Contents

Chapter 2

First Order Differential Equations

Contents

The subject of the chapter is the first order differential equation

$$y' = f(x, y).$$

The study includes closed-form solution formulas for special equations and some applications to science and engineering.

2.1 Quadrature Method

The **method of quadrature** refers to the technique of integrating both sides of an equation, hoping thereby to extract a solution formula.

The term **quadrature** originates in ancient geometry, where it means *finding area* of a plane figure, by constructing a square of equal area.[1] Numerical quadrature computes areas enclosed by plane curves from approximating rectangles, by

[1]See Katz, Victor J. (1998) *A History of Mathematics: An Introduction* (2nd edition) Addison Wesley Longman, ISBN 0321016181, and Wikipedia: http://en.wikipedia.org/wiki/Quadrature.

algorithms such as the rectangular rule and Simpson's rule. For symbolic problems, the task is overtaken by Newton's integral calculus. The naming convention follows computer algebra system `maple`.

Fundamental Theorem of Calculus

The foundation of the study of differential equations rests with Isaac Newton's discovery on instantaneous velocities. Details of the calculus background required appears in Appendix **??**, page **??**.

Theorem 2.1 (Fundamental Theorem of Calculus I)
Let G be continuous and let F be continuously differentiable on $[a, b]$. Then

(a) $F(b) - F(a) = \displaystyle\int_a^b \frac{dF}{dx}(x)dx$,

(b) $\displaystyle\frac{d}{dx}\int_a^x G(t)dt = G(x)$.

Theorem 2.2 (Fundamental Theorem of Calculus II)
Let $G(x)$ be continuous and let $y(x)$ be continuously differentiable on $[a, b]$. Then for some constant c,

(a) $y(x) = \displaystyle\int \frac{dy}{dx}dx + c$,

(b) $\displaystyle\frac{d}{dx}\int G(x)dx = G(x)$.

Part (a) of the fundamental theorem is used to find a candidate solution to a differential equation.

Part (b) of the fundamental theorem is used in differential equations to do an answer check.

The Method of Quadrature

The method is applied to differential equations $y' = f(x, y)$ in which f is independent of y. Then symbol y is absent from $f(x, y)$, which implies $f(x, y)$ is constant or else $f(x, y)$ depends only on the symbol x. The model differential equation then has the form $y' = F(x)$ where F is a given function of the single variable F.

(i) **To solve for $y(x)$ in $\dfrac{dy}{dx} = F(x)$, integrate on variable x across the equation, then use the Fundamental Theorem of Calculus.**

(ii) **Check the answer.**

Indefinite Integral Shortcut. Integrate across the equation with indefinite integrals, then collect all integration constants into symbol c.

Solution with Symbol c. Symbol c initially appears in the expression obtained for y. If no initial condition was given, then the answer for y is this expression, which contains the unresolved symbol c. Experts call this expression the **general solution**.

Solution with No symbol c. If an initial condition is given in the form $y = y_0$ at $x = x_0$ (same as $y(x_0) = y_0$), then symbol c can be resolved. For instance, if the answer is $y = 2(x-1) + c$ and the initial condition is $y(-1) = 3$, then $y = 2(x-1) + c$ with $x = -1, y = 3$ becomes $3 = 2(-1-1) + c$, and then $c = 7$. Experts call the xy–expression with c eliminated a **particular solution**.

Theorem 2.3 (Existence-Uniqueness for Quadrature Equations)
Let $F(x)$ be continuous on $a < x < b$. Assume $a < x_0 < b$ and $-\infty < y_0 < \infty$. Then the initial value problem

$$y' = F(x), \quad y(x_0) = y_0 \tag{1}$$

has on interval $a < x < b$ the unique solution

$$y(x) = y_0 + \int_{x_0}^{x} F(t)dt. \tag{2}$$

Details of proof appear on page 78.

Examples

Example 2.1 (Quadrature)
Solve $y' = 3e^x$, $y(0) = 0$.

Solution:
The final answer is $y = 3e^x - 3$. An answer check appears in the next example.

Details. The *shortcut* is applied.

$\frac{dy}{dx} = 3e^x$	Copy the differential equation.
$\int \frac{dy}{dx} dx = \int 3e^x dx$	Integrate across the equation on x.
$y(x) + c_1 = \int 3e^x dx$	Fundamental theorem of calculus, page 75.
$y(x) + c_1 = 3e^x + c_2$	Integral table.
$y(x) = 3e^x + c$	Where $c = c_2 - c_1$ is a constant.

The answer is $y = 3e^x + c$. The symbol c is to be resolved from the **initial condition** $y(0) = 0$, as follows.

$0 = y(0)$	Copy the initial condition (sides reversed).
$= (3e^x + c)\vert_{x=0}$	Insert $y = 3e^x + c$, the proposed solution.
$= 3e^0 + c$	Substitute $x = 0$.

$= 3 + c$	Use $e^0 = 1$.
$c = -1$	Equation $0 = 3 + c$ solved for c.

Candidate solution. Back-substitute the symbol c value $c = -1$ into the answer $y = 3e^x + c$ to obtain the candidate solution $y = 3e^x + (-3)$. This answer can contain errors, in general, due to integration and arithmetic mistakes.

Example 2.2 (Answer Check)
Given $y' = 3e^x$, $y(0) = 0$ and candidate solution $y(x) = 3e^x - 3$, display an answer check.

Solution: There are **two panels** in this answer check: **Panel 1**: differential equation check, **Panel 2**: initial condition check.

Panel 1. We check the answer $y = 3e^x - 3$ for the differential equation $y' = 3e^x$.

The steps are:

$\text{LHS} = y'$	Left side of the differential equation.
$= (3e^x - 3)'$	Substitute the expression for y.
$= 3e^x - 0$	Sum rule, constant rule and $(e^u)' = u'e^u$.
$= \text{RHS}$	Solution verified.

Panel 2. Let's check the answer $y = 3e^x - 3$ against the initial condition $y(0) = 0$. Expected is an immediate mental check that $e^0 = 1$ implies the correctness of $y(0) = 0$.

The steps will be shown in order to detail the algorithm for checking an initial condition. The algorithm applies when checking complex algebraic expressions. Abbreviated versions of the algorithm are used on simple expressions.

$\text{LHS} = y(0)$	Left side of the initial condition $y(0) = 0$.
$= (3e^x - 3)\|_{x=0}$	Notation $y(x_0)$ means substitute $x = x_0$ into the expression for y.
$= 3e^0 - 3$	Substitute $x = 0$ into the expression.
$= 0$	Because $e^0 = 1$.
$= \text{RHS}$	Initial condition verified.

River Crossing

A boat crosses a river at fixed speed with power applied perpendicular to the shoreline. Is it possible to estimate the boat's downstream location?

The answer is *yes*. The problem's variables are

x	Distance from shore,	w	Width of the river,
y	Distance downstream,	v_b	Boat velocity (dx/dt),
t	Time in hours,	v_r	River velocity (dy/dt).

The calculus chain rule $dy/dx = (dy/dt)/(dx/dt)$ is applied, using the symbols v_r and v_b instead of dy/dt and dx/dt, to give the *model equation*

$$\frac{dy}{dx} = \frac{v_r}{v_b}. \tag{3}$$

Stream Velocity. The downstream river velocity will be approximated by $v_r = kx(w-x)$, where $k > 0$ is a constant. This equation gives velocity $v_r = 0$ at the two shores $x = 0$ and $x = w$, while the **maximum stream velocity** at the center $x = w/2$ is (see page 79)

$$v_c = \frac{kw^2}{4}. \tag{4}$$

Special River-Crossing Model. The model equation (3) using $v_r = kx(w-x)$ and the constant k defined by (4) give the initial value problem

$$\frac{dy}{dx} = \frac{4v_c}{v_b w^2} x(w-x), \quad y(0) = 0. \tag{5}$$

The solution of (5) by the method of quadrature is

$$y = \frac{4v_c}{v_b w^2}\left(-\frac{1}{3}x^3 + \frac{1}{2}wx^2\right), \tag{6}$$

where w is the river's width, v_c is the river's midstream velocity and v_b is the boat's velocity. In particular, the boat's **downstream drift** on the opposite shore is $\frac{2}{3}w(v_c/v_b)$. See *Technical Details* page 79.

Example 2.3 (River Crossing)

A boat crosses a mile-wide river at 3 miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is 10 miles per hour. Find the transit time and the downstream drift to the opposite shore.

Solution: The answers, justified below, are 20 minutes and 20/9 miles.

Transit time. This is the time it takes to reach the opposite shore. The layman answer of 20 minutes is correct, because the boat goes 3 miles in one hour, hence 1 mile in 1/3 of an hour, perpendicular to the shoreline.

Downstream drift. This is the value $y(1)$, where y is the solution of equation (5), with $v_c = 10$, $v_b = 3$, $w = 1$, all distances in miles. The special model is

$$\frac{dy}{dx} = \frac{40}{3}x(1-x), \quad y(0) = 0.$$

The solution given by equation (6) is $y = \frac{40}{3}\left(-\frac{1}{3}x^3 + \frac{1}{2}x^2\right)$ and the downstream drift is then $y(1) = 20/9$ miles. This answer is 2/3 of the layman's answer of (1/3)(10) miles. The explanation is that the boat is pushed downstream at a variable rate from 0 to 10 miles per hour, depending on its position x.

Details and Proofs

Proof of Theorem 2.3:

Uniqueness. Let $y(x)$ be any solution of (1). It will be shown that $y(x)$ is given by the solution formula (2).

$y(x) = y(0) + \int_{x_0}^x y'(t)dt$	Fundamental theorem of calculus, page ??.
$= y_0 + \int_{x_0}^x F(t)dt$	Use (1). This verifies equation (2).

Answer Check. Let $y(x)$ be given by solution formula (2). It will be shown that $y(x)$ solves initial value problem (1).

$y'(x) = \left(y_0 + \int_{x_0}^x F(t)dt\right)'$	Compute the derivative from (2).
$= F(x)$	Apply the fundamental theorem of calculus.

The initial condition is verified in a similar manner:

$y(x_0) = y_0 + \int_{x_0}^{x_0} F(t)dt$	Apply (2) with $x = x_0$.
$= y_0$	The integral is zero: $\int_a^a F(x)dx = 0$.

■

Technical Details for (4): The maximum of a continuously differentiable function $f(x)$ on $0 \le x \le w$ can be found by locating the critical points (i.e., where $f'(x) = 0$) and then testing also the endpoints $x = 0$ and $x = w$. The derivative $f'(x) = k(w - 2x)$ is zero at $x = w/2$. Then $f(w/2) = kw^2/4$. This value is the maximum of f, because $f = 0$ at the endpoints.

Technical Details for (6): Let $a = \dfrac{4v_c}{v_b w^2}$. Then

$y = y(0) + \int_0^x y'(t)dt$	Method of quadrature.
$= 0 + a\int_0^x t(w-t)dt$	By (5), $y' = at(w-t)$.
$= a\left(-\frac{1}{3}x^3 + \frac{1}{2}wx^2\right)$.	Integral table.

To compute the downstream drift, evaluate $y(w) = a\dfrac{w^3}{6}$ or $y(w) = \dfrac{2w}{3}\dfrac{v_c}{v_b}$.

Exercises 2.1

Quadrature

Find a candidate solution for each initial value problem and verify the solution. See Example 2.1 and Example 2.2, page 76.

1. $y' = 4e^{2x}$, $y(0) = 0$.

2. $y' = 2e^{4x}$, $y(0) = 0$.

3. $(1+x)y' = x$, $y(0) = 0$.

4. $(1-x)y' = x$, $y(0) = 0$.

5. $y' = \sin 2x$, $y(0) = 1$.

6. $y' = \cos 2x$, $y(0) = 1$.

7. $y' = xe^x$, $y(0) = 0$.

8. $y' = xe^{-x^2}$, $y(0) = 0$.

9. $y' = \tan x$, $y(0) = 0$.

10. $y' = 1 + \tan^2 x$, $y(0) = 0$.

11. $(1+x^2)y' = 1$, $y(0) = 0$.

12. $(1+4x^2)y' = 1$, $y(0) = 0$.

13. $y' = \sin^3 x$, $y(0) = 0$.

14. $y' = \cos^3 x$, $y(0) = 0$.

15. $(1+x)y' = 1$, $y(0) = 0$.

16. $(2+x)y' = 2$, $y(0) = 0$.

17. $(2+x)(1+x)y' = 2$, $y(0) = 0$.

18. $(2+x)(3+x)y' = 3$, $y(0) = 0$.

19. $y' = \sin x \cos 2x$, $y(0) = 0$.

20. $y' = (1+\cos 2x)\sin 2x$, $y(0) = 0$.

River Crossing

A boat crosses a river of width w miles at v_b miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is v_c miles per hour. Find the transit time and the downstream drift to the opposite shore. See Example 2.3, page 78, and the details for (6).

21. $w = 1$, $v_b = 4$, $v_c = 12$

22. $w = 1$, $v_b = 5$, $v_c = 15$

23. $w = 1.2$, $v_b = 3$, $v_c = 13$

24. $w = 1.2$, $v_b = 5$, $v_c = 9$

25. $w = 1.5$, $v_b = 7$, $v_c = 16$

26. $w = 2$, $v_b = 7$, $v_c = 10$

27. $w = 1.6$, $v_b = 4.5$, $v_c = 14.7$

28. $w = 1.6$, $v_b = 5.5$, $v_c = 17$

Fundamental Theorem I

Verify the identity. Use the fundamental theorem of calculus part (b), page 75.

29. $\int_0^x (1+t)^3 dt = \frac{1}{4}\left((1+x)^4 - 1\right)$.

30. $\int_0^x (1+t)^4 dt = \frac{1}{5}\left((1+x)^5 - 1\right)$.

31. $\int_0^x te^{-t} dt = -xe^{-x} - e^{-x} + 1$.

32. $\int_0^x te^{t} dt = xe^{x} - e^{x} + 1$.

Fundamental Theorem II

Differentiate. Use the fundamental theorem of calculus part (b), page 75.

33. $\int_0^{2x} t^2 \tan(t^3) dt$.

34. $\int_0^{3x} t^3 \tan(t^2) dt$.

35. $\int_0^{\sin x} te^{t+t^2} dt$.

36. $\int_0^{\sin x} \ln(1+t^3) dt$.

Fundamental Theorem III

Integrate $\int_0^1 f(x)dx$. Use the fundamental theorem of calculus part (a), page 75. Check answers with computer or calculator assist. Some require a clever u-substitution or an integral table.

37. $f(x) = x(x-1)$

38. $f(x) = x^2(x+1)$

39. $f(x) = \cos(3\pi x/4)$

40. $f(x) = \sin(5\pi x/6)$

41. $f(x) = \dfrac{1}{1+x^2}$

42. $f(x) = \dfrac{2x}{1+x^4}$

43. $f(x) = x^2 e^{x^3}$

44. $f(x) = x(\sin(x^2) + e^{x^2})$

45. $f(x) = \dfrac{1}{\sqrt{-1+x^2}}$

46. $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

47. $f(x) = \dfrac{1}{\sqrt{1+x^2}}$

48. $f(x) = \dfrac{1}{\sqrt{1+4x^2}}$

49. $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

50. $f(x) = \dfrac{4x}{\sqrt{1-4x^2}}$

51. $f(x) = \dfrac{\cos x}{\sin x}$

52. $f(x) = \dfrac{\cos x}{\sin^3 x}$

53. $f(x) = \dfrac{e^x}{1+e^x}$

54. $f(x) = \dfrac{\ln|x|}{x}$

55. $f(x) = \sec^2 x$

56. $f(x) = \sec^2 x - \tan^2 x$

57. $f(x) = \csc^2 x$

58. $f(x) = \csc^2 x - \cot^2 x$

59. $f(x) = \csc x \cot x$

60. $f(x) = \sec x \tan x$

Integration by Parts

Integrate $\int_0^1 f(x)dx$ by parts, $\int u dv = uv - \int v du$. Check answers with computer or calculator assist.

61. $f(x) = xe^x$

62. $f(x) = xe^{-x}$

63. $f(x) = \ln|x|$

64. $f(x) = x\ln|x|$

65. $f(x) = x^2 e^{2x}$

66. $f(x) = (1+2x)e^{2x}$

67. $f(x) = x\cosh x$

68. $f(x) = x\sinh x$

69. $f(x) = x\arctan(x)$

70. $f(x) = x\arcsin(x)$

Partial Fractions

Integrate f by partial fractions. Check answers with computer or calculator assist.

71. $f(x) = \dfrac{x+4}{x+5}$

72. $f(x) = \dfrac{x-2}{x-4}$

73. $f(x) = \dfrac{x^2+4}{(x+1)(x+2)}$

74. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)}$

75. $f(x) = \dfrac{x+4}{(x+1)(x+2)}$

76. $f(x) = \dfrac{x-1}{(x+1)(x+2))}$

77. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x+5)}$

78. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x+3)}$

79. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x-1)}$

80. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x-1)}$

Special Methods

Integrate f by using the suggested u-substitution or method. Check answers with computer or calculator assist.

81. $f(x) = \dfrac{x^2+2}{(x+1)^2}$, $u = x+1$.

82. $f(x) = \dfrac{x^2+2}{(x-1)^2}$, $u = x-1$.

83. $f(x) = \dfrac{2x}{(x^2+1)^3}$, $u = x^2+1$.

84. $f(x) = \dfrac{3x^2}{(x^3+1)^2}$, $u = x^3+1$.

85. $f(x) = \dfrac{x^3+1}{x^2+1}$, use long division.

86. $f(x) = \dfrac{x^4+2}{x^2+1}$, use long division.

2.2 Separable Equations

An equation $y' = f(x,y)$ is called **separable** provided algebraic operations, usually multiplication, division and factorization, allow it to be written in a **separable form** $y' = F(x)G(y)$ for some functions F and G. This class includes the *quadrature equations* $y' = F(x)$. Separable equations and associated solution methods were discovered by G. Leibniz in 1691 and formalized by J. Bernoulli in 1694.

Finding a Separable Form

Given differential equation $y' = f(x,y)$, invent values x_0, y_0 such that $f(x_0,y_0) \neq 0$. Define F, G by the formulas

$$F(x) = \frac{f(x,y_0)}{f(x_0,y_0)}, \quad G(y) = f(x_0,y). \tag{1}$$

Because $f(x_0,y_0) \neq 0$, then (1) makes sense.

Theorem 2.4 (Separability Test)
Let F and G be defined by equation (1). Compute $F(x)G(y)$. Then

(a) $F(x)G(y) = f(x,y)$ implies $y' = f(x,y)$ is **separable**.

(b) $F(x)G(y) \neq f(x,y)$ implies $y' = f(x,y)$ is **not separable**.

Proof: Conclusion (b) follows from separability test I, *infra*. Conclusion (a) follows because two functions $F(x)$, $G(y)$ have been defined in equation (1) such that $f(x,y) = F(x)G(y)$ (definition of separable equation).

Invention and Application. Initially, let (x_0,y_0) be $(0,0)$ or $(1,1)$ or some suitable pair, for which $f(x_0,y_0) \neq 0$; then define F and G by (1). Multiply F and G to test the equation $FG = f$. The algebra will discover a factorization $f = F(x)G(y)$ without having to know algebraic tricks like factorizing multi-variable equations. But if $FG \neq f$, then the algebra *proves* the equation is not separable.

Non-Separability Tests

Test I — Equation $y' = f(x,y)$ is not separable if

$$f(x,y_0)f(x_0,y) - f(x_0,y_0)f(x,y) \neq 0 \tag{2}$$

for some pair of points (x_0,y_0), (x,y) in the domain of f.

Test II — The equation $y' = f(x,y)$ is not separable if either $f_x(x,y)/f(x,y)$ is non-constant in y or $f_y(x,y)/f(x,y)$ is non-constant in x.

Illustration. Consider $y' = xy + y^2$. *Test I* implies it is not separable, because $f(x,1)f(0,y) - f(0,1)f(x,y) = (x+1)y^2 - (xy+y^2) = x(y^2 - y) \neq 0$. *Test II* implies it is not separable, because $f_x/f = 1/(x+y)$ is not constant as a function of y.

Test I details. Assume $f(x,y) = F(x)G(y)$, then equation (2) fails because each term on the left side of (2) evaluates to $F(x)G(y_0)F(x_0)G(y)$ for all choices of (x_0, y_0) and (x, y) (hence contradiction $0 \neq 0$).

Test II details. Assume $f(x,y) = F(x)G(y)$ and F, G are sufficiently differentiable. Then $f_x(x,y)/f(x,y) = F'(x)/F(x)$ is independent of y and the fraction $f_y(x,y)/f(x,y) = G'(y)/G(y)$ is independent of x.

Separated Form Test

A **separated equation** $y'/G(y) = F(x)$ is recognized by this test:

Left Side Test. The left side of the equation has factor y' and it is independent of symbol x.

Right Side Test. The right side of the equation is independent of symbols y and y'.

Variables-Separable Method

Determined by the method are the following kinds of solution formulas.

Equilibrium Solutions. They are the constant solutions $y = c$ of $y' = f(x,y)$. Find them by substituting $y = c$ in $y' = f(x,y)$, followed by solving for c, then report the list of answers $y = c$ so found.

Non-Equilibrium Solutions. For separable equation $y' = F(x)G(y)$, it is a solution y with $G(y) \neq 0$. It is found by dividing by $G(y)$ and applying the method of quadrature.

The term *equilibrium* is borrowed from kinematics. Alternative terms are **rest solution** and **stationary solution**; all mean $y' = 0$ in calculus terms.

Spurious Solutions. If $F(x)G(y) = 0$ is solved instead of $G(y) = 0$, then both x and y solutions might be found. The x-solutions are ignored: they are not equilibrium solutions. Only solutions of the form $y =$ constant are called equilibrium solutions.

It is important to *check the solution* to a separable equation, because certain steps used to arrive at the solution may not be reversible.

For most applications, the two kinds of solutions suffice to determine all possible solutions. In general, a separable equation may have non-unique solutions to some initial value problem. To prevent this from happening, it can be assumed that F, G and G' are continuous; see the Picard-Lindelöf theorem, page **??**. If non-uniqueness does occur, then often the equilibrium and non-equilibrium solutions can be pieced together to represent all solutions.

Finding Equilibrium Solutions

The search for equilibria can be done without finding the separable form of $y' = f(x, y)$. It is enough to solve for y in the equation $f(x, y) = 0$, *subject to the condition that x is arbitrary.* An equilibrium solution y cannot depend upon x, because it is *constant.* If y turns out to depend on x, after solving $f(x, y) = 0$ for y, then this is sufficient evidence that $y' = f(x, y)$ is **not separable**. Some examples:

$y' = y\sin(x - y)$	It is *not separable*. The solutions of $y\sin(x - y) = 0$ are $y = 0$ and $x - y = n\pi$ for any integer n. The solution $y = x - n\pi$ is non-constant, therefore the equation cannot be separable.
$y' = xy(1 - y^2)$	It is *separable*. The equation $xy(1 - y^2) = 0$ has three equilibrium solutions $y = 0$, $y = 1$, $y = -1$. Equilibrium solutions must be constant solutions.

Algorithm. To find equilibrium solutions, formally substitute $y = c$ into the differential equation, then solve for c, and report all constant solutions $y = c$ so found. There can be zero solutions, or just one solution, or some finite number of solutions, or infinitely many solutions.

Shortcut. In a given problem, a formal substitution is not used, but instead y' is replaced by zero (the result when y = constant). For $y' = f(x, y)$, the equation $f(x, y) = 0$ is to be solved for y. For example, $y' = (x + 1)(y^2 - 4)$ becomes $0 = (x + 1)(y^2 - 4)$, equivalent to $y^2 - 4 = 0$ or $y = 2$, $y = -2$. The spurious solution $x = -1$ is ignored, because we are looking for constant solutions of the form $y = c$, which in this example are $y = 2$ and $y = -2$.

The problem of finding all equilibrium solutions is known to be technically unsolvable, that is, there is no proven algorithm for finding all the solutions of $G(y) = 0$. However, there are some very good numerical methods that apply, including **Newton's method** and the **bisection method**. Modern computer algebra systems make it practical to find equilibrium solutions, both symbolic (like $y = \pi$) and numeric (like $y = 3.14159$), in a single effort.

Finding Non-Equilibrium Solutions

A given solution $y(x)$ satisfying $G(y(x)) \neq 0$ throughout its domain of definition is called a non-equilibrium solution. Then division by $G(y(x))$ is allowed in the differential equation $y'(x) = F(x)G(y(x))$. The *method of quadrature* applies to the separated equation $y'/G(y(x)) = F(x)$. Some details:

$\int_{x_0}^{x} \dfrac{y'(t)dt}{G(y(t))} = \int_{x_0}^{x} F(t)dt$	Integrate both sides of the separated equation over $x_0 \leq t \leq x$.
$\int_{y_0}^{y(x)} \dfrac{du}{G(u)} = \int_{x_0}^{x} F(t)dt$	Apply on the left the change of variables $u = y(t)$. Define $y_0 = y(x_0)$.

$y(x) = W^{-1}\left(\int_{x_0}^{x} F(t)dt\right)$	Define $W(y) = \int_{y_0}^{y} du/G(u)$. Take inverses to isolate $y(x)$.

The calculation produces a formula which is strictly speaking a *candidate solution* y. It does not prove that the formula works in the equation: *checking the solution* is required.

Theoretical Inversion

The function W^{-1} appearing in the last step above is generally not given by a formula. Therefore, W^{-1} rarely appears explicitly in applications or examples. It is the *method* that is memorized:

> Prepare a separable differential equation by transforming it to separated form. Then apply the method of quadrature.

The separated form $y' = F(x)G(y)$ is checked by the separated form test, page 83. For example, $y' = (1 + x^2)y^3$ has $F = 1 + x^2$ and $G = y^3$; quadrature is applied to the divided equation $y'/y^3 = 1 + x^2$.

The theoretical basis for using W^{-1} is a calculus theorem which says that *a strictly monotone continuous function has a continuous inverse.* The fundamental theorem of calculus implies that $W(y)$ is continuous with nonzero derivative $W'(y) = 1/G(y)$. Therefore, $W(y)$ is strictly monotone. The cited calculus theorem implies that $W(y)$ has a continuously differentiable inverse W^{-1}.

Explicit and Implicit Solutions

The variables-separable method gives equilibrium solutions which are already **explicit**, that is:

Definition 2.1 (Explicit Solution)
A solution of $y' = f(x, y)$ is called **explicit** provided it is given by an equation

$$y = \text{an expression independent of } y.$$

To elaborate, on the left side must appear exactly the symbol y followed by an equal sign. Symbols y and $=$ are followed by an expression which does not contain the symbol y. Examples of explicit equations are $y = 0$, $y = -1$, $y = x + 2\pi$, $y = \sin x + x^2 + 10$. The definition is strict, for example $y + 1 = 0$ is **not explicit** because it fails to have y isolated left. Yes, it can be converted into an explicit equation $y = -1$.

Definition 2.2 (Implicit Solution)
A solution of $y' = f(x, y)$ is called **implicit** provided it is not explicit.

Equations like $2y = x$ are not explicit (they are called *implicit*) because the coefficient of y on the left is not 1. Similarly, $y = x + y^2$ is not explicit because the right side contains symbol y. Equation $y = e^\pi$ is explicit because the right side fails to contain symbol y (symbol x may be absent). Applications can leave the non-equilibrium solutions in *implicit* form $\int_{y_0}^{y(x)} du/G(u) = \int_{x_0}^{x} F(t)dt$, with serious effort being expended to do the indicated integrations.

In special cases, it is possible to find an explicit solution from the implicit one by algebraic methods. The required algebraic methods might appear to be unmotivated tricks. Computer algebra systems can make this step look like science instead of art.

Examples

Example 2.4 (Non-separable Equation)
Explain why $yy' = x - y^2$ is not separable.

Solution: It is tempting to try manipulations like adding y^2 to both sides of the equation, in an attempt to obtain a separable form, but every such trick fails. The failure of such attempts is evidence that the equation is perhaps not separable. Failure of attempts does not *prove* non-separability.

Test I applies to verify that the equation is not separable. Let $f(x, y) = x/y - y$ and choose $x_0 = 0$, $y_0 = 1$. Then $f(x_0, y_0) \neq 0$. Compute as follows:

$\text{LHS} = f(x, y_0)f(x_0, y) - f(x_0, y_0)f(x, y)$	Identity (2) left side.
$= f(x, 1)f(0, y) - f(0, 1)f(x, y)$	Use $x_0 = 0$, $y_0 = 1$.
$= (x - 1)(-y) - (-1)(x/y - y)$	Substitute $f(x, y) = x/y - y$.
$= -xy + x/y$	Simplify.

This expression fails to be zero for all (x, y) (e.g., $x = 1$, $y = 2$), therefore the equation is not separable, by *Test I*.

Test II also applies to verify the equation is not separable: $\dfrac{f_x}{f} = \dfrac{1/y}{f} = x - y^2$ is non-constant in x.

Example 2.5 (Separated Form Test Failure)
Given $yy' = 1 - y^2$, explain why the equivalent equation $yy' + y^2 = 1$, obtained by adding y^2 across the equation, fails the separated form test, page 83.

Solution: The *test* requires the left side of $yy' + y^2 = 1$ to contain the factor y'. It doesn't, so it fails the test. Yes, $yy' + y^2 = 1$ does pass the other checkpoints of the *test*: the left side is independent of x and the right side is independent of y and y'.

Example 2.6 (Separated Equation)
Find for $(x + 1)yy' = x - xy^2$ a separated equation using the *test*, page 83.

Solution: The equation usually reported is $\dfrac{yy'}{(1-y)(1+y)} = \dfrac{x}{x+1}$. It is found by factoring and division.

The given equation is factored into $(1+x)yy' = x(1-y)(1+y)$. To pass the *test*, the objective is to move all factors containing only y to the left and all factors containing only x to the right. This is technically accomplished using division by $(x+1)(1-y)(1+y)$.

To the result of the division is applied the *test* on page 83: the *left side* contains factor y' and otherwise involves the factor $y/(1-y^2)$, which depends only on y; the *right side* is $x/(x+1)$, which depends only on x. In short, the candidate separated equation passes the test.

There is another way to approach the problem, by writing the differential equation in standard form $y' = f(x,y)$ where $f(x,y) = x(1-y^2)/(1+x)$. Then $f(1,0) = 1/2 \neq 0$. Define $F(x) = f(x,0)/f(1,0)$, $G(y) = f(1,y)$. We verify $F(x)G(y) = f(x,y)$. A separated form is then $y'/G(y) = F(x)$ or $2y'/(1-y^2) = 2x/(1+x)$.

Example 2.7 (Equilibria)
Given $y' = x(1-y)(1+y)$, find all equilibria.

Solution: The constant solutions $y = -1$ and $y = 1$ are the equilibria, as will be justified.

Equilibria are found by substituting $y = c$ into the differential equation $y' = x(1-y)(1+y)$, which gives the equation

$$x(1-c)(1+c) = 0.$$

The formal college algebra solutions are $x = 0$, $c = -1$ and $c = 1$. However, we do not seek these college algebra answers! Equilibria are the solutions $y = c$ such that $x(1-c)(1+c) = 0$ *for all* x. The conditional *for all* x causes the algebra problem to reduce to just two equations: $0 = 0$ (from $x = 0$) and $(1-c)(1+c) = 0$ (from $x \neq 0$). We solve for $c = 1$ and $c = -1$, then report the two equilibrium solutions $y = 1$ and $y = -1$. Spurious algebraic solutions like $x = 0$ can appear, which must be removed from equilibrium solution reports.

Example 2.8 (Non-Equilibria)
Given $y' = x^2(1+y)$, $y(0) = y_0$, find all non-equilibrium solutions.

Solution: The unique solution is $y = (1+y_0)e^{x^3/3} - 1$. Details follow.

The separable form $y' = F(x)G(y)$ is realized for $F(x) = x^2$ and $G(y) = 1+y$. Sought are solutions with $G(y) \neq 0$, or simply $1+y \neq 0$.

$y' = x^2(1+y)$	Original equation.
$\dfrac{y'}{1+y} = x^2$	Divide by $1+y$. Separated form found.
$\int \dfrac{y'}{1+y} dx = \int x^2 dx$	Method of quadrature.
$\int \dfrac{du}{1+u} = \int x^2 dx$	Change variables $u = y(x)$ on the left.
$\ln\lvert 1+y(x)\rvert = x^3/3 + c$	Evaluate integrals. Implicit solution found.

Applications might stop at this point and report the *implicit solution*. This illustration continues, to find the *explicit solution* $y = (1+y_0)e^{x^3/3} - 1$.

$\lvert 1+y(x)\rvert = e^{x^3/3+c}$	By definition, $\ln u = w$ means $u = e^w$.
$1+y(x) = ke^{x^3/3+c}$	Drop absolute value, $k = \pm 1$.
$y(x) = ke^{x^3/3+c} - 1$	Candidate solution. Constants unresolved.

The initial condition $y(0) = y_0$ is used to resolve the constants c and k. First, $|1+y_0| = e^c$ from the first equation. Second, $1+y_0$ and $1+y(x)$ must have the same sign (they are never zero), so $k(1+y_0) > 0$. Hence, $1+y_0 = ke^c$, which implies the solution is $y = ke^c e^{x^3/3} - 1$ or $y = (1+y_0)e^{x^3/3} - 1$.

Example 2.9 (Equilibria)

Given $y' = x\sin(1-y)\cos(1+y)$, find all equilibrium solutions.

Solution: The infinite set of equilibria are justified below to be

$$y = 1 + n\pi, \quad y = -1 + (2n+1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

A separable form $y' = F(x)G(y)$ is verified by defining $F(x) = x$ and $G(y) = \sin(1-y)\cos(1+y)$. Equilibria $y = c$ are found by solving for c in the equation $G(c) = 0$, which is

$$\sin(1-c)\cos(1+c) = 0.$$

This equation is satisfied when the argument of the sine is an integer multiple of π or when the argument of the cosine is an odd integer multiple of $\pi/2$. The solutions are $c - 1 = 0, \pm\pi, \pm 2\pi, \ldots$ and $1 + c = \pm\pi/2, \pm 3\pi/2, \ldots$.

Multiple solutions and `maple`. Equations having multiple solutions may require **CAS** setup. Below, the first code fragment returns two solutions, $y = 1$ and $y = -1 + \pi/2$. The second returns all solutions.

```
# The default returns two solutions
G:=y->sin(1-y)*cos(1+y):
solve(G(y)=0,y);
# Special setup returns all solutions
_EnvAllSolutions := true:
G:=y->sin(1-y)*cos(1+y):
solve(G(y)=0,y);
```

Example 2.10 (Non-Equilibria)

Given $y' = x^2\sin(y)$, $y(0) = y_0$, justify that all non-equilibrium solutions are given by[2]

$$y = 2\,\text{Arctan}\left(\tan(y_0/2)e^{x^3/3}\right) + 2n\pi.$$

Solution: A separable form $y' = F(x)G(y)$ is defined by $F(x) = x^2$ and $G(y) = \sin(y)$. A non-equilibrium solution will satisfy $G(y) \neq 0$, or simply $\sin(y) \neq 0$. Define n by $y_0/2 = \text{Arctan}(\tan(y_0/2)) + n\pi$, where $|\text{Arctan}(u)| < \pi/2$. Then

[2] While $\theta = \arctan u$ gives any angle, $\theta = \text{Arctan}(u)$ gives $|\theta| < \pi/2$.

$y' = x^2\sin(y)$	The original equation.
$\csc(y)y' = x^2$	Separated form. Divided by $\sin(y) \neq 0$.
$\int \csc(y)y'dx = \int x^2 dx$	Quadrature using indefinite integrals.
$\int \csc(u)du = \int x^2 dx$	Change variables $u = y(x)$ on the left.
$\ln\lvert\csc y(x) - \cot y(x)\rvert = \frac{1}{3}x^3 + c$	Integral tables. Implicit solution found.

Trigonometric Identity. Integral tables make use of the identity $\tan(y/2) = \csc y - \cot y$, which is derived from the relations $2\theta = y$, $1-\cos 2\theta = 2\sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$. Tables offer an alternate answer for the last integral above, $\ln\lvert\tan(y/2)\rvert$.

The solution obtained at this stage is called an *implicit solution*, because y has not been isolated. It is possible to solve for y in terms of x, an *explicit solution*. The details:

$\lvert\csc y - \cot y\rvert = e^{x^3/3+c}$	By definition, $\ln u = w$ means $u = e^w$.
$\csc y - \cot y = ke^{x^3/3+c}$	Assign $k = \pm 1$ to drop absolute values.
$\dfrac{1-\cos y}{\sin y} = ke^{x^3/3+c}$	Then k has the same sign as $\sin(y)$, because $1 - \cos y \geq 0$.
$\tan(y/2) = ke^{x^3/3+c}$	Use $\tan(y/2) = \csc y - \cot y$.
$y = 2\text{Arctan}\left(ke^{x^3/3+c}\right) + 2n\pi$	Candidate solution, $n = 0, \pm 1, \pm 2, \ldots$

Resolving the Constants. Constants c and k are uniquely resolved for a given initial condition $y(0) = y_0$. Values $x = 0$ and $y = y_0$ determine constant c by the equation $\tan(y_0/2) = ke^c$ (two equations back). The condition $k\sin(y_0) > 0$ determines k, because $\sin y_0$ and $\sin y$ have identical signs. If n is defined by $y_0/2 = \text{Arctan}(\tan(y_0/2)) + n\pi$ and $K = ke^c = \tan(y_0/2)$, then the *explicit solution* is

$$y = 2\text{Arctan}\left(Ke^{x^3/3}\right) + 2n\pi, \quad K = \tan(y_0/2).$$

Trigonometric identities and `maple`. Using the identity $\csc y - \cot y = \tan(y/2)$, `maple` finds the same relation. Complications occur without it.

```
_EnvAllSolutions := true:
solve(csc(y)-cot(y)=k*exp(x^3/3+c),y);
solve(tan(y/2)=k*exp(x^3/3+c),y);
```

Example 2.11 (Independent of x)

Solve $y' = y(1 - \ln y)$, $y(0) = y_0$.

Solution: There is just one equilibrium solution $y = e \approx 2.718$. Not included is $y = 0$, because $y(1 - \ln y)$ is undefined for $y \leq 0$. Details appear below for the explicit solution (which includes $y = e$)

$$y = e^{1 - (1 - \ln y_0)e^{-x}}.$$

An equation $y' = f(x, y)$ independent of x has the form $y' = F(x)G(y)$ where $F(x) = 1$. Divide by $G(y)$ to obtain a separated form $y'/G(y) = 1$. In the present case, $G(y) = y(1-\ln y)$ is defined for $y > 0$. To require $G(y) \neq 0$ means $y > 0$, $y \neq e$. Non-equilibrium solutions will satisfy $y > 0$ and $y \neq e$.

$\dfrac{y'}{y(1-\ln y)} = 1$	Separated form. Assume $y > 0$ and $y \neq e$.
$\int \dfrac{y'}{y(1-\ln y)}dx = \int dx$	Method of quadrature.
$\int \dfrac{-du}{u} = \int dx$	Substitute $u = 1 - \ln y$ on the left. Chain rule $(\ln y)' = y'/y$ applied; $du = -y'dx/y$.
$-\ln \lvert 1 - \ln y(x) \rvert = x + c$	Evaluate the integral using $u = 1 - \ln y$. Implicit solution found.

The remainder of the solution contains college algebra details, to find from the *implicit solution* all *explicit solutions* y.

$\lvert 1 - \ln y(x) \rvert = e^{-x-c}$	Use $\ln u = w$ equivalent to $u = e^w$.
$1 - \ln y(x) = ke^{-x-c}$	Drop absolute value, $k = \pm 1$.
$\ln y(x) = 1 - ke^{-x-c}$	Solved for $\ln y$.
$y(x) = e^{1 - ke^{-x-c}}$	Candidate solution; c and k unresolved.

To resolve the constants, start with $y_0 > 0$ and $y_0 \neq e$. To determine k, use the requirement $G(y) \neq 0$ to deduce that $k(1-\ln y(x)) > 0$. At $x = 0$, it means $k|1-\ln y_0| = 1 - \ln y_0$. Then $k = 1$ for $0 < y_0 < e$ and $k = -1$ otherwise.

Let $y = y_0$, $x = 0$ to determine c through the equation $|1 - \ln y_0| = e^{-c}$. Combining with the value of k gives $1 - \ln y_0 = ke^{-c}$.

Assembling the answers for k and c produces the relations

$y = e^{1 - ke^{-x-c}}$	Candidate solution.
$= e^{1 - ke^{-c}e^{-x}}$	Exponential rule $e^{a+b} = e^a e^b$.
$= e^{1 - (1 - \ln y_0)e^{-x}}$	Explicit solution. Used $ke^{-c} = 1 - \ln y_0$.

Even though the solution has been found through legal methods, it remains to *verify the solution*. See the exercises.

Exercises 2.2

Separated Form Test

Test the given equation by the separated form test on page 83.

Report whether or not the equation *passes* or *fails*, as written. In this test, algebraic operations on the equation are disallowed. See Examples 2.4 and 2.5, page 86.

1. $y' = 2$

2. $y' = x$

3. $y' + y = 2$

4. $y' + 2y = x$

5. $yy' = 2 - x$

6. $2yy' = x + x^2$

7. $yy' + \sin(y') = 2 - x$

8. $2yy' + \cos(y) = x$

9. $2yy' = y'\cos(y) + x$

10. $(2y + \tan(y))y' = x$

Separated Equation

Determine the separated form $y'/G(y) = F(x)$ for the given separable equation. See Example 2.6, page 86.

11. $(1+x)y' = 2+y$

12. $(1+y)y' = xy$

13. $y' = \dfrac{x+xy}{(x+1)^2-1}$

14. $y' = \sin(x)\dfrac{1+y}{(x+2)^2-4}$

15. $xy' = y\sin(y)\cos(x)$

16. $x^2y' = y\cos(y)\tan(x)$

17. $y^2(x-y)y' = \dfrac{x^2-y^2}{x+y}$

18. $xy^2(x+y)y' = \dfrac{y^2-x^2}{x-y}$

19. $xy^2y' = \dfrac{y-x}{x-y}$

20. $xy^2y' = \dfrac{x^2-xy}{x-y}$

Equilibrium solutions

Determine the equilibria for the given equation. See Examples 2.7 and 2.9.

21. $y' = xy(1+y)$

22. $xy' = y(1-y)$

23. $y' = \dfrac{1+y}{1-y}$

24. $xy' = \dfrac{y(1-y)}{1+y}$

25. $y' = (1+x)\tan(y)$

26. $y' = y(1+\ln y)$

27. $y' = xe^y(1+y)$

28. $xy' = e^y(1-y)$

29. $xy' = e^y(1-y^2)(1+y)^3$

30. $xy' = e^y(1-y^3)(1+y^3)$

Non-Equilibrium Solutions

Find the non-equilibrium solutions for the given separable equation. See Examples 2.8 and 2.10 for details.

31. $y' = (xy)^{1/3}$, $y(0) = y_0$.

32. $y' = (xy)^{1/5}$, $y(0) = y_0$.

33. $y' = 1+x-y-xy$, $y(0) = y_0$.

34. $y' = 1+x+2y+2xy$, $y(0) = y_0$.

35. $y' = \dfrac{(x+1)y^3}{x^2(y^3-y)}$, $y(1) = y_0 \neq 0$.

36. $y' = \dfrac{(x-1)y^2}{x^3(y^3+y)}$, $y(0) = y_0$.

37. $2yy' = x(1-y^2)$

38. $2yy' = x(1+y^2)$

39. $(1+x)y' = 1-y$

40. $(1-x)y' = 1+y$, $y(0) = y_0$.

41. $\tan(x)y' = y$, $y(\pi/2) = y_0$.

42. $\tan(x)y' = 1+y$, $y(\pi/2) = y_0$.

43. $\sqrt{x}y' = \cos^2(y)$, $y(1) = y_0$.

44. $\sqrt{1-x}y' = \sin^2(y)$, $y(0) = y_0$.

45. $\sqrt{x^2-16}yy' = x$, $y(5) = y_0$.

46. $\sqrt{x^2-1}yy' = x$, $y(2) = y_0$.

47. $y' = x^2(1+y^2)$, $y(0) = 1$.

48. $(1-x)y' = x(1+y^2)$, $y(0) = 1$.

Independent of x

Solve the given equation, finding all solutions. See Example 2.11.

49. $y' = \sin y$, $y(0) = y_0$.

50. $y' = \cos y$, $y(0) = y_0$.

51. $y' = y(1+\ln y)$, $y(0) = y_0$.

52. $y' = y(2+\ln y)$, $y(0) = y_0$.

53. $y' = y(y-1)(y-2)$, $y(0) = y_0$.

54. $y' = y(y-1)(y+1)$, $y(0) = y_0$.

55. $y' = y^2+2y+5$, $y(0) = y_0$.

56. $y' = y^2+2y+7$, $y(0) = y_0$.

Details in the Examples

Collected here are verifications for details in the examples.

57. (Example 2.7) The equation $x(1-y)(1+y) = 0$ was solved in the example, but $x = 0$ was ignored, and only $y = -1$ and $y = 1$ were reported. Why?

58. (Example 2.8) An absolute value equation $|u| = w$ was replaced by $u = kw$ where $k = \pm 1$. Justify the replacement using the *definition* $|u| = u$ for $u \geq 0$, $|u| = -u$ for $u < 0$.

59. (Example 2.8) Verify directly that $y = (1+y_0)e^{x^3/3} - 1$ solves the initial value problem $y' = x^2(1+y)$, $y(0) = y_0$.

60. (Example 2.9) The relation $y = 1 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ describes the list $\ldots, 1-\pi, 1, 1+\pi, \ldots$. Write the list for the relation $y = -1 + (2n+1)\frac{\pi}{2}$.

61. (Example 2.9) Solve $\sin(u) = 0$ and $\cos(v) = 0$ for u and v. Supply graphs which show why there are infinity many solutions.

62. (Example 2.10) Explain why $y_0/2$ does not equal $\text{Arctan}(\tan(y_0/2))$. Give a calculator example.

63. (Example 2.10) Establish the identity $\tan(y/2) = \csc y - \cot y$.

64. (Example 2.11) Let $y_0 > 0$. Verify that $y = e^{1 - (1 - \ln y_0)e^{-x}}$ solves

$$y' = y(1 - \ln y), \quad y(0) = y_0.$$

2.3 Linear Equations

Definition 2.3 (Linear Differential Equation)
An equation $y' = f(x,y)$ is called **first-order linear** or a **linear equation** provided functions $p(x)$ and $r(x)$ can be defined to re-write the equation in the **standard form**

(1) $$y' + p(x)y = r(x).$$

In most applications, p and r are assumed to be continuous. Function $p(x)$ is called the **coefficient of** y. Function $r(x)$ (r abbreviates *right side*) is called the **non-homogeneous term** or the **forcing term**. Engineering texts call $r(x)$ the **input** and the solution $y(x)$ the **output**.

In examples, a linear equation is identified by matching:

$$\frac{dy}{dx} + \left(\begin{array}{l} p(x)\text{, an expression} \\ \text{independent of } y \end{array} \right) y = \left(\begin{array}{l} r(x)\text{, another expression} \\ \text{independent of } y \end{array} \right)$$

.

Calculus Test:

An equation $y' = f(x,y)$ with f continuously differentiable is **linear** provided $\dfrac{\partial f(x,y)}{\partial y}$ is independent of y.

If the test is passed, then standard linear form (1) is obtained by defining $r(x) = f(x,0)$ and $p(x) = -\partial f/\partial y(x,y)$. A brief calculation verifies this statement.

Key Examples

$L\dfrac{dI}{dt} + RI = E$ — The LR-circuit equation. Symbols L, R and E are respectively inductance, resistance and electromotive force, while $I(t)$ = current in amperes and t = time. **[1]**

$\dfrac{du}{dt} = -h(u - u_1)$ — Newton's cooling equation. In the roast model, the oven temperature is u_1 and the meat thermometer reading is $u(t)$, with t = time. **[2]**

Notes.

[1] Linear equation $y' + p(x)y = r(x)$ is realized with symbols y, x, p, r undergoing name changes. Define $x = t$, $y = I$, $p(x) = R/L$, $r(x) = E/L$.

[2] Linear equation $y' + p(x)y = r(x)$ is realized by re-defining symbols y, x, p, r. Start with the equation re-arranged algebraically to $\dfrac{du}{dt} + hu = hu_1$. Define $x = t$, $y = u$, $p(x) = h$, $r(x) = hu_1$.

Homogeneous Equation $y' + p(x)y = 0$

Homogeneous equations $y' + p(x)y = 0$ occur in applications devoid of external forces, like an LR-circuit with no battery in the circuit. Justified on page 101 is the fundamental result for such systems. See also the proof of Theorem 2.5 (a).

The general solution of $\dfrac{dy}{dx} + p(x)y = 0$ is the fraction

$$y(x) = \frac{\text{constant}}{\text{integrating factor}} = \frac{c}{W(x)}$$

where **integrating factor** $W(x)$ is defined by the equation

$$W(x) = e^{\int p(x)dx}.$$

An Illustration. The LR-circuit equation $\dfrac{dI}{dt} + 2I = 0$ is the model equation $y' + p(x)y = 0$ with $p(x) = 2$. Then $W(x) = e^{\int 2dx} = e^{2x}$, with integration constant set to zero. The general solution of $y' + 2y = 0$ is given by

$$y = \frac{c}{W(x)} = \frac{c}{e^{2x}} = ce^{-2x}.$$

The current is $I(t) = c\,e^{-2t}$, by the variable swap $x \to t$, $y \to I$.

Definition 2.4 (Integrating Factor)
An **integrating factor** $W(x)$ for equation $y' + p(x)y = r(x)$ is

$$W(x) = e^{\int p(x)dx}.$$

Lemma 2.1 (Integrating Factor Identity)
The integrating factor $W(x)$ satisfies the differential equation

$$W'(x) = p(x)W(x).$$

Lemma Details. Write $W = e^u$ where $u = \int p(x)dx$. By the fundamental theorem of calculus, $u' = p(x) =$ the integrand. Then the chain rule implies $W' = u'e^u = u'W = pW$.

A Shortcut. Factor $W(x)$ is generally expressed as a simplified expression, with integration constant set to zero and absolute value symbols removed. See the exercises for details about this simplification. For instance, integration in the special case $p(x) = 2$ formally gives $\int p(x)dx = \int 2dx = 2x + c_1$. Then the integrating factor becomes $W(x) = e^{\int 2dx} = e^{2x+c_1} = e^{2x}e^{c_1}$. Fraction $c/W(x)$ equals c_2/e^{2x}, where $c_2 = c/e^{c_1}$. The lesson is that we could have chosen $c_1 = 0$ to produce the same fraction. This is a shortcut, recognized as such, but it applies in examples to save effort.

Non-Homogeneous Equation $y' + p(x)y = r(x)$

Definition 2.5 (Homogeneous and Particular Solution)
Let $W(x)$ be an integrating factor constructed for $y'+p(x)y = r(x)$, that is, $W(x) = e^u$, where $u = \int p(x)dx$ is an antiderivative of $p(x)$.

Symbol y_h, called the **homogeneous solution**, is defined by the expression

$$y_h(x) = \frac{c}{W(x)}.$$

Symbol y_p, called a **particular solution**, is defined by the expression

$$y_p(x) = \frac{\int r(x)W(x)dx}{W(x)}$$

Theorem 2.5 (Homogeneous and Particular Solutions)
(a) Expression $y_h(x)$ is a solution of the homogeneous differential equation $y' + p(x)y = 0$.

(b) Expression $y_p(x)$ is a solution of the non-homogeneous differential equation $y' + p(x)y = r(x)$.

Proof:
(a) Define $y = c/W$. We prove $y' + py = 0$. Formula $y = c/W$ implies $(yW)' = (c)' = 0$. The product rule and the Lemma imply $(yW)' = y'W + yW' = y'W + y(pW) = (y' + py)W$. Then $(yW)' = 0$ implies $y' + py = 0$. The proof is complete.

(b) We prove $y' + py = r$ when y is replaced by the fraction y_p. Define $C(x) = \int r(x)W(x)dx$, so that $y = C(x)/W(x)$. The fundamental theorem of calculus implies $C'(x) = r(x)W(x)$. The product rule and the Lemma imply $C' = (yW)' = y'W + yW' = y'W + ypW = (y' + py)W$. Competition between the two equations for C' gives $rW = (y' + py)W)$. Cancel W to obtain $r = y' + py$. ■

Historical Note. The formula for $y_p(x)$ has the historical name **variation of constants** or **variation of parameters**. Both y_h and y_p have the same form C/W, with $C(x)$ constant for y_h and $C(x)$ equal to a function of x for y_p: **variation of constant** c in y_h produces the expression for y_p.

Experimental Viewpoint. The particular solution y_p depends on the forcing term $r(x)$, but the homogeneous solution y_h does not. Experimentalists view the computation of y_p as a *single experiment* in which the state y_p is determined by the forcing term $r(x)$ and zero initial data $y = 0$ at $x = x_0$. This particular experimental solution y_p^* is given by the definite integral formula

$$y_p^*(x) = \frac{1}{W(x)} \int_{x_0}^{x} r(x)W(x)dx. \tag{2}$$

Superposition. The sum of constant multiples of solutions to $y' + p(x)y = 0$ is again a solution. The next two theorems are **superposition** for $y'+p(x)y = r(x)$.

Theorem 2.6 (General Solution = Homogeneous + Particular)
Assume $p(x)$ and $r(x)$ are continuous on $a < x < b$ and $a < x_0 < b$. Let y be a solution of $y' + p(x)y = r(x)$ on $a < x < b$. Then y can be decomposed as $y = y_h + y_p$.

In short, a linear equation has the solution structure *homogeneous plus particular*.

The constant c in formula y_h and the integration constant in $\int W(x)rx)dx$ can always be selected to satisfy initial condition $y(x_0) = y_0$.

Theorem 2.7 (Difference of Solutions = Homogeneous Solution)
Assume $p(x)$ and $r(x)$ are continuous on $a < x < b$ and $a < x_0 < b$. Let y_1 and y_2 be two solutions of $y' + p(x)y = r(x)$ on $a < x < b$. Then $y = y_1 - y_2$ is a solution of the homogeneous differential equation

$$y' + p(x)y = 0.$$

In short, any two solutions of the non-homogeneous equation differ by some solution y_h of the homogeneous equation.

Integrating Factor Method

The technique called the **method of integrating factors** uses the replacement rule (justified on page 101)

(3) $$\text{Fraction } \frac{(YW)'}{W} \text{ replaces } Y' + p(x)Y, \text{ where } W = e^{\int p(x)dx}.$$

The fraction $(YW)'/W$ is called the **integrating factor fraction**.

The Integrating Factor Method

Standard Form	Rewrite $y' = f(x, y)$ in the form $y' + p(x)y = r(x)$ where p, r are continuous. The method applies only in case this is possible.
Find W	Find a simplified formula for $W = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently.
Prepare for Quadrature	Obtain the new equation $\dfrac{(yW)'}{W} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (3).
Method of Quadrature	Clear fractions to obtain $(yW)' = rW$. Apply the method of quadrature to get $yW = \int r(x)W(x)dx + C$. Divide by W to isolate the explicit solution $y(x)$.

In identity (3), functions p, Y and Y' are assumed continuous with p and Y *arbitrary* functions. Equation (3) is central to the method, because it collapses the two terms $y' + py$ into a single term $(Wy)'/W$; the method of quadrature applies to $(Wy)' = rW$. The literature calls the exponential factor W an **integrating factor** and equivalence (3) a **factorization** of $Y' + p(x)Y$.

Simplifying an Integrating Factor

Factor W is simplified by dropping constants of integration. To illustrate, if $p(x) = 1/x$, then $\int p(x)dx = \ln|x| + C$. The algebra rule $e^{A+B} = e^A e^B$ implies that $W = e^C e^{\ln|x|} = |x|e^C = (\pm e^C)x$, because $|x| = (\pm)x$. Let $c_1 = \pm e^C$. Then $W = c_1 W_1$ where $W_1 = x$. The fraction $(Wy)'/W$ reduces to $(W_1 y)'/W_1$, because c_1 cancels. In an application, we choose the simpler expression W_1. The illustration shows that exponentials in W can sometimes be eliminated.

Variation of Constants and $y' + p(x)y = r(x)$

Every solution of $y' + p(x)y = r(x)$ can be expressed as $y = y_h + y_p$, by choosing constants appropriately. The classical **variation of constants formula** puts initial condition zero on y_p and compresses all initial data into the constant c appearing in y_h. The general solution is given by

$$(4) \qquad y(x) = \frac{y(x_0)}{W(x)} + \frac{\int_{x_0}^{x} r(x)W(x)dx}{W(x)}, \quad W(x) = e^{\int_{x_0}^{x} p(s)ds}$$

Classifying Linear and Non-Linear Equations

Definition 2.6 (Non-linear Differential Equation)
An equation $y' = f(x, y)$ that fails to be linear is called **non-linear**.

Algebraic Complexity. A linear equation $y' = f(x, y)$ may appear to be non-linear, e.g., $y' = (\sin^2(xy) + \cos^2(xy))y$ simplifies to $y' = y$.

Computer Algebra System. These systems classify an equation $y' = f(x, y)$ as linear provided the identity $f(x, y) = f(x, 0) + f_y(x, 0)y$ is valid. Equivalently, $f(x, y) = r(x) - p(x)y$, where $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$.

Hand verification can use the same method. To illustrate, consider $y' = f(x, y)$ with $f(x, y) = (x - y)(x + y) + y(y - 2x)$. Compute $f(x, 0) = x^2$, $f_y(x, 0) = -2x$. Because f_y is independent of y, then $y' = f(x, y)$ is the linear equation $y' + p(x)y = r(x)$ with $p(x) = 2x$, $r(x) = x^2$.

Non-Linear Equation Tests. Elimination of an equation $y' = f(x, y)$ from the class of linear equations can be done from *necessary conditions*. The equality $f_y(x, y) = f_y(x, 0)$ implies two such conditions:

1. If $f_y(x, y)$ depends on y, then $y' = f(x, y)$ is not linear.

2. If $f_{yy}(x, y) \neq 0$, then $y' = f(x, y)$ is not linear.

For instance, either condition implies $y' = 1 + y^2$ is *not linear*.

Special Linear Equations

There are fast ways to solve certain linear differential equations that do not employ the linear integrating factor method.

Theorem 2.8 (Solving a Homogeneous Equation)
Assume $p(x)$ is continuous on $a < x < b$. Then the solution of the homogeneous differential equation $y' + p(x)y = 0$ is given by the formula

$$y(x) = \frac{\text{constant}}{\text{integrating factor}}. \tag{5}$$

Theorem 2.9 (Solving a Constant-Coefficient Equation)
Assume $p(x)$ and $r(x)$ are constants p, r with $p \neq 0$. Then the solution of the constant-coefficient differential equation $y' + py = r$ is given by the formula

$$\begin{aligned} y(x) &= \frac{\text{constant}}{\text{integrating factor}} + \text{equilibrium solution} \\ &= ce^{-px} + \frac{r}{p}. \end{aligned} \tag{6}$$

Proof: The homogeneous solution is a constant divided by the integrating factor, by Theorem 2.8. An equilibrium solution can be found by formally setting $y' = 0$, then solving for $y = r/p$. By superposition Theorem 2.6, the solution y must be the sum of these two solutions. The excluded case $p = 0$ results in a quadrature equation $y' = r$ which is routinely solved by the method of quadrature.

Examples

Example 2.12 (Shortcut: Homogeneous Equation)

Solve the homogeneous equation $2y' + x^2y = 0$.

Solution: By Theorem (2.8), the solution is a constant divided by the integrating factor. First, divide by 2 to get $y' + p(x)y = 0$ with $p(x) = \frac{1}{2}x^2$. Then $\int p(x)dx = x^3/6 + c$ implies $W = e^{x^3/6}$ is an integrating factor. The solution is $y = \dfrac{c}{e^{x^3/6}}$.

Example 2.13 (Shortcut: Constant-Coefficient Equation)

Solve the non-homogeneous constant-coefficient equation $2y' - 5y = -1$.

Solution: The method described here only works for first order constant coefficient differential equations. If $y' = f(x, y)$ is not linear or it fails to have constant coefficients, then the method fails.

The solution has two steps:

(1) Find the solution y_h of the homogeneous equation $2y' - 5y = 0$.
The answer is a constant divided by the integrating factor, which is $y = \frac{c}{e^{-5x/2}}$. First divide the equation by 2 to obtain the standard form $y' + (-5/2)y = 0$. Identify $p(x) = -5/2$, then $\int p(x)dx = -5x/2 + c$ and finally $W = e^{-5x/2}$ is the integrating factor. The answer is $y_h = c/W = ce^{5x/2}$.

(2) Find an equilibrium solution y_p for $2y' - 5y = -1$.
This answer is found by formally replacing y' by zero. Then $y_p = \frac{1}{5}$.

The answer is the sum of the answers from (1) and (2), by superposition, giving

$$y = y_h + y_p = ce^{5x/2} + \frac{1}{5}.$$

The method of this example is called the **superposition method shortcut**.

Example 2.14 (Integrating Factor Method)

Solve $2y' + 6y = e^{-x}$.

Solution: The solution is $y = \frac{1}{4}e^{-x} + ce^{-3x}$. An answer check appears in Example 2.16. The details:

$y' + 3y = 0.5e^{-x}$	Divide by 2 to get the standard form.
$W = e^{3x}$	Find the integrating factor $W = e^{\int 3dx}$.
$\dfrac{\left(e^{3x}y\right)'}{e^{3x}} = 0.5e^{-x}$	Replace the LHS of $y' + 3y = 0.5e^{-x}$ by the integrating factor quotient; see page 96.
$\left(e^{3x}y\right)' = 0.5e^{2x}$	Clear fractions. Prepared for quadrature
$e^{3x}y = 0.5\int e^{2x}dx$	Method of quadrature applied.
$y = 0.5\left(e^{2x}/2 + c_1\right)e^{-3x}$	Evaluate the integral. Divide by $W = e^{3x}$.
$= \frac{1}{4}e^{-x} + ce^{-3x}$	Final answer, $c = 0.5c_1$.

Example 2.15 (Superposition)

Find a particular solution of $y' + 2y = 3e^{x}$ with fewest terms.

Solution: The answer is $y = e^{x}$. The first step solves the equation using the integrating factor method, giving $y = e^{x} + ce^{-2x}$; details below. A particular solution with fewest terms, $y = e^{x}$, is found by setting $c = 0$.

Integrating factor method details:

$y' + 2y = 3e^{x}$	The standard form.
$W = e^{2x}$	Find the integrating factor $W = e^{\int 2dx}$.
$\dfrac{\left(e^{2x}y\right)'}{e^{2x}} = 3e^{x}$	Integrating factor identity applied to $y' + 2y = 3e^{x}$.
$e^{2x}y = 3\int e^{3x}dx$	Clear fractions and apply quadrature.
$y = \left(e^{3x} + c\right)e^{-2x}$	Evaluate the integral. Isolate y.

$= e^x + ce^{-2x}$ Solution found.

Remarks on Integral Formula (2). Computer algebra systems will compute the solution $y_p^* = e^x - e^{3x_0}e^{-2x}$ of equation (2). It has an extra term because of the condition $y = 0$ at $x = x_0$. The shortest particular solution e^x and the integral formula solution y_p^* differ by a homogeneous solution c_1e^{-2x}, where $c_1 = e^{3x_0}$. To shorten y_p^* to $y_p = e^x$ requires knowing the homogeneous solution, then apply superposition $y = y_p + y_h$ to extract a particular solution.

Example 2.16 (Answer Check)

Show the answer check details for $2y' + 6y = e^{-x}$ and candidate solution $y = \frac{1}{4}e^{-x} + ce^{-3x}$.

Solution: **Details**:

$\text{LHS} = 2y' + 6y$	Left side of the equation $2y'+6y = e^{-x}$.
$= 2(-\frac{1}{4}e^{-x} - 3ce^{-3x}) + 6(\frac{1}{4}e^{-x} + ce^{-3x})$	Substitute for y.
$= e^{-x} + 0$	Simplify terms.
$= \text{RHS}$	DE verified.

Example 2.17 (Finding y_h and y_p)

Find the homogeneous solution y_h and a particular solution y_p for the equation $2xy' + y = 4x^2$ on $x > 0$.

Solution: The solution by the integrating factor method is $y = 0.8x^2 + cx^{-1/2}$; details below. Then $y_h = cx^{-1/2}$ and $y_p = 0.8x^2$ give $y = y_h + y_p$.

The symbol y_p stands for *any* particular solution. It should be free of any arbitrary constants c.

Integral formula (2) gives a particular solution $y_p^* = 0.8x^2 - 0.8x_0^{5/2}x^{-1/2}$. It differs from the shortest particular solution $0.8x^2$ by a homogeneous solution $Kx^{-1/2}$.

Integrating factor method details:

$y' + 0.5y/x = 2x$	Standard form. Divided by $2x$.
$p(x) = 0.5/x$	Identify coefficient of y. Then $\int p(x)dx = 0.5\ln\lvert x\rvert + c$.
$W = e^{0.5\ln\lvert x\rvert+c}$	The integrating factor is $W = e^{\int p}$.
$W = e^{0.5\ln\lvert x\rvert}$	Choose integration constant zero.
$= \lvert x\rvert^{1/2}$	Used $\ln u^n = n\ln u$. Simplified W found.
$\dfrac{(x^{1/2}y)'}{x^{1/2}} = 2x$	Integrating factor identity applied on the left. Assumed $x > 0$.
$x^{1/2}y = 2\int x^{3/2}dx$	Clear fractions. Apply quadrature.
$y = \left(4x^{5/2}/5 + c\right)x^{-1/2}$	Evaluate the integral. Divide to isolate y.
$= \frac{4}{5}x^2 + cx^{-1/2}$	Solution found.

Example 2.18 (Classification)
Classify the equation $y' = x + \ln(xe^y)$ as linear or non-linear.

Solution: It's linear, with standard linear form $y' + (-1)y = x + \ln x$. To explain why, the term $\ln(xe^y)$ on the right expands into $\ln x + \ln e^y$, which in turn is $\ln x + y$, using logarithm rules. Because $e^y > 0$, then $\ln(xe^y)$ makes sense for only $x > 0$. Henceforth, assume $x > 0$.

Computer algebra test $f(x, y) = f(x, 0) + f_y(x, 0)y$. Expected is LHS−RHS $= 0$ after simplification. This example produced $\ln e^y - y$ instead of 0, evidence that limitations may exist.

```
assume(x>0):
f:=(x,y)->x+ln(x*exp(y)):
LHS:=f(x,y):
RHS:=f(x,0)+subs(y=0,diff(f(x,y),y))*y:
simplify(LHS-RHS);
```

If the test *passes*, then $y' = f(x, y)$ becomes $y' = f(x, 0) + f_y(x, 0)y$. This example gives $y' = x + \ln x + y$, which converts to the standard linear form $y' + (-1)y = x + \ln x$.

Details and Proofs

Justification of Homogeneous Solution $y = \dfrac{c}{W(x)}$:

Because $W = e^{\int p(x)dx}$, then $W' = p(x)W$ by the Fundamental Theorem of Calculus. Then $(e^u)' = u'e^u$ implies:

$$\frac{dy}{dx} + p(x)y = \frac{-cW'}{W^2} + \frac{cp(x)}{W} = \frac{-cp(x)W}{W^2} + \frac{cp(x)}{W} = 0$$

Justification of Factorization (3): It is assumed that $Y(x)$ is a given but otherwise arbitrary differentiable function. Equation (3) will be justified in its fraction-free form

(7) $$\left(Ye^{\mathbf{P}}\right)' = (Y' + pY)\,e^{\mathbf{P}}, \quad \mathbf{P}(x) = \int p(x)dx.$$

LHS $= \left(Ye^{\mathbf{P}}\right)'$	The left side of equation (7).
$= Y'e^{\mathbf{P}} + \left(e^{\mathbf{P}}\right)' Y$	Apply the product rule $(uv)' = u'v + uv'$.
$= Y'e^{\mathbf{P}} + pe^{\mathbf{P}}Y$	Use the chain rule $(e^u)' = u'e^u$ and $\mathbf{P}' = p$.
$= (Y' + pY)\,e^{\mathbf{P}}$	The common factor is $e^{\mathbf{P}}$.
$=$ RHS	The right hand side of equation (7).

Justification of Formula (4):

Existence. Because the formula is $y = y_h + y_p$ for particular values of c and the constant of integration, then y is a solution by superposition Theorem (2.6) and existence Theorem (2.5).

Uniqueness. It remains to show that the solution given by (4) is the only solution. Start by assuming Y is another, subtract them to obtain $u = y - Y$. Then $u' + pu = 0$, $u(x_0) = 0$. To show $y \equiv Y$, it suffices to show $u \equiv 0$.

According to the integrating factor method, the equation $u' + pu = 0$ is equivalent to $(uW)' = 0$. Integrate $(uW)' = 0$ from x_0 to x, giving $u(x)W(x) = u(x_0)W(x_0)$. Since $u(x_0) = 0$ and $W(x) \neq 0$, it follows that $u(x) = 0$ for all x. ■

About Picard's Theorem. The Picard-Lindelöf theorem, page **??**, implies existence-uniqueness, but only on a smaller interval, and furthermore it supplies no practical formula for the solution. Formula (4) is therefore an improvement over the results obtainable from the general theory.

Exercises 2.3

Integrating Factor Method

Apply the integrating factor method, page 96, to solve the given linear equation. See the examples starting on page 99 for details.

1. $y' + y = e^{-x}$

2. $y' + y = e^{-2x}$

3. $2y' + y = e^{-x}$

4. $2y' + y = e^{-2x}$

5. $2y' + y = 1$

6. $3y' + 2y = 2$

7. $2xy' + y = x$

8. $3xy' + y = 3x$

9. $y' + 2y = e^{2x}$

10. $2y' + y = 2e^{x/2}$

11. $y' + 2y = e^{-2x}$

12. $y' + 4y = e^{-4x}$

13. $2y' + y = e^{-x}$

14. $2y' + y = e^{-2x}$

15. $4y' + y = 1$

16. $4y' + 2y = 3$

17. $2xy' + y = 2x$

18. $3xy' + y = 4x$

19. $y' + 2y = e^{-x}$

20. $2y' + y = 2e^{-x}$

Superposition

Find a particular solution with fewest terms. See Example 2.15, page 99.

21. $3y' = x$

22. $3y' = 2x$

23. $y' + y = 1$

24. $y' + 2y = 2$

25. $2y' + y = 1$

26. $3y' + 2y = 1$

27. $y' - y = e^{x}$

28. $y' - y = xe^{x}$

29. $xy' + y = \sin x$ $(x > 0)$

30. $xy' + y = \cos x$ $(x > 0)$

31. $y' + y = x - x^2$

32. $y' + y = x + x^2$

General Solution

Find y_h and a particular solution y_p. Report the general solution $y = y_h + y_p$. See Example 2.17, page 100.

33. $y' + y = 1$

34. $xy' + y = 2$

35. $y' + y = x$

36. $xy' + y = 2x$

37. $y' - y = x + 1$

38. $xy' - y = 2x - 1$

39. $2xy' + y = 2x^2\ (x > 0)$

40. $xy' + y = 2x^2\ (x > 0)$

Classification

Classify as linear or non-linear. Use the test $f(x, y) = f(x, 0) + f_y(x, 0)y$ and a computer algebra system, when available, to check the answer. See Example 2.18, page 101.

41. $y' = 1 + 2y^2$

42. $y' = 1 + 2y^3$

43. $yy' = (1 + x) \ln e^y$

44. $yy' = (1 + x) (\ln e^y)^2$

45. $y' \sec^2 y = 1 + \tan^2 y$

46. $y' = \cos^2(xy) + \sin^2(xy)$

47. $y'(1 + y) = xy$

48. $y' = y(1 + y)$

49. $xy' = (x + 1)y - xe^{\ln y}$

50. $2xy' = (2x + 1)y - xye^{-\ln y}$

Shortcuts

Apply theorems for the homogeneous equation $y' + p(x)y = 0$ or for constant coefficient equations $y' + py = r$. Solutions should be done without paper or pencil, then write the answer and check it.

51. $y' - 5y = -1$

52. $3y' - 5y = -1$

53. $2y' + xy = 0$

54. $3y' - x^2y = 0$

55. $y' = 3x^4y$

56. $y' = (1 + x^2)y$

57. $\pi y' - \pi^2 y = -e^2$

58. $e^2y' + e^3y = \pi^2$

59. $xy' = (1 + x^2)y$

60. $e^xy' = (1 + e^{2x})y$

Proofs and Details

61. Prove directly without appeal to Theorem 2.6 that the difference of two solutions of $y' + p(x)y = r(x)$ is a solution of the homogeneous equation $y' + p(x)y = 0$.

62. Prove that y_p^* given by equation (2) and $y_p = W^{-1} \int r(x)W(x)dx$ given in the integrating factor method are related by $y_p = y_p^* + y_h$ for some solution y_h of the homogeneous equation.

63. The equation $y' = r$ with r constant can be solved by quadrature, without pencil and paper. Find y.

64. The equation $y' = r(x)$ with $r(x)$ continuous can be solved by quadrature. Find a formula for y.

2.4 Undetermined Coefficients

Studied here is the subject of undetermined coefficients for linear first order differential equations $y' + p(x)y = r(x)$. It finds a particular solution y_p *without* the integration steps present in variation of parameters (reviewed in an example and in exercises). The requirements and limitations:

1. Coefficient $p(x)$ of $y' + p(x)y = r(x)$ is constant.

2. The function $r(x)$ is a sum of constants times Euler solution atoms (defined below).

Definition 2.7 (Euler Solution Atom)
An **Euler base atom** is a term having one of the forms

$$1,\ e^{ax},\ \cos bx,\ \sin bx,\ e^{ax}\cos bx \quad \text{or} \quad e^{ax}\sin bx.$$

The symbols a and b are real constants, with $a \neq 0$ and $b > 0$.

An **Euler solution atom** equals x^n(Euler base atom). Symbol $n \geq 0$ is an integer.

Examples. The terms x^3, $x\cos 2x$, $\sin x$, e^{-x}, $x^6 e^{-\pi x}$ are Euler atoms. Conversely, if $r(x) = 4\sin x + 5xe^x$, then split the sum into terms and drop the coefficients 4 and 5 to identify Euler atoms $\sin x$ and xe^x; then $r(x)$ is a sum of constants times Euler solution atoms.

The Method

1. Repeatedly differentiate the Euler atoms in $r(x)$ until no new atoms appear. Multiply the distinct atoms so found by **undetermined coefficients** d_1, ..., d_k, then add to define a **trial solution** y.
2. **Correction rule**: if solution e^{-px} of $y' + py = 0$ appears in trial solution y, then replace in y matching Euler atoms e^{-px}, xe^{-px}, ... by xe^{-px}, x^2e^{-px}, ... (other Euler atoms in y are unchanged). The modified expression y is called the **corrected trial solution**.
3. Substitute y into the differential equation $y' + py = r(x)$. Match coefficients of Euler atoms left and right to write out linear algebraic equations for the undetermined coefficients d_1, ..., d_k.
4. Solve the equations. The trial solution y with evaluated coefficients d_1, ..., d_k becomes the particular solution y_p.

Undetermined Coefficients Illustrated

Solve

$$y' + 2y = xe^x + 2x + 1 + 3\sin x.$$

Solution:

Test Applicability. The right side $r(x) = xe^x + 2x + 1 + 3\sin x$ is a sum of terms constructed from the Euler atoms xe^x, x, 1, $\sin x$. The left side is $y' + p(x)y$ with $p(x) = 2$, a constant. Therefore, the method of undetermined coefficients applies to find y_p.

Trial Solution. The atoms of $r(x)$ are subjected to differentiation. The distinct Euler atoms so found are 1, x, e^x, xe^x, $\cos x$, $\sin x$ (split terms and drop coefficients to identify new atoms). Because the solution e^{-2x} of $y'+2y = 0$ does not appear in the list of atoms, then the correction rule does not apply. The corrected trial solution is the expression

$$y = d_1(1) + d_2(x) + d_3(e^x) + d_4(xe^x) + d_5(\cos x) + d_6(\sin x).$$

Equations for Undetermined Coefficients. To substitute the trial solution y into $y' + 2y$ requires a formula for y':

$$y' = d_2 + d_3e^x + d_4xe^x + d_4e^x - d_5 \sin x + d_6 \cos x.$$

Then

$$\begin{aligned} r(x) &= y' + 2y \\ &= d_2 + d_3e^x + d_4xe^x + d_4e^x - d_5 \sin x + d_6 \cos x \\ &\quad + 2d_1 + 2d_2x + 2d_3e^x + 2d_4xe^x + 2d_5 \cos x + 2d_6 \sin x \\ &= (d_2 + 2d_1)(1) + 2d_2(x) + (3d_3 + d_4)(e^x) + (3d_4)(xe^x) \\ &\quad + (2d_5 + d_6)(\cos x) + (2d_6 - d_5)(\sin x) \end{aligned}$$

Also, $r(x) \equiv 1 + 2x + xe^x + 3\sin x$. Coefficients of atoms on the left and right must match. For instance, constant term 1 in $r(x)$ matches the constant term in the expansion of $y' + 2y$, giving $1 = d_2 + 2d_1$. Writing out the matches, and swapping sides, gives the equations

$$\begin{aligned} 2d_1 + d_2 &= 1, \\ 2d_2 &= 2, \\ 3d_3 + d_4 &= 0, \\ 3d_4 &= 1, \\ 2d_5 + d_6 &= 0, \\ -d_5 + 2d_6 &= 3. \end{aligned}$$

Solve. The first four equations can be solved by back-substitution to give $d_2 = 1$, $d_1 = 0$, $d_4 = 1/3$, $d_3 = -1/9$. The last two equations are solved by elimination or Cramer's rule (reviewed in Chapter **??**) to give $d_6 = 6/5$, $d_5 = -3/5$.

Report y_p. The trial solution y with evaluated coefficients $d_1, \ldots, d_6$ becomes

$$y_p(x) = x - \frac{1}{9}e^x + \frac{1}{3}xe^x - \frac{3}{5}\cos x + \frac{6}{5}\sin x.$$

Remarks. The method of matching coefficients of atoms left and right is a subject of linear algebra, called *linear independence*. The method works because any finite list of atoms is known to be linearly independent. Further details for this technical topic appear in this text's linear algebra chapters.

A Correction Rule Illustration

Solve the equation

$$y' + 3y = 8e^x + 3x^2e^{-3x}$$

by the method of undetermined coefficients. Verify that the general solution $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = 2e^x + x^3e^{-3x}.$$

Solution: The right side $r(x) = 8e^x + 3x^2e^{-3x}$ is constructed from atoms e^x, x^2e^{-3x}. Repeated differentiation of these atoms identifies the new list of atoms e^x, e^{-3x}, xe^{-3x}, x^2e^{-3x}. The correction rule applies because the solution e^{-3x} of $y' + 3y = 0$ appears in the list. The atoms of the form x^me^{-3x} are multiplied by x to give the new list of atoms e^x, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}. Readers should take note that atom e^x is unaffected by the correction rule modification. Then the corrected trial solution is

$$y = d_1e^x + d_2xe^{-3x} + d_3x^2e^{-3x} + d_4x^3e^{-3x}.$$

The trial solution expression y is substituted into $y' + 3y = 2e^x + x^2e^{-3x}$ to give the equation

$$4d_1e^x + d_2e^{-3x} + 2d_3xe^{-3x} + 3d_4x^2e^{-3x} = 8e^x + 3x^2e^{-3x}.$$

Coefficients of atoms on each side of the preceding equation are matched to give the equations

$$\begin{aligned} 4d_1 \qquad\qquad\qquad &= 8, \\ d_2 \qquad\qquad &= 0, \\ 2d_3 \qquad &= 0, \\ 3d_4 &= 3. \end{aligned}$$

Then $d_1 = 2$, $d_2 = d_3 = 0$, $d_4 = 1$ and the particular solution is reported to be $y_p = 2e^x + x^3e^{-3x}$.

Remarks on the Method of Undetermined Coefficients

A mystery for the novice is the construction of the trial solution. **Why should it work**? Explained here is the reason behind the method of repeated differentiation to find the Euler atoms in the trial solution.

The theory missing is that the general solution y of $y' + py = r(x)$ is a sum of constants times Euler atoms (under the cited **limitations**). We don't try to prove this result, but use it to motivate the method.

The theory reduces the question of finding a trial solution to finding a sum of constants times Euler atoms. The question is: *which atoms*?

Consider this example: $y' - 3y = e^{3x} + xe^x$. The answer for y is revealed by finding a sum of constants times atoms such that y' and $-3y$ add termwise to $e^{3x} + xe^x$. The requirement eliminates all atoms from consideration except those containing exponentials e^{3x} and e^x.

Initially, we have to consider infinitely many atoms e^{3x}, xe^{3x}, x^2e^{3x}, ... and e^x, xe^x, x^2e^x, ... Such terms would also appear in y', but adding terms of this type to get $r(x) = e^{3x} + xe^x$ requires only the smaller list e^{3x}, xe^{3x}, e^x, xe^x. We have cut down the number of terms in y to four or less!

The algorithm presented here together with the correction rule strips down the number of terms to a minimum. Further details of the method appear in the chapter on scalar linear differential equations, page **??**.

Examples

Example 2.19 (Variation of Parameters Method)
Solve the equation $2y' + 6y = 4xe^{-3x}$ by the method of variation of parameters, verifying $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = x^2e^{-3x}.$$

Solution: Divide the equation by 2 to obtain the standard linear form

$$y' + 3y = 2xe^{-3x}.$$

Solution y_h. The homogeneous equation $y' + 3y = 0$ is solved by the shortcut formula $y_h = \dfrac{\text{constant}}{\text{integrating factor}}$ to give $y_h = ce^{-3x}$.

Solution y_p. Identify $p(x) = 3$, $r(x) = 2xe^{-3x}$ from the standard form. The mechanics: let $y' = f(x, y) \equiv 2xe^{-3x} - 3y$ and define $r(x) = f(x, 0)$, $p(x) = -f_y(x, y) = 3$. The variation of parameters formula is applied as follows. First, compute the integrating factor $W(x) = e^{\int p(x)dx} = e^{3x}$. Then

$$\begin{aligned} y_p(x) &= (1/W(x)) \int r(x)W(x)dx \\ &= e^{-3x} \textstyle\int 2xe^{-3x}e^{3x}dx \\ &= x^2e^{-3x}. \end{aligned}$$

It must be explained that all integration constants were set to zero, in order to obtain the shortest possible expression for y_p. Indeed, if $W = e^{3x+c_1}$ instead of e^{3x}, then the factors $1/W$ and W contribute constant factors $1/e^{c_1}$ and e^{c_1}, which multiply to one; the effect is to set $c_1 = 0$. On the other hand, an integration constant c_2 added to $\int r(x)W(x)dx$ adds the homogeneous solution c_2e^{-3x} to the expression for y_p. Because we seek the shortest expression which is a solution to the non-homogeneous differential equation, the constant c_2 is set to zero.

Example 2.20 (Undetermined Coefficient Method)
Solve the equation $2y' + 6y = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ by the method of undetermined coefficients, verifying $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Solution: The method applies, because the differential equation $2y' + 6y = 0$ has constant coefficients and the right side $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ is constructed from the list of atoms xe^{-x}, xe^{-3x}, $\sin x$.

List of Atoms. Differentiate the atoms in $r(x)$, namely xe^{-x}, xe^{-3x}, $\sin x$, to find the new list of atoms e^{-x}, xe^{-x}, e^{-3x}, xe^{-3x}, $\cos x$, $\sin x$. The solution e^{-3x} of $2y' + 6y = 0$ appears in the list: the correction rule applies. Then e^{-3x}, xe^{-3x} are replaced by xe^{-3x}, x^2e^{-3x} to give the corrected list of atoms e^{-x}, xe^{-x}, xe^{-3x}, x^2e^{-3x}, $\cos x$, $\sin x$. Please note that only two of the six atoms were corrected.

Trial solution. The corrected trial solution is

$$y = d_1e^{-x} + d_2xe^{-x} + d_3xe^{-3x} + d_4x^2e^{-3x} + d_5\cos x + d_6\sin x.$$

Substitute y into $2y' + 6y = r(x)$ to give

$$\begin{aligned} r(x) &= 2y' + 6y \\ &= (4d_1 + 2d_2)e^{-x} + 4d_2xe^{-x} + 2d_3e^{-3x} + 4d_4xe^{-3x} \\ &\quad +(2d_6 + 6d_5)\cos x + (6d_6 - 2d_5)\sin x. \end{aligned}$$

Equations. Matching atoms on the left and right of $2y' + 6y = r(x)$, given $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$, justifies the following equations for the undetermined coefficients; the solution is $d_2 = 1$, $d_1 = -1/2$, $d_3 = 0$, $d_4 = 1$, $d_6 = 3/4$, $d_5 = -1/4$.

$$\begin{aligned} 4d_1 + 2d_2 &= 0, \\ 4d_2 &= 4, \\ 2d_3 &= 0, \\ 4d_4 &= 4, \\ 6d_5 + 2d_6 &= 0, \\ -2d_5 + 6d_6 &= 5. \end{aligned}$$

Equations for variables d_5, d_6 were generated from trigonometric atoms. The 2×2 system has complex eigenvalues. The best method to find coefficients d_5, d_6 is not Gaussian elimination, but instead Cramer's Rule.

Report. The trial solution upon substitution of the values for the undetermined coefficients becomes

$$y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Exercises 2.4

Variation of Parameters I

Report the shortest particular solution given by the formula

$$y_p(x) = \frac{\int rW}{W}, \quad W = e^{\int p(x)dx}$$

1. $y' = x + 1$

2. $y' = 2x - 1$

3. $y' + y = e^{-x}$

4. $y' + y = e^{-2x}$

5. $y' - 2y = 1$

6. $y' - y = 1$

7. $2y' + y = e^{x}$

8. $2y' + y = e^{-x}$

9. $xy' = x + 1$

10. $xy' = 1 - x^2$

Variation of Parameters II

Define $W(t) = e^{\int_{x_0}^{t} p(x)dx}$. Compute $y_p^*(x) = \frac{\int_{x_0}^{x} r(t)W(t)\,dt}{W(x)}$

11. $y' = x + 1$, $y(0) = 0$

12. $y' = 2x - 1$, $x_0 = 0$

13. $y' + y = e^{-x}$, $x_0 = 0$

14. $y' + y = e^{-2x}$, $x_0 = 0$

15. $y' - 2y = 1$, $x_0 = 0$

16. $y' - y = 1$, $x_0 = 0$

17. $2y' + y = e^{x}$, $x_0 = 0$

18. $2y' + y = e^{-x}$, $x_0 = 0$

19. $xy' = x + 2$, $x_0 = 1$

20. $xy' = 1 - x^2$, $x_0 = 1$

Euler Solution Atoms

Report the list L of distinct Euler solution atoms found in function $f(x)$. Then $f(x)$ is a sum of constants times the Euler atoms from L.

21. $x + e^x$

22. $1 + 2x + 5e^x$

23. $x(1 + x + 2e^x)$

24. $x^2(2 + x^2) + x^2 e^{-x}$

25. $\sin x \cos x + e^x \sin 2x$

26. $\cos^2 x - \sin^2 x + x^2 e^x \cos 2x$

27. $(1 + 2x + 4x^5)e^x e^{-3x} e^{x/2}$

28. $(1 + 2x + 4x^5 + e^x \sin 2x)e^{-3x/4} e^{x/2}$

29. $\dfrac{x + e^x}{e^{-2x}} \sin 3x + e^{3x} \cos 3x$

30. $\dfrac{x + e^x \sin 2x + x^3}{e^{-2x}} \sin 5x$

Initial Trial Solution

Differentiate repeatedly $f(x)$ and report the list M of distinct Euler solution atoms which appear in f and all its derivatives. Then each of $f, f', \ldots$ is a sum of constants times Euler atoms in M.

31. $12 + 5x^2 + 6x^7$

32. $x^6/x^{-4} + 10x^4/x^{-6}$

33. $x^2 + e^x$

34. $x^3 + 5e^{2x}$

35. $(1 + x + x^3)e^x + \cos 2x$

36. $(x + e^x)\sin x + (x - e^{-x})\cos 2x$

37. $(x + e^x + \sin 3x + \cos 2x)e^{-2x}$

38. $(x^2 e^{-x} + 4\cos 3x + 5\sin 2x)e^{-3x}$

39. $(1 + x^2)(\sin x \cos x - \sin 2x)e^{-x}$

40. $(8 - x^3)(\cos^2 x - \sin^2 x)e^{3x}$

Correction Rule

Given the homogeneous solution y_h and an initial trial solution y, determine the final trial solution according to the correction rule.

41. $y_h(x) = ce^{2x}$, $y = d_1 + d_2 x + d_3 e^{2x}$

42. $y_h(x) = ce^{2x}$, $y = d_1 + d_2 e^{2x} + d_3 x e^{2x}$

43. $y_h(x) = ce^{0x}$, $y = d_1 + d_2 x + d_3 x^2$

44. $y_h(x) = ce^{x}$, $y = d_1 + d_2 x + d_3 x^2$

45. $y_h(x) = ce^{x}$, $y = d_1 \cos x + d_2 \sin x + d_3 e^x$

46. $y_h(x) = ce^{2x}$, $y = d_1 e^{2x}\cos x + d_2 e^{2x} \sin x$

47. $y_h(x) = ce^{2x}$, $y = d_1 e^{2x} + d_2 x e^{2x} + d_3 x^2 e^{2x}$

48. $y_h(x) = ce^{-2x}$, $y = d_1 e^{-2x} + d_2 x e^{-2x} + d_3 e^{2x} + d_4 x e^{2x}$

49. $y_h(x) = cx^2$, $y = d_1 + d_2 x + d_3 x^2$

50. $y_h(x) = cx^3$, $y = d_1 + d_2 x + d_3 x^2$

Trial Solution

Find the form of the **corrected** trial solution y but do not evaluate the undetermined coefficients.

51. $y' = x^3 + 5 + x^2 e^x(3 + 2x + \sin 2x)$

52. $y' = x^2 + 5x + 2 + x^3 e^x(2 + 3x + 5\cos 4x)$

53. $y' - y = x^3 + 2x + 5 + x^4 e^x(2 + 4x + 7\cos 2x)$

54. $y' - y = x^4 + 5x + 2 + x^3 e^x(2 + 3x + 5\cos 4x)$

55. $y' - 2y = x^3 + x^2 + x^3e^x(2e^x + 3x + 5\sin 4x)$

56. $y' - 2y = x^3e^{2x} + x^2e^x(3 + 4e^x + 2\cos 2x)$

57. $y' + y = x^2 + 5x + 2 + x^3e^{-x}(6x + 3\sin x + 2\cos x)$

58. $y' - 2y = x^5 + 5x^3 + 14 + x^3e^x(5 + 7xe^{-3x})$

59. $2y' + 4y = x^4 + 5x^5 + 2x^8 + x^3e^x(7 + 5xe^x + 5\sin 11x)$

60. $5y' + y = x^2 + 5x + 2e^{x/5} + x^3e^{x/5}(7 + 9x + 2\sin(9x/2))$

Undetermined Coefficients

Compute a particular solution y_p according to the method of undetermined coefficients. Expected details include:

(1) Initial trial solution
(2) Corrected trial solution
(3) Undetermined coefficient algebraic equations and solution
(4) Formula for y_p, coefficients evaluated

61. $y' + y = x + 1$

62. $y' + y = 2x - 1$

63. $y' - y = e^x + e^{-x}$

64. $y' - y = xe^x + e^{-x}$

65. $y' - 2y = 1 + x + e^{2x} + \sin x$

66. $y' - 2y = 1 + x + xe^{2x} + \cos x$

67. $y' + 2y = xe^{-2x} + x^3$

68. $y' + 2y = (2 + x)e^{-2x} + xe^x$

69. $y' = x^2 + 4 + xe^x(3 + \cos x)$

70. $y' = x^2 + 5 + xe^x(2 + \sin x)$

2.5 Linear Applications

This collection of applications for the linear equation $y' + p(x)y = r(x)$ includes mixing problems, especially brine tanks in single and multiple cascade, heating and cooling problems, radioactive isotope chains and elementary electric circuits.

The theory for brine cascades will be developed. Heating and cooling will be developed from Newton's cooling law. Radioactive decay theory appears on page **??**. Electric LR or RC circuits appear on page **??**.

Brine Mixing

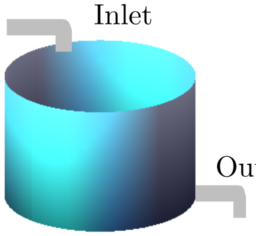


Figure 1. A Single Brine Tank.
The tank has one inlet and one outlet. The inlet supplies a brine mixture and the outlet drains the tank.

A given tank contains brine, which is a water and salt mixture. Input pipes supply other, possibly different brine mixtures at varying rates, while output pipes drain the tank. The problem is to determine the salt $x(t)$ in the tank at any time.

The basic chemical law to be applied is the **mixture law**

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}.$$

The law is applied under a simplifying assumption: *the concentration of salt in the brine is uniform throughout the fluid.* Stirring is one way to meet this requirement. Because of the uniformity assumption, the amount $x(t)$ of salt in kilograms divided by the volume $V(t)$ of the tank in liters gives salt **concentration**[3] $x(t)/V(t)$ kilograms per liter.

One Input and One Output

Let the input be $a(t)$ liters per minute with concentration C_1 kilograms of salt per liter. Let the output empty $b(t)$ liters per minute. The tank is assumed to contain V_0 liters of brine at $t = 0$. The tank gains fluid at rate $a(t)$ and loses fluid at rate $b(t)$, therefore $V(t) = V_0 + \int_0^t [a(r) - b(r)]dr$ is the volume of brine in the tank at time t. The *mixture law* applies to obtain (derived on page 121) the model linear differential equation

$$\frac{dx}{dt} = a(t)\, C_1 - b(t)\, \frac{x(t)}{V(t)}. \tag{1}$$

[3]Concentration is defined as amount per unit volume: **concentration** = $\dfrac{\textbf{amount}}{\textbf{volume}}$.

This equation is solved by the *linear integrating factor method*, page 96.

Two-Tank Mixing

Two tanks A and B are assumed to contain A_0 and B_0 liters of brine at $t = 0$. Let the input for the first tank A be $a(t)$ liters per minute with concentration C_1 kilograms of salt per liter. Let tank A empty at $b(t)$ liters per minute into a second tank B, which itself empties at $c(t)$ liters per minute.

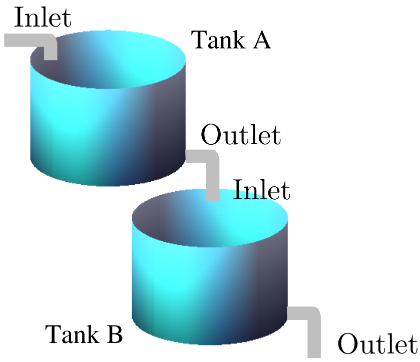


Figure 2. Two Brine Tanks. Tank A has one inlet, which supplies a brine mixture. The outlet of Tank A cascades into Tank B. The outlet of Tank B drains the two-tank system.

Let $x(t)$ be the number of kilograms of salt in tank A at time t. Similarly, $y(t)$ is the amount of salt in tank B. The *objective* is to find differential equations for the unknowns $x(t)$, $y(t)$.

Fluid loses and gains in each tank give rise to the brine volume formulas $V_A(t) = A_0 + \int_0^t [a(r) - b(r)]dr$ and $V_B(t) = B_0 + \int_0^t [b(r) - c(r)]dr$, respectively, for tanks A and B, at time t.

The *mixture law* applies to obtain the model linear differential equations

$$\begin{aligned}\frac{dx}{dt} &= a(t)\,C_1 - b(t)\,\frac{x(t)}{V_A(t)},\\ \frac{dy}{dt} &= b(t)\,\frac{x(t)}{V_A(t)} - c(t)\,\frac{y(t)}{V_B(t)}.\end{aligned}$$

The first equation is solved for an explicit solution $x(t)$ by the linear integrating factor method. Substitute the expression for $x(t)$ into the second equation, then solve for $y(t)$ by the linear integrating factor method.

Residential Heating and Cooling

The internal temperature $u(t)$ in a residence fluctuates with the outdoor temperature, indoor heating and indoor cooling. Newton's law of cooling for linear convection can be written as

$$\frac{du}{dt} = k(a(t) - u(t)) + s(t) + f(t), \tag{2}$$

where the various symbols have the interpretation below.

k	The insulation constant (see **Remarks on Insulation Constants**, 119). Typically $1/2 \le k < 1$, with $1 =$ no insulation, $0 =$ perfect insulation.
$a(t)$	The ambient outside temperature.
$s(t)$	Combined rate for all inside heat sources. Includes living beings, appliances and whatever uses energy.
$f(t)$	Inside heating or cooling rate.

Newton's cooling model applies to convection only, and not to heat transfer by radiation or conduction. A derivation of (2) appears on page 121. To solve equation (2), write it in standard linear form and use the integrating factor method on page 96.

No Sources

Assume the absence of heating inside the building, that is, $s(t) = f(t) = 0$. Let the outside temperature be constant: $a(t) = a_0$. Equation (2) simplifies to the Newton cooling equation on page **??**:

$$\frac{du}{dt} + ku(t) = ka_0. \tag{3}$$

From Theorem **??**, page **??**, the solution is

$$u(t) = a_0 + (u(0) - a_0)e^{-kt}. \tag{4}$$

This formula represents *exponential decay* of the interior temperature from $u(0)$ to a_0.

Half-Time Insulation Constant

Suppose it's 50°F outside and 70°F initially inside when the electricity goes off. How long does it take to drop to 60°F inside? The answer is *about 1–3 hours*, depending on the insulation.

The importance of 60°F is that it is halfway between the inside and outside temperatures of 70°F and 50°F. The range 1–3 hours is found from (4) by solving $u(T) = 60$ for T, in the extreme cases of poor or perfect insulation.

The more general equation $u(T) = (a_0 + u(0))/2$ can be solved. The answer is $T = \ln(2)/k$, called the **half-time insulation constant** for the residence. It measures the insulation quality, larger T corresponding to better insulation. For most residences, the half-time insulation constant ranges from 1.4 ($k = 0.5$) to 14 ($k = 0.05$) hours.

Winter Heating

The introduction of a furnace and a thermostat set at temperature T_0 (typically, 68°F to 72°F) changes the source term $f(t)$ to the special form

$$f(t) = k_1(T_0 - u(t)),$$

according to Newton's law of cooling, where k_1 is a constant. The differential equation (2) becomes

$$\frac{du}{dt} = k(a(t) - u(t)) + s(t) + k_1(T_0 - u(t)). \tag{5}$$

It is a first-order linear differential equation which can be solved by the integrating factor method.

Summer Air Conditioning

An air conditioner used with a thermostat leads to the same differential equation (5) and solution, because Newton's law of cooling applies to both heating and cooling.

Evaporative Cooling

In desert-mountain areas, where summer humidity is low, the **evaporative cooler** is a popular low-cost solution to cooling. The cooling effect is due to heat loss from the supply of outside air, caused by energy conversion during water evaporation. Cool air is pumped into the residence much like a furnace pumps warm air. An evaporative cooler may have no thermostat. The temperature $P(t)$ of the pumped air depends on the outside air temperature and humidity.

A Newton's cooling model for the inside temperature $u(t)$ requires a constant k_1 for the evaporative cooling term $f(t) = k_1(P(t) - u(t))$. If $s(t) = 0$ is assumed, then equation (2) becomes

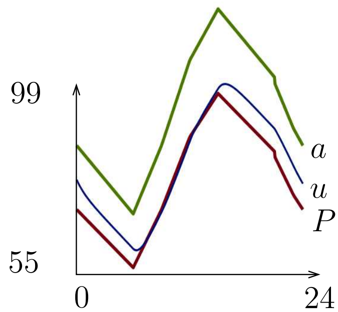
$$\frac{du}{dt} = k(a(t) - u(t)) + k_1(P(t) - u(t)). \tag{6}$$

This is a first-order linear differential equation, solvable by the integrating factor method.

During hot summer days the relation $P(t) = 0.85a(t)$ could be valid, that is, the air pumped from the cooler vent is 85% of the ambient outside temperature $a(t)$. Extreme temperature variations can occur in the fall and spring. In July, the reverse is possible, e.g., $100 < a(t) < 115$. Assuming $P(t) = 0.85a(t)$, the solution of (6) is

$$u(t) = u(0)e^{-kt-k_1t} + (k + 0.85k_1)\int_0^t a(r)e^{(k+k_1)(r-t)}dr.$$

Figure 3 shows the solution for a 24-hour period, using a sample profile $a(t)$, $k = 1/4$, $k_1 = 2$ and $u(0) = 69$. The residence temperature $u(t)$ is expected to be approximately between $P(t)$ and $a(t)$.



$$a(t) = \begin{cases} 75 - 2t & 0 \le t \le 6 \\ 39 + 4t & 6 < t \le 9 \\ 30 + 5t & 9 < t \le 12 \\ 54 + 3t & 12 < t \le 15 \\ 129 - 2t & 15 < t \le 21 \\ 170 - 4t & 21 < t \le 23 \\ 147 - 3t & 23 < t \le 24 \end{cases}$$

Figure 3. A 24-hour plot of P, u and temperature profile $a(t)$.

Examples

Example 2.21 (Pollution)
When industrial pollution in Lake Erie ceased, the level was five times that of its inflow from Lake Huron. Assume Lake Erie has perfect mixing, constant volume V and equal inflow/outflow rates of $0.73V$ per year. Estimate the time required to reduce the pollution in half.

Solution: The answer is about 1.34 years. An overview of the solution will be given, followed by technical details.

Overview. The brine-mixing model applies to pollution problems, giving a differential equation model for the pollution concentration $x(t)$,

$$x'(t) = 0.73Vc - 0.73x(t), \quad x(0) = 5cV,$$

where c is the inflow pollution concentration. The model has solution

$$x(t) = x(0)\left(0.2 + 0.8e^{-0.73t}\right).$$

Solving for the time T at which $x(T) = \frac{1}{2}x(0)$ gives $T = \ln(8/3)/0.73 = 1.34$ years.

Model details. The rate of change of $x(t)$ equals the concentration rate in minus the concentration rate out. The in-rate equals c times the inflow rate, or $c(0.73V)$. The out-rate equals $x(t)$ times the outflow rate, or $\frac{0.73V}{V}x(t)$. This justifies the differential equation. The statement $x(0)$="five times that of Lake Huron" means that $x(0)$ equals $5c$ times the volume of Lake Erie, or $5cV$.

Solution details. The differential equation can be re-written in equivalent form $x'(t) + 0.73x(t) = 0.73x(0)/5$. It has equilibrium solution $x_p = x(0)/5$. The homogeneous solution is $x_h = ke^{-0.73t}$, from the theory of growth-decay equations. Adding x_h and x_p gives the general solution x. To solve the initial value problem, substitute $t = 0$ and find $k = 4x(0)/5$. Substitute for k into $x = x(0)/5 + ke^{-0.73t}$ to obtain the reported solution.

Equation for T details. The equation $x(T) = \frac{1}{2}x(0)$ becomes $x(0)(0.2+0.8e^{-0.73T}) = x(0)/2$, which by algebra reduces to the exponential equation $e^{-0.73T} = 3/8$. Take logarithms to isolate $T = -\ln(3/8)/0.73 \approx 1.3436017$.

Example 2.22 (Brine Cascade)
Assume brine tanks A and B in Figure 4 have volumes 100 and 200 gallons, respectively. Let $A(t)$ and $B(t)$ denote the number of pounds of salt at time t, respectively, in tanks A and B. Pure water flows into tank A, brine flows out of tank A and into tank B, then brine flows out of tank B. All flows are at 4 gallons per minute. Given $A(0) = 40$ and $B(0) = 40$, find $A(t)$ and $B(t)$.

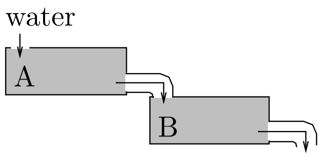


Figure 4. Cascade of two brine tanks.

Solution: The solutions for the brine cascade are (details below)

$$A(t) = 40e^{-t/25}, \quad B(t) = 120e^{-t/50} - 80e^{-t/25}.$$

Modeling. This is an instance of the two-tank mixing problem on page 112. The volumes in the tanks do not change and the input salt concentration is $C_1 = 0$. The equations are

$$\frac{dA}{dt} = -\frac{4A(t)}{100}, \quad \frac{dB}{dt} = \frac{4A(t)}{100} - \frac{4B(t)}{200}.$$

Solution $A(t)$ details.

$A' = -0.04A, \quad A(0) = 40$	Initial value problem to be solved.
$A = 40e^{-t/25}$	Solution found by the growth-decay model.

Solution $B(t)$ details.

$B' = 0.04A - 0.02B, \quad B(0) = 40$	Initial value problem to be solved.
$B' + 0.02B = 1.6e^{-t/25}$	Substitute for A. Get standard form.
$B' + 0.02B = 0, \quad B(0) = 40$	Homogeneous problem to be solved.
$B_h = 40e^{-t/50}$	Homogeneous solution. Growth-decay formula applied.
$B_p = e^{-t/50} \int_0^t 1.6e^{-r/25}e^{r/50}dr$	Variation of parameters solution.
$= 80e^{-t/50} - 80e^{-t/25}$	Evaluate integral.
$B = B_h + B_p$	Superposition.
$= 120e^{-t/50} - 80e^{-t/25}$	Final solution.

The solution can be checked in `maple` as follows.

```
de1:=diff(x(t),t)=-4*x(t)/100:
de2:=diff(y(t),t)=4*x(t)/100-4*y(t)/200:
ic:=x(0)=40,y(0)=40:
dsolve({de1,de2,ic},{x(t),y(t)});
```

Example 2.23 (Office Heating)
A worker shuts off the office heat and goes home at 5PM. It's 72°F inside and 60°F outside overnight. Estimate the office temperature at 8PM, 11PM and 6AM.

Solution:

The temperature estimates are 62.7-65.7°F, 60.6-62.7°F and 60.02-60.5°F. Details follow.

Model. The residential heating model applies, with no sources, to give $u(t) = a_0 + (u(0) - a_0)e^{-kt}$. Supplied are values $a_0 = 60$ and $u(0) = 72$. Unknown is constant k in the formula

$$u(t) = 60 + 12e^{-kt}.$$

Estimation of k. To make the estimate for k, assume the range $1/4 \le k \le 1/2$, which covers the possibilities of poor to excellent insulation.

Calculations. The estimates requested are for $t = 3$, $t = 6$ and $t = 13$. The formula $u(t) = 60 + 12e^{-kt}$ and the range $0.25 \le k \le 0.5$ gives the estimates

$$\begin{aligned} 62.68 \le 60 + 12e^{-3k} &\le 65.67, \\ 60.60 \le 60 + 12e^{-6k} &\le 62.68, \\ 60.02 \le 60 + 12e^{-13k} &\le 60.47. \end{aligned}$$

Example 2.24 (Spring Temperatures)

It's spring. The outside temperatures are between 45°F and 75°F and the residence has no heating or cooling. Find an approximation for the interior temperature fluctuation $u(t)$ using the estimate $a(t) = 60 - 15\cos(\pi(t-4)/12)$, $k = \ln(2)/2$ and $u(0) = 53$.

Solution: The approximation, justified below, is

$$u(t) \approx -8.5e^{-kt} + 60 + 1.5\cos\frac{\pi t}{12} - 12\sin\frac{\pi t}{12}.$$

Model. The residential model for no sources applies. Then

$$u'(t) = k(a(t) - u(t)).$$

Computation of $u(t)$. Let $\omega = \pi/12$ and $k = \ln(2)/2 \approx 0.35$ (poor insulation). The solution is

$$\begin{aligned} u &= u(0)e^{-kt} + \textstyle\int_0^t ka(r)e^{k(r-t)}dr && \text{Variation of parameters.} \\ &= 53e^{-kt} + \textstyle\int_0^t 15k(4 - \cos\omega(t-4))e^{k(r-t)}dr && \text{Insert } a(t) \text{ and } u(0). \\ &\approx -8.5e^{-kt} + 60 + 1.5\cos\omega t - 12\sin\omega t && \text{Used maple integration.} \end{aligned}$$

The `maple` code used for the integration appears below.

```
k:=ln(2)/2: u0:=53:
A:=r->k*(60-15*cos(Pi *(r-4)/12)):
U:=t->(u0+int(A(r)*exp(k*r),r=0..t))*exp(-k*t);
simplify(U(t));
```

Example 2.25 (Temperature Variation)

Justify that in the spring and fall, the interior of a residence might have temperature variation between 19% and 89% of the outside temperature variation.

Solution: The justification necessarily makes some assumptions, which are:

$a(t) = B - A\cos\omega(t-4)$	Assume $A > 0$, $B > 0$, $\omega = \pi/12$ and extreme temperatures at 4AM and 4PM.
$s(t) = 0$	No inside heat sources.
$f(t) = 0$	No furnace or air conditioner.
$0.05 \le k \le 0.5$	Vary from excellent ($k = 0.05$) to poor ($k = 0.5$) insulation.
$u(0) = B$	The average of the outside low and high.

Model. The residential model for no sources applies. Then

$$u'(t) = k(a(t) - u(t)).$$

Formula for u. Variation of parameters gives a compact formula:

$u = u(0)e^{-kt} + \int_0^t ka(r)e^{k(r-t)}dr$	See (4), page 97.
$= Be^{-kt} + \int_0^t k(B - A\cos\omega(t-4))e^{k(r-t)}dr$	Insert $a(t)$ and $u(0)$.
$= c_0 A e^{-kt} + B + c_1 A\cos\omega t + c_2 A\sin\omega t$	Evaluate. Values below.

The values of the constants in the calculation of u are

$$c_0 = 72k^2 - 6k\pi\sqrt{3}, \quad c_1 = \frac{6k\pi\sqrt{3} - 72k^2}{144k^2 + \pi^2}, \quad c_2 = \frac{-6k\pi - 72k^2\sqrt{3}}{144k^2 + \pi^2}.$$

The trigonometric formula $a\cos\theta + b\sin\theta = r\sin(\theta + \phi)$ where $r^2 = a^2 + b^2$ and $\tan\phi = a/b$ can be applied to the formula for u to rewrite it as

$$u = c_0 A e^{-kt} + B + A\sqrt{c_1^2 + c_2^2}\sin(\omega t + \phi).$$

The outside low and high are $B - A$ and $B + A$. The outside temperature variation is their difference $2A$. The exponential term contributes less than one degree after 12 hours. The inside low and high are therefore approximately $B - rA$ and $B + rA$ where $r = \sqrt{c_1^2 + c_2^2}$. The inside temperature variation is their difference $2rA$, which is r times the outside variation.

It remains to show that $0.19 \le r \le 0.89$. The equation for r has a simple representation:

$$r = \frac{12k}{\sqrt{144k^2 + \pi^2}}.$$

It has positive derivative dr/dk. Then extrema occur at the endpoints of the interval $0.05 \le k \le 0.5$, giving values $r = 0.19$ and $r = 0.89$, approximately. This justifies the estimates of 19% and 89%.

The `maple` code used for the integration appears below.

```
omega:=Pi/12:
F:=r->k*(B-A*cos(omega *(r-4))):
G:=t->(B+int(F(r)*exp(k*r),r=0..t))*exp(-k*t);
simplify(G(t));
```

Remarks on Insulation Constants. The insulation constant k in the Newton cooling model is usually between zero and one, with excellent insulation near zero and bad insulation near one. It is also called a **coupling constant**, because $k = 0$ means the temperature u is *decoupled* from the ambient temperature. The constant k depends in a complex way on geometry and insulation, therefore it is determined empirically, and not by a theoretical formula. Lab experiments with a thermocouple in an air-insulated vessel filled with about 300 ml of hot water (80 to 100 C) can determine insulation constants on the order of $k = 0.0003$ (units per second).

Printed on dual pane clear glass in the USA is a U-value of about 0.48. The U-value is equal to the reciprocal of the R-value (see below). You can think of it as the insulation constant k. The lower the U-value, the better the glass insulation quality.

For a solar water heater, $k = 0.00035$ is typical. This value is for an 80 gallon tank with R-15 insulation raised to 120 F during the day. Typically, the water temperature drops by only 3-4 F overnight.

The **thermal conductivity** symbol κ (Greek kappa) can be confused with the insulation constant symbol k, and it is a tragic error to substitute one for the other.

For USA R-values printed on insulation products, thermal conductivity is defined by the relation $U = \frac{1}{0.17611101838R} = \frac{\kappa}{L}$, where L is the material's thickness and U is the international U-factor in SI units. The U-factor value is the heat lost in Watts per square meter at a standard temperature difference of one degree Kelvin.

Example 2.26 (Radioactive Chain)

Let A, B and C be the amounts of three radioactive isotopes. Assume A decays into B at rate a, then B decays into C at rate b. Given $a \neq b$, $A(0) = A_0$ and $B(0) = 0$, find formulas for A and B.

Solution: The isotope amounts are (details below)

$$A(t) = A_0 e^{-at}, \quad B(t) = aA_0 \frac{e^{-at} - e^{-bt}}{b - a}.$$

Modeling. The reaction model will be shown to be

$$A' = -aA,\ A(0) = A_0, \quad B' = aA - bB,\ B(0) = 0.$$

The derivation uses the radioactive decay law on page **??**. The model for A is simple decay $A' = -aA$. Isotope B is *created* from A at a rate equal to the disintegration rate of A, or aA. But B itself undergoes disintegration at rate bB. The rate of increase of B is not aA but the difference of aA and bB, which accounts for lost material. Therefore, $B' = aA - bB$.

Solution Details for A.

$A' = -aA, \quad A(0) = A_0$	Initial value problem to solve.
$A = A_0 e^{-at}$	Use the *growth-decay* formula on page **??**.

Solution Details for B.

$B' = aA - bB, \quad B(0) = 0$	Initial value problem to solve.
$B' + bB = aA_0 e^{-at}, \quad B(0) = 0$	Insert $A = A_0 e^{-at}$. Standard form.
$B = e^{-bt} \int_0^t aA_0 e^{-ar} e^{br} dr$	Variation of parameters solution page **??**, which already satisfies $B(0) = 0$.

$$= aA_0 \frac{e^{-at} - e^{-bt}}{b-a}$$ Evaluate the integral for $b \neq a$.

Remark on radioactive chains. The sequence of radioactive decay processes creates at each stage a new element that may itself be radioactive. The chain ends when stable atoms are formed. For example, uranium-236 decays into thorium-232, which decays into radium-228, and so on, until stable lead-208 is created at the end of the chain. Analyzed here are 2 steps in such a chain.

Example 2.27 (Electric Circuits)

For the LR-circuit of Figure 5, show that $I_{SS} = E/R$ and $I_{tr} = I_0 e^{-Rt/L}$ are the steady-state and transient currents.

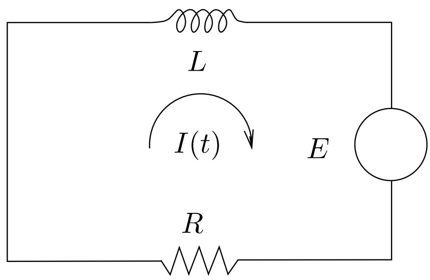


Figure 5. An LR-circuit with constant voltage E and zero initial current $I(0) = 0$.

Solution:

Model. The LR-circuit equation is derived from Kirchhoff's laws and the voltage drop formulas on page **??**. The only new element is the added electromotive force term $E(t)$, which is set equal to the algebraic sum of the voltage drops, giving the model

$$LI'(t) + RI(t) = E(t), \quad I(0) = I_0.$$

General solution. The details:

$I' + (R/L)I = E/L$	Standard linear form.
$I_p = E/R$	Set I=constant, solve for a particular solution I_p.
$I' + (R/L)I = 0$	Homogeneous equation. Solve for $I = I_h$.
$I_h = I_0 e^{-Rt/L}$	Growth-decay formula, page **??**.
$I = I_h + I_p$	Superposition.
$\quad = I_0 e^{-Rt/L} + E/R$	General solution found.

Steady-state solution. The steady-state solution is found by striking out from the general solution all terms that approach zero at $t = \infty$. Remaining after strike-out is $I_{SS} = E/R$.

Transient solution. The term *transient* refers to the terms in the general solution which approaches zero at $t = \infty$. Therefore, $I_{tr} = I_0 e^{-Rt/L}$.

Example 2.28 (Time constant)

Show that the current $I(t)$ in the LR-circuit of Figure 5 is at least 95% of the steady-state current E/R after three time constants, i.e., after time $t = 3L/R$.

Solution: Physically, the **time constant** L/R for the circuit is found by an experiment in which the circuit is initialized to $I = 0$ at $t = 0$, then the current I is observed until it reaches 63% of its steady-state value.

Time to 95% of I_{SS}. The solution is $I(t) = E(1 - e^{-Rt/L})/R$. Solving the inequality $1 - e^{-Rt/L} \geq 0.95$ gives

$0.95 \leq 1 - e^{-Rt/L}$	Inequality to be solved for t.
$e^{-Rt/L} \leq 1/20$	Move terms across the inequality.
$\ln e^{-Rt/L} \leq \ln(1/20)$	Take the logarithm across the inequality.
$-Rt/L \leq \ln 1 - \ln 20$	Apply logarithm rules.
$t \geq L\ln(20)/R$	Isolate t on one side.

The value $\ln(20) = 2.9957323$ leads to the rule: *after three times the time constant has elapsed, the current has reached 95% of the steady-state current.*

Details and Proofs

Brine-Mixing One-tank Proof: Equation $x'(t) = C_1 a(t) - b(t)x(t)/V(t)$, the brine-mixing equation, is justified for the one-tank model by applying the *mixture law* $dx/dt =$ input rate − output rate as follows.

$$\begin{aligned}\text{input rate} &= \left(a(t)\frac{\text{liters}}{\text{minute}}\right)\left(C_1\frac{\text{kilograms}}{\text{liter}}\right)\\ &= C_1 a(t)\frac{\text{kilograms}}{\text{minute}},\\ \text{output rate} &= \left(b(t)\frac{\text{liters}}{\text{minute}}\right)\left(\frac{x(t)}{V(t)}\frac{\text{kilograms}}{\text{liter}}\right)\\ &= \frac{b(t)x(t)}{V(t)}\frac{\text{kilograms}}{\text{minute}}.\end{aligned}$$

Residential Heating and Cooling Proof: Newton's law of cooling will be applied to justify the residential heating and cooling equation

$$\frac{du}{dt} = k(a(t) - u(t)) + s(t) + f(t).$$

Let $u(t)$ be the indoor temperature. The heat flux is due to three heat source rates:

$N(t) = k(a(t) - u(t))$	The Newton cooling rate.
$s(t)$	Combined rate for all inside heat sources.
$f(t)$	Inside heating or cooling rate.

The expected change in u is the sum of the rates N, s and f. In the limit, $u'(t)$ is on the left and the sum $N(t) + s(t) + f(t)$ is on the right. ■

Exercises 2.5

Concentration

A lab assistant collects a volume of brine, boils it until only salt crystals remain, then uses a scale to determine the crystal mass or weight.

Find the salt **concentration** of the brine in kilograms per liter.

1. One liter of brine, crystal mass 0.2275 kg

2. Two liters, crystal mass 0.32665 kg

3. Two liters, crystal mass 15.5 grams

4. Five pints, crystals weigh 1/4 lb

5. Eighty cups, crystals weigh 5 lb

6. Five gallons, crystals weigh 200 ounces

One-Tank Mixing

Assume one inlet and one outlet. Determine the amount $x(t)$ of salt in the tank at time t. Use the text notation for equation (1).

7. The inlet adds 10 liters per minute with concentration $C_1 = 0.023$ kilograms per liter. The tank contains 110 liters of distilled water. The outlet drains 10 liters per minute.

8. The inlet adds 12 liters per minute with concentration $C_1 = 0.0205$ kilograms per liter. The tank contains 200 liters of distilled water. The outlet drains 12 liters per minute.

9. The inlet adds 10 liters per minute with concentration $C_1 = 0.0375$ kilograms per liter. The tank contains 200 liters of brine in which 3 kilograms of salt is dissolved. The outlet drains 10 liters per minute.

10. The inlet adds 12 liters per minute with concentration $C_1 = 0.0375$ kilograms per liter. The tank contains 500 liters of brine in which 7 kilograms of salt is dissolved. The outlet drains 12 liters per minute.

11. The inlet adds 10 liters per minute with concentration $C_1 = 0.1075$ kilograms per liter. The tank contains 1000 liters of brine in which k kilograms of salt is dissolved. The outlet drains 10 liters per minute.

12. The inlet adds 14 liters per minute with concentration $C_1 = 0.1124$ kilograms per liter. The tank contains 2000 liters of brine in which k kilograms of salt is dissolved. The outlet drains 14 liters per minute.

13. The inlet adds 10 liters per minute with concentration $C_1 = 0.104$ kilograms per liter. The tank contains 100 liters of brine in which 0.25 kilograms of salt is dissolved. The outlet drains 11 liters per minute. Determine additionally the time when the tank is empty.

14. The inlet adds 16 liters per minute with concentration $C_1 = 0.01114$ kilograms per liter. The tank contains 1000 liters of brine in which 4 kilograms of salt is dissolved. The outlet drains 20 liters per minute. Determine additionally the time when the tank is empty.

15. The inlet adds 10 liters per minute with concentration $C_1 = 0.1$ kilograms per liter. The tank contains 500 liters of brine in which k kilograms of salt is dissolved. The outlet drains 12 liters per minute. Determine additionally the time when the tank is empty.

16. The inlet adds 11 liters per minute with concentration $C_1 = 0.0156$ kilograms per liter. The tank contains 700 liters of brine in which k kilograms of salt is dissolved. The outlet drains 12 liters per minute. Determine additionally the time when the tank is empty.

Two-Tank Mixing

Assume brine tanks A and B in Figure 4 have volumes 100 and 200 gallons, respectively. Let $x(t)$ and $y(t)$ denote the number of pounds of salt at time t, respectively, in tanks A and B. Distilled water flows into tank A, then brine flows out of tank A and into tank B, then out of tank B. All flows are at r gallons per minute. Given rate r and initial salt amounts $x(0)$ and $y(0)$, find $x(t)$ and $y(t)$.

17. $r = 4$, $x(0) = 40$, $y(0) = 20$.

18. $r = 3$, $x(0) = 10$, $y(0) = 15$.

19. $r = 5$, $x(0) = 20$, $y(0) = 40$.

20. $r = 5$, $x(0) = 40$, $y(0) = 30$.

21. $r = 8$, $x(0) = 10$, $y(0) = 12$.

22. $r = 8$, $x(0) = 30$, $y(0) = 12$.

23. $r = 9$, $x(0) = 16$, $y(0) = 14$.

24. $r = 9$, $x(0) = 22$, $y(0) = 10$.

25. $r = 7$, $x(0) = 6$, $y(0) = 5$.

26. $r = 7$, $x(0) = 13$, $y(0) = 26$

Residential Heating

Assume the Newton cooling model for heating and insulation values $1/4 \le k \le 1/2$. Follow Example 2.23, page 116.

27. The office heat goes off at 7PM. It's 74°F inside and 58°F outside overnight. Estimate the office temperature at 10PM, 1AM and 6AM.

28. The office heat goes off at 6:30PM. It's 73°F inside and 55°F outside overnight. Estimate the office temperature at 9PM, 3AM and 7AM.

29. The radiator goes off at 9PM. It's 74°F inside and 58°F outside overnight. Estimate the room temperature at 11PM, 2AM and 6AM.

30. The radiator goes off at 10PM. It's 72°F inside and 55°F outside overnight. Estimate the room temperature at 2AM, 5AM and 7AM.

31. The office heat goes on in the morning at 6:30AM. It's 57°F inside and 40° to 55°F outside until 11AM. Estimate the office temperature at 8AM, 9AM and 10AM. Assume the furnace provides a five degree temperature rise in 30 minutes with perfect insulation and the thermostat is set for 76°F.

32. The office heat goes on at 6AM. It's 55°F inside and 43° to 53°F outside until 10AM. Estimate the office temperature at 7AM, 8AM and 9AM. Assume the furnace provides a seven degree temperature rise in 45 minutes with perfect insulation and the thermostat is set for 78°F.

33. The hot water heating goes on at 6AM. It's 55°F inside and 50° to 60°F outside until 10AM. Estimate the room temperature at 7:30AM. Assume the radiator provides a four degree temperature rise in 45 minutes with perfect insulation and the thermostat is set for 74°F.

34. The hot water heating goes on at 5:30AM. It's 54°F inside and 48° to 58°F outside until 9AM. Estimate the room temperature at 7AM. Assume the radiator provides a five degree temperature rise in 45 minutes with perfect insulation and the thermostat is set for 74°F.

35. A portable heater goes on at 7AM. It's 45°F inside and 40° to 46°F outside until 11AM. Estimate the room temperature at 9AM. Assume the heater provides a two degree temperature rise in 30 minutes with perfect insulation and the thermostat is set for 90°F.

36. A portable heater goes on at 8AM. It's 40°F inside and 40° to 45°F outside until 11AM. Estimate the room temperature at 10AM. Assume the heater provides a two degree temperature rise in 20 minutes with perfect insulation and the thermostat is set for 90°F.

Evaporative Cooling

Define outside temperature (see Figure 3)

$$a(t) = \begin{cases} 75 - 2t & 0 \le t \le 6 \\ 39 + 4t & 6 < t \le 9 \\ 30 + 5t & 9 < t \le 12 \\ 54 + 3t & 12 < t \le 15 \\ 129 - 2t & 15 < t \le 21 \\ 170 - 4t & 21 < t \le 23 \\ 147 - 3t & 23 < t \le 24 \end{cases}.$$

Given k, k_1, $P(t) = wa(t)$ and $u(0) = 69$, then plot $u(t)$, $P(t)$ and $a(t)$ on one graphic.

$$u(t) = u(0)e^{-kt-k_1t} + (k + wk_1)\int_0^t a(r)e^{(k+k_1)(r-t)}dr.$$

37. $k = 1/4$, $k_1 = 2$, $w = 0.85$

38. $k = 1/4$, $k_1 = 1.8$, $w = 0.85$

39. $k = 3/8$, $k_1 = 2$, $w = 0.85$

40. $k = 3/8$, $k_1 = 2.4$, $w = 0.85$

41. $k = 1/4$, $k_1 = 3$, $w = 0.80$

42. $k = 1/4$, $k_1 = 4$, $w = 0.80$

43. $k = 1/2$, $k_1 = 4$, $w = 0.80$

44. $k = 1/2$, $k_1 = 5$, $w = 0.80$

45. $k = 3/8$, $k_1 = 3$, $w = 0.80$

46. $k = 3/8$, $k_1 = 4$, $w = 0.80$

Radioactive Chain

Let A, B and C be the amounts of three radioactive isotopes. Assume A decays into B at rate a, then B decays into C at rate b. Given a, b, $A(0) = A_0$ and $B(0) = B_0$, find formulas for A and B.

47. $a = 2$, $b = 3$, $A_0 = 100$, $B_0 = 10$

48. $a = 2$, $b = 3$, $A_0 = 100$, $B_0 = 100$

49. $a = 1$, $b = 4$, $A_0 = 100$, $B_0 = 200$

50. $a = 1$, $b = 4$, $A_0 = 300$, $B_0 = 100$

51. $a = 4$, $b = 3$, $A_0 = 100$, $B_0 = 100$

52. $a = 4$, $b = 3$, $A_0 = 100$, $B_0 = 200$

53. $a = 6$, $b = 1$, $A_0 = 600$, $B_0 = 100$

54. $a = 6$, $b = 1$, $A_0 = 500$, $B_0 = 400$

55. $a = 3$, $b = 1$, $A_0 = 100$, $B_0 = 200$

56. $a = 3$, $b = 1$, $A_0 = 400$, $B_0 = 700$

Electric Circuits

In the LR-circuit of Figure 5, assume $E(t) = A\cos wt$ and $I(0) = 0$. Solve for $I(t)$.

57. $A = 100$, $w = 2\pi$, $R = 1$, $L = 2$

58. $A = 100$, $w = 4\pi$, $R = 1$, $L = 2$

59. $A = 100$, $w = 2\pi$, $R = 10$, $L = 1$

60. $A = 100$, $w = 2\pi$, $R = 10$, $L = 2$

61. $A = 5$, $w = 10$, $R = 2$, $L = 3$

62. $A = 5$, $w = 4$, $R = 3$, $L = 2$

63. $A = 15$, $w = 2$, $R = 1$, $L = 4$

64. $A = 20$, $w = 2$, $R = 1$, $L = 3$

65. $A = 25$, $w = 100$, $R = 5$, $L = 15$

66. $A = 25$, $w = 50$, $R = 5$, $L = 5$

2.6 Kinetics

Studied are the following topics.

Newton's Laws	Free Fall with Constant Gravity
Linear Air Resistance	Nonlinear Air Resistance
Modeling	Parachutes
Lunar Lander	Escape Velocity

Newton's Laws

The ideal models of a particle or *point mass* constrained to move along the x-axis, or the motion of a projectile or satellite, have been studied from **Newton's second law**

(1) $$F = ma.$$

In the *mks system* of units, F is the force in **Newtons**, m is the mass in kilograms and a is the acceleration in meters per second per second.

The closely-related **Newton universal gravitation law**

(2) $$F = G\frac{m_1 m_2}{R^2}$$

is used in conjunction with (1) to determine the system's constant value g of gravitational acceleration. The masses m_1 and m_2 have centroids at a distance R. For the earth, $g = 9.8$ m/s^2 is commonly used; see Table 1.

Other commonly used unit systems are *cgs* and *fps*. Table 1 shows some useful equivalents.

Table 1. Units for *fps* and *mks* Systems

Unit name	*fps* unit	*mks* unit
Position	foot (ft)	meter (m)
Time	seconds (s)	seconds (s)
Velocity	feet/sec	meters/sec
Acceleration	feet/sec^2	meters/sec^2
Force	pound (lb)	Newton (N)
Mass	slug	kilogram (kg)
g	32.088 ft/s^2	9.7805 m/s^2

Other units in the various systems are in daily use. Table 2 shows some equivalents. An international synonym for **pound** is **libre**, with abbreviation **lb**. The origin of the word *pound* is migration of **libra pondo**, meaning *a pound in weight*. Dictionaries cite migrations *libra pondo* $\longrightarrow$ *pund* for German language, which is similar to English *pound*.

Table 2. Conversions for the *fps* and *mks* Systems

inch (in)	1/12 foot	2.54 centimeters
foot (ft)	12 inches	30.48 centimeters
centimeter (cm)	1/100 meter	0.39370079 inches
kilometer (km)	1000 meters	0.62137119 miles ($\approx 5/8$)
mile (mi)	5280 feet	1.609344 kilometers ($\approx 8/5$)
pound (lb)	$\approx$ 4.448 Newtons	
Newton (N)	$\approx$ 0.225 pounds	
kilogram (kg)	$\approx$ 0.06852 slugs	
slug	$\approx$ 14.59 kilograms	

Velocity and Acceleration

The position, velocity and acceleration of a particle moving along an axis are functions of time t. Notations vary; this text uses the following symbols, where primes denote t-differentiation.

$x = x(t)$	Particle **position** at time t.
$v = x'(t)$	Particle **velocity** at time t.
$a = x''(t)$	Particle **acceleration** at time t.
$x(0)$	**Initial position**.
$v(0)$	**Initial velocity**. Synonym $x'(0)$ is also used.

Free Fall with Constant Gravity

A body falling in a constant gravitational field might ideally move in a straight line, aligned with the gravitational vector. A typical case is the *lunar lander*, which falls freely toward the surface of the moon, its progress downward controlled by retrorockets. *Falling bodies*, e.g., an object launched up or down from a tall building, can be modeled similarly. For such ideal cases, in which air resistance and other external forces are ignored, the acceleration of the body is assumed to be a constant g and the differential equation model is

$$x''(t) = -g, \quad x(0) = x_0, \quad x'(0) = v_0. \tag{3}$$

The initial position x_0 and the initial velocity v_0 must be specified. The value of g in *mks* units is $g = 9.8$ m/s^2. The symbol x is the distance from the ground ($x = 0$); meters for *mks* units. The symbol t is the time in seconds. Falling body problems normally take $v_0 = 0$ and $x_0 > 0$, e.g., x_0 is the height of the building from which the body was dropped. Objects ejected downwards have $v_0 < 0$, which decreases the descent time. Objects thrown straight up satisfy $v_0 > 0$.

Equation (3) can be solved by the method of quadrature to give the explicit solution

$$x(t) = -\frac{g}{2}t^2 + x_0 + v_0 t. \tag{4}$$

See *Technical Details*, page 137, and the *method of quadrature*, page 74. Applications to free fall and the lunar lander appear in the examples, page 132.

Typical plots can be made by the following `maple` code.

```
X:=unapply(-9.8*t^2+100+(50)*t,t); #v(0)=50m/s,x(0)=100m
plot(X(t),t=0..7);
Y:=unapply(-9.8*t^2+100+(-5)*t,t); #v(0)=-5m/s,x(0)=100m
plot(Y(t),t=0..4);
```

Air Resistance Effects

The inclusion in a differential equation model of terms accounting for air resistance has historically two distinct models. The first is *linear resistance*, in which the force F due to air resistance is assumed to be proportional to the velocity v:

$$F \propto v. \tag{5}$$

It is known that linear resistance is appropriate only for slowly moving objects.[4] The second model is *nonlinear resistance*, modeled originally by Sir Isaac Newton himself as $F = kv^2$. The literature considers a generalized nonlinear resistance assumption

$$F \propto v|v|^p \tag{6}$$

where $0 < p \leq 1$ depends upon the *speed* of the object through the air; $p \approx 0$ is a low speed and $p \approx 1$ is a high speed. It will suffice for illustration purposes to treat just the two cases $F \propto v$ and $F \propto v|v|$.

Linear and Nonlinear Drag

For small spherical objects moving slowly through a viscous fluid, Sir George Gabriel Stokes derived an expression for the **linear drag force**:

$$-\frac{k}{m}v = \textbf{Stoke's drag force} = -6\,\pi\,\eta\,r\,v$$

The symbols: η = fluid viscosity and r = radius of the spherical object. References can use viscosity symbols ρ or μ instead of Stoke's symbol η.

Example: Falling raindrop
The radius is $r = 0.1$ to 0.3 mm and $\eta = 1.789x10^{-5}$ Kg/m/sec is the dynamic viscosity for 15 C air at sea level.

[4] More precisely, for Reynolds Number less than about 1000. The Reynolds Number is the ratio of inertial forces to viscous forces within a fluid.

Velocities $v = x'(t) >$ Mach 1 use **nonlinear drag force** $\pm kv^2$. Fluid theory gives $k = \frac{1}{2} C\,\eta\,A$ where $C =$ the **drag coefficient**, $\eta =$ **dynamic fluid viscosity**, $A =$ **frontal area** facing the fluid.

Example: 22 caliber high velocity long rifle bullet
Drag coefficient $C = 0.35$ to 0.4, air dynamic viscosity $\eta = 1.789x10^{-5}$ Kg/m/sec, frontal area $A = 0.25419304$ cm^2 . Nonlinear drag occurs for close targets. The bullet path below Mach 1 has a section of linear drag.

Linear Air Resistance

The model is determined by the sum of the forces due to air resistance and gravity, $F_{\text{air}} + F_{\text{gravity}}$, which by *Newton's second law* must equal $F = mx''(t)$, giving the differential equation

$$mx''(t) = -kx'(t) - mg. \tag{7}$$

In (7), the velocity is $v = x'(t)$ and k is a proportionality constant for the air resistance force $F \propto v$. The negative sign results from the assumed coordinates: x measures the distance from the ground ($x = 0$). We expect x to decrease, hence x' is negative. Equation (7) written in terms of the velocity $v = x'(t)$ becomes

$$v'(t) = -(k/m)v(t) - g. \tag{8}$$

This equation has a solution $v(t)$ which limits at $t = \infty$ to a **finite terminal velocity** $|v_\infty| = mg/k$; equation (9) below is justified in *Technical Details*, page 137. Physically, this limit is the **equilibrium solution** of (8), which is the observable steady state of the model. A quadrature applied to $x'(t) = v(t)$ using $v(t)$ in equation (9) solves (7). Then

$$\begin{aligned} v(t) &= -\frac{mg}{k} + \left(v(0) + \frac{mg}{k}\right) e^{-kt/m}, \\ x(t) &= x(0) - \frac{mg}{k}t + \frac{m}{k}\left(v(0) + \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right). \end{aligned} \tag{9}$$

Nonlinear Air Resistance

The model applies primarily to rapidly moving objects. It is obtained by the same method as the linear model, replacing the linear resistance term $kx'(t)$ by the nonlinear term $kx'(t)|x'(t)|$. The resulting model is

$$mx''(t) = -kx'(t)|x'(t)| - mg. \tag{10}$$

Velocity substitution $v = x'(t)$ gives first order equation

$$v'(t) = -(k/m)v(t)|v(t)| - g. \tag{11}$$

The model applies in particular to parachute flight and to certain projectile problems, like an arrow or bullet fired straight up.

Upward Launch. Separable equation (11) in the case $v(0) > 0$ for a launch upward becomes $v'(t) = -(k/m)v^2(t) - g$. The solution for $v(0) > 0$ is given below in (12); see *Technical Details* page 137. The equation $x'(t) = v(t)$ can be solved by quadrature. Then for some constants c and d

$$\begin{aligned} v(t) &= \sqrt{\frac{mg}{k}}\tan\left(\sqrt{\frac{kg}{m}}(c-t)\right), \\ x(t) &= d + \frac{m}{k}\ln\left|\cos\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right|. \end{aligned} \tag{12}$$

Downward Launch. The case $v(0) < 0$ for an object launched downward or dropped will use the equation $v'(t) = (k/m)v^2(t) - g$; see *Technical Details*, page 138. Then for some constants c and d

$$\begin{aligned} v(t) &= \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}(c-t)\right), \\ x(t) &= d - \frac{m}{k}\ln\left|\cosh\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right|. \end{aligned} \tag{13}$$

The **hyperbolic functions** appearing in (13) are *defined by*

$\cosh u = \frac{1}{2}\left(e^u + e^{-u}\right)$	Hyperbolic cosine.
$\sinh u = \frac{1}{2}\left(e^u - e^{-u}\right)$	Hyperbolic sine.
$\tanh u = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$	Hyperbolic tangent. Identity $\tanh u = \sinh u / \cosh u$.

The model applies to parachute problems in particular. Equation (13) and the limit formula $\lim_{|x|\to\infty} \tanh x = 1$ imply a *terminal velocity*

$$|v_\infty| = \sqrt{\frac{mg}{k}}.$$

The value is exactly the square root of the linear model terminal velocity. The falling body model (3) without air resistance effects allows the velocity to increase to unrealistic speeds. For instance, the terminal velocity of a raindrop falling from 3000 meters is about $25 - 35$ km/h, whereas the no air resistance model predicts about 870 km/h.

Modeling Remarks

It can be argued from air resistance models that projectiles spend more time falling to the ground than they spend reaching maximum height[5]; see Example 2.32. Simplistic models ignoring air resistance tend to over-estimate the maximum height of the projectile and the flight time; see Example 2.31. Falling bodies are predicted by air resistance models to have a *terminal velocity.*

Significant effects are ignored by the models of this text. Real projectiles are affected by spin and a flight path that is not planar. The **corkscrew** path of a bullet can cause it to miss a target, while a planar model predicts it will hit the target. The spin of a projectile can drastically alter its flight path and flight characteristics, as is known by players of table tennis, squash, court tennis, archery enthusiasts and gun club members.

Gravitational effects assumed constant may in fact not be constant along the flight path. This can happen in the soft touchdown problem for a lunar lander which activates retrorockets high above the moon's surface.

External effects like wind or the gravitational forces of nearby celestial bodies, ignored in simplistic models, may indeed produce significant effects. On the freeway, is it possible to throw an ice cube out the window ahead of your vehicle? Is it feasible to use forces from the moon to **assist** in the launch of an orbital satellite?

Parachutes

In a typical parachute problem, the jumper travels in a parabolic arc to the ground, buffeted about by up and down drafts in the atmosphere, but always moving in the direction determined by the airplane's flight. In short, a parachutist does not *fall* to the ground. Their flight path more closely resembles the path of a projectile and it is generally not a planar path.

Important to skydivers is an absolute limit to their speed, called the **terminal velocity**. It depends upon a number of physical factors, the dominant factor being body shape affecting area variable A of the **drag force**. See page . A parachutist with excess loose clothing will dive more slowly than when equipped with a tight lycra jump suit. When the parachute opens, the flight characteristics are dominated by physical factors of the open parachute.

The constant $k/m > 0$ is called the **drag factor**, where m is the mass and $k > 0$, appears in the resistive force equation $F = kv|v|$. In order for the parachute model to give a terminal velocity of 15 miles per hour, the drag factor must be approximately $k/m = 3/2$. Without the parachute, the skydiver can reach speeds of over 45 miles per hour, which corresponds to a drag factor $k/m < 1/2$.

[5]Racquetball, badminton, Lacrosse, tennis, squash, pickleball and table tennis players know about this effect and they use it in their game tactics and timing.

Who falls the greatest distance after 30 seconds, a 250-pound or a 110-pound parachutist? The answer is not always the layman's answer, because the 110-pound parachutist has *less* air resistance due to less body surface area but also *less* mass, making it difficult to compare the two drag factors.

Lunar Lander

A lunar lander is falling toward the moon's surface, in the radial direction, at a speed of 1000 miles per hour. It is equipped with retrorockets to retard the fall. In free space outside the gravitational effects of the moon the retrorockets provide a retardation thrust of 9 miles per hour per second of activation, e.g., 11 seconds of retrorocket power will slow the lander down by about 100 miles per hour.

A **soft touchdown** is made when the lander contacts the moon's surface falling at a speed of zero miles per hour. This ideal situation can be achieved by turning on the retrorockets at the right moment.

The lander is greatly affected by the gravitational field of the moon. Ignoring this field gives a gross overestimate for the activation time, causing the lander to reverse its direction and never reach the surface. The layman answer of $1000/9 \approx 112$ seconds to touchdown from an altitude of about 16 miles is incorrect by about 10 miles, causing the lander to crash at substantial speed into the lunar surface.

Escape velocity

Is it possible to fire a projectile from the earth's surface and reach the moon? The science fiction author Jules Verne, in his 1865 novel *From the Earth to the Moon*, seems to believe it is possible. Modern calculations give the initial **escape velocity** v_0 as about $25,000$ miles per hour. There is no record of this actually being tested, so the number $25,000$ remains a theoretical estimate.

This is a different problem than powered rocket flight. All the power must be applied initially, and it is not allowed to apply power during flight to the moon. Imagine instead a deep hole, in which a rocket is launched, the power being turned off just as the rocket exits the hole. The rocket has to coast to the moon, using just the velocity gained during launch.

Newton's law of universal gravitation gives $m_1 m_2 G/r^2$ as the magnitude of the force of attraction between two point-masses m_1, m_2 separated by distance r. The equation $g = Gm_2/R^2$ gives the acceleration due to gravity at the surface of the planet. For the earth, $g = 9.8$ meters per second per second and $R = 6,370,000$ meters.

A spherical projectile of mass m_1 hurled straight up from the surface of a planet moves in the radial direction. Ignoring air resistance and external gravitational forces, Newton's law implies the distance $y(t)$ traveled by the projectile satisfies

$$m_1 y''(t) = -\frac{m_1 m_2 G}{(y(t)+R)^2}, \quad y(0) = 0, \quad y'(0) = v_0, \tag{14}$$

where R is the radius of the planet, m_2 is its mass and G is the experimentally measured universal gravitation constant. Using $gR^2 = Gm_2$ and canceling m_1 in (14) gives

$$y''(t) = -\frac{gR^2}{(y(t)+R)^2}, \quad y(0) = 0, \quad y'(0) = v_0. \tag{15}$$

The projectile **escapes** the planet if $y(t) \to \infty$ as $t \to \infty$. The **escape velocity problem** asks which minimal value of v_0 causes escape.

To solve the escape velocity problem, multiply equation (15) by $y'(t)$, then integrate over $[0, t]$ and use the initial conditions $y(0) = 0$, $y'(0) = v_0$ to obtain

$$\frac{1}{2}\left((y'(t))^2 - (v_0)^2\right) = \frac{gR^2}{y(t)+R} - Rg.$$

The square term $(y'(t))^2$ being nonnegative gives the inequality

$$0 \le (v_0)^2 + \frac{2gR^2}{y(t)+R} - 2Rg.$$

If $y(t) \to \infty$, then $v_0^2 \ge 2Rg$, which gives the *escape velocity*

$$v_0 = \sqrt{2gR}. \tag{16}$$

For the earth, $v_0 \approx 11,174$ meters per second, which is slightly more than $25,000$ miles per hour.

Examples

Example 2.29 (Free Fall)

A ball is thrown straight up from the roof of a 100-foot building and allowed to fall to the ground. Assume initial velocity $v_0 = 32$ miles per hour. Estimate the maximum height of the ball and its flight time to the ground.

Solution: The maximum height H and flight time T are given by

$$H = 134.41 \text{ ft}, \quad T = 4.36 \text{ sec}.$$

Details: In fps units, $v_0 = 32(5280)/(3600) = 46.93$ ft/sec. Using solution (4) gives for $x_0 = 100$ and $v_0 = 46.93$

$$x(t) = -16t^2 + 100 + 46.93t.$$

Then $x(t) = H = \max$ when $x'(t) = 0$, which happens at $t = 46.93/32$. Therefore, $H = x(46.93/32) = 134.41$. The flight time T is given by the equation $x(T) = 0$ (the ground is $x = 0$). Solving this quadratic equation for $T > 0$ gives $T = 4.36$ seconds.

Example 2.30 (Lunar Lander)

A lunar lander falls to the moon's surface at $v_0 = -960$ miles per hour. The retrorockets in free space provide a deceleration effect on the lander of $a = 18,000$ miles per hour per hour. Estimate the retrorocket activation height above the surface which will give the lander zero touch-down velocity.

Solution: Presented here are two models, one which assumes the moon's gravitational field is constant and another which assumes it is variable. The results obtained for the activation height are different: 93.3 miles for the constant field model and 80.1 miles for the variable field model. The flight times to touchdown are estimated to be 11.7 minutes and 10.4 minutes, respectively.

Calculations use *mks* units: $v_0 = -429.1584$ meters per second and $a = 2.2352$ meters per second per second.

Constant field model. Let's assume constant gravitational acceleration $\mathcal{G}$ due to the moon. Other gravitational effects are ignored.

The acceleration value $\mathcal{G}$ is found in *mks* units from the formula

$$\mathcal{G} = \frac{G\,m_1}{R^2}.$$

Symbols: $m_1 = 7.36 \times 10^{22}$ kilograms and $R = 1.74 \times 10^6$ meters (1740 kilometers, 1081 miles), which are the mass and radius of the moon. Newton's universal gravitation constant is $G \approx 6.6726 \times 10^{-11}$ $\mathrm{N(m/kg)^2}$. Then $\mathcal{G} = 1.622087990$.

The lander itself has mass m. Let $r(t)$ be the distance from the lander to the surface of the moon. The value $r(0)$ is the height above the moon when the retrorockets are activated for the soft landing at time t_0. Then force analysis and Newton's second law implies the differential equation model

$$mr''(t) = ma - m\mathcal{G}, \quad r(t_0) = 0, \quad r'(t_0) = 0, \quad r'(0) = v_0.$$

The objective is to find $r(0)$. Cancel m, then integrate twice to obtain the quadrature solution

$$\begin{aligned} r'(t) &= (a - \mathcal{G})t + v_0, \\ r(t) &= (a - \mathcal{G})t^2/2 + v_0 t + r(0). \end{aligned}$$

Then $r'(t_0) = 0$ and $r(t_0) = 0$ give the equations

$$(a - \mathcal{G})t + v_0 = 0, \quad r(0) = -v_0 t_0 - (a - \mathcal{G})t_0^2/2.$$

The symbols in *mks* units: $a = 2.2352$, $v_0 = -429.1584$, $\mathcal{G} = 1.622087990$. Solving simultaneously provides the numerical answers

$$t_0 = 11.66 \text{ minutes}, \quad r(0) = 150.16 \text{ kilometers} = 93.3 \text{ miles}.$$

The conversion uses 1 mile = 1.609344 kilometers.

Variable field model. The constant field model will be modified to obtain this model. All notation developed above applies. We will replace the constant acceleration $\mathcal{G}$ by the variable acceleration $G\,m_1/(R + r(t))^2$. Then the model is

$$mr''(t) = ma - \frac{G\,m_1\,m}{(R + r(t))^2}, \quad r(t_0) = 0, \quad r'(t_0) = 0, \quad r'(0) = v_0.$$

Multiply this equation by $r'(t)/m$ and integrate. Then

$$\frac{(r'(t))^2}{2} = ar(t) + \frac{G\,m_1}{R + r(t)} + c, \quad \text{where} \quad c \equiv -\frac{G\,m_1}{R}.$$

We will find $r(0)$, the height above the moon. The equation to solve for $r(0)$ is found by substitution of $t = 0$ into the previous equation:

$$\frac{(r'(0))^2}{2} = ar(0) + \frac{Gm_1}{R + r(0)} - \frac{Gm_1}{R}.$$

After substitution of known values, the quadratic equation for $x = r(0)$ is:

$$92088.46615 = 2.2352\,x + \frac{2822179.310}{1 + x/1740000} - 2822179.310$$

Solving for the positive root gives $r(0) \approx 127.23$ kilometers or 79.06 miles. The analysis does not give the flight time t_0 directly, but it is approximately 10.4 minutes: see the exercises.

Answer check. A similar analysis is done in Edwards and Penney [**?**] for the case $a = 4$ meters per second per second, $v_0 = -450$ meters per second, with result $r(0) \approx 41.87$ kilometers. In their example, the retrorocket thrust is nearly doubled, resulting in a lower activation height. Substitute $v_0 = -450$ and $a = 4$ in the variable field model to obtain agreement: $r(0) \approx 41.90$ kilometers. The constant field model gives $r(0) \approx 42.58$ kilometers and $t_0 \approx 3.15$ minutes.

Example 2.31 (Flight Time and Maximum Height)

Show that the maximum height and the ascent time of a projectile are over-estimated by a model that ignores air resistance.

Solution: Treated here is the case of a projectile launched straight up from the ground $x = 0$ with velocity $v_0 > 0$. The ascent time is denoted t_1 and the maximum height M is then $M = x(t_1)$.

No air resistance. Consider the velocity model $v' = -g$, $v(0) = v_0$. The solution is $v = -gt + v_0$, $x = -gt^2/2 + v_0 t$. Then maximum height M occurs at $v'(t_1) = 0$ which gives $t_1 = v_0/g$ and $M = x(t_1) = t_1(v_0 - gt_1/2) = gv_0^2/2$.

Linear air resistance. Consider the model $v' = -\rho v - g$, $v(0) = v_0$. This is a Newton cooling equation in disguise, with solution given by equation (9), where $\rho = k/m$. Then t_1 is a function of (ρ, v_0) satisfying $ge^{\rho t_1} = v_0\rho + g$, hence t_1 is given by the equation

$$t_1(\rho, v_0) = \frac{1}{\rho}\ln\left|\frac{v_0\rho + g}{g}\right|. \tag{17}$$

The limit of $t_1 = t_1(\rho, v_0)$ as $\rho \to 0$ is the ascent time v_0/g of the no air resistance model. Verified in the exercises are the following.

Lemma 2.2 (Linear Ascent Time) The ascent time t_1 for linear air resistance satisfies $t_1(\rho, v_0) < v_0/g$.

The lemma implies that the rise time for linear air resistance is less than the rise time for no air resistance.

The inequality $v' = -\rho v - g < -g$ holds for $v > 0$, therefore $v(t) < -gt + v_0$ and $x(t) < -gt^2/2 + v_0 t =$ height for the no air resistance model. Thus the maximum height $x(t_1)$ is less than the maximum height for the no air resistance model, by Lemma 2.2; see the exercises page 141.

Nonlinear air resistance. The example is technically done, because it has been shown that the answers for t_1 and M decrease when using the linear model. Similar results can be stated for the nonlinear model $v' = \rho v|v| - g$; see the exercises page 141.

Example 2.32 (Modeling)

Argue from nonlinear air resistance models that a projectile takes more time to fall to the ground than it takes to reach maximum height.

Solution: The model will be the nonlinear model of the text, which historically goes back to Isaac Newton. The linear air resistance model, appropriate for slowly moving projectiles, is not considered in this example.

Let t_1 and t_2 be the ascent and fall times, so that the total flight time from the ground to maximum height and then to the ground again is $t_1 + t_2$.

The times t_1, t_2 are functions of the initial velocity $v_0 > 0$. As v_0 limits to zero, both t_1 and t_2 limit to zero. Inequality $t_2 dt_2/dv_0 - t_1 dt_1/dv_0 > 0$ is derived in Lemma 2.7 below. Integrate the inequality on variable v_0, then $\frac{1}{2}(t_2^2 - t_1^2) > 0$, from which it follows that $t_2 > t_1$ for $v_0 > 0$. Meaning: the projectile takes more time to fall to the ground (t_2) than it takes to reach maximum height (t_1).

Define nonlinear functions

$$f_1(v) = -(k/m)v^2 - g, \quad f_2(v) = (k/m)v^2 - g$$

The **ascent** or **rise** is controlled with velocity $v_1 > 0$ satisfying $v_1' = f_1(v_1)$, $v_1(0) = v_0 > 0$, $v_1(t_1) = 0$. The maximum height reached is $y_0 = \int_0^{t_1} v_1(t)dt$. The **descent** of **fall** is controlled with velocity $v_2(t)$ satisfying $v_2' = f_2(v_2)$, $v_2(t_1) = 0$. The flight ends at time $T = t_1 + t_2$, determined by $0 = y_0 + \int_{t_1}^{T} v_2(t)dt$.

Details of proof involve a number of technical results, some of which depend upon the formulas $f_1(v) = -(k/m)v^2 - g$, $f_2(v) = (k/m)v^2 - g$.

Lemma 2.3 The solution v_2 satisfies $v_2(t) = w(t - t_1)$, where w is defined by $w' = f_2(w)$, $w(0) = 0$. The solution w does not involve variables v_0, t_1, t_2.

Lemma 2.4 Assume f is continuously differentiable. Let $v(t, v_0)$ be the solution of $v' = f(v)$, $v(0) = v_0$. Then

$$\frac{dv}{dv_0} = e^{\int_0^t f'(v(t,v_0))dt}.$$

The function $z = dv/dv_0$ solves the linear problem $z' = f'(v(t, v_0))z$, $z(0) = 1$.

Lemma 2.5

$$\frac{dt_1}{dv_0} = \frac{1}{g}\, e^{-2k\int_0^{t_1} v_1(t,v_0)dt/m}.$$

Lemma 2.6

$$\frac{dt_2}{dv_0} = \frac{-1}{v_2(t_1 + t_2)} \int_0^{t_1} e^{-2k\int_0^t v_1(r,v_0)dr/m}\, dt.$$

Lemma 2.7

$$t_2 \frac{dt_2}{dv_0} - t_1 \frac{dt_1}{dv_0} > 0.$$

Proof of Lemma 2.7. Lemmas 2.3 to 2.6 will be applied. Define $w(t)$ by Lemma 2.3. Because $w' = f_2(w) = (k/m)w^2 - g$, then $f_2(w) \geq -g$ which implies $w(t) \geq w(0) - gt$. Using $w(0) = 0$ implies $v_2(t_1 + t_2) = w(t_2) \geq -gt_2$ and finally, using $w(t) < 0$ for $0 < t \leq t_2$,

$$\frac{1}{gt_2} \leq \frac{-1}{v_2(t_1 + t_2)}.$$

Multiply this inequality by $e^{u(t)}$, $u(t) = -2k \int_0^t v_1(r, v_0)dr/m$. Integrate over $t = 0$ to $t = t_1$. Then Lemma 2.6 implies

$$\frac{1}{gt_2} \int_0^{t_1} e^{u(t)} dt \leq \frac{dt_2}{dv_0}.$$

Because $u(t) > u(t_1)$, then

$$\frac{1}{gt_2} \int_0^{t_1} e^{u(t_1)} dt < \frac{dt_2}{dv_0}.$$

This implies by Lemma 2.5 the inequality

$$\frac{t_1}{t_2} \frac{dt_1}{dv_0} = \frac{t_1}{gt_2} e^{u(t_1)} < \frac{dt_2}{dv_0},$$

or $t_2 dt_2/dv_0 - t_1 dt_1/dv_0 > 0$. ■

Proof of Lemma 2.3. The function $z(t) = v_2(t + t_1)$ satisfies $z' = f_2(z)$, $z(0) = 0$ (an answer check for the reader). Function $w(t)$ is defined to solve $w' = f_2(w)$, $w(0) = 0$. By uniqueness, $z(t) \equiv w(t)$, or equivalently, $w(t) = v_2(t + t_1)$. Replace t by $t - t_1$ to obtain $v_2(t) = w(t - t_1)$.

Proof of Lemma 2.4. The exponential formula for dv_2/dv_0 is the unique solution of the first order initial value problem. It remains to show that the initial value problem is satisfied. Instead of doing the answer check, we motivate how to find the initial value problem. First, differentiate across the equation $v_2' = f_2(v_2)$ with respect to variable v_0 to obtain $z' = f_2'(v_2)z$ where $z = dv_2/dv_0$. Secondly, differentiate the relation $v_2(0, v_0) = v_0$ on variable v_0 to obtain $z(0) = 1$. The details of the answer check focus on showing Newton quotients converge to the given answer.

Proof of Lemma 2.5. Start with the determining equation $v_1(t_1, v_0) = 0$. Differentiate using the chain rule on variable v_0 to obtain the relation

$$v_1'(t_1, v_0)\frac{dt_1}{dv_0} + \frac{dv_1}{dv_0}(t_1, v_0) = 0.$$

Because $f_1'(u) = -2ku/m$, then the preceding lemma implies that dv_1/dv_0 is the same exponential function as in this Lemma. Also, $v_1(t_1, v_0) = 0$ implies $v_1'(t_1, v_0) = f_1(0) = -g$. Substitution gives the formula for dt_1/dv_0.

Proof of Lemma 2.6. Start with $y_0 = \int_0^{t_1} v_1(t, v_0)dt$ and $y(t) = y_0 + \int_{t_1}^t v_2(t)dt$. Then $0 = y(t_2 + t_1)$ implies that

$$\begin{aligned} 0 &= y(t_1 + t_2) \\ &= \int_0^{t_1} v_1(t, v_0)dt + \int_0^{t_2} v_2(t + t_1)dt \\ &= \int_0^{t_1} v_1(t, v_0)dt + \int_0^{t_2} w(t)dt. \end{aligned}$$

Because $w(t)$ is independent of t_1, t_2, v_0 and $v_1(t_1, v_0) = 0$, then differentiation on v_0 across the preceding formula gives

$$\begin{aligned} 0 &= \frac{d}{dv_0}\int_0^{t_1} v_1(t,v_0)dt + w(t_2)\frac{dt_2}{dv_0} \\ &= v_1(t_1,v_0)\frac{dt_1}{dv_0} + \int_0^{t_1}\frac{dv_1}{dv_0}(t,v_0)dt + w(t_2)\frac{dt_2}{dv_0} \\ &= 0 + \int_0^{t_1} e^{u(t)}dt + w(t_2)\frac{dt_2}{dv_0} \end{aligned}$$

where $u(t) = -2k\int_0^t v_1(r,v_0)dr/m$. Use $w(t_2) = v_2(t_2 + t_1)$ after division by $w(t_2)$ in the last display to obtain the formula.

Details and Proofs

Proof for Equation (4). The method of quadrature is applied as follows.

$x''(t) = -g$	The given differential equation.
$\int x''(t)dt = \int -gdt$	Quadrature step.
$x'(t) = -gt + c_1$	Fundamental theorem of calculus.
$\int x'(t)dt = \int(-gt + c_1)dt$	Quadrature step.
$x(t) = -g\frac{t^2}{2} + c_1t + c_2$	Fundamental theorem of calculus.

Using initial conditions $x(0) = x_0$ and $x'(0) = v_0$ it follows that $c_1 = v_0$ and $c_2 = x_0$. These steps verify the formula $x(t) = -gt^2/2 + x_0 + v_0t$.

Technical Details for Equation (9).

$v'(t) + (k/m)v(t) = -g$	Standard linear form.
$\frac{(Qv)'}{Q} = -g$	Integrating factor $Q = e^{kt/m}$.
$(Qv)' = -gQ$	Quadrature form.
$Qv = -mgQ/k + c$	Method of quadrature.
$v = -mg/k + c/Q$	Velocity equation.
$v = -\frac{mg}{k} + \left(v(0) + \frac{mg}{k}\right)e^{-kt/m}$	Evaluate c and use $Q = e^{kt/m}$.

The equation $x(t) = x(0) + \int_0^t v(r)dr$ gives the last relation in (9):

$$x(t) = x(0) - \frac{mg}{k}t + \frac{m}{k}\left(v(0) + \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right).$$

Technical Details for Equation (12), $v(0) > 0$.

$v'(t) = -(k/m)v^2(t) - g$	The upward launch equation.
$u'(t) = \sqrt{\frac{kg}{m}}(1 + u^2(t))$	Change of variables $u = \sqrt{\frac{k}{mg}}\,v$.
$\frac{u'(t)}{1+u^2(t)} = -\sqrt{\frac{kg}{m}}$	A separated form.
$\arctan(u(t)) = -\sqrt{\frac{kg}{m}}t + c_1$	Quadrature.
$u(t) = \tan\left(c_1 - \sqrt{\frac{kg}{m}}t\right)$	Take the tangent of both sides.

$v(t) = \sqrt{\frac{mg}{k}} \tan\left(\sqrt{\frac{kg}{m}}(c-t)\right)$	Define $c_1 = \sqrt{\frac{kg}{m}}c$.
$x(t) = \int v(t)dt$	Quadrature method.
$= d + \frac{m}{k} \ln\left|\cos\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right|$	Integration constant d.

Technical Details for Equation (13), $v(0) < 0$.

$v'(t) = (k/m)v^2(t) - g$	Downward launch equation.
$u'(t) = \sqrt{\frac{kg}{m}}\left(u^2(t) - 1\right)$	Change of variables $u = \sqrt{\frac{k}{mg}}\,v$.
$\frac{u'(t)}{u^2(t)-1} = \sqrt{\frac{kg}{m}}$	A separated form.
$-\operatorname{arctanh}(u) = 2t\sqrt{\frac{kg}{m}} + c_1$	Quadrature method and tables.
$u = \tanh\left(\sqrt{\frac{kg}{m}}(c-t)\right)$	Define c by $\sqrt{\frac{kg}{m}}\,c = -c_1$.
$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}(c-t)\right)$	Use $v = \sqrt{\frac{mg}{k}}\,u$.
$x(t) = \int v(t)dt$	Quadrature.
$= d - \frac{m}{k} \ln\left|\cosh\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right|$	Integration constant d.

Exercises 2.6

Newton's Laws

Review of units and conversions.

1. An object weighs 100 pounds. Find its mass in slugs and kilograms.

2. An object has mass 50 kilograms. Find its mass in slugs and its weight in pounds.

3. Convert from **fps** to **mks** systems: position 1000, velocity 10, acceleration 2.

4. Derive $g = \dfrac{Gm}{R^2}$, where m is the mass of the earth and R is its radius.

Velocity and Acceleration

Find the velocity x' and acceleration x''.

5. $x(t) = 16t^2 + 100$

6. $x(t) = 16t^2 + 10t + 100$

7. $x(t) = t^3 + t + 1$

8. $x(t) = t(t-1)(t-2)$

Free Fall with Constant Gravity

Solve using the model $x''(t) = -g$, $x(0) = x_0$, $x'(0) = v_0$.

9. A brick falls from a tall building, straight down. Find the distance it fell and its speed at three seconds.

10. An iron ingot falls from a tall building, straight down. Find the distance it fell and its speed at four seconds.

11. A ball is thrown straight up from the ground with initial velocity 66 feet per second. Find its maximum height.

12. A ball is thrown straight up from the ground with initial velocity 88 feet per second. Find its maximum height.

13. An arrow is shot straight up from the ground with initial velocity 23 meters per second. Find the flight time back to the ground.

14. An arrow is shot straight up from the ground with initial velocity 44 meters per second. Find the flight time back to the ground.

15. A car travels 140 kilometers per hour. Brakes are applied, with deceleration 10 meters per second per second. Find the distance the car travels before stopping.

16. A car travels 120 kilometers per hour. Brakes are applied, with deceleration 40 feet per second per second. Find the distance the car travels before stopping.

17. An arrow is shot straight down from a height of 500 feet, with initial velocity 44 feet per second. Find the flight time to the ground and its impact speed.

18. An arrow is shot straight down from a height of 200 meters, with initial velocity 13 meters per second. Find the flight time to the ground and its impact speed.

Linear Air Resistance

Solve using the linear air resistance model $mx''(t) = -kx'(t) - mg$. An equivalent model is $x'' = -\rho x' - g$, where $\rho = k/m$ is the drag factor.

19. An arrow is shot straight up from the ground with initial velocity 23 meters per second. Find the flight time back to the ground. Assume $\rho = 0.035$.

20. An arrow is shot straight up from the ground with initial velocity 27 meters per second. Find the maximum height. Assume $\rho = 0.04$.

21. A parcel is dropped from an aircraft at $32,000$ feet. It has a parachute that opens automatically after 25 seconds. Assume drag factor $\rho = 0.16$ without the parachute and $\rho = 1.45$ with it. Find the descent time to the ground.

22. A first aid kit is dropped from a helicopter at $12,000$ feet. It has a parachute that opens automatically after 15 seconds. Assume drag factor $\rho = 0.12$ without the parachute and $\rho = 1.55$ with it. Find the impact speed with the ground.

23. A motorboat has velocity v satisfying $1100v'(t) = 6000 - 110v$, $v(0) = 0$. Find the maximum speed of the boat.

24. A motorboat has velocity v satisfying $1000v'(t) = 4000 - 90v$, $v(0) = 0$. Find the maximum speed of the boat.

25. A parachutist falls until his speed is 65 miles per hour. He opens the parachute. Assume parachute drag factor $\rho = 1.57$. About how many seconds must elapse before his speed is reduced to within 1% of terminal velocity?

26. A parachutist falls until his speed is 120 kilometers per hour. He opens the parachute. Assume drag factor $\rho = 1.51$. About how many seconds must elapse before his speed is reduced to within 2% of terminal velocity?

27. A ball is thrown straight up with initial velocity 35 miles per hour. Find the ascent time and the descent time. Assume drag factor 0.042

28. A ball is thrown straight up with initial velocity 60 kilometers per hour. Find the ascent time and the descent time. Assume drag factor 0.042

Linear Ascent and Descent Times

Find the ascent time t_1 and the descent time t_2 for the linear model $x'' = -\rho x' - g$, $x(0) = 0$, $x'(0) = v_0$ where $\rho = k/m$ is the drag factor. Unit system **fps**. Computer algebra system expected.

29. $\rho = 0.01$, $v_0 = 50$

30. $\rho = 0.015$, $v_0 = 30$

31. $\rho = 0.02$, $v_0 = 50$

32. $\rho = 0.018$, $v_0 = 30$

33. $\rho = 0.022$, $v_0 = 50$

34. $\rho = 0.025$, $v_0 = 30$

35. $\rho = 1.5$, $v_0 = 50$

36. $\rho = 1.55$, $v_0 = 30$

37. $\rho = 1.6$, $v_0 = 50$

38. $\rho = 1.65$, $v_0 = 30$

39. $\rho = 1.45$, $v_0 = 50$

40. $\rho = 1.48$, $v_0 = 30$

Nonlinear Air Resistance

Assume ascent velocity v_1 satisfies $v_1' = -\rho v_1^2 - g$. Assume descent velocity v_2 satisfies $v_2' = \rho v_2^2 - g$. Motion from the ground $x = 0$. Let t_1 and t_2 be the ascent and descent times, so that $t_1 + t_2$ is the flight time. Let $g = 9.8$, $v_1(0) = v_0$, $v_1(t_1) = v_2(t_1) = 0$, units mks. Define M = maximum height and v_f = impact velocity. Computer algebra system expected.

41. Let $\rho = 0.0012$, $v_0 = 50$. Find t_1, t_2.

42. Let $\rho = 0.0012$, $v_0 = 30$. Find t_1, t_2.

43. Let $\rho = 0.0015$, $v_0 = 50$. Find t_1, t_2.

44. Let $\rho = 0.0015$, $v_0 = 30$. Find t_1, t_2.

45. Let $\rho = 0.001$, $v_0 = 50$. Find M, v_f.

46. Let $\rho = 0.001$, $v_0 = 30$. Find M, v_f.

47. Let $\rho = 0.0014$, $v_0 = 50$. Find M, v_f.

48. Let $\rho = 0.0014$, $v_0 = 30$. Find M, v_f.

49. Find t_1, t_2, M and v_f for $\rho = 0.00152$, $v_0 = 60$.

50. Find t_1, t_2, M and v_f for $\rho = 0.00152$, $v_0 = 40$.

Terminal Velocity

Find the terminal velocity for (a) a linear air resistance $a(t) = \rho v(t)$ and (b) a nonlinear air resistance $a(t) = \rho v^2(t)$. Use the model equation $v' = a(t) - g$ and the given drag factor ρ, **mks** units.

51. $\rho = 0.15$

52. $\rho = 0.155$

53. $\rho = 0.015$

54. $\rho = 0.017$

55. $\rho = 1.5$

56. $\rho = 1.55$

57. $\rho = 2.0$

58. $\rho = 1.89$

59. $\rho = 0.001$

60. $\rho = 0.0015$

Parachutes

A skydiver has velocity v_0 and height $5,500$ feet when the parachute opens. Velocity $v(t)$ is given by (a) linear resistance model $v' = -\rho v - g$ or (b) nonlinear resistance downward model $v' = \rho v^2 - g$. Given the drag factor ρ and the parachute-open velocity v_0, compute the elapsed time until the parachutist slows to within 2% of terminal velocity. Then find the flight time from parachute open to the ground. Report two values for (a) and two values for (b).

61. $\rho = 1.446$, $v_0 = -116$ ft/sec.

62. $\rho = 1.446$, $v_0 = -84$ ft/sec.

63. $\rho = 1.2$, $v_0 = -116$ ft/sec.

64. $\rho = 1.2$, $v_0 = -84$ ft/sec.

65. $\rho = 1.01$, $v_0 = -120$ ft/sec.

66. $\rho = 1.01$, $v_0 = -60$ ft/sec.

67. $\rho = 0.95$, $v_0 = -10$ ft/sec.

68. $\rho = 0.95$, $v_0 = -5$ ft/sec.

69. $\rho = 0.8$, $v_0 = -66$ ft/sec.

70. $\rho = 0.8$, $v_0 = -33$ ft/sec.

Lunar Lander

A lunar lander falls to the moon's surface at v_0 miles per hour. The retrorockets in free space provide a deceleration effect on the lander of a miles per hour per hour. Estimate the retrorocket activation height above the surface which will give the lander zero touch-down velocity. Follow Example 2.30, page 133.

71. $v_0 = -1000$, $a = 18000$

72. $v_0 = -980$, $a = 18000$

73. $v_0 = -1000$, $a = 20000$

74. $v_0 = -1000$, $a = 19000$

75. $v_0 = -900$, $a = 18000$

76. $v_0 = -900$, $a = 20000$

77. $v_0 = -1100$, $a = 22000$

78. $v_0 = -1100$, $a = 21000$

79. $v_0 = -800$, $a = 18000$

80. $v_0 = -800$, $a = 21000$

Escape velocity

Find the escape velocity of the given planet, given the planet's mass m and radius R.

81. (Planet A) $m = 3.1 \times 10^{23}$ kilograms, $R = 2.4 \times 10^7$ meters.

82. (Mercury) $m = 3.18 \times 10^{23}$ kilograms, $R = 2.43 \times 10^6$ meters.

83. (Venus) $m = 4.88 \times 10^{24}$ kilograms, $R = 6.06 \times 10^6$ meters.

84. (Mars) $m = 6.42 \times 10^{23}$ kilograms, $R = 3.37 \times 10^6$ meters.

85. (Neptune) $m = 1.03 \times 10^{26}$ kilograms, $R = 2.21 \times 10^7$ meters.

86. (Jupiter) $m = 1.90 \times 10^{27}$ kilograms, $R = 6.99 \times 10^7$ meters.

87. (Uranus) $m = 8.68 \times 10^{25}$ kilograms, $R = 2.33 \times 10^7$ meters.

88. (Saturn) $m = 5.68 \times 10^{26}$ kilograms, $R = 5.85 \times 10^7$ meters.

Lunar Lander Experiments

89. (Lunar Lander) Verify that the variable field model for Example 2.30 gives a soft landing flight model in MKS units

$$\begin{aligned} u''(t) &= 2.2352 - \frac{c_1}{(c_2 + u(t))^2}, \\ u(0) &= 127254.1306, \\ u'(0) &= -429.1584, \end{aligned}$$

where $c_1 = 4911033599000$ and $c_2 = 1740000$.

90. (Lunar Lander: Numerical Experiment) Using a computer, solve the flight model of the previous exercise. Determine the flight time $t_0 \approx 625.6$ seconds by solving $u(t) = 0$ for t.

Details and Proofs

91. (Linear Rise Time) Using the inequality $e^u > 1 + u$ for $u > 0$, show that the ascent time t_1 in equation (17) satisfies

$$g(1 + \rho t_1) < g e^{\rho t_1} = v_0 \rho + g.$$

Conclude that $t_1 < v_0/g$, proving Lemma 2.2.

92. (Linear Maximum) Verify that Lemma 2.2 plus the inequality $x(t) < -gt^2/2 + v_0 t$ imply $x(t_1) < g v_0^2/2$. Conclude that the maximum for $\rho > 0$ is less than the maximum for $\rho = 0$.

93. (Linear Rise Time) Consider the ascent time $t_1(\rho, v_0)$ given by equation (17). Prove that

$$\frac{dt_1}{d\rho} = \frac{\ln \frac{g}{v0\rho + g}}{\rho^2} + \frac{v0}{\rho(v0\rho + g)}.$$

94. (Linear Rise Time) Consider $dt_1(\rho, v_0)/d\rho$ given in the previous exercise. Let $\rho = gx/v_0$. Show that $dt_1/d\rho < 0$ by considering properties of the function $-(x+1)\ln(x+1) + x$. Then prove Lemma 2.2.

95. (Compare Rise Times) The ascent time for nonlinear model $v' = -g - \rho v^2$ is less than the ascent time for linear model $u' = -g - \rho u$. Verify for $\rho = 1$, $g = 32$ ft/sec/sec and initial velocity 50 ft/sec.

96. (Compare Fall Times) The descent time for nonlinear model $v' = \rho v^2 - g$, $v(0) = 0$ is greater than the descent time for linear model $u' = -\rho u - g$, $u(0) = 0$. Verify for $\rho = 1$, $g = 32$ ft/sec/sec and maximum heights both 100 feet.

2.7 Logistic Equation

The 1845 work of Pierre Francois Verhulst (1804–1849), Belgian demographer and mathematician, modified the classical growth-decay equation $y' = ky$ by replacing k by $a - by$ to obtain the **logistic equation**

$$y' = (a - by)y. \tag{1}$$

The solution of the logistic equation (1) is (details on page **??**)

$$y(t) = \frac{ay(0)}{by(0) + (a - by(0))e^{-at}}. \tag{2}$$

The logistic equation (1) applies not only to human populations but also to populations of fish, animals and plants, such as yeast, mushrooms or wildflowers. The y-dependent growth rate $k = a - by$ allows the model to have a finite *limiting population* a/b. The constant $M = a/b$ is called the **carrying capacity** by demographers. Verhulst introduced the terminology *logistic curves* for the solutions of (1).

To use the Verhulst model, a demographer must supply three population counts at three different times; these values determine the constants a, b and $y(0)$ in solution (2).

Logistic Models

Below are some variants of the basic logistic model known to researchers in medicine, biology and ecology.

Limited Environment. A container of $y(t)$ flies has a *carrying capacity* of N insects. A growth-decay model $y' = Ky$ with combined growth-death rate $K = k(N - y)$ gives the model $y' = k(N - y)y$.

Spread of a Disease. The initial size of the susceptible population is N. Then y and $N - y$ are the number of infectives and susceptibles. Chance encounters spread the incurable disease at a rate proportional to the infectives and the susceptibles. The model is $y' = ky(N - y)$. The spread of rumors has an identical model.

Explosion–Extinction. The number $y(t)$ of alligators in a swamp can satisfy $y' = Ky$ where the growth-decay symbol K is proportional to $y - N$ and N is a **threshold population**. The logistic model $y' = k(y - N)y$ gives **extinction** for initial populations smaller than M and a *doomsday* population **explosion** $y(t) \to \infty$ for initial populations greater than M. This model ignores harvesting.

Constant Harvesting. The number $y(t)$ of fish in a lake can satisfy a logistic model $y' = (a - by)y - h$, provided fish are **harvested** at a constant rate $h > 0$. This model can be written as $y' = k(M - y)(y - N)$ for small harvesting rates h, where M is the *carrying capacity* and N is the *threshold population*.

Variable Harvesting. The special logistic model $y' = (a - by)y - hy$ results by **harvesting** at a non-constant rate proportional to the present population y. The effect is to decrease the natural growth rate a by the constant amount h in the standard logistic model.

Restocking. The equation $y' = (a - by)y - h\sin(\omega t)$ models a logistic population that is periodically harvested and restocked with maximal rate $h > 0$. The period is $T = 2\pi/\omega$. The equation might model extinction for stocks less than some threshold population y_0, and otherwise a stable population that oscillates about an ideal carrying capacity a/b with period T.

Example 2.33 (Limited Environment)
Find the equilibrium solutions and the carrying capacity for the logistic equation $P' = 0.04(2 - 3P)P$. Then solve the equation.

Solution: The given differential equation can be written as the separable autonomous equation $P' = G(P)$ where $G(y) = 0.04(2 - 3P)P$. Equilibria are obtained as $P = 0$ and $P = 2/3$, by solving the equation $G(P) = 0.04(2 - 3P)P = 0$. The carrying capacity is the stable equilibrium $P = 2/3$; here we used the derivative $G'(P) = 0.04(2 - 6P)$ and evaluations $G'(0) > 0$, $G'(2/3) < 0$ to determine that $P = 2/3$ is a stable sink or funnel.

Example 2.34 (Spread of a Disease)
Find the number of infectives, the number of susceptibles and the rate of spread of the disease at $t = 4$ months for logistic model $y' = \frac{15}{1000}(10000 - y)y$, $y(0) = 200$.

Solution:

Answer: By month 4, about 8900 were infected, about 1100 were not infected and the disease was spreading at a rate of about 1450 per month.

Details: Write the differential equation in the form $y' = (a - by)y$ with $a = 15/10$, $b = \frac{15}{100000}$. Let $M = a/b = 10000$. The number of infectives after 4 months is $y(4)$ and the number of susceptibles is $M - y(4)$. The rate of spread of the disease is $y'(4)$.

Using formula (2) with $a = 15/10$, $b = \frac{15}{100000}$ and $y(0) = 200$ gives

$$y(t) = \frac{10000}{1 + 49e^{-3t/2}}.$$

Then the number of infectives at $t = 4$ is $y(4) = 8916.956640$. The number of susceptibles is $M - y(4) = 1083.043360$. The rate of spread of the disease is $y'(4) = 1448.617600$.

Example 2.35 (Explosion-Extinction)
Classify the model as **explosion** or **extinction**: $y' = 2(y - 100)y$, $y(0) = 200$.

Solution: Let $G(y) = 2(y-100)y$, then $G(y) = 0$ exactly for equilibria $y = 100$ and $y = 0$, at which $G'(y) = 4y - 200$ satisfies $G'(200) > 0$, $G'(0) < 0$. The initial value $y(0) = 200$ is above the equilibrium $y = 100$. Because $y = 100$ is a source, then $y \to \infty$, which implies the model is **explosion**.

A second, direct analysis can be made from the differential equation $y' = 2(y-100)y$: $y'(0) = 2(200-100)200 > 0$ means y increases from 200, causing $y \to \infty$ and explosion.

Example 2.36 (Constant Harvesting)
Find the carrying capacity M and the threshold population N for the harvesting equation $P' = (3-2P)P - 1$.

Solution: Carrying Capacity $M = 1$, Threshold Population $N = 1/2$.

Let $f(P) = -2(P-1)(P-1/2)$, which is the factored form of $(3-2P)P-1$, the right side of $P' = (3-2P)P-1$. Solve equation $f(P) = 0$ for $P = 1$, $P = 1/2$, the equilibrium solutions.

Requirements for **carrying capacity** M and **threshold population** N:

1. M and N are equilibrium solutions
2. M is a stable sink, a funnel in the phase portrait
3. If $P(0) > N$, then $\lim_{t\to\infty} P(t) = M$.

Stability test **??** on page **??** applies: if $f(M) = 0$ and $f'(M) < 0$, then equilibrium $P = M$ is a stable sink (a funnel). Calculate $G'(P) = 3 - 4P$. Test $P = 1$ and $P = 1/2$: $P = 1$ is a stable sink. Define $M = 1$, $N = 1/2$. Requirements **1** and **2** hold. To verify limit requirement **3**, write $G(P) = -2(P-1)(P-1/2) = -2(P-M)(P-N)$ and make a phase line diagram. Then use the **Three Drawing Rules** page **??**.

Example 2.37 (Variable Harvesting)
Re-model the variable harvesting equation $P' = (3-2P)P - P$ as $y' = (a-by)y$ and solve the equation by formula (2), page 142.

Solution: The equation is rewritten as $P' = 2P - 2P^2 = (2-2P)P$. This has the form of $y' = (a-by)y$ where $a = b = 2$. Then (2) implies

$$P(t) = \frac{2P_0}{2P_0 + (2-2P_0)e^{-2t}}$$

which simplifies to

$$P(t) = \frac{P_0}{P_0 + (1-P_0)e^{-2t}}.$$

Example 2.38 (Restocking)
Make a direction field graphic by computer for the restocking equation $P' = (1-P)P - 2\sin(2\pi t)$. Using the graphic, report (a) an estimate for the carrying capacity C and (b) approximations for the amplitude A and period T of a periodic solution which oscillates about $P = C$.

Solution: The computer algebra system `maple` is used with the code below to make Figure 6. An essential feature of the `maple` code is plotting of multiple solution curves. For instance, `[P(0)=1.3]` in the list `ics` of initial conditions causes the solution to the problem $P' = (1-P)P - 2\sin(2\pi t)$, $P(0) = 1.3$ to be added to the graphic.

The resulting graphic, which contains 13 solution curves, shows that all solution curves limit as $t \to \infty$ to what appears to be a unique periodic solution.

Using features of the `maple` interface, it is possible to determine by experiment estimates for the maxima $M = 1.26$ and minima $m = 0.64$ of the apparent periodic solution. Then (a) $C = (M+m)/2 = 0.95$, (b) $A = (M-m)/2 = 0.31$ and $T = 1$. The experimentally obtained period $T = 1$ matches the period of the term $-2\sin(2\pi t)$.

```
de:=diff(P(t),t)=(1-P(t))*P(t)-2*sin(2*Pi* t);
ics:=[[P(0)=1.4],[P(0)=1.3],[P(0)=1.2],[P(0)=1.1],[P(0)=0.1],
[P(0)=0.2],[P(0)=0.3],[P(0)=0.4],[P(0)=0.5],[P(0)=0.6],
[P(0)=0.7],[P(0)=0.8],[P(0)=0.9]];
opts:=stepsize=0.05,arrows=none:
DEtools[DEplot](de,P(t),t=-3..12,P=-0.1..1.5,ics,opts);
```

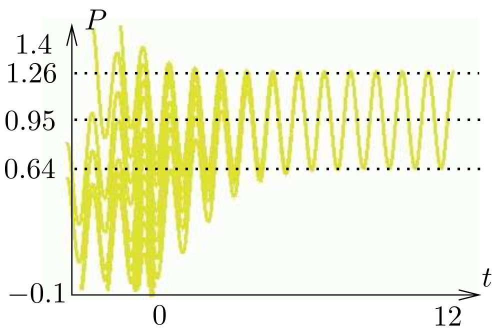


Figure 6. Solutions of $P' = (1-P)P - 2\sin(2\pi t)$.

The maximum is 1.26.
The minimum is 0.64.
Oscillation is about the line $P = 0.95$ with period 1.

Exercises 2.7

Limited Environment

Find the equilibrium solutions and the carrying capacity for each logistic equation.

1. $P' = 0.01(2-3P)P$

2. $P' = 0.2P - 3.5P^2$

3. $y' = 0.01(-3-2y)y$

4. $y' = -0.3y - 4y^2$

5. $u' = 30u + 4u^2$

6. $u' = 10u + 3u^2$

7. $w' = 2(2-5w)w$

8. $w' = -2(3-7w)w$

9. $Q' = Q^2 - 3(Q-2)Q$

10. $Q' = -Q^2 - 2(Q-3)Q$

Spread of a Disease

In each model, find the number of infectives and then the number of susceptibles at $t = 2$ months. Follow Example 2.34, page 143. A calculator is required for approximations.

11. $y' = (5/10 - 3y/100000)y$, $y(0) = 100$.

12. $y' = (13/10 - 3y/100000)y$, $y(0) = 200$.

13. $y' = (1/2 - 12y/100000)y$, $y(0) = 200$.

14. $y' = (15/10 - 4y/100000)y$, $y(0) = 100$.

15. $P' = (1/5 - 3P/100000)P$, $P(0) = 500$.

16. $P' = (5/10 - 3P/100000)P$, $P(0) = 600$.

17. $10P' = 2P - 5P^2/10000$, $P(0) = 500$.

18. $P' = 3P - 8P^2$, $P(0) = 10$.

Explosion–Extinction

Classify the model as **explosion** or **extinction**.

19. $y' = 2(y - 100)y$, $y(0) = 200$

20. $y' = 2(y - 200)y$, $y(0) = 300$

21. $y' = -100y + 250y^2$, $y(0) = 200$

22. $y' = -50y + 3y^2$, $y(0) = 25$

23. $y' = -60y + 70y^2$, $y(0) = 30$

24. $y' = -540y + 70y^2$, $y(0) = 30$

25. $y' = -16y + 12y^2$, $y(0) = 1$

26. $y' = -8y + 12y^2$, $y(0) = 1/2$

Constant Harvesting

Find the carrying capacity N and the threshold population M.

27. $P' = (3 - 2P)P - 1$

28. $P' = (4 - 3P)P - 1$

29. $P' = (5 - 4P)P - 1$

30. $P' = (6 - 5P)P - 1$

31. $P' = (6 - 3P)P - 1$

32. $P' = (6 - 4P)P - 1$

33. $P' = (8 - 5P)P - 2$

34. $P' = (8 - 3P)P - 2$

35. $P' = (9 - 4P)P - 2$

36. $P' = (10 - P)P - 2$

Variable Harvesting

Re-model the variable harvesting equation as $y' = (a - by)y$ and solve the equation by logistic solution (2) on page 142.

37. $P' = (3 - 2P)P - P$

38. $P' = (4 - 3P)P - P$

39. $P' = (5 - 4P)P - P$

40. $P' = (6 - 5P)P - P$

41. $P' = (6 - 3P)P - P$

42. $P' = (6 - 4P)P - P$

43. $P' = (8 - 5P)P - 2P$

44. $P' = (8 - 3P)P - 2P$

45. $P' = (9 - 4P)P - 2P$

46. $P' = (10 - P)P - 2P$

Restocking

Make a direction field graphic by computer following Example 2.38. Using the graphic, report (a) an estimate for the carrying capacity C and (b) approximations for the amplitude A and period T of a periodic solution which oscillates about $P = C$.

47. $P' = (2 - P)P - \sin(\pi t/3)$

48. $P' = (2 - P)P - \sin(\pi t/5)$

49. $P' = (2 - P)P - \sin(\pi t/7)$

50. $P' = (2 - P)P - \sin(\pi t/8)$

Richard Function

Ideas of L. von Bertalanffy (1934), A. Pütter (1920) and Verhulst were used by F. J. Richards (1957) to define a sigmoid function $Y(t)$ which generalizes the logistic function. It is suited for data-fitting models, for example forestry, tumor growth and stock-production problems. The Richard function is

$$Y(t) = A + \frac{K - A}{(1 + Qe^{-B(t-M)})^{1/\nu}},$$

where Y = weight, height, size, amount, etc., and t = time.

51. Differentiate for $\alpha > 0$, $\nu > 0$, the specialized Richard function

$$Y(t) = \frac{K}{(1 + Qe^{-\alpha\nu(t-t_0)})^{1/\nu}}$$

to obtain the sigmoid differential equation

$$Y'(t) = \alpha\left(1 - \left(\frac{Y}{K}\right)^{\nu}\right)Y.$$

The relation $Y(t_0) = \frac{K}{(1+Q)^{1/\nu}}$ implies $Q = -1 + \left(\frac{K}{Y(t_0)}\right)^{\nu}$.

52. Solve the differential equation $Y'(t) = \alpha\left(1 - \left(\frac{Y}{K}\right)^{\nu}\right)Y$ by means of the substitution $w = (Y/K)^{\nu}$, which gives a familiar logistic equation $w' = \alpha\nu(1 - w)w$.

2.8 Science and Engineering Applications

Assembled here are some classical applications of first order differential equations to problems of science and engineering.

Draining a Tank

Investigated here is a tank of water with orifice at the bottom emptying due to gravity; see Figure 7. The analysis applies to tanks of any geometrical shape.

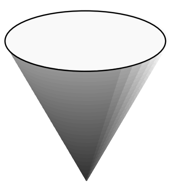


Figure 7. Draining a tank.
A tank empties from an orifice at the bottom. The fluid fills the tank to height y above the orifice, and it drains due to gravity.

Evangelista Torricelli (1608-1647), inventor of the **barometer**, investigated this physical problem using Newton's laws, obtaining the result in Lemma 2.8, proof on page 157.

Lemma 2.8 (Torricelli) A droplet falling freely from height h in a gravitational field with constant g arrives at the orifice with speed $\sqrt{2gh}$.

Tank Geometry. A simple but useful tank geometry can be constructed using *washers* of area $A(y)$, where y is the height above the orifice; see Figure 8.

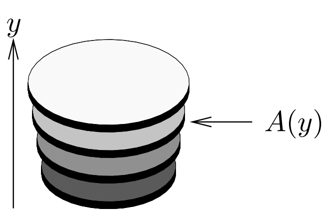


Figure 8. A tank constructed from washers.

Then the method of cross-sections in calculus implies that the *volume* $V(h)$ of the tank at height h is given by

$$(1) \qquad V(h) = \int_0^h A(y)dy, \quad \frac{dV}{dh} = A(h).$$

Torricelli's Equation. Torricelli's lemma applied to the tank fluid height $y(t)$ at time t implies, by matching drain rates at the orifice (see *Technical Details* page 156), that

$$(2) \qquad \frac{d}{dt}\left(V(y(t))\right) = -k\sqrt{y(t)}$$

for some proportionality constant $k > 0$. The *chain rule* gives the separable differential equation $V'(y(t))y'(t) = -k\sqrt{y(t)}$, or equivalently (see page 157), in terms of the **cross-sectional area** $A(y) = V'(y)$,

$$(3) \qquad y'(t) = -k\frac{\sqrt{y(t)}}{A(y(t))}.$$

Typical of the physical literature, the requirement $y(t) \geq 0$ is omitted in the model, but assumed implicitly. The model itself **exhibits non-uniqueness**: the tank can be drained hours ago or at instant $t = 0$ and result still in the solution $y(t) = 0$, interpreted as fluid height zero.

Stefan's Law

Heat energy can be transferred by **conduction**, **convection** or **radiation**. The following illustrations suffice to distinguish the three types of heat transfer.

Conduction. A soup spoon handle gains heat from the soup by exchange of kinetic energy at a molecular level.

Convection. A hot water radiator heats a room largely by *convection currents*, which move heated air upwards and denser cold air downwards to the radiator. In linear applications, **Newton's cooling law** applies.

Radiation. A car seat heated by the sun gets the heat energy from electromagnetic waves, which carry energy from the sun to the earth.

The rate at which an object emits or absorbs **radiant energy** is given by **Stefan's radiation law**

$$P = kT^4.$$

The symbol P is the power in watts (joules per second), k is a constant proportional to the surface area of the object and T is the temperature of the object in degrees Kelvin. Use $K = C + 273.15$ to convert Celsius to Kelvin.[6] The constant

[6] USA Fahrenheit F is Celsius $C = G + G/10 + G/100$, correct to 0.49 C for -40 to 120 F. Value $G = (F - 32)/2$ is a common **guess**. The idea uses $1/9 = 0.111\ldots$. **Example for** $F = 79$: Compute guess $G = (79 - 32)/2 = 23.5$. Then $C = 23.5 + 2.35 + 0.235 = 26.085$. The numbers added to G are decimal point shifts.

k in Stefan's law is decomposed as $k = \sigma A \mathcal{E}$. Here, $\sigma = 5.6696 \times 10^{-8} K^{-4}$ Watts per square meter (K=Kelvin), A is the surface area of the body in square meters and $\mathcal{E}$ is the **emissivity**, which is a constant between 0 and 1 depending on properties of the surface.

Constant room temperature. Suppose that a person with skin temperature T Kelvin sits unclothed in a room in which the thermometer reads T_0 Kelvin. The net heat flux P_{net} in joules per second (watts) is given by

$$P_{\text{net}} = k(T^4 - T_0^4). \tag{4}$$

If T and T_0 are constant, then $Q = kt(T^4 - T_0^4)$ can be used to estimate the total heat loss or gain in joules for a time period t. To illustrate, if the wall thermometer reads 20° Celsius, then $T_0 = 20 + 273.15$. Assume $A = 1.5$ square meters, $\mathcal{E} = 0.9$ and skin temperature 33° Celsius or $T = 33 + 273.15$. The total heat loss in 10 minutes is $Q = (10(60))(5.6696 \times 10^{-8})(1.5)(0.9)(305.15^4 - 293.15^4) = 64282$ joules. Over one hour, the total heat radiated is approximately $385,691$ joules, which is close to the total energy provided by a 6 ounce soft drink.[7]

Time-varying room temperature. Suppose that a person with skin temperature T degrees Kelvin sits unclothed in a room. Assume the thermometer initially reads 15° Celsius and then rises to 24° Celsius after t_1 seconds. The function $T_0(t)$ has values $T_0(0) = 15 + 273.15$ and $T_0(t_1) = 24 + 273.15$. In a possible physical setting, $T_0(t)$ reflects the reaction to the heating and cooling system, which is generally oscillatory about the thermostat setting. If the thermostat is off, then it is reasonable to assume a linear model $T_0(t) = at + b$, with $a = (T_0(t_1) - T_0(0))/t_1$, $b = T_0(0)$.

To compute the total heat radiated from the person's skin, we use the time-varying equation

$$\frac{dQ}{dt} = k(T^4 - T_0(t)^4). \tag{5}$$

The solution to (5) with $Q(0) = 0$ is formally given by the quadrature formula

$$Q(t) = k \int_0^t (T^4 - T_0(r)^4) dr. \tag{6}$$

For the case of a linear model $T_0(t) = at + b$, the total number of joules radiated from the person's skin is found by integrating (6), giving

$$Q(t_1) = kT^4 t_1 + k\frac{b^5 - (at_1 + b)^5}{5a}.$$

[7] American soft drinks are packaged in 12-ounce cans, twice the quantity cited. One calorie is *defined* to be 4.186 joules and one food **Calorie** is 1000 calories (a kilo-calorie) or 4186 joules. A boxed apple juice is about 6 ounces or 0.2 liters. Juice provides about 400 thousand joules in 0.2 liters. Product labels with 96 Calories mean 96 kilo-calories; it converts to $96(1000)(4.186) = 401,856$ joules.

Tsunami

A **seismic sea wave** due to an earthquake under the sea or some other natural event, called a **tsunami**, creates a wave on the surface of the ocean. The wave may have a height of less than 1 meter. These waves can have a very large wavelength, up to several hundred miles, depending upon the depth of the water where they were formed. The period is often more than one hour with wave velocity near 700 kilometers per hour. These waves contain a huge amount of energy. Their height increases as they crash upon the shore, sometimes to 30 meters high or more, depending upon water depth and underwater surface features. In the year 1737, a wave estimated to be 64 meters high hit Cape Lopatka, Kamchatka, in northeast Russia. The largest Tsunami ever recorded occurred in July of 1958 in Lituya Bay, Alaska, when a huge rock and ice fall caused water to surge up to 500 meters. For additional material on earthquakes, in particular the Sumatra and Chile earthquakes and resultant Tsunamis, see Chapter 11, **Systems of Differential Equations**.

Wave shape. A simplistic model for the shape $y(x)$ of a tsunami in the open sea is the differential equation [**?**, p. 81]

$$(y')^2 = 4y^2 - 2y^3. \tag{7}$$

This equation gives the *profile* $y(x)$ of one side of the 3D-wave, by cutting the 3D object with an xy-plane.

Equilibrium solutions. They are $y = 0$ and $y = 2$, corresponding to **no wave** and a **wall of water** 2 units above the ocean surface. There are *no solutions* for $y > 2$, because the two sides of (7) have in this case different signs.

Non-equilibrium solutions. They are given by

$$y(x) = 2 - 2\tanh^2(x + c). \tag{8}$$

The initial height of the wave is related to the parameter c by $y(0) = 2 - 2\tanh^2(c)$. Only initial heights $0 < y(0) < 2$ are physically significant. Due to the property $\lim_{u\to\infty} \tanh(u) = 1$ of the hyperbolic tangent, the wave height starts at $y(0)$ and quickly decreases to zero (sea level), as is evident from Figure 9.

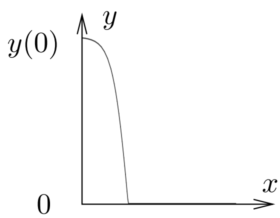


Figure 9. A tsunami profile.

Non-uniqueness. When $y(x_0) = 2$ for some $x = x_0$, then also $y'(x_0) = 0$, and this allows non-uniqueness of the solution y. An interesting solution different from equation (8) is the piecewise function

$$y(x) = \begin{cases} 2 - 2\tanh^2(x - x_0) & x > x_0, \\ 2 & x \le x_0. \end{cases} \tag{9}$$

This shape is an approximation to observed waves, in which the usual crest of the wave has been flattened. See Figure 12 on page 155.

Gompertz Tumor Equation

Researchers in tumor growth have shown that for some solid tumors the volume $V(t)$ of dividing cells at time t approaches a limiting volume M, even though the tumor volume may increase by 1000 fold. Gompertz is credited with an equation which fits the growth cycle of some solid tumors; the **Gompertzian relation** is

$$V(t) = V_0 e^{\frac{a}{b}\left(1-e^{-bt}\right)}. \tag{10}$$

The relation says that the doubling time for the total solid tumor volume *increases with time.* In contrast to a simple exponential model, which has a fixed doubling time and no volume limit, the limiting volume in the Gompertz model (10) is $M = V_0 e^{a/b}$.

Experts suggest to verify from Gompertz's relation (10) the formula

$$V' = ae^{-bt}V,$$

and then use this differential equation to argue why the tumor volume V approaches a limiting value M with a necrotic core; see *Technical Details for* (11), page 157.

A different approach is to make the substitution $y = V/V_0$ to obtain the differential equation

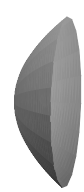
$$y' = (a - b\ln y)y, \tag{11}$$

which is almost a logistic equation, sometimes called the **Gompertz equation**. For details, see page 157. In analogy with logistic theory, low volume tumors should grow exponentially with rate a and then slow down like a population that is approaching the carrying capacity.

The exact mechanism for the slowing of tumor growth can be debated. One view is that the number of reproductive cells is related to available oxygen and nutrients present only near the surface of the tumor, hence this number decreases with time as the necrotic core grows in size.

Parabolic Mirror

Overhead projectors might use a high-intensity lamp located near a silvered reflector to provide a nearly parallel light source of high brightness. It is called a **parabolic mirror** because the surface of revolution is formed from a parabola, a fact which will be justified below.



The requirement is a shape $x = g(y)$ such that a light beam emanating from $C(0, 0)$ reflects at point on the curve into a second beam parallel to the x-axis; see Figure 10. The **optical law of reflection** implies that the angle of incidence equals the angle of reflection, the straight reference line being the tangent to the curve $x = g(y)$.

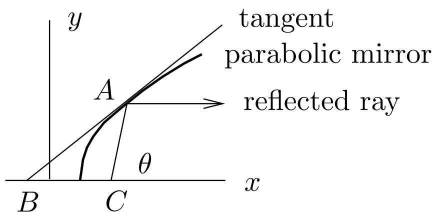


Figure 10. A parabolic mirror.

Symmetry suggests the restriction $y \geq 0$ will suffice to determine the shape. The assumption $y(0) = 1$ determines the y-axis scale.

The mirror shape $x = g(y)$ is shown in *Technical Details* page 157 to satisfy

$$\frac{dx}{dy} = \frac{x + \sqrt{x^2 + y^2}}{y}, \quad x(1) = 0. \tag{12}$$

This equation is equivalent for $y > 0$ to the separable equation $du/dy = \sqrt{u^2 + 1}$, $u(1) = 0$; see *Technical Details* page 157. Solving the separable equation (see page 157) gives the *parabola*

$$2x + 1 = y^2. \tag{13}$$

Logarithmic Spiral

The polar curve

$$r = r_0 e^{k\theta} \tag{14}$$

is called a **logarithmic spiral**. In equation (14), symbols r, θ are polar variables and r_0, k are constants. It will be shown that a logarithmic spiral has the following geometric characterization.

> A logarithmic spiral cuts each radial line from the origin at a constant angle.

The background required is the polar coordinate calculus formula

$$\tan(\alpha - \theta) = r\frac{d\theta}{dr} \tag{15}$$

where α is the angle between the x-axis and the tangent line at (r, θ); see Technical Details page 158. The angle α can also be defined from the calculus formula $\tan\alpha = dy/dx$.

The angle ϕ which a polar curve cuts a radial line is $\phi = \alpha - \theta$. By equation (15), the polar curve must satisfy the polar differential equation

$$r\frac{d\theta}{dr} = \frac{1}{k}$$

for constant $k = 1/\tan\phi$. This differential equation is separable with separated form

$$kd\theta = \frac{dr}{r}.$$

Solving gives $k\theta = \ln r + c$ or equivalently $r = r_0 e^{k\theta}$, for $c = -\ln r_0$. Hence equation (14) holds. All steps are reversible, therefore a logarithmic spiral is characterized by the geometrical description given above.

Examples

Example 2.39 (Conical Tank)
A conical tank with xy-projection given in Figure 11 is realized by rotation about the y-axis. An orifice at $x = y = 0$ is created at time $t = 0$. Find an approximation for the drain time and the time to empty the tank to half-volume, given 10% drains in 20 seconds.

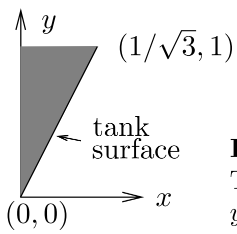


Figure 11. Conical tank xy-projection.
The tank is obtained by rotation of the shaded triangle about the y-axis. The cone has height 1.

Solution: The answers are approximately 238 seconds and 104 seconds. The incorrect drain time estimate of ten times the given 20 seconds is wrong by 19 percent. Doubling the half-volume time to find the drain time is equally invalid (both 200 and 208 are incorrect).

Tank cross-section $A(y)$. From Figure 11, the line segment along the tank surface has equation $y = \sqrt{3}x$; the equation was found from the two points $(0,0)$ and $(1/\sqrt{3}, 1)$ using the point-slope form of a line. A washer then has area $A(y) = \pi x^2$ or $A(y) = \pi y^2/3$.

Tank half-volume V_h. The half-volume is given by

$$\begin{aligned} V_h &= \frac{1}{2}V(1) && \text{Full volume is } V(1). \\ &= \frac{1}{2}\int_0^1 A(y)dy && \text{Apply } V(h) = \textstyle\int_0^h A(y)dy. \\ &= \frac{\pi}{18} && \text{Evaluate integral, } A(y) = \pi y^2/3. \end{aligned}$$

Torricelli's equation. The differential equation (3) becomes

$$y'(t) = -\frac{3k}{\pi\sqrt{y^3(t)}}, \quad y(0) = 1, \tag{16}$$

with k to be determined. The solution by separation of variables is

$$y(t) = \left(1 - \frac{15k}{2\pi}\,t\right)^{2/5}. \tag{17}$$

The details:

$y^{3/2}y' = -\frac{3k}{\pi}$	Separated form.
$\frac{2}{5}y^{5/2} = -\frac{3kt}{\pi} + C$	Integrate both sides.
$y^{5/2} = -\frac{15kt}{2\pi} + 1$	Isolate y, then use $y(0) = 1$.
$y = \left(1 - \frac{15kt}{2\pi}\right)^{2/5}$	Take roots.

Determination of k. Let $V_0 = V(1)/10$ be the volume drained after $t_0 = 20$ seconds. Then t_0, V_0 and k satisfy

$V_0 = V(1) - V(y(t_0))$	Volume from height $y(t_0)$ to $y(0)$.
$= \frac{\pi}{9}\left(1 - y^3(t_0)\right)$	
$= \frac{\pi}{9}\left(1 - \left(1 - \frac{15k}{2\pi}t_0\right)^{6/5}\right)$	Substitute (17).
$k = \frac{2\pi}{15t_0}\left(1 - \left(1 - \frac{9V_0}{\pi}\right)^{5/6}\right)$	Solve for k.
$= \frac{2\pi}{15t_0}\left(1 - 0.9^{5/6}\right)$	

Drain times. The volume is $V_h = \pi/18$ at time t_1 given by $\pi/18 = V(t_1)$ or in detail $\pi/18 = \pi y^3(t_1)/9$. This requirement simplifies to $y^3(t_1) = 1/2$. Then

$\left(1 - \frac{15kt_1}{2\pi}\right)^{6/5} = \frac{1}{2}$	Insert the formula for $y(t)$.
$1 - \frac{15kt_1}{2\pi} = \frac{1}{2^{5/6}}$	Take the $5/6$ power of both sides.
$t_1 = \frac{2\pi}{15k}\left(1 - 2^{-5/6}\right)$	Solve for t_1.
$= t_0\frac{1 - 2^{-5/6}}{1 - 0.9^{5/6}}$	Insert the formula for k.
≈ 104.4	Half-tank drain time in seconds.

The drain time t_2 for the full tank is not twice this answer but $t_2 \approx 2.28t_1$ or 237.9 seconds. The result is justified by solving for t_2 in the equation $y(t_2) = 0$, which gives $t_2 = \frac{2\pi}{15k} = \frac{t_1}{1 - 2^{-5/6}} = \frac{t_0}{1 - 0.9^{5/6}}$.

Example 2.40 (Stefan's Law)

An inmate sits unclothed in a room with skin temperature 33° Celsius. The Celsius room temperature is given by $C(r) = 14 + 11r/20$ for r in minutes. Assume in Stefan's law $k = \sigma A\mathcal{E} = 6.349952 \times 10^{-8}$. Find the number of joules lost through the skin in the first 20 minutes.

Solution: The theory implies that the answer is $Q(t_1)$ where $t_1 = (20)(60)$ is in seconds and $Q' = kT^4 - kT_0^4$. Equation $r = t/60$ converts seconds t to minutes r. Let $T = 33 + 273.15$ and $T_0(t) = C(t/60) + 273.15$. Then

$$Q(t_1) = k \int_0^{t_1} (T^4 - (T_0(t))^4) dt \approx 110,0095 \quad \text{joules.}$$

Example 2.41 (Tsunami)

Find a piecewise solution, which represents a Tsunami wave profile, similar to equation (9), on page 150. Graph the solution on $|x - x_0| \le 2$.

$$(y')^2 = 8y^2 - 4y^3, \quad x_0 = 1.$$

Solution: Equilibrium solutions $y = 0$ and $y = 2$ are found from the equation $8y^2 - 4y^3 = 0$, which has factored form $4y^2(2 - y) = 0$.

Non-equilibrium solutions with $y' \ge 0$ and $0 < y < 2$ satisfy the first order differential equation

$$y' = 2y\sqrt{2 - y}.$$

Consulting a computer algebra system gives the solution

$$y(x) = 2 - 2\tanh^2(\sqrt{2}(x - x_0)).$$

Treating $-y' = 2y\sqrt{2 - y}$ similarly results in exactly the same solution.

Hand solution. Start with the substitution $u = \sqrt{2 - y}$. Then $u^2 = 2 - y$ and $2uu' = -y' = -2yu = -2(2 - u^2)u$, giving the separable equation $u' = u^2 - 2$. Reformulate it as $u' = (u - a)(u + a)$ where $a = \sqrt{2}$. Normal partial fraction methods apply to find an implicit solution involving the inverse hyperbolic tangent. Some integral tables tabulate the integral involved, therefore partial fractions can be technically avoided. Solving for u in the implicit equation gives the hyperbolic tangent solution $u = \sqrt{2}\tanh(\sqrt{2}(x - x_0))$. Then $y = 2 - u^2$ produces the answer reported above. The piecewise solution, which represents an ocean Tsunami wave, is given by

$$y(x) = \begin{cases} 2 & x \le 1, & \textbf{back-wave} \\ 2 - 2\tanh^2(\sqrt{2}(x - 1)) & 1 < x < \infty. & \textbf{wave front} \end{cases}$$

The figure can be made by hand. A computer algebra graphic appears in Figure 12, with `maple` code as indicated.

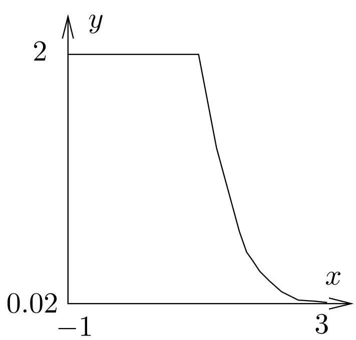


Figure 12. Tsunami wave profile. The back-wave is at height 2. The front wave has height given by the hyperbolic tangent term, which approaches zero as $x \to \infty$. The `maple` code:

```
g:=x->2-2*tanh(sqrt(2)*(x-1))^2;
f:=x->piecewise(x<1,2,g(x));
plot(f,-1..3);
```

Example 2.42 (Gompertz Equation)
First, solve the Gompertz tumor equation, and then make (a) a phase line diagram and (b) a direction field.

$$y' = (8 - 2\ln y)y.$$

Solution: The only equilibrium solution computed from $G(y) \equiv (8 - 2\ln y)y = 0$ is $y = e^4 \approx 54.598$, because $y = 0$ is not in the domain of the right side of the differential equation.

Non-equilibrium solutions require integration of $1/G(y)$. Evaluation using a computer algebra system gives the implicit solution

$$-\frac{1}{2}\ln(8 - 2\ln(y)) = x + c.$$

Solving this equation for y in terms of x results in the explicit solution

$$y(x) = c_1 e^{-\frac{1}{2}e^{-2x}}, \quad c_1 = e^{4-\frac{1}{2}e^{-2c}}.$$

The `maple` code for these two independent tasks appears below.

```
p:=int(1/((8-2*ln(y))*y),y);
solve(p=x+c,y);
```

The phase line diagram in Figure 13 requires the equilibrium $y = e^4$ and formulas $G(y) = (8 - 2\ln y)y$, $G'(y) = 8 - 2\ln y - 2$. Then $G'(e^4) = -2$ implies G changes sign from positive to negative at $y = e^4$, making $y = e^4$ a stable sink or funnel.

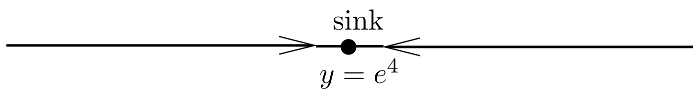


Figure 13. Gompertz phase line diagram.
The unique equilibrium at $y = e^4$ is a stable sink.

A computer-generated direction field appears in Figure 14, using the following `maple` code. Visible is the funnel structure at the equilibrium point.

```
de:=diff(y(x),x)=y(x)*(8-2*ln(y(x)));
with(DEtools):
DEplot(de,y(x),x=0..4,y=1..70);
```

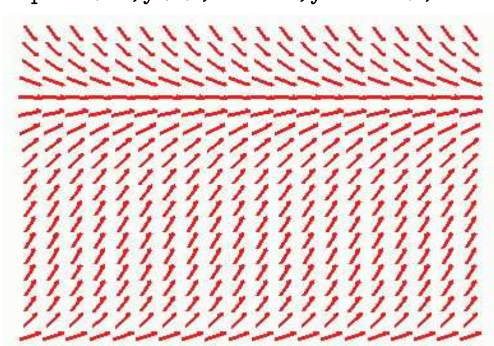


Figure 14. A Gompertz direction field.

Details and Proofs

Technical Details for (2): The derivation of $\frac{d}{dt}(V(y(t))) = -k\sqrt{y(t)}$ uses Torricelli's speed formula $|v| = \sqrt{2gy(t)}$. The volume change in the tank for an orifice of cross-sectional area a is $-av$. Therefore, $dV(y(t))/dt = -a\sqrt{2gy(t)}$. Succinctly, $dV(y(t))/dt = -k\sqrt{y(t)}$. This completes the verification.

Technical Details for (3): The equation $y'(t) = -k\dfrac{\sqrt{y(t)}}{A(y(t))}$ is equivalent to equation $A(y(t))\,y'(t) = -k\sqrt{y(t)}$. Equation $dV(y(t))/dt = V'(y(t))y'(t)$ obtained by the chain rule, definition $A(y) = V'(y)$, and equation (2) give result (3).

Technical Details for (2.8): To be verified is the Torricelli orifice equation $|v| = \sqrt{2gh}$ for the speed $|v|$ of a droplet falling from height h. Let's view the droplet as a point mass m located at the droplet's centroid. The distance $x(t)$ from the droplet to the orifice satisfies a falling body model $mx''(t) = -mg$. The model has solution $x(t) = -gt^2/2 + x(0)$, because $x'(0) = 0$. The droplet arrives at the orifice in time t given by $x(t) = 0$. Because $x(0) = h$, then $t = \sqrt{2h/g}$. The velocity v at this time is $v = x'(t) = -gt = -\sqrt{2gh}$. A technically precise derivation can be done using kinetic and potential energy relations; some researchers prefer energy method derivations for Torricelli's law. Formulas for the orifice speed depend upon the shape and size of the orifice. For common drilled holes, the speed is a constant multiple $c\sqrt{2gh}$, where $0 < c < 1$.

Technical Details for (11): Assume $V = V_0 e^{\mu(t)}$ and $\mu(t) = a(1 - e^{-bt})/b$. Then $\mu' = ae^{-bt}$ and

$$
\begin{aligned}
V' &= V_0\mu'(t)e^{\mu(t)} && \text{Calculus rule } (e^u)' = u'e^u. \\
&= \mu'(t)V && \text{Use } V = V_0 e^{\mu(t)}. \\
&= ae^{-bt}V && \text{Use } \mu' = ae^{-bt}.
\end{aligned}
$$

The equation $V' = ae^{-bt}V$ is a growth equation $y' = ky$ where k decreases with time, causing the doubling time to increase. One biological explanation for the increase in the mean generation time of the tumor cells is aging of the reproducing cells, causing a slower dividing time. The correctness of this explanation is debatable.

Let $y = V/V_0$. Then

$$
\begin{aligned}
\frac{y'}{y} &= \frac{V'}{V} && \text{The factor } 1/V_0 \text{ cancels.} \\
&= ae^{-bt} && \text{Differential equation } V' = ae^{-bt}V \text{ applied.} \\
&= a - b\mu(t) && \text{Use } \mu(t) = a(1 - e^{-bt})/b. \\
&= a - b\ln(V/V_0) && \text{Take logs across } V/V_0 = e^{\mu(t)} \text{ to find } \mu(t). \\
&= a - b\ln y && \text{Use } y = V/V_0.
\end{aligned}
$$

Hence $y' = (a - b\ln y)y$. When $V \approx V_0$, then $y \approx 1$ and the growth rate $a - b\ln y$ is approximately a. Hence the model behaves like the exponential growth model $y' = ay$ when the tumor is small. The tumor grows subject to $a - b\ln y > 0$, which produces the volume restraint $\ln y = a/b$ or $V_{\max} = V_0 e^{a/b}$.

Technical Details for (12): Polar coordinates r, θ will be used. The geometry in the parabolic mirror Figure 10 shows that triangle ABC is isosceles with angles α, α and

$\pi - 2\alpha$. Therefore, $\theta = 2\alpha$ is the angle made by segment CA with the x-axis (C is the origin $(0,0)$).

$y = r\sin\theta$	Polar coordinates.
$= 2r\sin\alpha\cos\alpha$	Use $\theta = 2\alpha$ and $\sin 2x = 2\sin x\cos x$.
$= 2r\tan\alpha\cos^2\alpha$	Identity $\tan x = \sin x/\cos x$ applied.
$= 2r\dfrac{dy}{dx}\cos^2\alpha$	Use calculus relation $\tan\alpha = dy/dx$.
$= r\dfrac{dy}{dx}(1+\cos 2\alpha)$	Identity $2\cos^2 x - 1 = \cos 2x$ applied.
$= \dfrac{dy}{dx}(r+x)$	Use $x = r\cos\theta$ and $2\alpha = \theta$.

For $y > 0$, equation (12) can be solved as follows.

$\dfrac{dx}{dy} = \dfrac{x}{y} + \sqrt{(x/y)^2+1}$	Divide by y on the right side of (12).
$y\dfrac{du}{dy} = \sqrt{u^2+1}$	Substitute $u = x/y$ (u cancels).
$\int \dfrac{du}{\sqrt{u^2+1}} = \int \dfrac{dy}{y}$	Integrate the separated form.
$\sinh^{-1} u = \ln y$	Integral tables. The integration constant is zero because $u(1) = 0$.
$\dfrac{x}{y} = \sinh(\ln y)$	Let $u = x/y$ and apply sinh to both sides.
$= \dfrac{1}{2}\left(e^{\ln y} - e^{-\ln y}\right)$	Definition $\sinh u = (e^u - e^u)/2$.
$= \dfrac{1}{2}(y - 1/y)$	Identity $e^{\ln y} = y$.

Clearing fractions in the last equality gives $2x + 1 = y^2$, a parabola of the form $X = Y^2$.

Technical Details for (15): Given polar coordinates r, θ and $\tan\alpha = dy/dx$, it will be shown that $r\,d\theta/dr = \tan(\alpha - \theta)$. Details require the formulas

$$(18)\qquad \begin{aligned} x &= r\cos\theta, & \frac{dx}{dr} &= \cos\theta - r\frac{d\theta}{dr}\sin\theta, \\ y &= r\sin\theta, & \frac{dy}{dr} &= \sin\theta + r\frac{d\theta}{dr}\cos\theta. \end{aligned}$$

Then

$\tan\alpha = \dfrac{dy}{dx}$	Definition of derivative.
$= \dfrac{dy/dr}{dx/dr}$	Chain rule.
$= \dfrac{\sin\theta + r\frac{d\theta}{dr}\cos\theta}{\cos\theta - r\frac{d\theta}{dr}\sin\theta}$	Apply equation (18).
$= \dfrac{\tan\theta + r\frac{d\theta}{dr}}{1 - r\frac{d\theta}{dr}\tan\theta}$	Divide by $\cos\theta$.

Let $X = rd\theta/dr$ and cross-multiply to eliminate fractions. Then the preceding relation implies $(1 - X\tan\theta)\tan\alpha = \tan\theta + X$ and finally

$$r\frac{d\theta}{dr} = X \qquad \text{Definition of } X.$$

$$= \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta} \qquad \text{Solve for } X \text{ in } (1 - X\tan\theta)\tan\alpha = \tan\theta + X.$$

$$= \tan(\alpha - \theta) \qquad \text{Apply identity } \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}.$$

Physicists and engineers often justify formula (15) referring to Figure 15. Such diagrams are indeed the initial intuition required to *guess* formulas like (15).

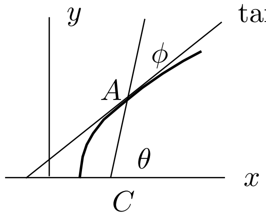


Figure 15. Polar differential triangle. Angle ϕ is the *signed angle* between the radial vector and the tangent line.

Exercises 2.8

Tank Draining

1. A cylindrical tank 6 feet high with 6-foot diameter is filled with gasoline. In 15 seconds, 5 gallons drain out. Find the drain times for the next 20 gallons and the half-volume.

2. A cylindrical tank 4 feet high with 5-foot diameter is filled with gasoline. The half-volume drain time is 11 minutes. Find the drain time for the full volume.

3. A conical tank is filled with water. The tank geometry is a solid of revolution formed from $y = 2x$, $0 \le x \le 5$. The units are in feet. Find the drain time for the tank, given the first 5 gallons drain out in 12 seconds.

4. A conical tank is filled with oil. The tank geometry is a solid of revolution formed from $y = 3x$, $0 \le x \le 5$. The units are in meters. Find the half-volume drain time for the tank, given the first 5 liters drain out in 10 seconds.

5. A spherical tank of diameter 12 feet is filled with water. Find the drain time for the tank, given the first 5 gallons drain out in 20 seconds.

6. A spherical tank of diameter 9 feet is filled with solvent. Find the half-volume drain time for the tank, given the first gallon drains out in 3 seconds.

7. A hemispherical tank of diameter 16 feet is filled with water. Find the drain time for the tank, given the first 5 gallons drain out in 25 seconds.

8. A hemispherical tank of diameter 10 feet is filled with solvent. Find the half-volume drain time for the tank, given the first gallon drains out in 4 seconds.

9. A parabolic tank is filled with water. The tank geometry is a solid of revolution formed from $y = 2x^2$, $0 \le x \le 2$. The units are in feet. Find the drain time for the tank, given the first 5 gallons drain out in 12 seconds.

10. A parabolic tank is filled with oil. The tank geometry is a solid of revolution formed from $y = 3x^2$, $0 \le x \le 2$. The units are in meters. Find the half-volume drain time for the tank, given the first 4 liters drain out in 16 seconds.

Torricelli's Law and Uniqueness

It it known that Torricelli's law gives a differential equation for which Picard's existence-uniqueness theorem is inapplicable for initial data $y(0) = 0$.

11. Explain why Torricelli's equation $y' = k\sqrt{y}$ plus initial condition $y(0) = 0$ fails to satisfy the hypotheses in Picard's theorem. Cite all failed hypotheses.

12. Consider a typical Torricelli's law equation $y' = k\sqrt{y}$ with initial condition $y(0) = 0$. Argue physically that the depth $y(t)$ of the tank for $t < 0$ can be zero for an arbitrary duration of time t near $t = 0$, even though $y(t)$ is not zero for all t.

13. Display infinitely many solutions $y(t)$ on $-5 \le t \le 5$ of Torricelli's equation $y' = k\sqrt{y}$ such that $y(t)$ is not identically zero but $y(t) = 0$ for $0 \le t \le 1$.

14. Does Torricelli's equation $y' = k\sqrt{y}$ plus initial condition $y(0) = 0$ have a solution $y(t)$ defined for $t \ge 0$? Is it unique? Apply Picard's theorem and Peano's theorem, if possible.

Clepsydra: Water Clock Design

A surface of revolution is used to make a container of height h feet for a water clock. An increasing curve $y = f(x)$ on $0 \le x \le 1$ is revolved around the y-axis to make the container shape, e.g., $y = x$ makes a conical tank. Water drains by gravity out of diameter d orifice at $(0, 0)$. The tank water level must fall at a constant rate of r inches per hour, important for marking a time scale on the tank. Find d and $f(x)$, given h and r.

15. $h = 5$ feet, $r = 4$ inches/hour. Answers: $f(x) = 5x^4$, $d = 0.05460241726 \approx 3/64$ inch.

16. $h = 4$, $r = 4$

17. $h = 3$, $r = 6$

18. $h = 4$, $r = 3$

19. $h = 3$, $r = 2$

20. $h = 4$, $r = 1$

Stefan's Law

An unclothed prison inmate is handcuffed to a chair. The inmate's skin temperature is 33° Celsius. Find the number of Joules of heat lost by the inmate's skin after t_0 minutes, given skin area A in square meters, Kelvin room temperature $T_0(r) = C(r/60) + 273.15$ and Celsius room temperature $C(t)$. Variables: t minutes, r seconds. Use equation $\frac{dQ}{dt} = k(T^4 - T_0(t)^4)$ page 149. Assume emissivity $\sigma = 5.6696 \times 10^{-8} K^{-4}$ Watts per square meter, K=degrees Kelvin.

21. $\mathcal{E} = 0.9$, $A = 1.5$, $t_0 = 10$, $C(t) = 24 + 7t/t_0$

22. $\mathcal{E} = 0.9$, $A = 1.7$, $t_0 = 12$, $C(t) = 21 + 10t/12$

23. $\mathcal{E} = 0.9$, $A = 1.4$, $t_0 = 10$, $C(t) = 15 + 15t/t_0$

24. $\mathcal{E} = 0.9$, $A = 1.5$, $t_0 = 12$, $C(t) = 15 + 14t/t_0$

On the next two exercises, use a computer algebra system (CAS). Same assumptions as Exercise 21.

25. $\mathcal{E} = 0.8$, $A = 1.4$, $t_0 = 15$, $C(t) = 15 + 15\sin\pi(t - t_0)/12$

26. $\mathcal{E} = 0.8$, $A = 1.4$, $t_0 = 20$, $C(t) = 15 + 14\sin\pi(t - t_0)/12$

Tsunami Wave Shape

Plot the piecewise solution

$$(19) \quad y(x) = 2 - \begin{cases} 2\tanh^2(x - x_0) & x > x_0, \\ 0 & x \le x_0. \end{cases}$$

See Figure 12 page 155.

27. $x_0 = 2$, $|x - x_0| \le 2$

28. $x_0 = 3$, $|x - x_0| \le 4$.

Tsunami Wavefront

Find non-equilibrium solutions for the given differential equation.

29. $(y')^2 = 12y^2 - 10y^3$.

30. $(y')^2 = 13y^2 - 12y^3$.

31. $(y')^2 = 8y^2 - 2y^3$.

32. $(y')^2 = 7y^2 - 4y^3$.

Gompertz Tumor Equation

Solve the Gompertz tumor equation $y' = (a - b\ln y)y$.

33. $a = 1,\ b = 1$

34. $a = 1,\ b = 2$

35. $a = -1,\ b = 1$

36. $a = -1,\ b = 2$

37. $a = 4,\ b = 1$

38. $a = 5,\ b = 1$

2.9 Exact Equations and Level Curves

A **level curve** or a **conservation law** is an equation of the form

$$U(x, y) = c.$$

Hikers like to think of U as the *altitude* at position (x, y) on the map and $U(x, y) = c$ as the *curve* which represents the easiest walking path, that is, altitude does not change along that route. The altitude is **conserved** along the route, hence the terminology *conservation law*.

Other examples of level curves are *isobars* and *isotherms*. An **isobar** is a planar curve where the atmospheric pressure is constant. An **isotherm** is a planar curve along which the temperature is constant.

Definition 2.8 (Potential)
The function $U(x, y)$ in a conservation law is called a **potential**. The **dynamical equation** is the first order differential equation

$$Mdx + Ndy = 0, \quad M = U_x(x, y), \quad N = U_y(x, y). \tag{1}$$

The *dynamics* or *changes* in x and y are described by (1). To **solve** $Mdx+Ndy = 0$ means this: find a conservation law $U(x, y) = c$ so that (1) holds. Formally, (1) is found by *implicit differentiation* of $U(x, y) = c$; see *Technical Details*, page 165.

The Potential Problem and Exactness

The **potential problem** assumes given a dynamical equation $Mdx + Ndy = 0$ and seeks to find a potential $U(x, y)$ from the set of equations

$$\begin{aligned} U_x &= M(x, y), \\ U_y &= N(x, y). \end{aligned} \tag{2}$$

If some potential $U(x, y)$ satisfies equation (2), then $Mdx + Ndy = 0$ is said to be **exact**. It is a consequence of the mixed partial equality $U_{xy} = U_{yx}$ that the existence of a solution U implies $M_y = N_x$. Surprisingly, this condition is also sufficient.

Theorem 2.10 (Exactness)
Let $M(x, y)$, $N(x, y)$ and their first partials be continuous in a rectangle D. Assume $M_y(x, y) = N_x(x, y)$ in D and (x_0, y_0) is a point of D. Then the equation $Mdx + Ndy = 0$ is exact with potential U given by the formula

$$U(x, y) = \int_{x_0}^{x} M(t, y)dt + \int_{y_0}^{y} N(x_0, s)ds. \tag{3}$$

The proof is on page 165.

The Method of Potentials

Formula (3) has technical problems because it requires two integrations. The integrands have a *parameter*: they are *parametric integrals*. Integration effort can be reduced by using the **method of potentials** for $Mdx + Ndy = 0$, which applies equation (3) with $x_0 = y_0 = 0$ in order to simplify integrations.

Test $M_y = N_x$	Compute the partials M_y and N_x, then test the equality $M_y = N_x$. Proceed if equality holds.
Trial Potential	Let $U = \int_0^x M(x,y)dx + \int_0^y N(0,y)dy$. Evaluate both integrals.
Test $U(x,y)$	Compute U_x and U_y, then test both $U_x = M$ and $U_y = N$. This step finds integration errors.

Examples

Example 2.43 (Exactness Test)
Test $Mdx + Ndy = 0$ for the existence of a potential U, given $M = 2xy + y^3 + y$ and $N = x^2 + 3xy^2 + x$,

Solution: Theorem 2.10 implies that $Mdx + Ndy = 0$ has a potential U exactly when $M_y = N_x$. It suffices to compute the partials and show they are equal.

$$\begin{aligned} M_y &= \partial_y(2xy + y^3 + y) \\ &= 2x + 3y^2 + 1, \end{aligned} \qquad \begin{aligned} N_x &= \partial_x(x^2 + 3xy^2 + x) \\ &= 2x + 3y^2 + 1. \end{aligned}$$

Example 2.44 (Conservation Law Test)
Test whether $U = x^2y + xy^3 + xy$ is a potential for $Mdx + Ndy = 0$, given $M = 2xy + y^3 + y$, $N = x^2 + 3xy^2 + x$.

Solution: By definition, it suffices to test the equalities $U_x = M$ and $U_y = N$.

$$\begin{aligned} U_x &= \partial_x(x^2y + xy^3 + xy) \\ &= 2xy + y^3 + y \\ &= M, \end{aligned} \qquad \begin{aligned} U_y &= \partial_y(x^2y + xy^3 + xy) \\ &= x^2 + 3xy^2 + x \\ &= N. \end{aligned}$$

Example 2.45 (Method of Potentials)
Solve $y' = -\dfrac{2xy + y^3 + y}{x^2 + 3xy^2 + x}$.

Solution: The implicit solution $x^2y + xy^3 + xy = c$ will be justified.

The equation has the form $Mdx+Ndy = 0$ where $M = 2xy+y^3+y$ and $N = x^2+3xy^2+x$. It is exact, by Theorem 2.10, because the partials $M_y = 2x+3y^2+1$ and $N_x = 2x+3y^2+1$ are equal.

The method of potentials applies to find the potential $U = x^2y + xy^3 + xy$ as follows.

$U = \int_0^x M(x,y)dx + \int_0^y N(0,y)dy$	Formula for U, Theorem 2.10.
$= \int_0^x \left(2xy + y^3 + y\right) dx + \int_0^y (0)dy$	Insert M and N.
$= x^2y + xy^3 + xy$	Evaluate integral.

Observe that $N(x,y)$ simplifies to zero at $x = 0$, which reduces the actual work in half. Any choice other than $x_0 = 0$ in Theorem 2.10 increases the labor.

To *test the solution*, compute the partials of U, then compare them to M and N; see Example 2.44.

Example 2.46 (Exact Equation)

Solve $\dfrac{x+y}{(1-x)^2}dx + \dfrac{x}{1-x}dy = 0.$

Solution: The implicit solution $\dfrac{xy+x}{1-x} + \ln|x-1| = c$ will be justified.

Assume given the exactness of the equation $Mdx + Ndy = 0$, where $M = (x+y)/(1-x)^2$ and $N = x/(1-x)$. Apply Theorem 2.10:

$U = \int_0^x M(x,y)dx + \int_0^y N(0,y)dy$	Method of potentials.
$= \int_0^x \dfrac{x+y}{(1-x)^2}dx + \int_0^y (0)dy$	Substitute for M, N.
$= \int_0^x \left(\dfrac{y+1}{(x-1)^2} + \dfrac{1}{x-1}\right)dx$	Partial fractions.
$= \dfrac{xy+x}{1-x} + \ln|x-1|$	Evaluate integral.

Additional examples, including the context for the preceding example, appear in the next section.

Remarks on the Method of Potentials

Indefinite integrals $\int M(x,y)dx$ and $\int N(0,y)dy$ can be used, provided the two integration answers are zero at $x = 0$ and $y = 0$, respectively. Swapping the roles of x and y gives $U = \int_0^y N(x,y)dy + \int_0^x M(x,0)dx$, a form which may have easier integrations.

Can the test $M_y = N_x$ be skipped? True, it is enough to verify that the potential works (the last step). If the last step fails, then the first step must be done to resolve the error.

The equation $ydx + 2xdy = 0$ fails $M_y = N_x$ and the trial potential $U = xy$ fails $U_x = M$, $U_y = N$. In the equivalent form $x^{-1}dx + 2y^{-1}dy = 0$, the method of potentials does not apply directly, because $(0,0)$ is outside the domain of continuity. Nevertheless, the trial potential $U = \ln x + 2\ln y$ passes the test $U_x = M$, $U_y = N$. Such pleasant accidents account for the popularity of the method of potentials.

It is prudent in applications of Theorem 2.10 to test $x_0 = y_0 = 0$ in M and N, to detect a discontinuity. If detected, then another vertex x_0, y_0 of the unit square, e.g., $x_0 = y_0 = 1$, might suffice.

Details and Proofs

Justification of equation (1) uses the calculus *chain rule*

$$\frac{d}{dt}U(x(t), y(t)) = U_x(x(t), y(t))x'(t) + U_y(x(t), y(t))y'(t)$$

and differential notation $dx = x'(t)dt$, $dy = y'(t)dt$. To justify (1), let $(x(t), y(t))$ be some parameterization of the level curve, then differentiate on t across the equation $U(x(t), y(t)) = c$ and apply the chain rule.

Proof of Theorem 2.10

Background result. The proof assumes the following identity:

$$\frac{\partial}{\partial y}\int_{x_0}^{x} M(t, y)dt = \int_{x_0}^{x} M_y(t, y)dt.$$

The identity is obtained by forming the Newton quotient $(G(y + h) - G(y))/h$ for the derivative of $G(y) = \int_{x_0}^{x} M(t, y)dt$ and then taking the limit as h approaches zero. Technically, the limit must be taken inside an integral sign, which for success requires continuity of the partial M_y.

Details. It has to be shown that the implicit relation $U(x, y) = c$ with U defined by (3) is a solution of the exact equation $Mdx + Ndy = 0$, that is, the relations $U_x = M$, $U_y = N$ hold. The partials are calculated from the background result as follows.

$U_x = \partial_x \int_{x_0}^{x} M(t, y)dt$	Use (3), in which the second integral does not depend on x.
$= M(x, y)$,	Fundamental theorem of calculus.
$U_y = \partial_y \int_{x_0}^{x} M(t, y)dt$ $+ \partial_y \int_{y_0}^{y} N(x_0, s)ds$	Use (3).
$= \int_{x_0}^{x} M_y(t, y)dt + N(x_0, y)$	Apply the background result and the fundamental theorem.
$= \int_{x_0}^{x} N_x(t, y)dt + N(x_0, y)$	Substitute $M_y = N_x$.
$= N(x, y)$	Fundamental theorem of calculus.

■

Power Series Proof of Theorem 2.10 It will be assumed that M and N have power series expansions about $x = y = 0$. Let $U_1 = \int M(x, y)dx$ and $U_2 = \int N(x, y)dy$ with $U_1(0, y) = U_2(x, 0) = 0$. The series forms of U_1 and U_2 will be

$$U_1 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}x^i y^j + \sum_{i=1}^{\infty} a_i x^i,$$
$$U_2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} d_{ij}x^i y^j + \sum_{j=1}^{\infty} b_j y^j.$$

The identities $\partial_y\partial_x U_1 = M_y = N_x = \partial_x\partial_y U_2$ imply that $c_{ij} = d_{ij}$, using term-by-term differentiation. The trial potential is $U = U_1 + \sum_{j=1}^{\infty} b_j y^j$ or $U = U_2 + \sum_{i=1}^{\infty} a_i x^i$. From these relations it follows that $U_x = M$ and $U_y = N$. Therefore, $Mdx + Ndy = 0$ is exact with potential U.

Exercises 2.9

Exactness Test

Test the equality $M_y = N_x$ for the given equation, as written, and report *exact* when true. Do not try to solve the differential equation. See Example 2.43, page 163.

1. $(y - x)dx + (y + x)dy = 0$

2. $(y + x)dx + (x - y)dy = 0$

3. $(y + \sqrt{xy})dx + (-y)dy = 0$

4. $(y + \sqrt{xy})dx + xydy = 0$

5. $(x^2 + 3y^2)dx + 6xydy = 0$

6. $(y^2 + 3x^2)dx + 2xydy = 0$

7. $(y^3 + x^3)dx + 3xy^2dy = 0$

8. $(y^3 + x^3)dx + 2xy^2dy = 0$

9. $2xydx + (x^2 - y^2)dy = 0$

10. $2xydx + (x^2 + y^2)dy = 0$

Conservation Law Test

Test conservation law $U(x, y) = c$ for a solution to $Mdx + Ndy = 0$. See Example 2.44, page 163.

11. $2xydx + (x^2 + 3y^2)dy = 0$,
$x^2y + y^3 = c$

12. $2xydx + (x^2 - 3y^2)dy = 0$,
$x^2y - y^3 = c$

13. $(3x^2 + 3y^2)dx + 6xydy = 0$,
$x^3 + 3xy^2 = c$

14. $(x^2 + 3y^2)dx + 6xydy = 0$,
$x^3 + 3xy^2 = c$

15. $(y - 2x)dx + (2y + x)dy = 0$,
$xy - x^2 + y^2 = c$

16. $(y + 2x)dx + (-2y + x)dy = 0$,
$xy + x^2 - y^2 = c$

Exactness Theorem

Find an implicit solution $U(x, y) = c$. See Examples 2.45-2.46, page 163.

17. $(y - 4x)dx + (4y + x)dy = 0$

18. $(y + 4x)dx + (4y + x)dy = 0$

19. $(e^y + e^x)dx + (xe^y)dy = 0$

20. $(e^{2y} + e^x)dx + (2xe^{2y})dy = 0$

21. $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$

22. $(1 + ye^{-xy})dx + (xe^{-xy} - 4y)dy = 0$

23. $(2x + \arctan y)dx + \dfrac{x}{1+y^2}\,dy = 0$

24. $(2x + \arctan y)dx + \dfrac{x+2y}{1+y^2}\,dy = 0$

25. $\dfrac{2x^5 + 3y^3}{x^4y}dx - \dfrac{2y^3 + x^5}{x^3y^2}dy = 0$

26. $\dfrac{2x^4 + y^2}{x^3y}dx - \dfrac{2x^4 + y^2}{2x^2y^2}dy = 0$

27. $Mdx + Ndy = 0$, $M = e^x \sin y + \tan y$,
$N = e^x \cos y + x \sec^2 y$

28. $Mdx + Ndy = 0$, $M = e^x \cos y + \tan y$,
$N = -e^x \sin y + x \sec^2 y$

29. $\left(x^2 + \ln y\right) dx + \left(y^3 + x/y\right) dy = 0$

30. $\left(x^3 + \ln y\right) dx + \left(y^3 + x/y\right) dy = 0$

2.10 Special equations

Homogeneous-A Equation

A first order equation of the form $y' = F(y/x)$ is called a **homogeneous class A equation**. The substitution $u = y/x$ changes it into an equivalent first order separable equation $xu' + u = F(u)$. Solutions of $y' = F(y/x)$ and $xu' + u = F(u)$ are related by the equation $y = xu$.

Homogeneous-C Equation

Let $R(x, y)$ be a rational function constructed from *two affine functions*:

$$R(x, y) = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}.$$

A first order equation of the form $y' = G(R(x, y))$ is called a **homogeneous class C equation** . If the system

$$a_1 a + b_1 b = c_1, \quad a_2 a + b_2 b = c_2$$

has a solution (a, b), then the change of variables $x = X - a$, $y = Y - b$ effectively eliminates the terms c_1 and c_2. Accordingly, the equation $y' = G(R(x, y))$ converts into a homogeneous class A equation

$$Y' = G\left(\frac{a_1 + b_1 Y/X}{a_2 + b_2 Y/X}\right).$$

This equation type was solved in the previous paragraph. Justification follows from $y' = Y'$ and $R(X - a, Y - b) = (a_1 X + b_1 Y)/(a_2 X + b_2 Y)$.

Bernoulli's Equation

The equation $y' + p(x)y = q(x)y^n$ is called the **Bernoulli differential equation**. If $n = 1$ or $n = 0$, then this is a linear equation. Otherwise, the substitution $u = y/y^n$ changes it into the linear first order equation $u' + (1-n)p(x)u = (1-n)q(x)$.

Integrating Factors and Exact Equations

An equation $\mathsf{M}dx + \mathsf{N}dy = 0$ is said to have an **integrating factor** $Q(x, y)$ if multiplication across the equation by Q produces an exact equation $Mdx + Ndy = 0$. The definition implies $M = Q\mathsf{M}$, $N = Q\mathsf{N}$ and $M_y = N_x$. The search for Q is only interesting when $\mathsf{M}_y \neq \mathsf{N}_x$.

A systematic approach to finding Q includes a list of **trial integrating factors**, which are known to work for special equations:

$Q = x^a y^b$	Require $xy\,(\mathbf{M}_y - \mathbf{N}_x) = ay\mathbf{N} - bx\mathbf{M}$. This integrating factor can introduce *extraneous solutions* $x = 0$ or $y = 0$.
$Q = e^{ax+by}$	Require $\mathbf{M}_y - \mathbf{N}_x = a\mathbf{N} - b\mathbf{M}$.
$Q = e^{\int \mu(x)dx}$	Require $\mu = (\mathbf{M}_y - \mathbf{N}_x)/N$ to be independent of y.
$Q = e^{\int \nu(y)dy}$	Require $\nu = (\mathbf{N}_x - \mathbf{M}_y)/M$ to be independent of x.

Examples

Example 2.47 (Homogeneous-A)
Solve $yy' = 2x + y^2/x$

Solution: The *implicit solution* will be shown to be

$$y^2 = cx^2 + 4x^2 \ln x.$$

The equation $yy' = 2x + y^2/x$ is not separable, linear nor exact. Division by y gives the homogeneous-A form $y' = 2/u + u$ where $u = y/x$. Then

$xu' + u = 2/u + u$	Form $xu' + u = F(u)$.
$xu' = 2/u$	Separable form.
$u^2 = c + 4\ln x$	Implicit solution u.
$y^2 = x^2u^2$	Change of variables $y = xu$.
$\quad = cx^2 + 4x^2 \ln x$	Substitute $u^2 = c + 4\ln x$.

Check the implicit solution against $yy' = 2x + y^2/x$ as follows.

LHS $= yy'$	Left side of $yy' = 2x + y^2/x$.
$= \frac{1}{2}(y^2)'$	Calculus identity.
$= \frac{1}{2}(cx^2 + 4x^2 \ln x)'$	Substitute.
$= cx + 4x\ln x + 2x$	Differentiate.
$= 2x + y^2/x$	Use $y^2 = cx^2 + 4x^2 \ln x$.
= RHS.	Equality verified.

Example 2.48 (Homogeneous-C)
Solve $y' = \dfrac{x + y + 3}{x - y + 5}$.

Solution: The *implicit solution* will be shown to be

$$2\ln(x+4) + \ln\left(\left(\frac{y-1}{x+4}\right)^2 + 1\right) - 2\arctan\left(\frac{y-1}{x+4}\right) = c.$$

The equation would be of type homogeneous-A, if not for the constants 3 and 5 in the fraction $(x+y+3)/(x-y+5)$. The method applies a translation of coordinates $x = X - a$, $y = Y - b$ as below.

$x+y+3 = X+Y,$ $x-y+5 = X-Y$	Require the translation to remove the constant terms.
$3 = a+b,$ $5 = a-b$	Substitute $X = x+a$, $Y = y+b$ and simplify.
$a = 4,\ b = -1$	Unique solution of the system.
$\dfrac{dY}{dX} = \dfrac{X+Y}{X-Y}$	Translated type homogeneous-A equation.
$X\dfrac{du}{dX} + u = \dfrac{1+u}{1-u}$	Use $u = Y/X$ to eliminate Y.
$\dfrac{1-u}{1+u^2}\dfrac{du}{dX} = \dfrac{1}{X}$	Separated form.

The separated form is integrated as $\int du/(1+u^2) - \int u du/(1+u^2) = \int dX/X$. Evaluation gives the implicit solution

$$\arctan(u) - \frac{1}{2}\ln\left(u^2+1\right) = C + \ln X.$$

Changing variables $x = X - 4$, $y = Y + 1$ and consolidating constants produces the announced solution.

To check the solution by maple assist, use the following code, which tests $U(x,y) = c$ against $y' = f(x,y)$. The test succeeds if odetest returns zero.

```
# Maple
U:=(x,y)->2*ln(x+4)+ln(((y-1)/(x+4))^2+1)-2*arctan((y-1)/(x+4));
f:=(x,y)->(x+y+3)/(x-y+5); DE:=diff(y(x),x)=f(x,y(x));
odetest(U(x,y(x))=c,DE);
```

Example 2.49 (Bernoulli Substitution)
Solve $y' + 2y = y^2$.

Solution: It will be shown that the solution is $y = \dfrac{1}{1+Ce^x}$.

The equation can be solved by other methods, notably separation of variables. Bernoulli's substitution $u = y/y^n$ will be applied to find the equivalent first order linear differential equation, as follows.

$u' = (y/y^2)'$	Bernoulli's substitution, $n = 2$.
$= -y^{-2}y'$	Chain rule.
$= -1 + y^{-1}$	Use $y' + 2y = y^2$.
$= -1 + u$	Use $u = y/y^2$.

This linear equation $u' = -1 + u$ has equilibrium solution $u_p = 1$ and homogeneous solution $u_h = Ce^x$. Therefore, $u = u_h + u_p$ gives $y = u^{-1} = 1/(1 + Ce^x)$.

Example 2.50 (Integrating factor $Q = x^a y^b$)
Solve $(3y + 4xy^2)dx + (4x + 5x^2y)dy = 0$.

Solution: The implicit solution $x^3y^4 + x^4y^5 = c$ will be justified.

The equation is not exact as written. To explain why, let $\mathbf{M} = 3y + 4xy^2$ and $\mathbf{N} = 4x + 5x^2y$. Then $\mathbf{M}_y = 8xy + 3$, $\mathbf{N}_x = 10xy + 4$ which implies $\mathbf{M}_y \neq \mathbf{N}_x$ (not exact).

The factor $Q = x^a y^b$ will be an integrating factor for the equation provided a and b are chosen to satisfy $xy\,(\mathbf{M}_y - \mathbf{N}_x) = ay\mathbf{N} - bx\mathbf{M}$. This requirement becomes $xy\,(-2xy - 1) = ay(4x + 5x^2y) - bx(3y + 4xy^2)$. Comparing terms across the equation gives the 2×2 system of equations

$$\begin{array}{rcrcr} 4a & - & 3b & = & -1, \\ 5a & - & 4b & = & -2. \end{array}$$

The unique solution by Cramer's determinant rule is

$$a = \frac{\begin{vmatrix} -1 & -3 \\ -2 & -4 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 5 & -4 \end{vmatrix}} = 2, \quad b = \frac{\begin{vmatrix} 4 & -1 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 5 & -4 \end{vmatrix}} = 3.$$

Then $Q = x^2y^3$ is the required integrating factor. After multiplication by Q, the original equation becomes the exact equation

$$(3x^2y^4 + 4x^3y^5)dx + (4x^3y^3 + 5x^4y^4)dy = 0.$$

The method of potentials applied to $M = 3x^2y^4 + 4x^3y^5$ and $N = 4x^3y^3 + 5x^4y^4$ finds the potential U as follows.

$U = \int_0^x M(x,y)dx + \int_0^y N(0,y)dy$	Method of potentials formula.
$= \int_0^x (3x^2y^4 + 4x^3y^5)dx + \int_0^y (0)dy$	Insert M and N.
$= x^3y^4 + x^4y^5$	Evaluate integral.

Example 2.51 (Integrating factor $Q = e^{ax+by}$)
Solve $(e^x + e^y)\,dx + (e^x + 2e^y)\,dy = 0$.

Solution: The implicit solution $2e^{3x+3y} + 3e^{2x+4y} = c$ will be justified. A constant $5/6$ appears in the integrations below, mysteriously absent in the solution, because $5/6$ has been absorbed into the constant c.

Let $\mathbf{M} = e^x + e^y$ and $\mathbf{N} = e^x + 2e^y$. Then $\mathbf{M}_y = e^y$ and $\mathbf{N}_x = e^x$ (not exact). The condition for $Q = e^{ax+by}$ to be an integrating factor is $\mathbf{M}_y - \mathbf{N}_x = a\mathbf{N} - b\mathbf{M}$, which becomes the requirement

$$e^y - e^x = a\,(e^x + 2e^y) - b\,(e^x + e^y)\,.$$

The equations are satisfied provided (a, b) is a solution of the 2×2 system of equations

$$\begin{array}{rcrcr} a & - & b & = & -1, \\ 2a & - & b & = & 1. \end{array}$$

The unique solution is $a = 2$, $b = 3$, by elimination. The original equation multiplied by the integrating factor $Q = e^{2x+3y}$ is the exact equation $Mdx + Ndy = 0$, where $M = e^{3x+3y} + e^{2x+4y}$ and $N = e^{3x+3y} + 2e^{2x+4y}$. The method of potentials applies to find the potential U, as follows.

$$\begin{aligned} U &= \int_0^x M(x,y)dx + \int_0^y N(0,y)dy && \text{Method of potentials.} \\ &= \int_0^x \left(e^{3x+3y} + e^{2x+4y}\right)dx + \int_0^y \left(e^{3y} + 2e^{4y}\right)dy && \text{Insert } M \text{ and } N. \\ &= \tfrac{1}{3}e^{3x+3y} + \tfrac{1}{2}e^{2x+4y} - \tfrac{5}{6} && \text{Evaluate integral.} \end{aligned}$$

Example 2.52 (Integrating factor $Q = Q(x)$)
Solve $(x + y)dx + (x - x^2)dy = 0$.

Solution: The implicit solution $\dfrac{xy + x}{1 - x} + \ln|x - 1| = c$ will be justified.

Let $\mathbf{M} = x + y$, $\mathbf{N} = x - x^2$. Then $\mathbf{M}_y = 1$ and $\mathbf{N}_x = 1 - 2x$ (not exact). Then

$$\begin{aligned} \mu &= \frac{\mathbf{M}_y - \mathbf{N}_x}{\mathbf{N}} && \text{Hope } \mu \text{ depends on } x \text{ alone.} \\ &= 2/(1-x) && \text{Substitute } \mathbf{M}, \mathbf{N}\text{; success.} \\ Q &= e^{\int \mu(x)dx} && \text{Integrating factor.} \\ &= e^{-2\ln|1-x|} && \text{Substitute for } \mu \text{ and integrate.} \\ &= (1-x)^{-2} && \text{Simplified factor found.} \end{aligned}$$

Multiplication of $\mathbf{M}dx + \mathbf{N}dy = 0$ by Q gives the corresponding exact equation

$$\frac{x + y}{(1 - x)^2}dx + \frac{x}{1 - x}dy = 0.$$

The method of potentials applied to $M = (x + y)/(1 - x)^2$, $N = x/(1 - x)$ finds the implicit solution as follows.

$$\begin{aligned} U &= \int_0^x M(x,y)dx + \int_0^y N(0,y)dy && \text{Method of potentials.} \\ &= \int_0^x \frac{x+y}{(1-x)^2}dx + \int_0^y (0)dy && \text{Substitute for } M, N. \\ &= \int_0^x \left(\frac{y+1}{(x-1)^2} + \frac{1}{x-1}\right)dx && \text{Partial fractions.} \\ &= \frac{xy + x}{1 - x} + \ln|x - 1| && \text{Evaluate integral.} \end{aligned}$$

Example 2.53 (Integrating factor $Q = Q(y)$)
Solve $(y - y^2)dx + (x + y)dy = 0$.

Solution: Interchange the roles of x and y, then apply the previous example, to obtain the implicit solution $\dfrac{xy + y}{1 - y} + \ln|y - 1| = c$.

This example happens to fit the case when the integrating factor is a function of y alone. The details parallel the previous example.

Details and Proofs

The exactness condition $M_y = N_x$ for $M = Q\mathbf{M}$ and $N = Q\mathbf{N}$ becomes in the case $Q = x^a y^b$ the relation

$$bx^a y^{b-1}\mathbf{M} + x^a y^b \mathbf{M}_y = ax^{a-1}y^b\mathbf{N} + x^a y^b \mathbf{N}_x$$

from which rearrangement gives $xy(\mathbf{M}_y - \mathbf{N}_x) = ay\mathbf{N} - bx\mathbf{M}$. The case $Q = e^{ax+by}$ is similar.

Consider $Q = e^{\int \mu(x)dx}$. Then $Q' = \mu Q$. The exactness condition $M_y = N_x$ for $M = Q\mathbf{M}$ and $N = Q\mathbf{N}$ becomes $Q\mathbf{M}_y = \mu Q\mathbf{N} + Q\mathbf{N}_x$ and finally

$$\mu = \frac{\mathbf{M}_y - \mathbf{N}_x}{\mathbf{N}}.$$

The similar case $Q = e^{\int \nu(y)dy}$ is obtained from the preceding case, by swapping the roles of x, y.

Exercises 2.10

Homogeneous-A Equations

Find f such that the equation can be written in the form $y' = f(y/x)$. Solve for y using a computer algebra system.

1. $xy' = y^2/x$

2. $x^2y' = x^2 + y^2$

3. $yy' = \dfrac{xy^2}{x^2+y^2}$

4. $yy' = \frac{2xy^2}{x^2+y^2}$

5. $y' = \dfrac{1}{x+y}$

6. $y' = y/x + x/y$

7. $y' = (1 + y/x)^2$

8. $y' = 2y/x + x/y$

9. $y' = 3y/x + x/y$

10. $y' = 4y/x + x/y$

Homogeneous-C Equations

Given $y' = f(x, y)$, decompose $f(x, y) = G(R(x, y))$ where $R(x, y) = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, then convert to Homogeneous-A. Investigate solving $y' = f(x, y)$ by computer.

11. $y' = -\frac{(y+1)x}{y^2+2\,y+1+x^2}$

12. $y' = 2\,\dfrac{(1+y)\,x}{x^2+y^2+2\,y+1}$

13. $y' = \dfrac{(1+x)\,y}{x^2+4\,y^2+2\,x+1}$

14. $y' = \dfrac{1+x}{y+1+x}$

15. $y' = \dfrac{1+y}{x+y+1}$

16. $x(y+1)y' = x^2 + y^2 + 2y + 1$

17. $y' = \dfrac{x^2-y^2-2\,y-1}{(1+y)\,x}$

18. $y' = \dfrac{(y+2\,x)^2}{x^2}$

19. $y' = \dfrac{x^2+xy+y^2+5\,x+4\,y+7}{(x+2)\,(3+y+x)}$

20. $y' = -\dfrac{x^2-xy-y^2+5\,x-5\,y+5}{(3+x)\,(4+y+x)}$

Bernoulli's Equation

Identify the exponent n in Bernoulli's equation $y' + p(x)y = q(x)y^n$ and solve for $y(x)$.

21. $y^{-2}y' = 1 + x$

22. $yy' = 1 + x$

23. $y^{-2}y' + y^{-1} = 1 + x$

24. $yy' + y^2 = 1 + x$

25. $y' + y = y^{1/3}$

26. $y' + y = y^{1/5}$

27. $y' - y = y^{-1/2}$

28. $y' - y = y^{-1/3}$

29. $yy' + y^2 = e^x$

30. $y' + y = e^{2x}y^2$

Integrating Factor $x^a y^b$

Report an implicit solution for the given equation $Mdx + Ndy = 0$, using an integrating factor $Q = x^a y^b$. Follow Example 2.50, page 169. Computer assist expected.

31. $M = 3xy - 6y^2$, $N = 4x^2 - 15xy$

32. $M = 3xy - 10y^2$, $N = 4x^2 - 25xy$

33. $M = 2\,y - 12\,xy^2$, $N = 4\,x - 20\,x^2y$

34. $M = 2\,y - 21\,xy^2$, $N = 4\,x - 35\,x^2y$

35. $M = 3\,y - 32\,xy^2$, $N = 4\,x - 40\,x^2y$

36. $M = 3\,y - 20\,xy^2$, $N = 4\,x - 25\,x^2y$

37. $M = 12\,y - 30\,x^2y^2$, $N = 12\,x - 25\,x^3y$

38. $M = 12\,y + 90\,x^2y^2$, $N = 12\,x + 75\,x^3y$

39. $M = 15\,y + 90\,xy^2$, $N = 12\,x + 75\,x^2y$

40. $M = 35\,y + 30\,xy^2$, $N = 28\,x + 25\,x^2y$.

Integrating Factor e^{ax+by}

Report an implicit solution $U(x, y) = c$ for the given equation $Mdx + Ndy = 0$ using an integrating factor $Q = e^{ax+by}$. Follow Example 2.51, page 170.

41. $M = e^x + 2e^{2y}$, $N = e^x + 5e^{2y}$

42. $M = 3e^x + 2e^y$, $N = 4e^x + 5e^y$

43. $M = 12\,e^x + 2$, $N = 20\,e^x + 5$

44. $M = 12\,e^x + 2\,e^{-y}$, $N = 24\,e^x + 5\,e^{-y}$

45. $M = 12\,e^y + 2\,e^{-x}$, $N = 24\,e^y + 5\,e^{-x}$

46. $M = 12\,e^{-2\,y} + 2\,e^{-x}$, $N = 12\,e^{-2\,y} + 5\,e^{-x}$

47. $M = 16\,e^y + 2\,e^{-2\,x+3\,y}$, $N = 12\,e^y + 5\,e^{-2\,x+3\,y}$

48. $M = 16\,e^{-y} + 2\,e^{-2\,x-3\,y}$, $N = -12\,e^{-y} - 5\,e^{-2\,x-3\,y}$

49. $M = -16 - 2\,e^{2\,x+y}$, $N = 12 + 4\,e^{2\,x+y}$

50. $M = -16\,e^{-3\,y} - 2\,e^{2\,x}$, $N = 8\,e^{-3\,y} + 5\,e^{2\,x}$

Integrating Factor $Q(x)$

Report an implicit solution $U(x, y) = c$ for the given equation, using an integrating factor $Q = Q(x)$. Follow Example 2.52, page 171.

51. $(x + 2y)dx + (x - x^2)dy = 0$

52. $(x + 3y)dx + (x - x^2)dy = 0$

53. $(2x + y)dx + (x - x^2)dy = 0$

54. $(2x + y)dx + (x + x^2)dy = 0$

55. $(2x + y)dx + (-x - x^2)dy = 0$

56. $(x + y)dx + (-x - x^2)dy = 0$

57. $(x + y)dx + (-x - 2x^2)dy = 0$

58. $(x + y)dx + (x + 5x^2)dy = 0$

59. $(x + y)dx + (3x)dy = 0$

60. $(x + y)dx + (7x)dy = 0$

Integrating Factor $Q(y)$

61. $(y - y^2)dx + (x + y)dy = 0$

62. $(y - y^2)dx + (2x + y)dy = 0$

63. $(y - y^2)dx + (2x + 3y)dy = 0$

64. $(y + y^2)dx + (2x + 3y)dy = 0$

65. $(y + y^2)dx + (x + 3y)dy = 0$

66. $(y + 5y^2)dx + (x + 3y)dy = 0$

67. $(y + 3y^2)dx + (x + 3y)dy = 0$

68. $(2y + 5y^2)dx + (7x + 11y)dy = 0$

69. $(2y + 5y^2)dx + (x + 7y)dy = 0$

70. $(3y + 5y^3)dx + (7x + 9y)dy = 0$

PDF Sources

Text, Solutions and Corrections

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