Bifurcation Points

Let X and Y be real Banach spaces and let $F : X \times \mathbf{R} \to Y$ be a mapping with $F(0, \lambda) = 0$, $\lambda \in \mathbf{R}$, and F of class C^2 in a neighborhood of $\{0\} \times \mathbf{R}$.

We shall be interested in obtaining existence of solutions $u \neq 0$ of the equation $F(u, \lambda) = 0$.

A number λ_0 a **bifurcation value** or a point $(0, \lambda_0)$ is a **bifurcation point** provided every neighborhood of $(0, \lambda_0)$ in $X \times \mathbf{R}$ contains solutions of $F(u, \lambda) = 0$ with $u \neq 0$.

Theorem. If the point $(0, \lambda_0)$ is a bifurcation point for the equation $F(u, \lambda) = 0$, then the Fréchet derivative $F_u(0, \lambda_0)$ cannot be a linear homeomorphism of X to Y.

Fredholm Operators

Definition. A linear operator $L : X \to Y$ is called a **Fredholm operator** provided:

- $\text{ker}(L) = \{x : L(x) = 0\}$ is finite dimensional.
- $\text{im}(L) = \{y : y = L(x)\}$ is closed in Y.
- $\text{coker}L = Y/\text{im}(L)$ is finite dimensional.

Lemma. Let $F_u(0, \lambda_0)$ be a Fredholm operator with kernel V and cokernel Z. Then there exists a closed subspace W of X and a closed subspace T of Y (not necessarily unique) such that $X = V \oplus W$ and $Y = Z \oplus T$. The operator $F_u(0, \lambda_0)$ restricted to W is a linear homeomorphism of W onto T.

Splitting Equations

Let $L = F_u(0, \lambda_0)$, $V = \text{ker}(L)$, $Z = \text{coker}(L)$. Due to the decompositions $X = V \oplus W$, $Y = Z \oplus T$ the equation $F(u, \lambda) = 0$ is split into two equations $F_1 = 0$ and F_2=0. The first is solved by implicit function theory and the second by finite-dimensional methods. In terms of linear projections $P : X \to V$ and $Q : Y \to Z$, the splitting equations may be written as

$$\begin{aligned}
&(I - Q)F(u_1 + u_2, \lambda) = 0, \\
&QF(u_1 + u_2, \lambda) = 0, \\
&u_1 = Pu, \quad u_2 = (I - P)u.
\end{aligned}$$

Theorem (Implicit Equation). Assume that $F_u(0, \lambda_0)$ is a Fredholm operator with W non-trivial. Then there exist $\epsilon > 0$, $\delta > 0$ and a unique function $u_2(u_1, \lambda)$ defined for $|\lambda - \lambda_0| + \|u_1\| < \epsilon$ with $\|u_2(u_1, \lambda)\| < \delta$ which solves the equation

$$(I - Q)F(u_1 + u_2(u_1, \lambda), \lambda) = 0.$$

Proof: The implicit function theorem will be applied. First of all, $u_1 = 0$, $\lambda = \lambda_0$, $u_2 = 0$ solve the equation. The Fréchet derivative with respect to u_2 at this point is the identity mapping on W. Therefore, implicit function theory applies to give $u_2(u_1, \lambda)$.

Bifurcation Equations

Solve for the finite-dimensional variables u_1, λ whenever $|\lambda - \lambda_0| + \|u_1\| < \epsilon$ in the finite-dimensional equation

$$QF(u_1 + u_2(u_1, \lambda), \lambda) = 0.$$

Bifurcation at a Simple Eigenvalue

Theorem (Simple Kernel and Cokernel). Let $L = F_u(0, \lambda_0)$ and assume the kernel and cokernel of L have dimension one with spanning elements ϕ and 1, respectively. Let $Q : Y \to \text{coker}(L)$ be a linear projection. If

$$QF_{u\lambda}(0, \lambda_0)(\phi, 1) \neq 0,$$

then $(0, \lambda_0)$ is a bifurcation point and there exists a unique curve $u = u(\alpha)$, $\lambda = \lambda(\alpha)$, defined near $\alpha = 0$ so that $u(0) = 0$, $u(\alpha) \neq 0$ for $\alpha \neq 0$, $\lambda(0) = \lambda_0$ and $F(u(\alpha), \lambda(\alpha)) = 0$. The curve equations have for some function $\mu(\alpha)$ the special form

$$\begin{aligned} u(\alpha) &= \alpha\phi + u_2(\alpha\phi, \lambda), \\ \lambda(\alpha) &= \lambda_0 + \mu(\alpha). \end{aligned}$$

Simple Eigenvalue Example

The 2π-periodic solution problem

$$\begin{aligned} &u'' + \lambda(u + u^3) = 0, \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{aligned}$$

can be placed in the framework above by defining X to be the subspace of $C^2[0, 2\pi]$ with 2π-periodic boundary conditions. The space Y is the subspace of $C[0, 2\pi]$ with $u(0) = u(2\pi)$. Operator F is $F(u, \lambda) = u'' + \lambda(u + u^3)$. The Fréchet derivative $L = F_u(0, \lambda_0)$ has the one-dimenional requirement only for $\lambda_0 = 0$. We show $Qh = (2\pi)^{-1} \int_0^{2\pi} h(t)dt$ and $\mathrm{im}(L) = \{h : \int_0^{2\pi} h = 0\}$. The simple eigenvalue theorem applies. The problem has a 2π-periodic solution branch $u(\alpha)$, $\lambda(\alpha)$ near $\alpha = 0$.