Degree Theory in R^1

Let $f : \overline{\Omega} \to R$ be continuously differentiable where Ω is a bounded open set in R, possibly disconnected. Assume $f(x) \neq 0$ on the boundary of Ω, $f'(x) \neq 0$ whenever $f(x) = 0$. Define

$$\text{degree}(f, \Omega, 0) = \sum_{f(x)=0} \frac{f'(x)}{|f'(x)|}.$$

This function is well-defined, because the sum in the definition is finite.

The algebraic sum is an integer, but this value does not reveal the number of roots. However, a nonzero degree implies $f(x) = 0$ has at least one root in Ω. This degree formula is not defined for functions with multiple roots, e.g., $f(x) = x^2$. The degree has a continuity property, e.g., $f(x) = (x-a)(x-b)(x-c)$ for $|a| + |b| + |c|$ small and $a \neq b \neq c$ satisfies $\text{degree}(f, (-1, 1), 0) = 1$.

Degree Theory in R^n

Let $f : \overline{\Omega} \to R^n$ be continuously differentiable where Ω is a bounded open set in R^n. Assume $y \in R^n$, $f(x) \neq y$ on the boundary of Ω, $\det f'(x) \neq 0$ whenever $f(x) = y$. Define

$$\mathrm{d}(f, \Omega, y) = \sum_{f(x)=y} \frac{\det f'(x)}{|\det f'(x)|}.$$

This function is well-defined, because the sum in the definition is finite.

The algebraic sum is an integer, but this value does not reveal the number of roots. However, a nonzero degree implies $f(x) = y$ has at least one root in Ω.

Lemma. Let $\phi : [0, \infty) \to R^1$ and $r > 0$ be given such that $\phi(0) = 0$, $\phi(t) = 0$ for $t \geq r$, $\int_{R^n} \phi(|x|)dx = 1$. Then for all sufficiently small $r > 0$, $\mathrm{d}(f, \Omega, y) = \int_\Omega \phi(|f(x) - y|) \det f'(x)dx$.

Lemma 4
Let $f : \Omega \to R^n$ belong to $C^1(\overline{\Omega})$. Let $y \in R^n$, $f(x) \neq y$ on $\partial\Omega$, $\det f'(x) \neq 0$ for $f(x) = y$. Let $r > 0$ be such that $|f(x) - y| > r$ for $x \in \partial\Omega$. Let $\phi : [0, \infty) \to \mathbf{R}$ be continuous and satisfy $\phi(0) = 0$, $\phi(s) = 0$ for $s \geq r$ and $\int_0^\infty s^{n-1}\phi(s)ds = 0$. Then

$$\int_\Omega \phi(|f(x) - y|) \text{ det } f'(x)dx = 0.$$

Lemma 5
Let $f : \Omega \to R^n$ belong to $C^1(\overline{\Omega})$. Let $y \in R^n$, $f(x) \neq y$ on $\partial\Omega$, $\det f'(x) \neq 0$ for $f(x) = y$. Choose $r \in R^1$, $0 < r \leq \min_{x\in\partial\Omega} |f(x) - y|$. Let $\phi : [0, \infty) \to \mathbf{R}$ be continuous, $\phi(0) = 0$, $\phi(s) = 0$ for $s \geq r$ and $\int_{\mathbf{R}^n} \phi(|x|)dx = 1$. Then for all such ϕ, the integrals

$$\int_\Omega \phi(|f(x) - y|) \text{ det } f'(x)dx$$

have a common value.

Lemma 6
Let f_1 and f_2 be of class C^1 on the closure of a bounded open set Ω in R^n, $f_1(x) \neq y$ and $f_2(x) \neq y$ on $\partial\Omega$, and $\det f_1'(x) \neq 0$ for $f_1(x) = y$, $\det f_2'(x) \neq 0$ for $f_2(x) = y$. Let $\epsilon > 0$ be given such that $|f_1(x) - y| > 7\epsilon$ and $|f_2(x) - y| > 7\epsilon$ for $x \in \partial\Omega$, and for $x \in \bar{\Omega}$, $|f_1(x) - f_2(x)| < \epsilon$. Then

$$\mathrm{d}(f_1, \Omega, y) = \mathrm{d}(f_2, \Omega, y).$$

Corollary 7
Let f_1 and f_2 be of class C^1 on the closure of a bounded open set Ω in R^n, $f_1(x) \neq y$ and $f_2(x) \neq y$ on $\partial\Omega$, and $\det f_1'(x) \neq 0$ for $f_1(x) = y$, $\det f_2'(x) \neq 0$ for $f_2(x) = y$. Then for $\epsilon > 0$ sufficiently small, $|f_1(x) - f_2(x)| < \epsilon$, $x \in \bar{\Omega}$, implies

$$\mathrm{d}(f_1, \Omega, y) = \mathrm{d}(f_2, \Omega, y).$$

Lemma 8 and Corollary 9

Sard's Theorem. Assume Ω is a bounded open set in $\mathbf{R}^n$, $f : \overline{\Omega} \to \mathbf{R}^n$ is continuously differentiable and $f(x) \neq y$ on $\partial\Omega$. Then for all small $\epsilon > 0$ there exists $h \in \mathbf{R}^n$ with $0 < |h| < \epsilon$ such that $\det f'(x) \neq 0$ for all $x \in \Omega$ satisfying $f(x) = y + h$.

Alternatively, let F be the set of points $h \in R^n$ such that $\det f'(x) \neq 0$ at a solution $x \in \Omega$ of the equation $f(x) = y + h$. Then Sard's theorem says that F is dense in a neighborhood of zero.

Brouwer Degree

Definition. Let $f \in C(\overline{\Omega}, R^n)$ satisfy $f(x) \neq y$ on $\partial\Omega$. Define $\mathrm{d}(f, \Omega, y)$ to be the limit as $g \to f$ of $\mathrm{d}(g, \Omega, y)$ where $g \in C^1(\overline{\Omega}, R^n)$, $g(x) \neq y$ on $\partial\Omega$ and $\det g'(x) \neq 0$ when $g(x) = y$.

Lemma. The Brouwer degree for a function $f \in C^1(\overline{\Omega}, R^n)$ with $f(x) \neq y$ on $\partial\Omega$ can be represented by the relation

$$\mathrm{d}(f, \Omega, y) = \int_\Omega \phi(|x|) \det f'(x)\, dx$$

where $r < \inf_{x \in \partial\Omega} |f(x) - y|$, $\phi \in C([0, \infty), R)$, $\phi(0) = 0$, $\phi(s) = 0$ for $s > r$ and

$$\int_{R^n} \phi(|x|)\, dx = 1.$$

Brouwer Degree Properties

Let $f \in C(\bar{\Omega}, \mathbf{R}^n)$ with $f(x) \neq y$ on $\partial\Omega$.

Solution property. If $\mathrm{d}(f, \Omega, y) \neq 0$, then the equation $f(x) = y$ has a solution in Ω.

Continuity property. For some $\epsilon > 0$, $g \in C(\bar{\Omega}, \mathbf{R}^n)$ and $\hat{y} \in \mathbf{R}$ with $\|f - g\| + |y - \hat{y}| < \epsilon$ implies

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(g, \Omega, \hat{y}).$$

Briefly, the Brouwer degree is a continuous function of its arguments f and y into the integers equipped with the discrete topology.

Properties – continued

Homotopy invariance property. Let $h : [a,b]\times \bar{\Omega} \to \mathbf{R}^n$ be continuous such that $h(t,x) \neq y$, $(t,x) \in [a,b] \times \partial\Omega$. Then $\mathrm{d}(h(t,\cdot),\Omega,y) =$ constant for $a \leq t \leq b$.

This property implies that $f(x) = h(a,x)$ and $g(x) = h(b,x)$ have the same Brouwer degree, therefore homotopy invariance provides an elegant tool for computing the degree of a mapping. Some applications:

Rouche's Criterion. Let $g \in C(\bar{\Omega}, \mathbf{R}^n)$ be such that $|f(x) - g(x)| < |f(x) - y|$, $x \in \partial\Omega$. Then $\mathrm{d}(f,\Omega,y) = \mathrm{d}(g,\Omega,y)$.

Boundary Dependence Property. The equality $\mathrm{d}(f,\Omega,y) = \mathrm{d}(g,\Omega,y)$ holds for any $g \in C(\overline{\Omega}, R^n)$ such that $g(x) = f(x)$ on $\partial\Omega$.

Properties – continued

Additivity property. Let the bounded open set Ω be the union of m disjoint open sets $\Omega_1, \cdots, \Omega_m$. Assume $f(x) \neq y$ for $x \in \cup_{i=1}^m \partial\Omega_i$. Then

$$\mathrm{d}(f, \Omega, y) = \sum_{i=1}^m \mathrm{d}(f, \Omega_i, y).$$

Excision property. Let K be a closed subset of $\bar{\Omega}$ such that $f(x) \neq y$ for $x \in \partial\Omega \cup K$. Then

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(f, \Omega \setminus K, y).$$

Cartesian product formula. Assume the open bounded set Ω is a product $\Omega_1 \times \Omega_2$ with Ω_1 open in $\mathbf{R}^p$ and Ω_2 open in $\mathbf{R}^q$, $p + q = n$. For $x \in \mathbf{R}^n$ write $x = (x_1, x_2)$, $x_1 \in \mathbf{R}^p$, $x_2 \in \mathbf{R}^q$. Write $f(x) = (f_1(x_1), f_2(x_2))$ where $f_1 \in C(\bar{\Omega}_1, \mathbf{R}^p)$, $f_2 \in C(\bar{\Omega}_2, \mathbf{R}^q)$. Let $y = (y_1, y_2) \in \mathbf{R}^n$ satisfy $y_1(x_1) \neq y_1$ and $y_2(x_2) \neq y_2$ for $x_1 \in \partial\Omega_1$, $x_2 \in \partial\Omega_2$. Then

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(f_1, \Omega_1, y_1)\mathrm{d}(f_2, \Omega_2, y_2).$$

Borsuk's theorem. Let Ω be a bounded open neighborhood of $0 \in \mathbf{R}^n$ such that

$$x \in \Omega \quad \text{implies} \quad -x \in \Omega.$$

Let $f \in C(\bar{\Omega}, \mathbf{R}^n)$ satisfy $f(x) = -f(-x)$ (f is an odd map) and assume $f(x) \neq y$ for $x \in \partial\Omega$. Then $\mathrm{d}(f, \Omega, 0)$ is an odd integer.

Brouwer's Fixed Point Theorem.
I. Let $r > 0$ be given and assume $\Omega = \{x \in \mathbf{R}^n : |x| < r\}$. Let $f \in C(\bar{\Omega}, \mathbf{R}^n)$ satisfy $f(x) \in \overline{\Omega}$ for $x \in \overline{\Omega}$. Then the equation $f(x) = x$ has a solution $x \in \bar{\Omega}$.
II. If K is a compact convex set in R^n and $F : K \to K$ is continuous, then the equation $F(x) = x$ has a solution $x \in K$.

Result **I** implies result **II** by construction of a map $f(x) = F(r(x))$ which maps a ball containing K into K itself. The map $r(x)$ is a Dugundji extension of the identity map on K. A fixed point $x = F(r(x))$ implies $r(x) = x$, so x is a fixed point of F.

Degree in a Banach Space

Definition. Let E be a real Banach space, Ω a bounded open set in E and $F : \bar{\Omega} \to E$ a continuous mapping of the form $f(x) = x + F(x)$ where $F \in C(\bar{\Omega}, E_1)$ with E_1 finite dimensional. Given $y \in E$, let $\bar{E}$ be the linear span of y and E_1 with basis vectors e_1, ..., e_n. Define a linear homeomorphism $T : \bar{E} \to R^n$ by $T(e_i) =$ column i of the $n \times n$ identity matrix. Then the **degree** of f is defined by the identity

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(TfT^{-1}, T(\Omega \cap \bar{E}), T(y)).$$

Lemma 23. The degree is well-defined: the right side of the definition does not depend on which finite-dimensional space E_1 is selected, nor upon the choice of basis for $\bar{E} = \mathrm{span}\,(y, E_1)$.

Leray–Schauder Degree

The degree will be defined for a continuous mapping $f(x) = x + F(x)$ on a bounded open set Ω in a Banach space E. Initially, F will map $\bar{\Omega}$ into a finite–dimensional space. Finally, F will map bounded sets to precompact sets, which is the setting for **Leray–Schauder degree**.

Lemma 24. Let Ω be a bounded open set in the Banach space E and assume $f(x) = x + F(x)$ with $F : \bar{\Omega} \to E$ completely continuous. Suppose $f(x) \neq y$ for $x \in \partial\Omega$. Then there exists an integer d with the following property: If $h(x) = x + H(x)$ with $H : \bar{\Omega} \to E$ finite dimensional and

$$\sup_{x\in\Omega} \|f(x) - h(x)\| < \inf_{x\in\partial\Omega} \|f(x) - y\|,$$

then $h(x) \neq y$ for $x \in \partial\Omega$ and $\mathrm{d}(h, \Omega, y) = d$.

Schauder Projection

Let M be a compact subset of the Banach space E, covered by spheres of radius $\epsilon > 0$ with centers at $y_1, \ldots, y_n$. Let co(S) denote the **convex hull** of the set S. Define functions $\mu_i : M \to [0, \infty)$ by $\mu_i(y) = \epsilon - \|y - y_i\|$ in the sphere at y_i of radius ϵ and $\mu_i(y) = 0$ otherwise. Define

$$\lambda_i(y) = \frac{\mu_i(y)}{\sum_{j=1}^n \mu_j(y)}, \qquad P_\epsilon(y) = \sum_{i=1}^n \lambda_i(y) y_i.$$

The operator $P_\epsilon : M \to \text{co}(y_1, \ldots, y_n)$ is called the **Schauder projection** on M determined by ϵ, $y_1, \ldots, y_n$.

Lemma 25. The Schauder projection has these three properties:

1. The Schauder projection P_ϵ is continuous.
2. The range of P_ϵ is in the span of $y_1, \ldots, y_n$.
3. For $y \in M$, $\|P_\epsilon(y) - y\| \leq \epsilon$.

Leray–Schauder degree

Lemma 26. Let $f(x) = x + F(x)$ where $F : \bar{\Omega} \to E$ is completely continuous. Let $f(x) \neq y$ for $x \in \partial\Omega$. Let $\epsilon > 0$ be such that $\epsilon < \inf_{x\in\partial\Omega} \|f(x) - y\|$. Let P_ϵ be a Schauder projection operator determined by ϵ and points $\{y_1, \cdots, y_n\} \subset \overline{F(\bar{\Omega})}$. Then $\mathrm{d}(\mathrm{id} + P_\epsilon F, \Omega, y) = d$, where d is the integer whose existence is established by Lemma 24.

Definition. The integer d of Lemma 26 is called the **Leray-Schauder degree** of f relative to Ω and the point y and it is denoted by $\mathrm{d}(f, \Omega, y)$.

Leray-Schauder Degree Properties

The Leray–Schauder degree has the solution, continuity, homotopy invariance, additivity, and excision properties similar to the Brouwer degree; the Cartesian product formula also holds.

Borsuk's Theorem. Let Ω be a bounded symmetric open neighborhood of $0 \in E$ and let $f : \bar{\Omega} \to E$ be a completely continuous odd perturbation of the identity with $f(x) \neq 0$ for $x \in \partial\Omega$. Then $\mathrm{d}(f, \Omega, 0)$ is an odd integer.

Schauder's Fixed Point Theorem.

(a) Let K be a compact convex subset of E and let $F : K \to K$ be continuous. Then F has a fixed point in K.

(b) Let K be a closed, bounded, convex subset of E and let F be a completely continuous mapping such that $F : K \to K$. Then F has a fixed point in K.