Chapter IX: Invariant Sets

Let $D \subset \mathbf{R}^N$ be open and connected and let $f : D \to \mathbf{R}^N$ be locally Lipschitz continuous.

Consider the initial value problem $u' = f(u)$, $u(0) = u_0 \in D$.

Definition. An **invariant subset** $M \subset D$ is defined by the property that the solution $u(t)$ stays in M whenever $u_0 \in M$, for all $t \in I_{u_0}$, the maximal t-interval of existence.

Examples of invariant sets.
1. An equilibrium point, $f(u_0) = 0$.
2. A periodic solution, $u(0) = u(T)$.

Definition. The **flow determined by** f is the mapping $u : I_{u_0} \times D \to D$ defined by $(t, u_0) \mapsto u(t, u_0)$. The set $U = \cup_{v \in D} I_v \times \{v\}$, which is a subset of $\mathbf{R} \times \mathbf{R}^N$, is the **natural domain**.

Lemma 1. The flow determined by f has the following properties:
1. The flow $u : U \mapsto D$ is continuous.
2. $u(0, u_0) = u_0$ for all $u_0 \in D$.
3. If $u_0 \in D$, $s \in I_{u_0}$ and $t \in I_{u(s,u_0)}$, then $s + t \in I_{u_0}$ and $u(s + t, u_0) = u(t, u(s, u_0))$.

Definition. A mapping having the three properties of the lemma is called a **flow** on D. A flow may be defined without the underlying differential equation.

Orbits and Flows

If u is a flow for f and S is a subset of $I_{u_0} = (t^-_{u_0}, t^+_{u_0})$, then $u(S, u_0)$ is the set

$$\{u(t, u_0) : \ t \in S\}.$$

Given $v \in U$, denote by
$\gamma(v) = u(I_v, v)$ the **orbit**,
$\gamma^+(v) = u([0, t^+_v), v)$ the **positive semiorbit**,
$\gamma^-(v) = u((t^-_v, 0]), v)$ the **negative semiorbit**.

Definition. Call $v \in D$ satisfying $f(v) = 0$ a **stationary point** or **critical point** of the flow.

Lemma. If $v \in D$ is a stationary point of the flow u, then $I_v = \mathbf{R}$ and $\gamma(v) = \gamma^+(v) = \gamma^-(v) = \{v\}$.

Definition. Call $v \in D$ satisfying $u(0, v) = u(T, v)$ for some $T > 0$ a **periodic point** of period T of the flow. If in addition, $u(0, v) \neq u(t, v)$, $0 < t < T$, then T is called the **minimal period**.

Proposition 2. Let u be the flow determined by f and let $v \in D$. Then either:

1. v is a stationary point, or
2. v is a periodic point having a minimal positive period, or
3. the flow $u(\cdot, v)$ is injective.

If $\gamma^{+}(v)$ is relatively compact, then $t^{+}(v) = +\infty$. If $\gamma^{-}(v)$ is relatively compact, then $t^{-}(v) = -\infty$. If $\gamma(v)$ is relatively compact, then $I_v = \mathbf{R}$.

Proof.
The three options are mutually exclusive: a constant solution has no minimal period and an injective map cannot satisfy $u(t_1, v) = u(t_2, v)$ for $t_1 \neq t_2$. The first part of the proof assumes 1 and 2 do not hold and then 3 is proved.

The formulas for $t^{-}(v)$, $t^{+}(v)$ result from application of the extension theory for solutions of $u' = f(u)$. For example, if u remains bounded for $t \geq 0$, then the theory implies that the trajectory $(t, u(t))$ reaches the boundary at $t = \infty$.

Definition. A subset $M \subset D$ is **positively invariant** with respect to the the flow u determined by f whenever $\gamma^+(v) \subset M$, for all $v \in M$. A subset $M \subset D$ is **negatively invariant** provided $\gamma^-(v) \subset M$, for all $v \in M$. A subset $M \subset D$ is **invariant** provided it is both positively and negatively invariant.

Proposition 3. Let u be the flow determined by f and let $V \subset D$. Then there exists a smallest positively invariant subset M, $V \subset M \subset D$, and there exists a largest invariant set $\tilde{M}$, $\tilde{M} \subset V$. Also there exists a largest negatively invariant subset M, $V \supset M$, and there exists a smallest invariant set $\tilde{M}$, $\tilde{M} \supset V$. As a consequence V, contains a largest invariant subset and it is contained in a smallest invariant set.

Corollary 4.
(i) If a set M is positively invariant with respect to the flow u, then so are $\overline{M}$ and int(M).
(ii) A closed set M is positively invariant with respect to the flow u if and only if for every $v \in \partial M$ there exists $\epsilon > 0$ such that $u([0, \epsilon), v) \subset M$.
(iii) A set M is positively invariant if and only if comp(M), the complement of M, is negatively invariant.
(iv) If a set M is invariant, then so is ∂M. If ∂M is invariant, then so are $\overline{M}$, $\mathbf{R} \setminus M$, and int(M).

Theorem 5. Let $M \subset D$ be a closed set. Then M is positively invariant with respect to the flow u determined by f if and only if for every $v \in M$

$$\liminf_{t \to 0+} \frac{\text{dist}(v + tf(v), M)}{t} = 0.$$

Proof of Theorem 5. Taylor's expansion $u(t,v) = v + tf(v) + o(t)$ proves the necessity. To prove sufficiency, assume the identity holds and define $w(t) = \text{dist}(u(t,v), M) = |u(t,v) - v_t|$, where $v_t \in M$ and $\lim_{t\to 0+} v_t = v$. Using a Lipschitz constant L for f it follows that

$$w(t+s) \le w(t) + sLw(t) + \text{dist}(v_t + tf(v_t), M),$$

hence $D_+ w(t) \le Lw(t)$. This implies $w(t) = 0$ near $t = 0$, completing the proof.

Theorem 6. Consider $\phi \in C^1(D, R)$ with $\nabla\phi(v) \ne 0$ when $\phi(v) = 0$. Let $M = \phi^{-1}(-\infty, 0]$. Then M is positively invariant with respect to the flow determined by f if and only if $\nabla\phi(v) \cdot f(v) \le 0$ for all $v \in \partial M = \phi^{-1}(0)$.

Limit Sets

Definition. The **positive limit set** of v is the set $\Gamma^+(v)$ of all limits $w = \lim_{n\to\infty} u(t_n, v)$ where $\{t_n\}$ is a sequence with limit t_v^+. The **negative limit set** of v is the set $\Gamma^-(v)$ of all limits $w = \lim_{n\to\infty} u(t_n, v)$ where $\{t_n\}$ is a sequence with limit t_v^-. If t_v^+ is finite, then $\Gamma^+(v) \subset \partial D$. If t_v^- is finite, then $\Gamma^-(v) \subset \partial D$.

Proposition 7.
(i) $\overline{\gamma^+(v)} = \gamma^+(v) \cup \Gamma^+(v)$.
(ii) $\Gamma^+(v) = \cap_{w\in\gamma^+(v)} \overline{\gamma^+(w)}$.
(iii) If $\gamma^+(v)$ is bounded, then $\Gamma^+(v) \neq \emptyset$ and compact.
(iv) If $\Gamma^+(v) \neq \emptyset$ and bounded, then $\lim_{t\to t_v^+} \mathrm{dist}\Big(u(t,v), \Gamma^+(v)\Big) = 0$.
(v) $\Gamma^+(v) \cap D$ is an invariant set.

Proof of proposition 7–(v). To show that $\Gamma^+(v) \cap D$ is an invariant set, it will be shown that solutions through points of $\Gamma^+(v)$ are defined for all time and their orbit remains in $\Gamma^+(v)$.

Let $w \in \Gamma^+(v) \cap D$. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$, such that $u(t_n, v) \to w$. For each $n \geq 1$, the function $u_n(t) = u(t + t_n, v)$ is the unique solution of $u' = f(u)$, $u(0) = u(t_n, v)$, and hence the maximal interval of existence of u_n will contain the interval $[-t_n, \infty)$. Since $u(t_n, v) \to w$, there will exist a subsequence of $\{u_n(t)\}$, which we relabel as $\{u_n(t)\}$ converging to the solution, call it y, of $u' = f(u)$, $u(0) = w$. We note that given any compact interval $[a, b]$ the sequence $\{u_n(t)\}$ will be defined on $[a, b]$ for n sufficiently large and hence y will be defined on $[a, b]$. Since this interval is arbitrary it follows that y is defined on $(-\infty, \infty)$. Furthermore for any t_0

$$y(t_0) = \lim_{n \to \infty} u_n(t_0) = \lim_{n \to \infty} u(t_0 + t_n, v),$$

and hence $y(t_0) \in \Gamma^+(v)$.

Theorem 8. If $\gamma^+(v)$ is contained in a compact subset $K \subset D$, then $\Gamma^+(v) \neq \emptyset$ is a compact connected set, i.e., a continuum.

Proof. Proposition 7 implies $\Gamma^+(v)$ is compact. To be shown is connectedness. Suppose it is not. Then there exist nonempty disjoint compact sets M and N such that $\Gamma^+(v) = M \cup N$. Let $\delta = \inf\{|v - w| : v \in M, w \in N\} > 0$. Since $M \subset \Gamma^+(v)$ and $N \subset \Gamma^+(v)$, there exist values of t arbitrarily large such that $\text{dist}(u(t,v), M) < \frac{\delta}{2}$ and values of t arbitrarily large such that $\text{dist}(u(t,v), N) < \frac{\delta}{2}$ and hence there exists a sequence $\{t_n \to \infty\}$ such that $\text{dist}(u(t_n,v), M) = \frac{\delta}{2}$. The sequence $\{u(t_n,v)\}$ must have a convergent subsequence and hence has a limit point which is in neither M nor N, a contradiction.

LaSalle's Theorem

Let $\phi : D \to \mathbf{R}$ be a C^1 function. The notation $\phi'(v) = \nabla\phi(v) \cdot f(v)$ will be used.

Lemma 9. Assume that $\nabla\phi(v) \cdot f(v) \leq 0$, for all $v \in D$. Then for all $v \in D$, ϕ is constant on the set $\Gamma^+(v) \cap D$.

Proof. The function $\phi(u(t, v))$ is nonincreasing in t and therefore there is an inequality $\phi(u(s_n, v)) \leq \phi(u(t_k, v))$ valid for each t_k when s_n is sufficiently large. This inequality implies that $\phi(w_2) \leq \phi(w_1)$ for $w_1, w_2 \in \Gamma^+(v) \cap D$. Therefore, swapping roles of w_1 and w_2 gives $\phi(w_1) \leq \phi(w_2)$, hence $\phi(w) =$constant.

Theorem 10. Let there exist a compact set $K \subset D$ such that $\nabla\phi(v) \cdot f(v) \leq 0$, for all $v \in K$. Let $\tilde{K} = \{v \in K : \phi'(v) = 0\}$ and let M be the largest invariant set contained in $\tilde{K}$. Then for all $v \in D$ such that $\gamma^+(v) \subset K$

$$\lim_{t\to\infty} \text{dist}\,(u(t,v), M) = 0.$$

Proof. Let $v \in D$ such that $\gamma^+(v) \subset K$, then, using the previous lemma, we have that ϕ is constant on $\Gamma^+(v)$, which is an invariant set and hence contained in M.

Theorem 11 (LaSalle's Theorem). Assume that $D = \mathbf{R}^N$ and let $\nabla\phi(x) \cdot f(x) \leq 0$, for all $x \in \mathbf{R}^N$. Furthermore suppose that ϕ is bounded below and that $\phi(x) \to \infty$ as $|x| \to \infty$. Let $E = \{v : \phi'(v) = 0\}$, then

$$\lim_{t\to\infty} \text{dist}\,(u(t,v), M) = 0,$$

for all $v \in \mathbf{R}^N$, where M is the largest invariant set contained in E.

Two Dimensional Systems

Definition. A point $v \in D$ is a **regular point** if it is not a critical point of f, that is, $f(v) \neq 0$. A compact straight line segment $\ell \subset D$ through v is a **transversal through** v, provided ℓ contains only regular points and if for all $w \in \ell$, $f(w)$ is not parallel to the direction of ℓ.

Lemma 12. Let $v \in D$ be a regular point of f. Then there exists a transversal ℓ containing v in its relative interior. An orbit associated with f which crosses ℓ must cross always in the same direction.

Lemma 13. Let v be an interior point of some transversal ℓ. Then for every $\epsilon > 0$ there exists a circular disc D_ϵ with center at v such that for every $w \in D_\epsilon$, $u(t, w)$ will cross ℓ in time t, $|t| < \epsilon$.

Proof of Lemma 12. Let v be a regular point of f. Choose a neighborhood V of v consisting of regular points only. Let $\eta \in \mathbf{R}^2$ be any direction not parallel to $f(v)$, i.e., $\eta \times f(v) \neq 0$, (here $\times$ is the cross product in $\mathbf{R}^3$). We may restrict V further such that $\eta \times f(w) \neq 0$, for all $w \in V$, and is bounded away from 0 on V. We then may take ℓ to be the intersection of the straight line through v with direction η and $\overline{V}$. The proof is completed by observing that $\eta \times f(w) = (0, 0, |\eta||f(w)| \sin \theta)$, where θ is the angle between η and $f(w)$.

Proof of Lemma 13. Let $v \in \text{int}(\ell)$ and let $\ell = \{z : z = v + s\eta,\ s_0 \leq s \leq s_1\}$. Let B be a disc centered at v containing only regular points of f. Let $L(t, w) = au^1(t, w) + bu^2(t, w) + c$, where $u(t, w)$ is the solution with initial condition w and $au^1 + bu^2 + c = 0$ is the equation of the straight line containing ℓ. Then $L(0, v) = 0$, and $\frac{\partial L}{\partial t}(0, v) = (a, b) \cdot f(v) \neq 0$. We hence may apply the implicit function theorem to complete the proof.

Lemma 14. Let ℓ be a transversal and let $\Gamma = \{w = u(t,v) : a \leq t \leq b\}$ be a closed arc of an orbit u associated with f which has the property that $u(t_1,v) \neq u(t_2,v)$, $a \leq t_1 < t_2 \leq b$. Then if Γ intersects ℓ it does so at a finite number of points whose order on Γ is the same as the order on ℓ. If the orbit is periodic it intersects ℓ at most once.

The proof relies on the Jordan curve theorem:

Theorem (Jordan). If J is a curve in R^2 given by a continuous function $g : [0,1] \to R^2$ such that $g(0) = g(1)$ and $g(t) \neq g(s)$ for $0 < t < s < 1$, then the complement of J is the union of a unbounded open connected set Ext(J) and a bounded open connected set Int(J) such that J is the boundary of each set.

Lemma 15. Let $\gamma^+(v)$ be a semiorbit which does not intersect itself and let $w \in \Gamma^+(v)$ be a regular point of f. Then any transversal containing w in its interior contains no other points of $\Gamma^+(v)$ in its interior.

Lemma 16. Let $\gamma^+(v)$ be a semiorbit which does not intersect itself and which is contained in a compact set $K \subset D$ and let all points in $\Gamma^+(v)$ be regular points of f. Then $\Gamma^+(v)$ contains a periodic orbit.

Proof. Let $w \in \Gamma^+(v)$. it follows from Proposition 7 that $\Gamma^+(v)$ is an invariant set and hence that $\gamma^+(w) \subset \Gamma^+(v)$, and thus also $\Gamma^+(w) \subset \Gamma^+(v)$. Let $z \in \Gamma^+(w)$, and let ℓ be a transversal containing z in its relative interior. It follows that the semiorbit $\gamma^+(w)$ must intersect ℓ and by the above for an infinte number of values of t. On the other hand, the previous lemma implies that all these point of intersection must be the same.

Theorem 17 (Poincaré–Bendixson). Let $\gamma^+(v)$ be a semiorbit which does not intersect itself and which is contained in a compact set $K \subset D$ and let all points in $\Gamma^+(v)$ be regular points of f. Then $\Gamma^+(v)$ is the orbit of a periodic solution u_T with smallest positive period T.

Proof. It follows from Lemma 16 that every point in $\Gamma^+(v)$ is a point on some periodic orbit of a minimal positive period. On the other hand, it also follows from earlier results that $\Gamma^+(v)$ is a compact connected set. Hence, if for some $w \in \Gamma^+(v)$, $\gamma(w) \neq \Gamma^+(v)$, then $\Gamma^+(v) \setminus \gamma(w)$ must be a relatively open set with $A = \overline{\Gamma^+(v) \setminus \gamma(w)} \cap \gamma(w) \neq \emptyset$. One now easily obtains a contradiction by examining transversals through points of A. One hence concludes that in fact under the hypotheses of Lemma 16, the limit set $\Gamma^+(v)$ is a periodic orbit.

Theorem 18. Let Γ be a periodic orbit of $u' = f(u)$ which together with its interior is contained in a compact set $K \subset D$. Then there exists at least one singular point of f in the interior of D.

Proof. Let $\Omega = \text{interior}\Gamma$. Then f is continuous on $\overline{\Omega}$ and does not vanish on $\Gamma = \partial\Omega$. Let us assume that f has no stationary points in Ω. Then for each $w \in \Omega$, $\Gamma^+(w)$ is a periodic orbit. We partially order the collection $\{\Gamma_\alpha\}_{\alpha \in I}$, where I is an index set, of all periodic orbits which are contained in $\overline{\Omega}$, by saying that

$$\Gamma_\alpha \leq \Gamma_\beta \ \Leftrightarrow \text{interior}\Gamma_\alpha \subset \text{interior}\Gamma_\beta.$$

One now employs the Hausdorff minimum principle together with LaSalle's Theorem to arrive at a contradiction.