Ch X: A Hopf Bifurcation Theorem

This chapter is devoted to a version of the classical **Hopf bifurcation** theorem which establishes the existence of nontrivial periodic orbits of autonomous systems of differential equations which depend upon a parameter and for which the stability properties of the trivial solution changes as the parameter is varied.

The proof is based on the **method of Lyapunov-Schmidt** presented in Chapter II.

Theorem (Hopf Bifurcation). Assume that f satisfies the following conditions:

(1) $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is a C^2 mapping such that $f(0, \alpha) = 0,\ \alpha \in \mathbf{R}$.

(2) For some given value of $\alpha = \alpha_0$, $i = \sqrt{-1}$ and $-i$ are eigenvalues of $f_u(0, \alpha_0)$ and $\pm ni$, $n = 0, 2, 3, \cdots$ are not eigenvalues of $f_u(0, \alpha_0)$;

(3) in a neighborhood of α_0 there is a curve of eigenvalues and eigenvectors

$$f_u(0, \alpha)a(\alpha) = \beta(\alpha)a(\alpha)$$
$$a(\alpha_0) \neq 0,\ \beta(\alpha_0) = i,\ \mathrm{Re}\frac{d\beta}{d\alpha}|_{\alpha_0} \neq 0.$$

$\cdots$ continued on next slide $\cdots$

Hopf Bifurcation − continued

Then there exist positive numbers ϵ and η and a C^1 function $(u, \rho, \alpha) : (-\eta, \eta) \to C^1_{2\pi} \times \mathbf{R} \times \mathbf{R}$, where $C^1_{2\pi}$ is the space of 2π periodic C^1 $\mathbf{R}^n-$ valued functions, such that $(u(s), \rho(s), \alpha(s))$ solves the equation $\frac{du}{d\tau} + \rho f(u, \alpha) = 0$, nontrivially, i.e., $u(s) \neq 0$, $s \neq 0$ and $\rho(0) = 1$, $\alpha(0) = \alpha_0$, $u(0) = 0$.

Furthermore, if (u_1, α_1) is a nontrivial solution of $\frac{du}{dt} + f(u, \alpha) = 0$ of period $2\pi\rho_1$, with $|\rho_1 - 1| < \epsilon$, $|\alpha_1 - \alpha_0| < \epsilon$, $|u_1(t)| < \epsilon$, $t \in [0, 2\pi\rho_1]$, then there exists $s \in (-\eta, \eta)$ such that $\rho_1 = \rho(s)$, $\alpha_1 = \alpha(s)$ and $u_1(\rho_1 t) = u(s)(\tau + \theta)$, $\tau = \rho_1 t \in [0, 2\pi]$, $\theta \in [0, 2\pi)$.

Van der Pol Oscillator Example

Consider the nonlinear oscillator

$$x'' + x - \alpha(1 - x^2)x' = 0.$$

This equation will be shown to have for for certain small values of α nontrivial periodic solutions with periods close to 2π.

Details. Transform the equation into a system via $u_1 = x$, $u_2 = x'$, $u = (u_1, u_2)^T$ to obtain $u' + \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix} u + \begin{pmatrix} 0 \\ u_1^2 u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $f(u, \alpha) = \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix} u + \begin{pmatrix} 0 \\ u_1^2 u_2 \end{pmatrix}$ and $f_u(0, \alpha) = \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix}$.

Continued on next slide $\cdots$

Details continued

The eigenvalues of $f_u(0,\alpha)$ satisfy $\beta(\alpha+\beta)+1=0$. Let $\alpha_0=0$, then $\beta(0)=\pm i$, and computing $\frac{d\beta}{d\alpha}=\beta'$ we obtain $2\beta\beta'+\beta'\alpha+\beta=0$, or $\beta'=\frac{-\beta}{\alpha+2\beta}=-\frac{1}{2}$, for $\alpha=0$. Thus by Theorem II of chapter III there exists $\eta>0$ and continuous functions $\alpha(s)$, $\rho(s)$, $s\in(-\eta,\eta)$ such that $\alpha(0)=0$, $\rho(0)=1$ and the differential equation has for $s\neq 0$ a nontrivial solution $x(s)$ with period $2\pi\rho(s)$. This solution is unique up to phase shift.