Chapter VIII: Stability

Definition (stability). Assume f continuous and $f(t,0) = 0$. The trivial solution of $u' = f(t,u)$ is:

(i) **stable** (s) on $[t_0, \infty)$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that any solution of $v' = f(t,v)$ with $|v(t_0)| < \delta$ exists on $[t_0, \infty)$ and satisfies $|v(t)| < \epsilon,\ t_0 \leq t < \infty$;

(ii) **asymptotically stable** (a.s) on $[t_0, \infty)$, if it is stable and $\lim_{t\to\infty} v(t) = 0$, where v is as in (i);

(iii) **unstable** (us), if it is not stable;

(iv) **uniformly stable** (u.s) on $[t_0, \infty)$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that any solution of $v' = f(t,v)$ with $|v(t_1)| < \delta,\ t_1 \geq t_0$ exists on $[t_1, \infty)$ and satisfies $|v(t)| < \epsilon,\ t_1 \leq t < \infty$;

(v) **uniformly asymptotically stable** (u.a.s), if it is uniformly stable and there exists $\delta > 0$ such that for all $\epsilon > 0$ there exists $T = T(\epsilon)$ such that any solution of $v' = f(t, v)$ with $|v(t_1)| < \delta,\ t_1 \geq t_0$ exists on $[t_1, \infty)$ and satisfies $|v(t)| < \epsilon,\ t_1 + T \leq t < \infty$;

(v) **strongly stable** (s.s) on $[t_0, \infty)$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that any solution of $v' = f(t, v)$ with $|v(t_1)| < \delta$ exists on $[t_0, \infty)$ and satisfies $|v(t)| < \epsilon,\ t_0 \leq t < \infty$.

Proposition 2. The following implications are valid for non-autonomous and autonomous equations:

$u' = f(t, u)$	$u' = F(u)$
u.a.s $\Rightarrow$ a.s $\Downarrow$ $\Downarrow$ s.s $\Rightarrow$ u.s $\Rightarrow$ s.	u.a.s $\Leftrightarrow$ a.s $\Downarrow$ $\Downarrow$ u.s $\Leftrightarrow$ s.

Stability of linear equations

Theorem 4. Let Φ be a fundamental matrix solution of $u' = A(t)u$. Then the equation is :

(i) **stable** iff there exists $K > 0$ such that $|\Phi(t)| \leq K$, $t_0 \leq t < \infty$;

(ii) **uniformly stable** iff there exists $K > 0$ such that $|\Phi(t)\Phi^{-1}(s)| \leq K$, $t_0 \leq s \leq t < \infty$;

(iii) **strongly stable** iff there exists $K > 0$ such that $|\Phi(t)| \leq K$, $|\Phi^{-1}(t)| \leq K$, $t_0 \leq t < \infty$;

(iv) **asymptotically stable** iff $\lim_{t\to\infty} |\Phi(t)| = 0$;

(v) **uniformly asymptotically stable** iff there exist $K > 0$, $\alpha > 0$ such that $|\Phi(t)\Phi^{-1}(s)| \leq Ke^{-\alpha(t-s)}$, $t_0 \leq s \leq t < \infty$.

Corollary 5. The equation $u' = Au$ is:
(i) **stable** iff every eigenvalue of A has non-positive real part and those with zero real part are semisimple.
(ii) **strongly stable** iff all eigenvalues of A have zero real part and are semisimple.
(iii) **asymptotically stable** iff all eigenvalues of A have negative real part.

Theorem 6. The equation $u' = A(t)u$ is **unstable** whenever $\limsup_{t\to\infty} \int_{t_0}^{t} \text{trace} A(s)ds = \infty$. If $u' = A(t)u$ is stable, then it is **strongly stable** if and only if $\liminf_{t\to\infty} \int_{t_0}^{t} \text{trace} A(s)ds > -\infty$.

Let $|\cdot|$ be a norm on R^N and let $\|\cdot\|$ satisfy for $N \times N$ matrices A, B the properties
(1) $\|Ax\| \leq \|A\||x|$,
(2) $\|A+B\| \leq \|A\| + \|B\|$,
(3) $\|cA\| = |c|\|A\|$.
Usually, (1), (2), (3) will be obtained by assuming (4) $\|A\| = \sup_{|x|=1} |Ax|$.

Definition 7. For an $N \times N$ matrix A, let $Q(A,t) = (\|I+tA\| - \|I\|)/t$. The **Lozinskii–Dahlquist** measure is $\mu(A) = \lim_{t \to 0+} Q(A,t)$.

Proposition 8. Assume (1), (2), (3). Then
1. $Q(A,t)$ is nondecreasing in t;
2. $|\mu(A)| \leq |Q(A,t)| \leq \|A\|$;
3. $\mu(\alpha A) = \alpha\mu(A),\ \alpha \geq 0$;
4. $\mu(A+B) \leq \mu(A) + \mu(B)$;
5. $|\mu(A) - \mu(B)| \leq \|A-B\|$.

Lemma. Let $|\cdot|$ be an R^N-norm and define $\|A\| = \sup_{|x|=1} |Ax|$. If $u' = A(t)u$ and $r(t) = |u(t)|$, then the right derivative $r'_+(t)$ exists and satisfies $r'_+(t) \leq \mu(A(t))\, r(t)$. Similarly, $r'_-(t)$ exists and satisfies $r'_-(t) \geq -\mu(-A(t))\, r(t)$.

Proposition 9. Let $|\cdot|$ be an R^N-norm and define $\|A\| = \sup_{|x|=1}|Ax|$. Let $A(t)$ be a continuous $N \times N$ matrix and let $u' = A(t)u$ on $t_0 \le t < \infty$. Then

$$|u(t)|e^{-\int_{t_0}^{t}\mu(A(s))ds} \le |u(t_0)| \le |u(t)|e^{\int_{t_0}^{t}\mu(-A(s))ds}.$$

The left side of this inequality is nonincreasing and the right side is nondecreasing, on $t_0 \le t < \infty$.

Corollary 10. If A is a constant $N \times N$ matrix, then $e^{-t\mu(-A)} \le |e^{tA}| \le e^{t\mu(A)}$.

Theorem 11. Let $|\cdot|$ be an R^N-norm and define $\|A\| = \sup_{|x|=1}|Ax|$. The system $u' = A(t)u$ is:

1. **unstable** if $\liminf_{t\to\infty}\int_{t_0}^{t}\mu(-A(s))ds = -\infty$;
2. **stable** if $\limsup_{t\to\infty}\int_{t_0}^{t}\mu(A(s))ds < \infty$;
3. **asymptotically stable** if $\lim_{t\to\infty}\int_{t_0}^{t}\mu(A(s))ds = -\infty$;
4. **uniformly stable** if $\mu(A(t)) \le 0,\ t \ge t_0$;
5. **uniformly asymptotically stable** if $\mu(A(t)) \le -\alpha < 0,\ t \ge t_0$.

Stability of $u' = A(t)u + f(t,u)$

Assume $f : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous and $|f(t,x)| \le \gamma(t)|x|$, $x \in \mathbf{R}^N$, where γ is positive and continuous. Let the $N \times N$ matrix A be continuous on $\mathbf{R}$. Denote by $\Phi(t)$ a fundamental matrix solution of $u' = A(t)u$. Variation of constants implies solutions of $u' = A(t)u + f(t,u)$ satisfy

$$u(t) = \Phi(t)\left(\Phi^{-1}(t_0)u(t_0) + \int_{t_0}^{t} \Phi^{-1}(s) f(s,u(s))ds\right).$$

Theorem 12. Assume $\int^{\infty} \gamma(s)ds < \infty$ and $|\Phi(t)\Phi^{-1}(s)| \le K$, $t_0 \le s \le t < \infty$. Then there exists a positive constant $L = L(t_0)$ such that any solution of $u' = A(t)u + f(t,u)$ is defined for $t \ge t_0$ and satisfies $|u(t)| \le L|u(t_1)|$, $t \ge t_1 \ge t_0$. Further, $\lim_{t\to\infty} |\Phi(t)| = 0$ implies $\lim_{t\to\infty} |u(t)| = 0$. Therefore, $u' = A(t)u + f(t,u)$ is u.s., u.a.s. or s.s. whenever the same is true for $u' = A(t)u$.

Theorem 14. Assume $|\Phi(t)\Phi^{-1}(s)| \le Ke^{-\alpha(t-s)}$, $t_0 \le s \le t < \infty$, and γ is constant, with $\beta = \alpha - \gamma K > 0$. Then any solution of $u' = A(t)u + f(t,u)$ exists for $t \ge t_0$ and satisfies $|u(t)| \le Ke^{-\beta(t-t_1)}|u(t_1)|$, $t \ge t_1 \ge t_0$. Further, $u' = A(t)u + f(t,u)$ is u.a.s. whenever the same is true for $u' = A(t)u$.

Lyapunov stability example

Let $|a| < 1$ and k be constants and consider the example

$$x' = ax - y + kx\left(x^2 + y^2\right),$$
$$y' = x - ay + ky\left(x^2 + y^2\right).$$

Lyapunov theory employs a **guiding function** v, in this case $v(x,y) = x^2 - 2axy + y^2$. The level curves $v(x,y) = c$ meet orbits $(x(t), y(t))$ transversally toward the origin, indirectly establishing asymptotic stability of the example.

Lyapunov stability

Assume $f(t,u)$ is continuous for $t \in R$, $u \in R^N$ and $f(t,0) = 0$. Let $v : R \times R^N \to R$ be given (a guiding function) with $v(t,0) = 0$.

Definition 16. The functional v is called:

1. **positive definite**, if there exists a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$, with $\phi(0) = 0$, $\phi(r) \neq 0$, $r \neq 0$ and $\phi(|x|) \leq v(t,x)$, $x \in \mathbf{R}^N$, $t \geq t_0$;

2. **radially unbounded**, if it is positive definite and $\lim_{r\to\infty} \phi(r) = \infty$;

3. **decrescent**, if it is positive definite and there exists a continuous increasing function $\psi : [0,\infty) \to [0,\infty)$, with $\psi(0) = 0$, and $\psi(|x|) \geq v(t,x)$, $x \in \mathbf{R}^N$, $t \geq t_0$.

Theorem 17. Let there exist a positive definite functional v and $\delta_0 > 0$ such that for every solution of $u' = f(t,u)$ with $|u(t_0)| \le \delta_0$, the function $v^*(t) = v(t,u(t))$ is nonincreasing with respect to t. Then the trivial solution of $u' = f(t,u)$ is stable.

Proof: Inequality $|u(t)| < \epsilon$ results from proving $\phi(|u(t)|) < \phi(\epsilon)$. The latter follows from $\phi(|u(t)|) \le v(t,u(t)) \le v(t_0,u_0) < \phi(\epsilon)$, valid for all initial conditions $u(t_0) = u_0$ near zero.

Theorem 18. Let there exist a positive definite functional v which is decrescent and $\delta_0 > 0$ such that for every solution of $u' = f(t,u)$ with $|u(t_1)| \le \delta_0$, $t_1 \ge t_0$ the function $v^*(t) = v(t,u(t))$ is nonincreasing with respect to t, then the trivial solution of $u' = f(t,u)$ is uniformly stable.

Proof: Stability follows from Theorem 17. The chain of inequalities $\phi(|u(t)|) \le v(t,u(t)) \le v(t_1,u_0) \le \psi(|u_0|)$ plus $\psi(0) = 0$ implies uniform stability.

2D Uniform Stability example

Let $a(t)$, $b(t)$ be continuous with $b(t) \leq 0$. Consider the two dimensional system

$$\begin{aligned} x' &= a(t)y + b(t)x\left(x^2+y^2\right), \\ y' &= -a(t)x + b(t)y\left(x^2+y^2\right). \end{aligned}$$

Uniform stability will be established by applying Theorem 18.

Choose guiding function $v(x,y) = x^2+y^2$. Compute

$$\begin{aligned} \frac{dv^*}{dt} &= \frac{\partial v}{\partial t} + \nabla v \cdot f(t,u), \\ &= 2b(t)\left(x^2+y^2\right)^2 \\ &\leq 0. \end{aligned}$$

The function v is positive definite and decrescent. Theorem 18 applies.

Theorem 19 (Instability). The trivial solution of $u' = f(t, u)$ is unstable, provided there exists a continuous functional $v : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ with these properties:

1. There exists a continuous increasing function $\psi : [0, \infty) \to [0, \infty)$, such that $\psi(0) = 0$ and $|v(t, x)| \le \psi(|x|)$;

2. For all $\delta > 0$, and $t_1 \ge t_0$, there exists x_0, $|x_0| < \delta$ such that $v(t_1, x_0) < 0$;

3. If $u' = f(t, u)$, $u(t) = x$, then

$$\lim_{h \to 0+} \frac{v(t + h, u(t + h)) - v(t, x)}{h} \le -c(|x|),$$

where c is a continuous increasing function with $c(0) = 0$.

Theorem 20 (Asymptotic stability). Let there exist a positive definite functional $v(t,x)$ such that

$$\frac{dv^*}{dt} = \frac{dv(t,u(t))}{dt} \leq -c(v(t,u(t))$$

for every solution of $u' = f(t,u)$ with $|u(t_0)| \leq \delta_0$, where c a continuous increasing function with $c(0) = 0$. Then the trivial solution of $u' = f(t,u)$ is asymptotically stable. If v is also decrescent, then the trivial solution is uniformly asymptotically stable.

Proof: The plan is to prove that $u(t)$ exists on $t \geq t_0$ and $\lim_{t\to\infty} \phi(|u(t)|) = 0$. Then $\lim_{t\to\infty} |u(t)| = 0$, proving asymptotic stability.

Perturbed linear systems

Consider the equation $u' = Au + g(t,u)$, where A is a constant $N \times N$ matrix and $g : [t_0, \infty) \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous with $g(t,x) = o(|x|)$, uniformly for $t \in [t_0, \infty)$.

We seek a quadratic guiding function $v(x) = x^T Bx$, where B is a constant $N \times N$ matrix. The matrix $C = A^T B + BA$ appears in stability criteria, because of the following computation:

$$\begin{aligned}\frac{dv^*}{dt} &= \frac{dv(t,u(t))}{dt} \\ &= u^T \left(A^T B + BA\right) u + \\ &\quad g^T(t,u)Bu + u^T Bg(t,u).\end{aligned}$$

Proposition 21. Let A be a constant $N \times N$ matrix having the property that for any eigenvalue λ of A, $-\lambda$ is not an eigenvalue of A. Then for any $N \times N$ matrix C, there exists a unique $N \times N$ matrix B such that $C = A^T B + BA$.

Corollary 22. Let A be a constant $N \times N$ matrix. Then for any $N \times N$ matrix C, there exists $\mu > 0$ and a unique $N \times N$ matrix B such that $2\mu B + C = A^T B + BA$.

Proof: Apply Proposition 21 to $A - \mu I$ and C, for $0 < \mu < \mu_0$, where μ_0 is small.

Corollary 23. Let A be a constant $N \times N$ matrix having the property that all eigenvalues λ of A have negative real parts. Then for any negative definite $N \times N$ matrix C, there exists a unique positive definite $N \times N$ matrix B such that $C = A^T B + BA$.

Corollary 24. A necessary and sufficient condition that an $N \times N$ matrix A have all of its eigenvalues with negative real part is that there exists a unique positive definite matrix B such that $A^T B + BA = -I$.

Definition. An $N \times N$ matrix A is called **critical** if all its eigenvalues have nonpositive real part and there exists at least one eigenvalue with zero real part. It is alled **noncritical** otherwise.

Theorem 26. Assume A is a noncritical $N \times N$ matrix and let $g : [t_0, \infty) \times R^N \to R^N$ be continuous with $g(t, x) = o(|x|)$ uniformly in t. Then the stability behavior of the trivial solution of $u' = Au + g(t, u)$ is the same as that of the trivial solution of $u' = Au$, i.e., the trivial solution of $u' = Au + g(t, u)$ is uniformly asymptotically stable if all eigenvalues of A have negative real part and it is unstable if A has an eigenvalue with positive real part.