Initial Value Problem

Let D be an open connected subset of $\mathbf{R}\times\mathbf{R}^N$, $N \geq 1$, and let $f : D \to \mathbf{R}^N$ be a continuous mapping.

Definition (Solution). Let I be an interval in $\mathbf{R}$. A function $u \in C^1(I,\mathbf{R}^N)$ such that $(t,u(t)) \in D$ and $u'(t) = f(t,u(t))$, $t \in I$, is called a **solution**.

Definition (IVP). Let $(t_0,u_0) \in D$. An **initial value problem** is the problem of finding an open interval I containing t_0 and a function $u \in C^1(I,\mathbf{R}^N)$ such that $(t,u(t)) \in D$, $u'(t) = f(t,u(t))$ for $t \in I$, and $u(t_0) = u_0$.

Theorem. Let I be an open interval containing t_0. Assume $u \in C^1(I,\mathbf{R}^N)$ and $(t,u(t)) \in D$ for $t \in I$. Then u is a solution of the initial value problem $u'(t) = f(t,u(t))$, $u(t_0) = u_0$ if and only if $u(t) = u(t_0) + \int_{t_0}^t f(s,u(s))ds$.

Picard–Lindelöf Theorem
Let $f : D \to R^N$ satisfy a *local Lipschitz condition*, that is, $|f(t,u) - f(t,v)| \le L|u - v|$ for $(t,u), (t,v)$ in each compact set $K \subset D$, with L depending on K.

If $(t_0, u_0) \in D$, then there exists an interval $I : |t - t_0| < r$ and a function $u \in C^1(I, R^N)$ such that $u'(t) = f(t, u(t))$ for all $t \in I$, and $u(t_0) = u_0$. Furthermore, any two such functions u defined on I must be equal, that is, the solution of the IVP is unique.

Remark. The proof actually constructs a parallelepiped Q inside D which contains the graph of the solution curve. Construct Q as $|t - t_0| \le a$, $|u - u_0| \le b$, with $Q \subset D$. Then define $m = \max |f(t,u)|$ on Q and let $r = \min(a, b/m)$. Then $|u(t) - u_0| \le b$ for $|t - t_0| \le r$, hence the solution curve remains inside Q.

Picard Iteration

The solution $u(t)$ given by Picard's theorem can be written in the limit form

$$u(t) = \lim_{n\to\infty} u_n(t)$$

or the equivalent series form

$$u(t) = u_0 + \sum_{n=0}^{\infty} (u_{n+1}(t) - u_n(t)).$$

Here, $u_0(t) \equiv u_0$ and $u_n(t)$ is defined to be the nth Picard iterate for the IVP, given explicitly by

$$u_n(t) = u_0 + \int_0^t f(s, u_{n-1}(s))ds.$$

The series is the only known explicit formula for the solution provided by Picard's theorem. However, you are warned that it is currently thought by experts to be completely impractical. Nevertheless, the formula for $u(t)$ remains important to applications, because this theoretical solution is computed by all numerical schemes.

Computer Algebra Systems

Closed-form solutions to differential equations can be found using computer algebra systems like **maple**. Implied is the summation of the Picard series solution to the resulting formula, which must represent the unique solution.

Intuition about differential equations can be improved by becoming familiar with the extensive number of equation types which have closed-form solutions. Noteworthy are:

quadrature, separable, linear,

Bernoulli, homogeneous A, B, C, G,

exact, Riccati, d'Alembert, Abel, Chini.

For more types, consult **maple** help for `dsolve`.

Cauchy-Peano Theorem
Let $f : D \to R^N$ be continuous. If $(t_0, u_0) \in D$, then there exists an interval $I : |t - t_0| < r$ and a function $u \in C^1(I, R^N)$ such that $u'(t) = f(t, u(t))$ for all $t \in I$, and $u(t_0) = u_0$.

Example. The IVP $u' = 3u^{2/3}$, $u(0) = 0$ has two solutions, $u = 0$ and $u = x^3$. Therefore, the solution predicted by Peano's theorem need not be unique.

Example. The IVP $u' = H(t)$, $u(0) = 0$ has a unique solution $u(t) = tH(t)$ (H is Heaviside's unit step). However, both Picard's theorem and Peano's theorem fail to apply, because $f(t, u) = H(t)$ is discontinuous.

Carathéodory Conditions

Let Q be a parallelepiped in D, defined by inequalities $|u - u_0| \leq b$, $|t - t_0| \leq a$. A function $f : D \to R^N$ is said to satisfy Carathéodory conditions on Q provided (1) $f(t, u)$ is measurable in t for each fixed u and continuous in u for almost all t, (2) $|f(t, u)| \leq m(t)$ for $(t, u) \in Q$, for some function $m \in L^1$ on $|t - t_0| < a$.

Theorem (Carathédory). Let f satisfy for each parallelepiped Q contained in D a Carathéodory condition. If $(t_0, u_0) \in D$, then there exists an interval $I : |t - t_0| < r$ and a function $u \in AC(I, R^N)$ such that $u'(t) = f(t, u(t))$ for almost all $t \in I$, and $u(t_0) = u_0$.

Extension of Solutions

Lemma 5. Assume $f : D \to R^N$ is continuous and f is bounded by a constant m on a subdomain $D_0 \subset D$. Let $u(t)$ be a solution of $u' = f(t, u)$ with $(t, u(t)) \in D_0$ on $a < t < b$. Then $u(t)$ satisfies a Lipschitz condition $|u(t_1) - u(t_2)| \le m|t_1 - t_2|$ and hence $u(t)$ has one-sided limits at $t = a$ and $t = b$: $\lim_{t \to a+} u(t)$ and $\lim_{t \to b-} u(t)$ exist and are finite.

Definition. A solution $u(t)$ of $u' = f(t, u)$ has **maximal interval of existence** $(\omega-, \omega+)$ provided $u(t)$ cannot be continued as a solution to the right of $t = \omega+$ nor to the left of $t = \omega-$.

Theorem 6 (Extension). Let $f : D \to R^N$ be continuous and assume $u(t)$ is a solution of $u' = f(t, u)$ defined on some t-interval. Then $u(t)$ may be extended as a solution to a maximal interval of existence $(\omega-, \omega+)$ and the solution curve $(t, u(t)) \to \partial D$ as t approaches the endpoints $\omega-$, $\omega+$.

Partially and Linearly Ordered Sets

Let $(P, \leq)$ be a **partially ordered set**, that is, a set P and a relation $\leq$ in $P \times P$ satisfying
(1) $x \leq x$ [reflexive],
(2) If $x \leq y$, $y \leq x$, then $x = y$ [antisymmetric]
(3) $x \leq y$ and $y \leq z$ implies $x \leq z$ [transitive]
If also
(4) $x, y \in P$ implies $x \leq y$ or $y \leq x$ [trichotomy], then $\leq$ is called a **linear ordering** on P. The notation $x < y$ means $x \leq y$ and $x \neq y$. An element $m \in P$ is **maximal** if $x \in P$ and $m \leq x$ implies $x = m$. A **chain** in a partially ordered set $(P, \leq)$ is a set $C \subset P$ which is linearly ordered by $\leq$.

Hausdorff Maximum Principle

Every nonvoid partially ordered set $(P, \leq)$ contains a linearly ordered subset C such that if D is linearly ordered and $C \subset D$, then $D = C$. Briefly, C is a maximal chain.

Finite Blowup of Solutions

Corollary 7. Assume $f : [t_0, t_0 + a] \times R^N \to R^N$ is continuous and let $u(t)$ be a solution of $u' = f(t, u)$ defined on a maximal interval of existence $I \subset [t_0, t_0 + a]$ with $t_0 \in I$. Then either $I = [t_0, t_0 + a]$ or else $I = [t_0, \omega+)$ with $\omega+ < t_0 + a$ and $|u(t)| \to \infty$ as $t \to \omega+$.

Corollary 8. Assume $f : (a, b) \times R^N \to R^N$ is continuous and $|f(t, u)| \leq \alpha(t)|u| + \beta(t)$ where $\alpha, \beta \in L^1(a, b)$ are nonnegative continuous functions. Then the maximal interval of existence of each solution of $u' = f(t, u)$ is (a, b).

Dependence upon t_0 and u_0

Theorem 9. Let D be an open connected subset of $\mathbf{R} \times \mathbf{R}^N$, $N \geq 1$. Assume that $f : D \to \mathbf{R}^N$ is a continuous mapping and that $u' = f(t, u)$, $u(t_0) = u_0$ has a unique solution $u(t) = u(t, t_0, u_0)$, for every $(t_0, u_0) \in D$. Then the solution depends continuously on (t_0, u_0), in the following sense: If $\{(t_n, u_n)\} \subset D$ converges to $(t_0, u_0) \in D$, then given $\epsilon > 0$, there exists n_ϵ and an interval I_ϵ such that for all $n \geq n_\epsilon$, the solution $u_n(t) = u(t, t_n, u_n)$, exists on I_ϵ and $\max_{t \in I_\epsilon} |u(t) - u_n(t)| \leq \epsilon$.

Dependence upon t_0, u_0 and f

Corollary 10. Assume that $f_n : D \to \mathbf{R}^N$, $n = 1, 2, \cdots$, are continuous mappings and that $u' = f_n(t, u)$, $u(t_n) = u_n$ has a unique solution $u_n(t) = u(t, t_n, u_n)$, for every $(t_n, u_n) \in D$. Then the solution depends continuously on (t_0, u_0), in the following sense: If $\{(t_n, u_n)\} \subset D$ converges to $(t_0, u_0) \in D$, and f_n converges to f, uniformly on compact subsets of D, then given $\epsilon > 0$, there exists n_ϵ and an interval I_ϵ such that for all $n \geq n_\epsilon$, the solution $u_n(t) = u(t, t_n, u_n)$, exists on I_ϵ and $\max_{t \in I_\epsilon} |u(t) - u_n(t)| \leq \epsilon$.

Differentiation of $u(t, t_0, u_0)$ in u_0

Theorem (differentiability). Assume that $f : D \to \mathbf{R}^N$ and its Fréchet derivative $D_u f(t, u)$ are continuous. Then the initial value problem $u' = f(t, u)$, $u(t_0) = u_0$ has a unique solution $u(t) = u(t, t_0, u_0)$ of class C^1 in the variables t, t_0 and u_0. Further, if $J(t) = D_u f(t, u)$ with $u = u(t, t_0, u_0)$, then $D_{u_0} u(t, t_0, u_0)\mathbf{h}$ is the solution y of the initial value problem $y' = J(t)y$, $y(t_0) = \mathbf{h}$ and $D_{t_0} u(t, t_0, u_0)$ is the vector given by the product $-D_{u_0} u(t, t_0, u_0) f(t_0, u_0)$.

Dini Derivatives

$$
\begin{aligned}
D^{+}u(t) &= \limsup_{h\to 0+} \tfrac{u(t+h)-u(t)}{h}, \\
D_{+}u(t) &= \liminf_{h\to 0+} \tfrac{u(t+h)-u(t)}{h}, \\
D^{-}u(t) &= \limsup_{h\to 0-} \tfrac{u(t+h)-u(t)}{h}, \\
D_{-}u(t) &= \liminf_{h\to 0-} \tfrac{u(t+h)-u(t)}{h},
\end{aligned}
$$

The operations lim sup and lim inf are taken componentwise.

Definition (Kamke Functions). A function $f : \mathbf{R}^N \to \mathbf{R}^N$ is said to be of type K (after Kamke) on a set $S \subset \mathbf{R}^N$, whenever $f^i(x) \leq f^i(y)$ for all $x, y \in S$, $x \leq y$, $x^i = y^i$.

Estimates of Solutions

Theorem 13. Assume $f : [a,b] \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous and f is of type K for each fixed t. Let $u : [a,b] \to \mathbf{R}^N$ be a solution of $u' = f(t,u)$.

If $v : [a,b] \to \mathbf{R}^N$ is continuous and satisfies $D^- v(t) > f(t, v(t))$ for $a < t \leq b$ and $v(a) > u(a)$, then $v(t) > u(t)$ for $a \leq t \leq b$.

If $z : [a,b] \to \mathbf{R}^N$ is continuous and satisfies $D_- z(t) < f(t, z(t))$ for $a \leq t < b$ and $z(a) < u(a)$, then $z(t) < u(t)$ for $a \leq t \leq b$.

Definition. A solution u^* of $u' = f(t,u)$ is called a **right maximal solution** on interval I if for each $t_0 \in I$ and any solution u with $u(t_0) \leq u^*(t_0)$ satisfies $u(t) \leq u^*(t)$ for $t_0 \leq t \in I$. A **right minimal solution** is defined similarly.

Existence of Maximal Solutions

Theorem 15. Assume $f : D \to \mathbf{R}^N$ is continuous and of type K for each fixed t. Then the initial value problem $u' = f(t, u)$, $u(t_0) = u_0$ has a unique right maximal (minimal) solution for each $(t_0, u_0) \in D$.

Proof: The idea of the proof is to apply the previous theorem to approximations of the IVP obtained by replacing f by $f+\epsilon$ and u_0 by $u_0+\epsilon$.

Theorem 16. Assume $f : [a, b] \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous and of type K for each fixed t. Let $v, z : [a, b] \to \mathbf{R}^N$ be continuous and satisfy $D^+v(t) \geq f(t, v(t))$, $D_+z(t) \leq f(t, z(t))$, $z(t) \leq v(t)$ for $a \leq t < b$. Then for every u_0, $z(a) \leq u_0 \leq v(a)$, there exists a solution u of $u' = f(t, u)$, $u(a) = u_0$ such that $z(t) \leq u(t) \leq v(t)$, $a \leq t \leq b$.

Definition. The function z in theorem 16 is called a **sub-solution**; the function v is called a **super-solution**.

A Priori Bounds

Corollary 17. Assume the hypotheses of Theorem 16 and let f satisfy a local Lipschitz condition. Assume furthermore that $z(a) \leq z(b)$, $v(a) \geq v(b)$. Then the problem $u' = f(t, u)$, $u(a) = u(b)$ has a solution u with $z(t) \leq u(t) \leq v(t)$, $a \leq t \leq b$.

Theorem 18. Assume that $F : [a, b] \times \mathbf{R}_+ \to \mathbf{R}_+$ is continuous and $f : [a, b] \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous also and $|f(t, x)| \leq F(t, |x|)$, $a \leq t \leq b$, $x \in \mathbf{R}^N$. Let $u : [a, b] \to \mathbf{R}^N$ be a solution of $u' = f(t, u)$ and let $v : [a, b] \to \mathbf{R}_+$ be the continuous and right maximal solution of $v'(t) = F(t, v(t))$, $a \leq t \leq b$, $v(a) \geq |u(a)|$. Then $v(t) \geq |u(t)|$, $a \leq t \leq b$.

Uniqueness

Theorem 19. Assume that $F : (a, b) \times \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous mapping and that $f : (a, b) \times \mathbf{R}^N \to \mathbf{R}^N$ is a continuous also and $|f(t, x) - f(t, y)| \leq F(t, |x - y|)$, $a \leq t \leq b$, $x, y \in \mathbf{R}^N$. Let $F(t, 0) \equiv 0$ and let, for any $c \in (a, b)$, $w \equiv 0$ be the only solution of $w' = F(t, w)$ on (a, c) such that $w(t) = 0(\mu(t))$, $t \to a$ where μ is a given positive and continuous function. Then $u' = f(t, u)$ cannot have distinct solutions u, v such that $|u(t) - v(t)| = 0(\mu(t))$, $t \to a$.