^x - e^{-x/2})$. The kernel is $\mathcal{K}(x,t) = U(x-t) = \frac{2}{3}(e^{x-t} - e^{-(x-t)/2})$.

Example 7.3 (Variation of Parameters)
Solve $y'' = |x|$ by Cauchy kernel methods, verifying $y = c_1 + c_2 x + |x|^3/6$.

Solution: First, an independent method will be described, in order to provide a check on the solution. The method involves splitting the equation into two problems $y'' = x$ and $y'' = -x$. The two polynomial answers $y = x^3/6$ on $x > 0$ and $y = -x^3/6$ on $x < 0$, obtained by quadrature, are re-assembled to obtain a single formula $y_p(x) = |x|^3/6$ valid on $-\infty < x < \infty$.

The Cauchy kernel method will be applied to verify the general solution $y = c_1 + c_2 x + |x|^3/6$.

Homogeneous solution. Theorem **??** for constant equations, applied to $y'' = 0$, gives $y_h = c_1 + c_2 x$. Suitable independent solutions are $y_1(x) = 1$, $y_2(x) = x$.

Cauchy kernel for $y'' = 0$. It is computed by formula, $\mathcal{K}(x,t) = ((1)(x) - (t)(1))/(1)$ or $\mathcal{K}(x,t) = x - t$.

Variation of parameters. The solution is $y_p(x) = |x|^3/6$, by Theorem 7.7, details below.

$$\begin{aligned}
y_p(x) &= \textstyle\int_0^x \mathcal{K}(x,t)|t|dt && \text{Theorem 7.7, page 569.} \\
&= \textstyle\int_0^x (x-t)t\,dt && \text{Substitute } \mathcal{K} = x - t \text{ and } |t| = t \text{ for } x > 0. \\
&= x\textstyle\int_0^x t\,dt - \int_0^x t^2 dt && \text{Split into two integrals.} \\
&= x^3/6 && \text{Evaluate for } x > 0.
\end{aligned}$$

If $x < 0$, then the evaluation differs only by $|t| = -t$ in the integrand. This gives $y_p(x) = -x^3/6$ for $x < 0$. The two formulas can be combined into $y_p(x) = |x|^3/6$, valid for $-\infty < x < \infty$.

Example 7.4 (Two Methods)
Solve $y'' - y = e^x$ by undetermined coefficients and by variation of parameters. Explain any differences in the answers.

Solution:

Homogeneous solution. The characteristic equation $r^2 - 1 = 0$ for $y'' - y = 0$ has roots ± 1. The homogeneous solution is $y_h = c_1 e^x + c_2 e^{-x}$.

Undetermined Coefficients Summary. The general solution is reported to be $y = y_h + y_p = c_1 e^x + c_2 e^{-x} + xe^x/2$.

Kümmer's polynomial $\times$ exponential method applies to give $y = e^x Y$ and $[(D+1)^2 - 1]Y = 1$. The latter simplifies to $Y'' + 2Y' = 1$, which has polynomial solution $Y = x/2$. Then $y_p = xe^x/2$.

Variation of Parameters Summary. The homogeneous solution $y_h = c_1 e^x + c_2 e^{-x}$ found above implies $y_1 = e^x$, $y_2 = e^{-x}$ is a suitable independent pair of solutions, because their Wronskian is $W = -2$

The Cauchy kernel is given by $\mathcal{K}(x,t) = \frac{1}{2}(e^{x-t} - e^{t-x})$, details below. The shortcut Theorem 7.6 also applies with $U(x) = \sinh(x) = (e^x - e^{-x})/2$. Variation of parameters Theorem 7.7 gives $y_p(x) = \int_0^x \mathcal{K}(x,t)e^t dt$. It evaluates to $y_p(x) = xe^x/2 - (e^x - e^{-x})/4$, details below.

Differences. The two methods give respectively $y_p = xe^x/2$, and $y_p = xe^x/2 - (e^x - e^{-x})/4$. The solutions $y_p = xe^x/2$ and $y_p = xe^x/2 - (e^x - e^{-x})/4$ differ by the homogeneous solution $(e^x - e^{-x})/4$. In both cases, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2}xe^x,$$

because terms of the homogeneous solution can be absorbed into the arbitrary constants c_1, c_2.

Computational Details.

$$\begin{aligned}
\mathcal{K}(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} && \text{Definition page 569.}\\
&= \frac{e^t e^{-x} - e^x e^{-t}}{e^t(-e^{-t}) - e^t e^{-t}} && \text{Substitute.}\\
&= \frac{1}{2}(e^{x-t} - e^{t-x}) && \text{Cauchy kernel found.}
\end{aligned}$$

$$\begin{aligned}
y_p(x) &= \int_0^x \mathcal{K}(x,t)e^t dt && \text{Theorem 7.7, page 569.}\\
&= \frac{1}{2}\int_0^x (e^{x-t} - e^{t-x})e^t dt && \text{Substitute } \mathcal{K} = \tfrac{1}{2}(e^{x-t} - e^{t-x}).\\
&= \frac{1}{2}e^x \int_0^x dt - \frac{1}{2}\int_0^x e^{2t-x} dt && \text{Split into two integrals.}\\
&= \frac{1}{2}xe^x - \frac{1}{4}(e^x - e^{-x}) && \text{Evaluation completed.}
\end{aligned}$$

Proofs and Technical Details

Proofs for page 569, Cauchy Kernel Properties:

The equation $\mathcal{K}(x,t) = C_1(t)y_1(x) + C_2(t)y_2(x)$ is an algebraic identity, using the definitions of C_1 and C_2. Then $\mathcal{K}(x,x)$ is a fraction with numerator $y_1(x)y_2(x) - y_1(x)y_2(x) = 0$, giving the second identity $\mathcal{K}(x,x) = 0$.

The partial derivative formula $\mathcal{K}_x(x,t) = C_1(t)y_1'(x)+C_2(t)y_2'(x)$ is obtained by ordinary differentiation on x in the previous identity. Then $\mathcal{K}_x(x,x)$ is a fraction with numerator $y_1(x)y_2'(x) - y_1'(x)y_2(x)$, which exactly cancels the denominator, giving the identity $\mathcal{K}_x(x,x) = 1$.

The second derivative formula $\mathcal{K}_{xx}(x,t) = C_1(t)y_1''(x) + C_2(t)y_2''(x)$ results by ordinary differentiation on x in the formula for $\mathcal{K}_x$. The differential equation $a\mathcal{K}_{xx}+b\mathcal{K}_x+c\mathcal{K} = 0$ is satisfied, because $\mathcal{K}$ in the variable x is a linear combination of y_1 and y_2, which are given to be solutions.

Proof of Theorem 7.6, Cauchy Kernel Shortcut:
Let $y(x) = \mathcal{K}(x,t) - U(x-t)$ for fixed t. It will be shown that y is a solution and $y(t) = y'(t) = 0$. Already known from page 569 is the relation $a\mathcal{K}_{xx}(x,t) + b\mathcal{K}_x(x,t) + c\mathcal{K}(x,t) = 0$. By assumption, $aU''(x-t) + bU'(x-t) + cU(x-t) = 0$. By the chain rule, both terms in y satisfy the differential equation, hence y is a solution. At $x = t$, $y(t) = \mathcal{K}(t,t) - U(0) = 0$ and $y'(t) = K_x(t,t) - U'(0) = 0$ (see page 569). Then y is a solution of the homogeneous equation with zero initial conditions. By uniqueness, $y(x) \equiv 0$, which proves $\mathcal{K}(x,t) = U(x-t)$.

Proof of Theorem 7.7, Variation of Parameters:
Let $F(t) = f(t)/a(t)$. It will be shown that y_p as given has two continuous derivatives given by the integral formulas

$$y_p'(x) = \int_{x_0}^{x} \mathcal{K}_x(x,t)F(t)dt, \quad y_p''(x) = \int_{x_0}^{x} \mathcal{K}_{xx}(x,t)F(t)dt + F(x).$$

Then

$$ay_p'' + by_p' + cy_p = \int_{x_0}^{x} (a\mathcal{K}_{xx} + b\mathcal{K}_x + c\mathcal{K})F(t)dt + aF.$$

The equation $a\mathcal{K}_{xx} + b\mathcal{K}_x + c\mathcal{K} = 0$, page 569, shows the integrand on the right is zero. Therefore $ay_p'' + by_p' + cy_p = f(x)$, which would complete the proof.

Needed for the calculation of the derivative formulas is the fundamental theorem of calculus relation $\left(\int_{x_0}^{x} G(t)dt\right)' = G(x)$, valid for continuous G. The product rule from calculus can be applied directly, because y_p is a sum of products:

$$\begin{aligned}
y_p' &= \left(y_1(x)\int_{x_0}^{x} C_1 F dt + y_2(x)\int_{x_0}^{x} C_2 F dt\right)' \\
&= y_1'\int_{x_0}^{x} C_1 F dt + y_2'\int_{x_0}^{x} C_2 F dt + y_1(x)C_1(x)F(x) + y_2(x)C_2(x)F(x) \\
&= y_1'\int_{x_0}^{x} C_1 F dt + y_2'\int_{x_0}^{x} C_2 F dt + \mathcal{K}(x,x)F(x) \\
&= \int_{x_0}^{x} \mathcal{K}_x(x,t)F(t)dt
\end{aligned}$$

The terms contributed by differentiation of the integrals add to zero because $\mathcal{K}(x,x) = 0$ (page 569).

$$\begin{aligned}
y_p'' &= \left(y_1'(x)\int_{x_0}^{x} C_1 F dt + y_2'(x)\int_{x_0}^{x} C_2 F dt\right)' \\
&= y_1''\int_{x_0}^{x} C_1 F dt + y_2''\int_{x_0}^{x} C_2 F dt + y_1'(x)C_1(x)F(x) + y_2'(x)C_2(x)F(x) \\
&= y_1''\int_{x_0}^{x} C_1 F dt + y_2''\int_{x_0}^{x} C_2 F dt + \mathcal{K}_x(x,x)F(x) \\
&= \int_{x_0}^{x} \mathcal{K}_{xx}(x,t)F(t)dt + F(x)
\end{aligned}$$

The terms contributed by differentiation of the integrals add to $F(x)$ because $\mathcal{K}_x(x,x) = 1$ (page 569).

Exercises 7.5

Cauchy Kernel

Find the Cauchy kernel $\mathcal{K}(x,t)$ for the given homogeneous differential equation.

1. $y'' - y = 0$

2. $y'' - 4y = 0$

3. $y'' + y = 0$

4. $y'' + 4y = 0$

5. $4y'' + y' = 0$

6. $y'' + y' = 0$

7. $y'' + y' + y = 0$

8. $y'' - y' + y = 0$

Variation of Parameters

Find the general solution $y_h + y_p$ by applying a variation of parameters formula.

9. $y'' = x^2$

10. $y'' = x^3$

11. $y'' + y = \sin x$

12. $y'' + y = \cos x$

13. $y'' + y' = \ln|x|$

14. $y'' + y' = -\ln|x|$

15. $y'' + 2y' + y = e^{-x}$

16. $y'' - 2y' + y = e^{x}$

7.6 Undetermined Coefficients Library

The study of undetermined coefficients continues for the problem

$$ay'' + by' + cy = f(x). \tag{1}$$

As in previous sections, $f(x)$ is assumed to be a sum of constants times **Euler solution atoms** and the symbols a, b, c are constants. Recorded here are special methods for efficiently solving (1). **Linear algebra is not required** in any of the special methods: only calculus and college algebra are assumed as background.

The special methods provide a justification for the trial solution method, presented earlier in this text.

The Easily-Solved Equations

The algebra problem for undetermined coefficients can involve many unknowns. It is recommended to reduce the size of the algebra problem by breaking the differential equation into several simpler differential equations. A particular solution y_p of (1) can be expressed as a sum

$$y_p = y_1 + \cdots + y_n$$

where each y_k solves a related easily-solved differential equation.

The idea can be quickly communicated for $n = 3$. The superposition principle applied to the three equations

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= f_1(x), \\ ay_2'' + by_2' + cy_2 &= f_2(x), \\ ay_3'' + by_3' + cy_3 &= f_3(x) \end{aligned} \tag{2}$$

shows that $y = y_1 + y_2 + y_3$ is a solution of

$$ay'' + by' + cy = f_1 + f_2 + f_3. \tag{3}$$

If each equation in (2) is easily solved, then solving equation (3) is also easy: *add the three answers for the easily solved problems.*

To use the idea, it is necessary to start with $f(x)$ and determine a decomposition $f = f_1 + f_2 + f_3$ so that equations (2) are easily solved.

Each **Easily-Solved equation** is engineered to have right side in one of the four forms below:

$$\begin{array}{ll} p(x) & \textbf{polynomial}, \\ p(x)e^{kx} & \textbf{polynomial} \times \textbf{exponential}, \\ p(x)e^{kx}\cos mx & \textbf{polynomial} \times \textbf{exponential} \times \textbf{cosine}, \\ p(x)e^{kx}\sin mx & \textbf{polynomial} \times \textbf{exponential} \times \textbf{sine}. \end{array} \tag{4}$$

To illustrate, consider

$$ay'' + by' + cy = x + xe^x + x^2 \sin x - \pi e^{2x} \cos x + x^3. \tag{5}$$

The right side is decomposed as follows, in order to define the easily solved equations:

$ay_1'' + by_1' + cy_1 = x + x^3$	Polynomial.
$ay_2'' + by_2' + cy_2 = xe^x$	Polynomial $\times$ exponential.
$ay_3'' + by_3' + cy_3 = x^2 \sin x$	Polynomial $\times$ exponential $\times$ sine.
$ay_4'' + by_4' + cy_4 = -\pi e^{2x} \cos x$	Polynomial $\times$ exponential $\times$ cosine.

There are $n = 4$ equations. In the illustration, x^3 is included with x, but it could have caused creation of a fifth equation. To decrease effort, minimize the number n of easily solved equations. *One final checkpoint*: the right sides of the n equations must add to the right side of (5).

Library of Special Methods

It is assumed that the differential equation is already in easily-solved form: the library methods are designed to apply directly. If an equation requires a decomposition into easily-solved equations, then the desired solution is then the sum of the answers to the decomposed equations.

Equilibrium and Quadrature Methods

The special case of $ay'' + by' + cy = k$ where k is a constant occurs so often that an efficient method has been isolated to find y_p. It is called the **equilibrium method**, because in the simplest case y_p is a constant solution or an *equilibrium solution.* The method in words:

> Verify that the right side of the differential equation is constant. Cancel on the left side all derivative terms except for the lowest order and then solve for y by quadrature.

The method works to find a solution, because if a derivative $y^{(n)}$ is constant, then all higher derivatives $y^{(n+1)}$, y^{n+2}, etc., are zero. A precise description follows for second order equations.

Differential Equation	Cancelled DE	Particular Solution
$ay'' + by' + cy = k$, $c \neq 0$	$cy = k$	$y_p = \dfrac{k}{c}$
$ay'' + by' = k$, $b \neq 0$	$by' = k$	$y_p = \dfrac{k}{b}x$
$ay'' = k$, $a \neq 0$	$ay'' = k$	$y_p = \dfrac{k}{a}\dfrac{x^2}{2}$

The equilibrium method also applies to nth order linear differential equations $\sum_{i=0}^{n} a_i y^{(i)} = k$ with constant coefficients $a_0, \ldots, a_n$ and constant right side k.

A special case of the equilibrium method is the *simple quadrature method*, illustrated in Example 7.5 page 582. The method is used in elementary physics to solve falling body problems.

The Polynomial Method

The method applies to find a particular solution of $ay'' + by' + cy = p(x)$, where $p(x)$ represents a polynomial of degree $n \geq 1$. Such equations **always have a polynomial solution**; see Theorem 7.8 page 581.

Let a, b and c be given with $a \neq 0$. Differentiate the differential equation successively until the right side is constant:

$$
\begin{array}{lllllll}
ay'' & + & by' & + & cy & = & p(x), \\
ay''' & + & by'' & + & cy' & = & p'(x), \\
ay^{iv} & + & by''' & + & cy'' & = & p''(x), \\
 & & & & & \vdots & \\
ay^{(n+2)} & + & by^{(n+1)} & + & cy^{(n)} & = & p^{(n)}(x).
\end{array} \tag{6}
$$

Apply the equilibrium method to the *last equation* in order to find a polynomial **trial solution**

$$y(x) = d_m \frac{x^m}{m!} + \cdots + d_0.$$

It will emerge that $y(x)$ always has $n+1$ terms, but its degree can be either n, $n+1$ or $n+2$. The **undetermined coefficients** $d_0, \ldots, d_m$ are resolved by setting $x = 0$ in equations (6). The Taylor polynomial relations $d_0 = y(0), \ldots, d_m = y^{(m)}(0)$ give the equations

$$
\begin{array}{lllllll}
ad_2 & + & bd_1 & + & cd_0 & = & p(0), \\
ad_3 & + & bd_2 & + & cd_1 & = & p'(0), \\
ad_4 & + & bd_3 & + & cd_2 & = & p''(0), \\
 & & & & & \vdots & \\
ad_{n+2} & + & bd_{n+1} & + & cd_n & = & p^{(n)}(0).
\end{array} \tag{7}
$$

These equations can always be solved by **back-substitution**; linear algebra is not required. Three cases arise, according to the number of zero roots of the characteristic equation $ar^2+br+c=0$. The values $m=n, n+1, n+2$ correspond to zero, one or two roots $r=0$.

No root $r=0$. Then $c \neq 0$. There were n integrations to find the trial solution, so $d_{n+2}=d_{n+1}=0$. The unknowns are d_0 to d_n. The system can be solved by back-substitution to uniquely determine $d_0, \ldots, d_n$. The resulting polynomial $y(x)$ is the desired solution $y_p(x)$.

One root $r=0$. Then $c=0$, $b \neq 0$. The unknowns are $d_0, \ldots, d_{n+1}$. There is no condition on d_0; simplify the trial solution by taking $d_0=0$. Solve (7) for unknowns d_1 to d_{n+1} as in the no root case.

Double root $r=0$. Then $c=b=0$ and $a \neq 0$. The equilibrium method gives a polynomial trial solution $y(x)$ involving $d_0, \ldots, d_{n+2}$. There are no conditions on d_0 and d_1. Simplify y by taking $d_0=d_1=0$. Solve (7) for unknowns d_2 to d_{n+2} as in the no root case.

College algebra back-substitution applied to (7) is illustrated in Example 7.7, page 583. A complete justification of the polynomial method appears in the proof of Theorem 7.8, page 588.

Recursive Polynomial Hybrid

A *recursive method* based upon quadrature appears in Example 7.8, page 584. This method, independent from the *polynomial method* above, is useful when the number of equations in (6) is two or three.

Some researchers (see [**?**]) advertise the recursive method as easy to remember, easy to use and faster than other methods. In this textbook, the method is advertised as a **hybrid**: equations in (6) are written down, but equations (7) are not. Instead, the undetermined coefficients are found recursively, by repeated quadrature and back-substitution.

Classroom testing of the recursive polynomial method reveals it is best suited to algebraic helmsmen with flawless talents. The method should be applied when conditions suggest rapid and reliable computation details. Error propagation possibilities dictate that polynomial solutions of degree 4 or larger be suspect and subjected to an answer check.

Polynomial × Exponential Method

The method applies to special equations $ay''+by'+cy=p(x)e^{kx}$ where $p(x)$ is a polynomial. The idea, due to Kümmer, uses the transformation $y=e^{kx}Y$ to obtain the auxiliary equation

$$[a(D+k)^2 + b(D+k) + c]Y = p(x), \quad D = \frac{d}{dx}.$$

The polynomial method applies to find Y. Multiplication by e^{kx} gives y.

Computational details are in Example 7.9, page 584. Justification appears in Theorem 7.9. In words, to find the differential equation for Y:

> In the differential equation, replace D by $D+k$ and cancel e^{kx} on the RHS.

Polynomial × Exponential × Cosine Method

The method applies to equations $ay'' + by' + cy = p(x)e^{kx}\cos(mx)$ where $p(x)$ is a polynomial. Kümmer's transformation $y = e^{kx}\,\mathcal{R}\mathrm{e}(e^{imx}Y)$ gives the auxiliary problem

$$[a(D+z)^2 + b(D+z) + c]Y = p(x), \quad z = k + im, \quad D = \frac{d}{dx}.$$

The polynomial method applies to find Y. Symbol $\mathcal{R}\mathrm{e}$ extracts the real part of a complex number. Details are in Example 7.10, page 585. The formula is justified in Theorem 7.10. In words, to find the equation for Y:

> In the differential equation, replace D by $D+k+im$ and cancel $e^{kx}\cos mx$ on the RHS.

Polynomial × Exponential × Sine Method

The method applies to equations $ay'' + by' + cy = p(x)e^{kx}\sin(mx)$ where $p(x)$ is a polynomial. Kümmer's transformation $y = e^{kx}\,\mathcal{I}\mathrm{m}(e^{imx}Y)$ gives the auxiliary problem

$$[a(D+z)^2 + b(D+z) + c]Y = p(x), \quad z = k + im, \quad D = \frac{d}{dx}.$$

The polynomial method applies to find Y. Symbol $\mathcal{I}\mathrm{m}$ extracts the imaginary part of a complex number. Details are in Example 7.11, page 586. The formula is justified in Theorem 7.10. In words, to find the equation for Y:

> In the differential equation, replace D by $D+k+im$ and cancel $e^{kx}\sin mx$ on the RHS.

Kümmer's Method

The methods known above as the polynomial × exponential method, the polynomial × exponential × cosine method, and the polynomial × exponential × sine method, are collectively called **Kümmer's method**, because of their origin.

Trial Solution Shortcut

The library of special methods leads to a justification for the **trial solution method**, a method which has been popularized by leading differential equation textbooks published over the past 50 years.

Trial Solutions and Kümmer's Method

Assume given $ay'' + by' + cy = f(x)$ where $f(x) =$(polynomial)$e^{kx}\cos mx$, then the method of Kümmer predicts

$$y = e^{kx}\,\mathcal{R}\text{e}\left(Y(x)(\cos mx + i\sin mx)\right),$$

where $Y(x)$ is a polynomial solution of a different, **associated differential equation**. In the simplest case, $Y(x) = \sum_{j=0}^{n} A_j x^j + i\sum_{j=0}^{n} B_j x^j$, a polynomial of degree n with complex coefficients, matching the degree of the polynomial in $f(x)$. Expansion of the Kümmer formula for y plus definitions $a_j = A_j - B_j$, $b_j = B_j + A_j$ gives a **trial solution**

$$y = \left(\cos(mx)\sum_{j=0}^{n} a_j x^j + \sin(mx)\sum_{j=0}^{n} b_j x^j\right) e^{kx}. \tag{8}$$

The undetermined coefficients are $a_0, \ldots, a_n, b_0, \ldots, b_n$. Exactly the same trial solution results when $f(x) =$(polynomial)$e^{kx}\sin mx$. If $m = 0$, then the trigonometric functions do not appear and the trial solution is either a polynomial $(k = 0)$ or else a polynomial times an exponential.

The characteristic equation for the associated differential equation has root $r = 0$ exactly when $r = k+m\sqrt{-1}$ is a root of $ar^2+br+c = 0$. Therefore, Y, and hence y, must be multiplied by x for each time $k+m\sqrt{-1}$ is a root of $ar^2+br+c = 0$. In the trial solution method, this requirement is met by multiplication by x until the trial solution no longer contains a term of the homogeneous solution. Certainly both correction rules produce exactly the same final trial solution.

Shortcuts using (8) have been known for some time. The results can be summarized in words as follows.

> If the right side of $ay''+by'+cy = f(x)$ is a polynomial of degree n times $e^{kx}\cos(mx)$ or $e^{kx}\sin(mx)$, then an initial trial solution y is given by relation (8), with undetermined coefficients $a_0, \ldots, a_n, b_0,$

$\ldots$, b_n. Correct the trial solution y by multiplication by x, once for each time $r = k + m\sqrt{-1}$ is a root of the characteristic equation $ar^2 + br + c = 0$.

The Correction Rule

The **Final Trial Solution** is found by this rule:

> Given an initial trial solution y for $au'' + by' + cy = f(x)$, from Table 1 below, correct y by multiplication by x, once for each time that $r = k + m\sqrt{-1}$ is a root of the characteristic equation $ar^2 + br + c = 0$. This is equivalent to multiplication by x until the trial solution no longer contains a term of the homogeneous solution.

Once the **final trial solution** y is determined, then y is substituted into the differential equation. The undetermined coefficients are found by matching terms of the form $x^j e^{kx}\cos(mx)$ and $x^j e^{kx}\sin(mx)$, which appear on the left and right side of the equation after substitution.

A Table Lookup Method

Table 1 below summarizes the form of an initial trial solution in special cases, according to the form of $f(x)$.

Table 1. A Table Method for Trial Solutions.
The table predicts the **Initial Trial Solution** y in the method of undetermined coefficients. Then the **Correction Rule** is applied to find the **final trial solution**. Symbol n is the degree of the polynomial in column 1.

Form of $f(x)$	Values	Initial Trial Solution
constant	$k = m = 0$	$y = a_0$ =constant
polynomial	$k = m = 0$	$y = \sum_{j=0}^{n} a_j x^j$
combination of $\cos mx$ and $\sin mx$	$k = 0$, $m > 0$	$y = a_0 \cos mx + b_0 \sin mx$
(polynomial)e^{kx}	$m = 0$	$y = \left(\sum_{j=0}^{n} a_j x^j\right) e^{kx}$
(polynomial)$e^{kx}\cos mx$ or (polynomial)$e^{kx}\sin mx$	$m > 0$	$y = \left(\sum_{j=0}^{n} a_j x^j\right) e^{kx}\cos mx + \left(\sum_{j=0}^{n} b_j x^j\right) e^{kx}\sin mx$

Details for lines 2-3 of Table 1 appear in Examples 7.6 and 7.13.

Alternate Trial Solution Shortcut

The method avoids the root testing of the correction rule, at the expense of repeated substitutions. The simplicity of the method is appealing, but a few computations reveal that the correction rule is a more practical method.

> Let y be the initial trial solution of Table 1. Substitute it into the differential equation. It will either compute y_p, or else some coefficients cannot be determined. In the latter case, multiply y by x and repeat, until a solution y_p is found.

Key Theorems

Theorem 7.8 (Polynomial Solutions)
Assume a, b, c are constants, $a \neq 0$. Let $p(x)$ be a polynomial of degree d. Then $ay'' + by' + cy = p(x)$ has a polynomial solution y of degree d, $d+1$ or $d+2$. Precisely, these three cases hold:

Case 1. $ay'' + by' + cy = p(x)$ $c \neq 0$. Then $y = y_0 + \cdots + y_d \dfrac{x^d}{d!}$.

Case 2. $ay'' + by' = p(x)$ $b \neq 0$. Then $y = \left(y_0 + \cdots + y_d \dfrac{x^d}{d!}\right) x$.

Case 3. $ay'' = p(x)$ $a \neq 0$. Then $y = \left(y_0 + \cdots + y_d \dfrac{x^d}{d!}\right) x^2$.

Proof on page 588.

Theorem 7.9 (Polynomial $\times$ Exponential)
Assume a, b, c, k are constants, $a \neq 0$, and $p(x)$ is a polynomial. If Y is a solution of $[a(D+k)^2 + b(D+k) + c]Y = p(x)$, then $y = e^{kx}Y$ is a solution of $ay'' + by' + cy = p(x)e^{kx}$.

Proof on page 588.

Theorem 7.10 (Polynomial $\times$ Exponential $\times$ Cosine or Sine)
Assume a, b, c, k, m are real, $a \neq 0$, $m > 0$. Let $p(x)$ be a real polynomial and $z = k + im$. If Y is a solution of $[a(D+z)^2 + b(D+z) + c]Y = p(x)$, then $y = e^{kx}\mathcal{R}\mathrm{e}(e^{imx}Y)$ is a solution of $ay'' + by' + cy = p(x)e^{kx}\cos(mx)$ and $y = e^{kx}\mathcal{I}\mathrm{m}(e^{imx}Y)$ is a solution of $ay'' + by' + cy = p(x)e^{kx}\sin(mx)$.

Proof on page 589.

The theorems form the theoretical basis for the method of undetermined coefficients.

Examples and Methods

Example 7.5 (Simple Quadrature)

Solve for y_p in $y'' = 2 - x + x^3$ using the fundamental theorem of calculus, verifying $y_p = x^2 - x^3/6 + x^5/20$.

Solution: Two integrations using the fundamental theorem of calculus give $y = y_0 + y_1 x + x^2 - x^3/6 + x^5/20$. The terms $y_0 + y_1 x$ represent the homogeneous solution y_h. Therefore, $y_p = x^2 - x^3/6 + x^5/20$ is reported. The method works in general for $ay'' + by' + cy = p(x)$, provided $b = c = 0$, that is, in **case 3** of Theorem 7.8. Some explicit details:

$\int y''(x)dx = \int(2 - x + x^3)dx$	Integrate across on x.
$y' = y_1 + 2x - x^2/2 + x^4/4$	Fundamental theorem.
$\int y'(x)dx = \int(y_1 + 2x - x^2/2 + x^4/4)dx$	Integrate across again on x.
$y = y_0 + y_1 x + x^2 - x^3/6 + x^5/20$	Fundamental theorem.

Example 7.6 (Undetermined Coefficients: A Hybrid Method)

Solve for y_p in the equation $y'' - y' + y = 2 - x + x^3$ by the method of undetermined coefficients, verifying $y_p = -5 - x + 3x^2 + x^3$.

Solution: Let's begin by *calculating the trial solution* $y = d_0 + d_1 x + d_2 x^2/2 + x^3$. This is done by differentiation of $y'' - y' + y = 2 - x + x^3$ until the right side is constant:

$$y^v - y^{iv} + y''' = 6.$$

The equilibrium method solves the truncated equation $0 + 0 + y''' = 6$ by quadrature to give $y = d_0 + d_1 x + d_2 x^2/2 + x^3$.

The **undetermined coefficients** d_0, d_1, d_2 will be found by a classical technique in which the trial solution y is back-substituted into the differential equation. We begin by computing the derivatives of y:

$y = d_0 + d_1 x + d_2 x^2/2 + x^3$	Calculated above; see Theorem 7.8.
$y' = d_1 + d_2 x + 3x^2$	Differentiate.
$y'' = d_2 + 6x$	Differentiate again.

The relations above are back-substituted into the differential equation $y'' - y' + y = 2 - x + x^3$ as follows:

$$\begin{aligned} 2 - x + x^3 &= y'' - y' + y && \text{Write the DE backwards.} \\ &= [d_2 + 6x] \\ &\quad - [cd_1 + d_2 x + 3x^2] && \text{Substitute for } y,\ y',\ y''. \\ &\quad + [d_0 + d_1 x + d_2 x^2/2 + x^3] \\ &= [c_2 - c_1 + c_0] \\ &\quad + [6 - d_2 + c_1]x && \text{Collect on powers of } x. \\ &\quad + [-3 + d_2/2]x^2 \\ &\quad + [1]x^3 \end{aligned}$$

The coefficients d_0, d_1, d_2 are found by **matching powers** on the LHS and RHS of the expanded equation:

$$\begin{array}{rcll} 2 & = & [d_2 - d_1 + c_0] & \text{match constant term,} \\ -1 & = & [6 - d_2 + d_1] & \text{match } x\text{-term,} \\ 0 & = & [-3 + d_2/2] & \text{match } x^2\text{-term.} \end{array} \tag{9}$$

These equations are solved by back-substitution, starting with the last equation and proceeding to the first equation. The answers are successively $d_2 = 6$, $d_1 = -1$, $d_0 = -5$. For more detail on back-substitution, see the next example. Substitution into $y = d_0 + d_1 x + d_2 x^2/2 + x^3$ gives the particular solution $y_p = -5 - x + 3x^2 + x^3$.

Example 7.7 (Undetermined Coefficients: Taylor's Method)

Solve for y_p in the equation $y'' - y' + y = 2 - x + x^3$ by Taylor's method, verifying $y_p = -5 - x + 3x^2 + x^3$.

Solution: Theorem 7.8 implies that there is a polynomial solution $y = d_0 + d_1 x + d_2 x^2/2 + d_3 x^3/6$. The **undetermined coefficients** d_0, d_1, d_2, d_3 will be found by a technique related to **Taylor's method** in calculus. The Taylor technique requires differential equations obtained by successive differentiation of $y'' - y' + y = 2 - x + x^3$, as follows.

$y'' - y' + y = 2 - x + x^3$	The original.
$y''' - y'' + y' = -1 + 3x^2$	Differentiate the original once.
$y^{iv} - y''' + y'' = 6x$	Differentiate the original twice.
$y^{v} - y^{iv} + y''' = 6$	Differentiate the original three times. The process stops when the right side is constant.

Set $x = 0$ in the above differential equations. Then substitute the Taylor polynomial derivative relations

$$y(0) = d_0,\ y'(0) = d_1,\ y''(0) = d_2,\ y'''(0) = d_3.$$

It is also true that $y^{iv}(0) = y^{v}(0) = 0$, since y is a cubic. This produces the following equations for *undetermined coefficients* d_0, d_1, d_2, d_3:

$$\begin{array}{rcr} d_2 - d_1 + d_0 & = & 2 \\ d_3 - d_2 + d_1 & = & -1 \\ -d_3 + d_2 & = & 0 \\ d_3 & = & 6 \end{array}$$

These equations are solved by **back-substitution**, working in reverse order. No experience with linear algebra is required, because this is strictly a low-level college algebra method. Successive back-substitutions, working from the last equation in reverse order, give the answers

$d_3 = 6$,	Use the fourth equation first.
$d_2 = d_3$	Solve for d_2 in the third equation.
$\quad = 6$,	Back-substitute d_3.
$d_1 = -1 + d_2 - d_3$	Solve for d_1 in the second equation.

$= -1,$	Back-substitute d_2 and d_3.
$d_0 = 2 + d_1 - d_2$	Solve for d_0 in the first equation.
$= -5.$	Back-substitute d_1 and d_2.

The result is $d_0 = -5$, $d_1 = -1$, $d_2 = 6$, $d_3 = 6$. Substitution into $y = d_0 + d_1 x + d_2 x^2/2 + d_3 x^3/6$ gives the particular solution $y_p = -5 - x + 3x^2 + x^3$.

Example 7.8 (Polynomial Method: Recursive Hybrid)

In the equation $y'' - y' = 2 - x + x^3$, verify $y_p = -7x - 5x^2/2 - x^3 - x^4/4$ by the polynomial method, using a recursive hybrid.

Solution: A **Recursive Method** will be applied, based upon the fundamental theorem of calculus, as in Example 7.5.

Step 1. Differentiate $y'' - y' = 2 - x + x^3$ until the right side is constant, to obtain

Equation 1: $y'' - y' = 2 - x + x^3$	The original.
Equation 2: $y''' - y'' = -1 + 3x^2$	Differentiate the original once.
Equation 3: $y^{iv} - y''' = 6x$	Differentiate the original twice.
Equation 4: $y^{v} - y^{iv} = 6$	Differentiate the original three times. The process stops when the right side is constant.

Step 2. There are 4 equations. Theorem 7.8 implies that there is a polynomial solution y of degree 4. Then $y^v = 0$.

The last equation $y^v - y^{iv} = 6$ then gives $y^{iv} = -6$, which can be solved for y''' by the fundamental theorem of calculus. Then $y''' = -6x + c$. Evaluate c by requiring that y satisfy equation 3: $y^{iv} - y''' = 6x$. Substitution of $y''' = -6x + c$, followed by setting $x = 0$ gives $-6 - c = 0$. Hence $c = -6$. The conclusion: $y''' = -6x - 6$.

Step 3. Solve $y''' = -6x - 6$, giving $y'' = -3x^2 - 6x + c$. Evaluate c as in *Step 2* using equation 2: $y''' - y'' = -1 + 3x^2$. Then $-6 - c = -1$ gives $c = -5$. The conclusion: $y'' = -3x^2 - 6x - 5$.

Step 4. Solve $y'' = -3x^2 - 6x - 5$, giving $y' = -x^3 - 3x^2 - 5x + c$. Evaluate c as in *Step 2* using equation 1: $y'' - y' = 2 - x + x^3$. Then $-5 - c = 2$ gives $c = -7$. The conclusion: $y' = -x^3 - 3x^2 - 5x - 7$.

Step 5. Solve $y' = -x^3 - 3x^2 - 5x - 7$, giving $y = -x^4/4 - x^3 - 5x^2/2 - 7x + c$. Just one solution is sought, so take $c = 0$. Then $y = -7x - 5x^2/2 - x^3 - x^4/4$. Theorem 7.8 also drops the constant term, because it is included in the homogeneous solution y_h. While this method duplicates all the steps in Example 7.7, it remains attractive due to its simplistic implementation. The method is best appreciated when it terminates at step 2 or 3.

Example 7.9 (Polynomial × Exponential)

Solve for y_p in $y'' - y' + y = (2 - x + x^3)e^{2x}$, verifying that $y_p = e^{2x}(x^3/3 - x^2 + x + 1/3)$.

Solution: Let $y = e^{2x}Y$ and $[(D+2)^2-(D+2)+1]Y = 2-x+x^3$, as per the *polynomial* $\times$ *exponential method*, page 577. The equation $Y''+3Y'+3Y = 2-x+x^3$ will be solved by the polynomial method of Example 7.7.

Differentiate $Y''+3Y'+3Y = 2-x+x^3$ until the right side is constant.

$$\begin{aligned}
&Y''+3Y'+3Y = 2-x+x^3\\
&Y'''+3Y''+3Y' = -1+3x^2\\
&Y^{iv}+3Y'''+3Y'' = 6x\\
&Y^{v}+3Y^{iv}+3Y''' = 6
\end{aligned}$$

The last equation, by the equilibrium method, implies Y is a polynomial of degree 4, $Y = d_0+d_1x+d_2x^2/2+d_3x^3/6$. Set $x=0$ and $d_i = Y^{(i)}(0)$ in the preceding equations to get the system

$$\begin{aligned}
d_2+3d_1+3d_0 &= 2\\
d_3+3d_2+3d_1 &= -1\\
d_4+3d_3+3d_2 &= 0\\
d_5+3d_4+3d_3 &= 6
\end{aligned}$$

in which $d_4 = d_5 = 0$. Solving by back-substitution gives the answers $d_3 = 2$, $d_2 = -2$, $d_1 = 1$, $d_0 = 1/3$. Then $Y = x^3/3 - x^2 + x + 1/3$.

Finally, Kümmer's transformation $y = e^{2x}Y$ implies $y = e^{2x}(x^3/3 - x^2 + x + 1/3)$.

Example 7.10 (Polynomial $\times$ Exponential $\times$ Cosine)

Solve in $y''-y'+y = (3-x)e^{2x}\cos(3x)$ for y_p, verifying that $y_p = \frac{1}{507}((26x-107)e^{2x}\cos(3x)+(115-39x)e^{2x}\sin(3x))$.

Solution: Let $z = 2+3i$. If Y satisfies $[(D+z)^2-(D+z)+1]Y = 3-x$, then $y = e^{2x}\mathcal{R}\mathrm{e}(e^{3ix}Y)$, by the method on page 578. The differential equation simplifies into $Y''+(3+6i)Y'+(9i-6)Y = 3-x$. It will be solved by the recursion method of Example 7.8.

Step 1. Differentiate $Y''+(3+6i)Y'+(9i-6)Y = 3-x$ until the right side is constant, to obtain $Y'''+(3+6i)Y''+(9i-6)Y' = -1$. The conclusion: $Y' = 1/(6-9i)$.

Step 2. Solve $Y' = 1/(6-9i)$ for $Y = x/(6-9i)+c$. Evaluate c by requiring Y to satisfy the original equation $Y''+(3+6i)Y'+(9i-6)Y = 3-x$. Substitution of $Y' = x/(6-9i)+c$, followed by setting $x=0$ gives $0+(3+6i)/(6-9i)+(9i-6)c = 3$. Hence $c = (-15+33i)/(6-9i)^2$. The conclusion: $Y = x/(6-9i)+(-15+33i)/(6-9i)^2$.

Step 3. Use variable $y = e^{2x}\mathcal{R}\mathrm{e}(e^{3ix}Y)$ to complete the solution. This is the point where complex arithmetic must be used. Let $y = e^{2x}\mathcal{Y}$ where $\mathcal{Y} = \mathcal{R}\mathrm{e}(e^{3ix}Y)$. Some details:

$Y = \dfrac{x}{6-9i} + \dfrac{-15+33i}{(6-9i)^2}$	The plan: write $Y = Y_1 + iY_2$.
$= x\dfrac{6+9i}{6^2+9^2} + \dfrac{(-15+33i)(6+9i)^2}{(6^2+9^2)^2}$	Use $1/Z = \overline{Z}/\lvert Z\rvert^2$, $Z = a+ib$, $\overline{Z} = a-ib$, $\lvert Z\rvert = a^2+b^2$.
$= \dfrac{2x}{39} + \dfrac{xi}{13} + \dfrac{-2889-3105i}{117^2}$	Use $6^2+9^2 = 117 = (9)(13)$.
$= \dfrac{26x-107}{507} + i\dfrac{39x-115}{507}$	Split off real and imaginary.

$$Y_1 = \frac{26x-107}{507}, \quad Y_2 = \frac{39x-115}{507} \qquad \text{Decomposition found.}$$

$$\begin{aligned}
\mathcal{Y} &= \mathcal{R}\mathrm{e}((\cos 3x + i\sin 3x)(Y_1 + iY_2)) && \text{Use } e^{3ix} = \cos 3x + i \sin 3x. \\
&= Y_1 \cos 3x - Y_2 \sin 3x && \text{Take the real part.} \\
&= \frac{26x-107}{507}\cos 3x + \frac{115-39x}{507}\sin 3x && \text{Substitute for } Y_1,\ Y_2.
\end{aligned}$$

The solution $y = e^{2x}\mathcal{Y}$ multiplies the above display by e^{2x}. This verifies the formula $y_p = \frac{1}{507}((26x-107)e^{2x}\cos(3x) + (115-39x)e^{2x}\sin(3x))$.

Example 7.11 (Polynomial × Exponential × Sine)

Solve in $y'' - y' + y = (3-x)e^{2x}\sin(3x)$ for y_p, verifying that a particular solution is $y_p = \frac{1}{507}\left((39x-115)e^{2x}\cos(3x) + (26x-107)e^{2x}\sin(3x)\right)$.

Solution: Let $z = 2+3i$. Kümmer's transformation $y = e^{2x}\,\mathcal{I}\mathrm{m}(e^{3ix}Y)$ as on page 578 implies that Y satisfies $[(D+z)^2-(D+z)+1]Y = 3-x$. This equation has been solved in the previous example: $Y = Y_1 + iY_2$ with $Y_1 = (26x-107)/507$ and $Y_2 = (39x-115)/507$. Let $\mathcal{Y} = \mathcal{I}\mathrm{m}(e^{3ix}Y)$. Then

$$\begin{aligned}
\mathcal{Y} &= \mathcal{I}\mathrm{m}((\cos 3x + i\sin 3x)(Y_1 + iY_2)) && \text{Expand complex factors.} \\
&= Y_2 \cos 3x + Y_1 \sin 3x && \text{Extract the imaginary part.} \\
&= \frac{(39x-115)\cos 3x + (26x-107)\sin 3x}{507} && \text{Substitute for } Y_1 \text{ and } Y_2.
\end{aligned}$$

The solution $y = e^{2x}\mathcal{Y}$ multiplies the display by e^{2x}. This verifies the formula $y = \frac{1}{507}\left((39x-115)e^{2x}\cos(3x) + (26x-107)e^{2x}\sin(3x)\right)$.

Example 7.12 (Undetermined Coefficient Library Methods)

Solve $y'' - y' + y = 1 + e^x + \cos(x)$, verifying

$$y = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2) + 1 + e^x - \sin(x).$$

Solution: There are $n = 3$ easily solved equations: $y_1'' - y_1' + y_1 = 1$, $y_2'' - y_2' + y_2 = e^x$ and $y_3'' - y_3' + y_3 = \cos(x)$. The plan is that each such equation is solvable by one of the **library methods**. Then $y_p = y_1 + y_2 + y_3$ is the sought particular solution.

Equation 1: $y_1'' - y_1' + y_1 = 1$. It is solved by *the equilibrium method*, which gives immediately solution $y_1 = 1$.

Equation 2: $y_2'' - y_2' + y_2 = e^x$. Then $y_2 = e^xY$ and $[(D+1)^2 - (D+1) + 1]Y = 1$, by the *polynomial × exponential method.* The equation simplifies to $Y'' + Y' + Y = 1$. Obtain $Y = 1$ by the *equilibrium method*, then $y_2 = e^x$.

Equation 3: $y_3'' - y_3' + y_3 = \cos(x)$. Then $[(D+i)^2 - (D+i) + 1]Y = 1$ and $y_3 = \mathcal{R}\mathrm{e}(e^{ix}Y)$, by the *polynomial × exponential × cosine method.* The equation simplifies to $Y'' + (2i-1)Y' - iY = 1$. Obtain $Y = i$ by the *equilibrium method.* Then $y_3 = \mathcal{R}\mathrm{e}(e^{ix}Y)$ implies $y_3 = -\sin(x)$.

Solution y_p. The particular solution is given by addition, $y_p = y_1 + y_2 + y_3$. Therefore, $y_p = 1 + e^x - \sin(x)$.

Solution y_h. The homogeneous solution y_h is the linear equation solution for $y''-y'+y=0$, obtained from Theorem **??**, which uses the characteristic equation $r^2-r+1=0$. The latter has roots $r=(1\pm i\sqrt{3})/2$ and then $y_h=c_1e^{x/2}\cos(\sqrt{3}x/2)+c_2e^{x/2}\sin(\sqrt{3}x/2)$ where c_1 and c_2 are arbitrary constants.

General Solution. Add y_h and y_p to obtain the general solution

$$y=c_1e^{x/2}\cos(\sqrt{3}x/2)+c_2e^{x/2}\sin(\sqrt{3}x/2)+1+e^x-\sin(x).$$

Example 7.13 (Sine–Cosine Trial solution)
Verify for $y''+4y=\sin x-\cos x$ that $y_p(x)=5\cos x+3\sin x$.

Solution: The lookup table method suggests to substitute $y=d_1\cos x+d_2\sin x$ into the differential equation. The correction rule does not apply, because the homogeneous solution terms involve $\cos 2x$, $\sin 2x$. Use $u''=-u$ for $u=\sin x$ or $u=\cos x$ to obtain the relation

$$\begin{aligned}\sin x-\cos x &= y''+4y\\ &= (-d_1+4)\cos x+(-d_2+4)\sin x.\end{aligned}$$

Comparing sides, matching sine and cosine terms, gives

$$\begin{aligned}-d_1+4 &= -1,\\ -d_2+4 &= 1.\end{aligned}$$

Solving, $d_1=5$ and $d_2=3$. The trial solution $y=d_1\cos x+d_2\sin x$ becomes $y_p(x)=5\cos x+3\sin x$.

Historical Notes

The method of undetermined coefficients presented on page **??** uses the idea of a **trial solution**. Textbooks that present this method appear in the references, especially Edwards–Penney [**?**] and Kreyszig [**?**].

If the right side $f(x)$ is a polynomial, then the trial solution is a polynomial $y=d_0+\cdots+d_kx^k$ with unknown coefficients. It is substituted into the non-homogeneous differential equation to determine the coefficients d_0, ..., d_k, as in Example 7.6. The Taylor method in Example 7.7 implements the same ideas. In the some textbook presentations, the three key theorems of this section are replaced by Table 1 and the **Correction Rule** on page 580. Attempts have been made to integrate the correction rule into the table itself; see Edwards–Penney [**?**], [**?**].

The *method of annihilators* has been used as an alternative approach; see Kreider–Kuller–Ostberg–Perkins [**?**]. The approach gives a deeper insight into higher order differential equations. It requires knowledge of linear algebra and a small nucleus of differential operator calculus.

The idea to employ a recursive polynomial method seems to appear first in a paper by Love [**?**]. A generalization and expansion of details appears in [**?**]. The method is certainly worth learning, but doing so does not excuse one from learning other methods. The recursive method is a worthwhile hybrid method for special circumstances.

Proofs and Technical Details

Proof of Theorem 7.8: The three cases correspond to zero, one or two roots $r = 0$ for the characteristic equation $ar^2 + br + c = 0$. The missing constant and x-terms in **case 2** and **case 3** are justified by including them in the homogeneous solution y_h, instead of in the particular solution y_p.

Assume $p(x)$ has degree d and succinctly write down the successive derivatives of the differential equation as

$$ay^{(2+k)} + by^{(1+k)} + cy^{(k)} = p^{(k)}(x), \quad k = 0, \ldots, d. \tag{10}$$

Assume, to consider simultaneously all three cases, that

$$y = y_0 + y_1 + \cdots + y_{m+d}\frac{x^{m+d}}{(m+d)!}$$

where $m = 0, 1, 2$ corresponding to cases 1,2,3, respectively. It has to be shown that there are coefficients $y_0, \ldots, y_{m+d}$ such that y is a solution of $ay'' + by' + cy = p(x)$.

Let $x = 0$ in equations (10) and use the definition of polynomial y to obtain the equations

$$ay_{2+k} + by_{1+k} + cy_k = p^{(k)}(0), \quad k = 0, \ldots, d. \tag{11}$$

In **case 1** ($c \neq 0$), $m = 0$ and the last equation in (11) gives $y_{m+d} = p^{(d)}(0)/c$. Back-substitution succeeds in finding the other coefficients, in reverse order, because $y^{(d+1)}(0) = y^{(d+2)}(0) = 0$, in this case. *Define* the constants y_0 to y_d to be the solutions of (11). Define $y_{d+1} = y_{d+2} = 0$.

In **case 2** ($c = 0$, $b \neq 0$), $m = 1$ and the last equation in (11) gives $y_{m+d} = p^{(d)}(0)/b$. Back-substitution succeeds in finding the other coefficients, in reverse order, because $y^{(d+2)}(0) = 0$, in this case. However, y_0 is undetermined. Take it to be zero, then *define* y_1 to y_{d+1} to be the solutions of (11). Define $y_{d+2} = 0$.

In **case 3** ($c = b = 0$), $m = 2$ and the last equation in (11) gives $y_{m+d} = p^{(d)}(0)/a$. Back-substitution succeeds in finding the other coefficients, in reverse order. However, y_0 and y_1 are undetermined. Take them to be zero, then *define* y_2 to y_{d+2} to be the solutions of (11).

It remains to prove that the polynomial y so defined is a solution of the differential equation $ay'' + by' + cy = p(x)$. Begin by applying quadrature to the last differentiated equation $ay^{(2+d)} + by^{(1+d)} + cy^{(d)} = p^{(d)}(x)$. The result is $ay^{(1+d)} + by^{(d)} + cy^{(d-1)} = p^{(d-1)}(x) + C$ with C undetermined. Set $x = 0$ in this equation. Then relations (11) say that $C = 0$. This process can be continued until $ay'' + by' + cy = q(x)$ is obtained, hence y is a solution.

Proof of Theorem 7.9: Kümmer's transformation $y = e^{kx}Y$ is differentiated twice to give the formulas

$$\begin{aligned} y &= e^{kx}Y, \\ y' &= ke^{kx}Y + e^{kx}Y' \\ &= e^{kx}(D+k)Y, \\ y'' &= k^2e^{kx}Y + 2ke^{kx}Y' + e^{kx}Y'' \\ &= e^{kx}(D+k)^2Y. \end{aligned}$$

Insert them into the differential equation $a(D+k)^2Y + b(D+k)Y + cY = p(x)$. Then multiply through by e^{kx} to remove the common factor e^{-kx} on the left, giving $ay'' + by' + cy = p(x)e^{kx}$. This completes the proof.

Proof of Theorem 7.10: Abbreviate $ay'' + by' + cy$ by Ly. Consider the complex equation $Lu = p(x)e^{zx}$, to be solved for $u = u_1 + iu_2$. According to Theorem 7.9, u can be computed as $u = e^{zx}Y$ where $[a(D+z)^2 + b(D+z) + c]Y = p(x)$. Take the real and imaginary parts of $u = e^{zx}Y$ and $Lu = p(x)e^{zx}$. Then $u_1 = \mathcal{R}\mathrm{e}(e^{zx}Y)$ and $u_2 = \mathcal{I}\mathrm{m}(e^{zx}Y)$ satisfy $Lu_1 = \mathcal{R}\mathrm{e}(p(x)e^{zx}) = p(x)\cos(mx)e^{kx}$ and $Lu_2 = \mathcal{I}\mathrm{m}(p(x)e^{zx}) = p(x)\sin(mx)e^{kx}$. ■

Exercises 7.6

Polynomial Solutions

Determine a polynomial solution y_p for the given differential equation. Apply Theorem 7.8, page 581, and model the solution after Examples 7.5, 7.6, 7.7 and 7.8.

1. $y'' = x$

2. $y'' = x - 1$

3. $y'' = x^2 - x$

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

14. $y'' - y = x^3$

Polynomial-Exponential Solutions

Determine a solution y_p for the given differential equation. Apply Theorem 7.9, page 581, and model the solution after Example 7.9.

15. $y'' + y = e^x$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

18. $y'' = e^{-2x}$

19. $y'' - y = (x+1)e^{2x}$

20. $y'' - y = (x-1)e^{-2x}$

21. $y'' - y' = (x+3)e^{2x}$

22. $y'' - y' = (x-2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2+3)e^{3x}$

24. $y'' - 3y' + 2y = (x^2-2)e^{-3x}$

Sine and Cosine Solutions

Determine a solution y_p for the given differential equation. Apply Theorem 7.10, page 581, and model the solution after Examples 7.10 and 7.11.

25. $y'' = \sin(x)$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

28. $y'' + y = \cos(x)$

29. $y'' = (x+1)\sin(x)$

30. $y'' = (x+1)\cos(x)$

31. $y'' - y = (x+1)e^x\sin(2x)$

32. $y'' - y = (x+1)e^x\cos(2x)$

33. $y'' - y' - y = (x^2+x)e^x\sin(2x)$

34. $y'' - y' - y = (x^2+x)e^x\cos(2x)$

Undetermined Coefficients Algorithm

Determine a solution y_p for the given differential equation. Apply the polynomial algorithm, page 576, and model the solution after Example 7.12.

35. $y'' = x + \sin(x)$

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2e^x + xe^x\cos(2x)$

46. $y'' + y' - y = x^2e^{-x} + xe^x\sin(2x)$

Additional Proofs

The exercises below fill in details in the text. The hints are in the proofs in the textbook. No solutions will be given for the odd exercises.

47. (Theorem 7.8)
Supply the missing details in the proof of Theorem 7.8 for case 1. In particular, give the details for back-substitution.

48. (Theorem 7.8)
Supply the details in the proof of Theorem 7.8 for case 2. In particular, give the details for back-substitution and explain fully why it is possible to select $y_0 = 0$.

49. (Theorem 7.8)
Supply the details in the proof of Theorem 7.8 for case 3. In particular, explain why back-substitution leaves y_0 and y_1 undetermined, and why it is possible to select $y_0 = y_1 = 0$.

50. (Superposition)
Let Ly denote $ay'' + by' + cy$. Show that solutions of $Lu = f(x)$ and $Lv = g(x)$ add to give $y = u + v$ as a solution of $Ly = f(x) + g(x)$.

51. (Easily Solved Equations)
Let Ly denote $ay'' + by' + cy$. Let $Ly_k = f_k(x)$ for $k = 1, \ldots, n$ and define $y = y_1 + \cdots + y_n$, $f = f_1 + \cdots + f_n$. Show that $Ly = f(x)$.

PDF Sources

Text, Solutions and Corrections

Author: Grant B. Gustafson, University of Utah, Salt Lake City 84112.

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