## Continuation Principle

Let $E$ be a real Banach space. Let $O \subset E \times [a,b]$ be an open bounded set in the relative topology of $E \times [a,b]$. Define $O_\lambda = \{u \in E : (u,\lambda) \in O\}$.

**Generalized Homotopy Principle**.
Let $F : \bar{O} \to E$ be a completely continuous mapping. Define $f(u,\lambda) = u - F(u,\lambda)$ and assume that $f(u,\lambda) \neq 0$ for $(u,\lambda)$ in the boundary of $O$. Then for $a \leq \lambda \leq b$

$$\mathrm{d}(f(\cdot,\lambda), O_\lambda, 0) = \text{constant}.$$

**Leray–Schauder Continuation**.
Assume $O$ and $f$ as in the generalized homotopy principle. Assume $\mathrm{d}(f(\cdot,a), O_a, 0) \neq 0$ and define $S = \{(u,\lambda) \in \bar{O} : f(u,\lambda) = 0\}$. Then there exists a closed connected set $C \subset S$ such that $C_a \cap O_a \neq \emptyset$ and $C_b \cap O_b \neq \emptyset$.

**Example**

Let $I = [0,1]$ and let $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ be continuous. Consider the nonlinear Dirichlet problem

$$\begin{cases} u'' + g(x,u) & = & 0, \quad 0 \le x \le 1, \\ u(x) & = & 0, \quad x \in \partial I. \end{cases}$$

Let there exist constants $a < 0 < b$ such that $g(x,a) > 0 > g(x,b)$, $x \in \Omega$. Then the Dirichlet problem has a solution $u \in C^2([0,1],\mathbf{R})$ such that $a < u(x) < b$, $x \in I$.

**Proof**: Let $E = C[0,1]$. Write the problem as $u = \lambda LG(u)$, $u \in E$, where $w = LG(u)$ solves $w'' + g(x,u(x)) = 0$, $w(0) = w(1) = 0$. Let $O$ be all $(u,\lambda)$ with $u \in E$, $0 \le \lambda \le 1$, $a < u(x) < b$. We show $u = \lambda LG(u)$ implies $(u,\lambda) \notin \partial O$ and $\mathrm{d}(I - \lambda LG, O_\lambda, 0) = 1$. Now apply the Leray-Schauder continuation theorem.

## Globalization of Implicit Functions

Assume that $F : E \times \mathbf{R} \to E$ is a completely continuous mapping and consider the equation $f(u, \lambda) \equiv u - F(u, \lambda) = 0$. Let $f(u_0, \lambda_0) = 0$ and suppose $f_u(u_0, \lambda_0)$ is a homeomorphism.

The implicit function theorem implies that the equation

$$f(u, \lambda) = 0$$

has a solution $u = u(\lambda)$ defined in a neighborhood of $\lambda = \lambda_0$ such that $u(\lambda_0) = u_0$. If $O$ is a suitable small neighborhood of $u = u_0$, then the proof of the implicit function theorem shows that $\text{d}(f(\cdot, \lambda_0), O, 0) \neq 0$.

The *globalization* refers to the existence of a continuum of solutions $(u, \lambda)$ emanating from $(u_0, \lambda_0)$, extending to both $\lambda = \infty$ and $\lambda = -\infty$.

**Global Implicit Function Theorem**
Let $O$ be a bounded open subset of $E$ and assume that the equation $f(u, \lambda_0) = 0$ has a unique solution $u = u_0$ in $O$. Suppose that $\mathrm{d}(f(\cdot, \lambda_0), O, 0) \neq 0$.

Define

$$\mathbf{S}^+ = \{(u, \lambda) \in E \times [\lambda_0, \infty) : f(u, \lambda) = 0\}.$$

Then there exists a continuum $C^+ \subset \mathbf{S}^+$ such that $C^+_{\lambda_0} \cap O = \{u_0\}$ and either $C^+_{\lambda_0} \cap (E \backslash \overline{O}) \neq \emptyset$ or else $C^+$ is unbounded in $E \times [\lambda_0, \infty)$.

Similarly, define

$$\mathbf{S}^- = \{(u, \lambda) \in E \times (-\infty, \lambda_0] : f(u, \lambda) = 0\}.$$

Then there exists a continuum $C^- \subset \mathbf{S}^-$ such that $C^-_{\lambda_0} \cap O = \{u_0\}$ and either $C^-_{\lambda_0} \cap (E \backslash \overline{O}) \neq \emptyset$ or else $C^-$ is unbounded in $E \times (-\infty, \lambda_0]$.

## Cones and Positive Maps

Let $E$ be a real Banach space. A **cone** $K$ is a closed convex subset of $E$ such that

(1) if $u \in K$ and $t \geq 0$, then

(2) $K \cap \{-K\} = \{0\}$.

A cone $K$ induces a partial order defined by $u \leq v$ if and only if $v - u \in K$. A linear operator $L : E \to E$ which maps $K$ into itself is called **positive**.

**Theorem**. Let $E$ be a real Banach space with a cone $K$ and let $L : E \to E$ be a positive compact linear operator. Assume there exists $w \in K,\ \ w \neq 0$ and a constant $m > 0$ such that $w \leq mLw$, where $\leq$ is the partial order induced by $K$. Then there exists $\lambda_0 > 0$ and $u \in K$, $||u|| = 1$, such that $u = \lambda_0 Lu$.

**Krein-Rutman Theorem**

If $K$ is a cone with nonvoid interior and $L$ maps $K\backslash\{0\}$ into int($K$), then $L$ is called a **strongly positive operator**.

**Theorem**. Let $E$ have a cone $K$ with nonvoid interior int($K$). Let $L$ be a strongly positive compact linear operator. Then there exists a unique $\lambda_0 > 0$ with the following properties:
(1) There exists $u \in \text{int}(K)$ with $u = \lambda_0 L u$.
(2) If $v = \lambda L v$ with $\lambda \neq \lambda_0$ and $v \neq 0$, then $v \notin K \cup \{-K\}$ and $\lambda_0 < |\lambda|$.

## Branching from the Trivial Solution

Let $E$ be a real Banach space and assume $F : E \times \mathbf{R} \to E$ is completely continuous with $F(0, \lambda) \equiv 0$, for all $\lambda \in \mathbf{R}$.

Let $f(u, \lambda) = u - F(u, \lambda)$. Then the equation $f(u, \lambda) = 0$ has the trivial solution $u = 0$ for all $\lambda$.

To demonstrate the existence of global *branches* of nontrivial solutions bifurcating from the trivial branch, the main tools will be Leray-Schauder degree theory and Whyburn's lemma.

**Lemma (Whyburn)**. Let $A$ and $B$ be disjoint closed sets in a compact metric space $K$. Then either there exists a compact connected set $C$ in $K$ with $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$ or else there are two open sets $U$, $V$ in $K$ with $A \subset U$, $B \subset V$, $\bar{U} \cap V = \emptyset = U \cap \bar{V}$.

## Global Bifurcation

Let $E$ be a real Banach space and assume $F : E \times \mathbf{R} \to E$ is completely continuous with $F(0, \lambda) \equiv 0$, for all $\lambda \in \mathbf{R}$. Let $f(u, \lambda) = u - F(u, \lambda)$.

**Theorem**. Let real numbers $a < b$ be given with $u = 0$ an isolated solution of $f(u, a) = 0$ and also $f(u, b) = 0$. Assume that neither $a$ nor $b$ are bifurcation points, that $B_r(0) = \{u \in E : ||u|| < r\}$ is an isolating neighborhood of the trivial solution and

$$\mathrm{d}(f(\cdot, a), B_r(0), 0) \neq \mathrm{d}(f(\cdot, b), B_r(0), 0).$$

Let $S_0 = \{0\} \times [a, b]$ and define

$$\mathbf{S} = \overline{\{(u, \lambda) : f(u, \lambda) = 0,\ u \neq 0\}} \cup S_0.$$

Let $\mathbf{C} \subset \mathbf{S}$ be the maximal connected subset of $\mathbf{S}$ which contains $S_0$. Then

(i) $\mathbf{C}$ is unbounded in $E \times \mathbf{R}$, or else

(ii) $\mathbf{C} \cap \{0\} \times (\mathbf{R} \backslash [a, b]) \neq \emptyset$.

**Example 13**

Let $f(u,\lambda) = u(u^2 + \lambda^2 - 1)$. Let $S_1$ be the circle $u^2 + \lambda^2 = 1$, $u \neq 0$. The only bifurcation points are at $u = 0$, $\lambda = \pm 1$, therefore choices for $a$, $b$ in the theorem should be points near $\lambda = -1$ or $\lambda = 1$. The set $S = S_1 \cup S_0$ is bounded, therefore the theorem concludes that the continuum $C = S_1 \cup \{(0,1),(0,-1)\}$ wraps back onto the $\lambda$-axis.

**Example 14**

Let $f(u,\lambda) = u(1 - \lambda + \sin(1/u))$. Define $S_1$ to be the solution set of $\lambda - 1 = \sin(1/u)$ for $u \neq 0$. Every value $\lambda$ from 0 to 2 produces a bifurcation point, therefore $a < 0$ and $b > 2$ is required. The degree requirement is met. The continuum $C = S_1 \cup \{0\} \times [0,2]$ in the theorem is unbounded.

## Compact Linear Maps

**Proposition 15**. Assume $F(u, \lambda) = \lambda Bu + o(\|u\|)$ as $\|u\| \to 0$ with $B$ a compact linear map. If $(0, \lambda_0)$ is a bifurcation point from the trivial solution for $f(u, \lambda) = 0$, then $\lambda_0$ is a characteristic value of $B$.

**Theorem 16**. Assume $F(u, \lambda) = \lambda Bu + o(\|u\|)$ as $\|u\| \to 0$ with $B$ a compact linear map. Let $\lambda_0$ be a characteristic value of $B$ of odd algebraic multiplicity. Then there exists a continuum $\mathbf{C}$ of nontrivial solutions of $f(u, \lambda) = 0$ which bifurcates from the set of trivial solutions at $(0, \lambda_0)$ and $\mathbf{C}$ is either unbounded in $E \times \mathbf{R}$ or else $\mathbf{C}$ also bifurcates from the trivial solution set at $(0, \lambda_1)$, where $\lambda_1$ is another characteristic value of $B$.

**Example 17**

The scalar system $x = \lambda x + y^3$, $y = \lambda y - x^3$ has only the trivial solution $x = y = 0$ for all $\lambda$. The value $\lambda_0 = 1$ is a characteristic value of $B$ of multiplicity two and this characteristic value does not yield a bifurcation point.

**Example 18**

The boundary value problem $u'' + \lambda \sin u = 0$, $u(0) = u(\pi) = 0$ is equivalent to an operator equation $u = \lambda F(u)$ where $F$ maps $E = C[0, \pi]$ into itself. After analysis of the Fréchet derivative of $F$ it is found that the bifurcation points are found from the eigenvalues $\lambda = k^2$ ($k = 1, 2, \ldots$) of $F'(0)$. The eigenspaces are one dimensional and the theorem applies to yield a bifurcation point for each characteristic value.