## Chapter VII: Periodic Solutions

Let $A(t)$ and $g(t)$ be continuous and $T$-periodic. Let $\Phi(t)$ be a fundamental matrix for $u' = A(t)u$, with $\Phi(0) = I$, so that $\Phi(t) = C(t)e^{Rt}$ for some class $C^1$ nonsingular $T$-periodic matrix $C(t)$ and constant matrix $R$ satisfying $e^{RT} = \Phi(T)$.

The solution of $u' = A(t)u + g(t)$ by variation of constants is given by

$$u(t) = \Phi(t)u(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds.$$

Then

$$u(T) = e^{RT}\left(u(t_0) + \int_0^T \Phi^{-1}(s)g(s)ds\right).$$

**Proposition 1**. The equation $u' = A(t)u + g(t)$ has a unique $T-$periodic solution for every $T-$periodic forcing term $g$ if and only if $e^{TR} - I$ is nonsingular.

The periodic solution $u$ is given by $u(t) = Qg(t)$ where $Q$ is defined by the formula

$$Qg(t) = \Phi(t)(I - e^{TR})^{-1} e^{TR} \int_0^T \Phi^{-1}(s)g(s)ds$$

$$+ \Phi(t) \int_0^t \Phi^{-1}(s)g(s)ds.$$

**Proposition 2 (Fixed points)**. Let $f(t,u)$ be continuous and $T$-periodic in $t$. Assume $I - e^{TR}$ nonsingular. Let $E = \{u \in C([0,T], \mathbf{R}^N) : u(0) = u(T)\}$ with $\|u\| = \max_{t \in [0,T]} |u(t)|$. Define $F(u)(t) = f(t, u(t))$ and $S = QF$. Then $u' = A(t)u + f(t,u)$ has a $T$-periodic solution $u$ whenever the operator $S$ has a fixed point in the space $E$.

**Definition**. An equation $u' = A(t)u + f(t,u)$ with $A(t)$ and $f(t,u)$ continuous and $T$-periodic in $t$ is called **nonresonant** if $u' = A(t)u$ has only the trivial $T$-periodic solution $u \equiv 0$. Otherwise, it is called **resonant**.

**Theorem 3**. Let $A(t)$ and $f(t,u)$ be continuous and $T$-periodic in $t$. Define $P(r) = \max\{|f(t,u)| : 0 \le t \le T,\ |u| \le r\}$. Assume $I - e^{TR}$ is nonsingular and

$$\liminf_{r\to\infty} \frac{P(r)}{r} = 0.$$

Then $u' = A(t)u + f(t,u)$ has a $T-$ periodic solution $u$.

**Corollary 4**. For $\epsilon$ small, $u' = A(t)u + \epsilon f(t,u)$ has a $T$-periodic solution $u$.

**Proof of Theorem 3**: Let $S = QF$ as in Proposition 2. Then $S$ is completely continuous and $\|Su\| \le KP(r)$ for $\|u\| \le r$. Schauder's fixed point theorem applies to $S$ on a large ball in $E$.

**Continua of $T$-periodic solutions.**

**Theorem 5.** Let $A(t)$ and $f(t,u)$ be continuous and $T$-periodic in $t$. Assume $I - e^{TR}$ is nonsingular. Let

$$\mathbf{S}^+ = \{(u,\epsilon) \in E \times [0,\infty) : u' = A(t)u + \epsilon f(t,u)\}.$$

Then there exists a continuum $C^+ \subset \mathbf{S}^+$ such that

(1) $C_0^+ \cap E = \{0\}$,
(2) $C^+$ is unbounded in $E \times [0,\infty)$.

Similarly, let

$$\mathbf{S}^- = \{(u,\epsilon) \in E \times (-\infty,0] : u' = A(t)u + \epsilon f(t,u)\}.$$

Then there exists a continuum $C^- \subset \mathbf{S}^-$ such that

(3) $C_0^- \cap E = \{0\})$,
(4) $C^-$ is unbounded in $E \times (-\infty,0]$.

## Resonant Equations

Consider the periodic BVP $u' = f(t,u)$, $u(0) = u(T)$ where $f$ is continuous and $T$-periodic in $t$, as a perturbation of $u' = 0$.

**Lemma 6**. Let $E = \{u \in C([0,T], \mathbf{R}^N)\}$ with $\|u\| = \max_{t\in[0,T]} |u(t)|$ and define $(Su)(t) = u(T) + \int_0^t f(s,u(s))ds$, a map from $E$ into $E$. The map $S$ is completely continuous and fixed points $u = Su$ are solutions of $u' = f(t,u)$, $u(0) = u(T)$.

**Theorem 7**. Assume that $f(t,u)$ is continuous and there exists a bounded open set $\Omega \subset \mathbf{R}^N$ such that the $\mathbf{R}^N$–map $g(x) = -\int_0^T f(s,x)ds$ does not vanish for $x \in \partial\Omega$. Further assume that the Brouwer degree $\mathrm{d}(g, \Omega, 0)$ is nonzero. Then the problem $u' = \epsilon f(t,u)$, $u(0) = u(T)$ has a solution for all sufficiently small $\epsilon$.

**Corollary 8**. Assume the hypotheses of Theorem 7. Let all solutions of $u' = \epsilon f(t,u)$, $u(0) = u(T)$ satisfy $u \notin \partial G$ where $G = \{u \in E : u(t) \in \Omega\}$ and $0 < \epsilon \leq 1$. Then $u' = f(t,u)$ has a $T$-periodic solution.

**Proof of Theorem 7**. Let $S(u,\lambda,\epsilon)$ equal

$$u(T)+\int_0^{T+\lambda(t-T)} \epsilon f(s,\lambda(u(s)-u(T))+u(T))ds.$$

We prove

**7-A**. $S(\cdot,\lambda,\epsilon)$ is continuous and compact.

**7-B**. $Id - S(\cdot,\lambda,\epsilon) \neq 0$ on $\partial G$ for small $\epsilon$.

**7-C**. $\mathrm{d}(Id - S(\cdot,\lambda,\epsilon), G, 0) \neq 0$.

Then $u = S(u,1,\epsilon)$ by **7-C** and the solution property of degree. Setting $t = 0$ in this equation gives $u(0) = u(T)$. By differentiation, $u' = \epsilon f(t,u(t))$. This completes the proof.

## A Liénard equation

Corollary 8 applies to prove the existence of a $T$-periodic solution to special Liénard equations.

**Theorem 9**. Let $h$ be continuous and assume $e(t)$ is continuous $T$-periodic with $T < 2\pi$. Then $x'' + h(x)x' + x = e(t)$ has a $T$-periodic solution $x(t)$.

**Proof outline**. The condition $\int_0^T e(t)dt = 0$ can be assumed by changing variables. Let $\Omega$ be $|x| < R$, $|y| < R$ for large $R$. Then $G$ is the set of continuous functions $t \to (x(t), y(t))$ with $|x(t)| < R$, $|y(t)| < R$. The conditions in Corollary 8 are satisfied by proving solutions to $x'' + \epsilon h(x)x' + \epsilon^2 x = \epsilon e(t)$ satisfy $|x(t)| < M$, $|x'(t)| < M$ for some constant $M$, $0 < \epsilon \leq 1$.

## Partial resonance

**Theorem 10**. Let $f(t,u,v)$, $h(t,u,v)$ be $T$-periodic in $t$ and continuous for all $t \in R^1$, $u \in R^p$, $v \in R^q$ with values in $R^p$. Assume $B$ is $q \times q$ and $v' = Bv$ has the unique $T$-periodic solution $v = 0$. Let $g(x) = -\int_0^T f(s,x,0)ds$ for $x \in R^p$ and assume $g(x) \neq 0$ on the boundary of an open set $\Omega \subset R^p$. Then for all small $\epsilon$ there is a solution $u$, $v$ to the problem $u' = \epsilon f(t,u,v)$, $v' = By + \epsilon h(t,u,v)$, $u(0) = u(T)$, $v(0) = v(T)$.

**Corollary 11**. If in the proof of Theorem 10, for $0 < \epsilon \leq 1$, $u$, $v$ solutions of $u' = \epsilon f(t,u,v)$, $v' = By + \epsilon h(t,u,v)$, $u(0) = u(T)$, $v(0) = v(T)$ implies $u$, $v \notin \partial G$, then $u' = f(t,u,v)$, $v' = By + h(t,u,v)$ has a $T$-periodic solution pair.