Spaces of continuous functions

Let Ω be an open subset of $\mathbf{R}^n$. Define $C^0(\Omega, \mathbf{R}^m)$ to be the set of all continuous $f : \Omega \to \mathbf{R}^m$. The norm in $C^0(\Omega, \mathbf{R}^m)$ is defined by $\|f\|_0 = \sup_{x\in\Omega} |f(x)|$, where $|\cdot|$ is a norm in $\mathbf{R}^m$.

Let E be the space of all $f \in C^0(\Omega, \mathbf{R}^m)$ such that $\|f\|_0 < \infty$. Then E is a Banach space.

Let Ω' be an open set with $\bar{\Omega} \subset \Omega'$. Define $C^0(\bar{\Omega}, \mathbf{R}^m)$ to be the set of all restrictions to $\bar{\Omega}$ of functions $f \in C^0(\Omega', \mathbf{R}^m)$. The space $C^0(\bar{\Omega}, \mathbf{R}^m)$ is a Banach space.

Spaces of differentiable functions - 1

Let $\beta = (i_1, \cdots, i_n)$ be a multiindex, i.e. $i_k \in \mathbf{Z}$ (the nonnegative integers), $1 \leq k \leq n$. We let $|\beta| = \sum_{k=1}^{n} i_k$.

Let Ω be an open subset of $\mathbf{R}^n$ and assume $f : \Omega \to \mathbf{R}^m$. Then the partial derivative of f of order $\beta, D^\beta f(x)$, is given by

$$D^\beta f(x) = \frac{\partial^{|\beta|} f(x)}{\partial^{i_1} x_1 \cdots \partial^{i_n} x_n},$$

where $x = (x_1, \cdots, x_n)$.

Spaces of differentiable functions - 2

Define $C^j(\Omega, \mathbf{R}^m)$ to be the set of all $f : \Omega \to \mathbf{R}^m$ such that $D^\beta f$ is continuous for all β, $|\beta| \le j$. Define the norm on $C^j(\Omega, \mathbf{R}^m)$ by $\|f\|_j = \sum_{k=0}^{j} \max_{|\beta| \le k} \|D^\beta f\|_0$. Then the set E of all $f \in C^j(\Omega, \mathbf{R}^m)$ such that $\|f\|_j < +\infty$ is a Banach space.

The space $C^j(\bar{\Omega}, \mathbf{R}^m)$ is defined in a manner similar to the space $C^0(\bar{\Omega}, \mathbf{R}^m)$. If Ω is bounded, then $C^j(\bar{\Omega}, \mathbf{R}^m)$ is a Banach space.

Hölder spaces – 1/2

Let Ω be an open set in $\mathbf{R}^n$. A function $f : \Omega \to \mathbf{R}^m$ is called Hölder continuous with exponent α, $0 < \alpha \le 1$, at a point $x \in \Omega$, if

$$\sup_{y \neq x} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty,$$

and Hölder continuous with exponent $\alpha, 0 < \alpha \le 1$, on Ω if it is Hölder continuous with the same exponent α at every $x \in \Omega$. For such f we define

$$H^\alpha_\Omega(f) = \sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{1}$$

If $f \in C^j(\Omega, \mathbf{R}^m)$ with each $D^\beta f$, $|\beta| = j$, Hölder continuous with exponent α on Ω, then we say $f \in C^{j,\alpha}(\Omega, \mathbf{R}^m)$.

Hölder spaces – 2/2

Define the norm on $C^{j,\alpha}(\Omega, \mathbf{R}^m)$ by

$$\|f\|_{j,\alpha} = \|f\|_j + \max_{|\beta|=j} H^\alpha_\Omega(D^\beta f).$$

The space E of all $f \in C^{j,\alpha}(\Omega, \mathbf{R}^m)$ such that $\|f\|_{j,\alpha} < \infty$ is a Banach space.

Define the space $C^{j,\alpha}(\overline{\Omega}, \mathbf{R}^m)$ in analogy with $C^j(\overline{\Omega}, \mathbf{R}^m)$. If Ω is bounded, then $C^{j,\alpha}(\overline{\Omega}, \mathbf{R}^m)$ is a Banach space.

Conventions: $C^{j,0}(\Omega, \mathbf{R}^m)$ is written $C^j(\Omega, \mathbf{R}^m)$ and $C^{j,0}(\bar{\Omega}, \mathbf{R}^m)$ is written $C^j(\bar{\Omega}, \mathbf{R}^m)$.

Functions with compact support

Let Ω be an open subset of $\mathbf{R}^n$. A function $f : \Omega \to \mathbf{R}^m$ is said to have compact support in Ω if the set

$$\begin{aligned}\operatorname{supp} f &= \text{closure}\{x \in \Omega : f(x) \neq 0\} \\ &= \overline{\{x \in \Omega : f(x) \neq 0\}}\end{aligned}$$

is compact.

Define $C_0^{j,\alpha}(\Omega, \mathbf{R}^m)$ to be the set of all $f \in C^{j,\alpha}(\Omega, \mathbf{R}^m)$ such that $\operatorname{supp} f$ is a compact subset of Ω. Define $C_0^{j,\alpha}(\bar{\Omega}, \mathbf{R}^m)$ similarly.

If Ω is bounded, then the space $C_0^{j,\alpha}(\bar{\Omega}, \mathbf{R}^m)$ is a Banach space. It consists of all $f \in C^{j,\alpha}(\bar{\Omega}, \mathbf{R}^m)$ such that $f(x) = 0$ for $x \in \partial\Omega$.

L^p spaces

Let Ω be a Lebesgue measurable subset of $\mathbf{R}^n$ and let $f : \Omega \to \mathbf{R}^m$ be a measurable function. For $1 \leq p < \infty$, define $\|f\|_{L^p} = (\int_\Omega |f(x)|^p dx)^{1/p}$, and for $p = \infty$, define $\|f\|_{L^\infty} = \text{essup}_{x\in\Omega}|f(x)|$.

The essential supremum **essup** is defined by

$\inf\{\alpha : \text{measure}\{x \in \Omega : |f(x)| > \alpha\} = 0\}$.

Define $L^p(\Omega, \mathbf{R}^m) = \{f : \|f\|_{L^p} < +\infty\}$ for $1 \leq p \leq \infty$. Then $L^p(\Omega, \mathbf{R}^m)$ is a Banach space.

Let $u \cdot v$ denote the inner product of u and v in $\mathbf{R}^n$. The space $L^2(\Omega, \mathbf{R}^m)$ is a Hilbert space with inner product defined by

$\langle f, g\rangle = \int_\Omega f(x) \cdot g(x) dx$.

Weak derivatives

Let Ω be an open subset of $\mathbf{R}^n$. A function $f : \Omega \to \mathbf{R}^m$ is said to belong to class $L^p_{loc}(\Omega, \mathbf{R}^m)$, if for every compact subset $\Omega' \subset \Omega$, $f \in L^p(\Omega', \mathbf{R}^m)$.

Let $\beta = (\beta_1, \ldots, \beta_n)$ be a multi–index. A function $v \in L^1_{loc}(\Omega, \mathbf{R}^m)$ is called the β^{th} **weak derivative** of f if it satisfies for all $\phi \in C_0^\infty(\Omega)$ the relation

$$\int_\Omega v\phi dx = (-1)^{|\beta|} \int_\Omega f D^\beta \phi dx.$$

Write $v = D^\beta f$; up to a set of measure zero, v is uniquely determined.

Sobolev spaces

We say that $f \in W^k(\Omega, \mathbf{R}^m)$, if f has weak derivatives up to order k. Define $W^{k,p}(\Omega, \mathbf{R}^m)$ to be the set of all $f \in W^k(\Omega, \mathbf{R}^m)$ such that $D^\beta f \in L^p(\Omega, \mathbf{R}^m)$, $|\beta| \leq k$. The vector space $W^{k,p}(\Omega, \mathbf{R}^m)$ equipped with the norm $\|f\|_{W^{k,p}} = \left(\int_\Omega \sum_{|\beta|\leq k} |D^\beta f|^p dx\right)^{1/p}$ is a Banach space which has $C_0^k(\Omega, \mathbf{R}^m)$ as a subspace.

Denote by $W_0^{k,p}(\Omega, \mathbf{R}^m)$ the closure of $C_0^k(\Omega, \mathbf{R}^m)$ in the space $W^{k,p}(\Omega, \mathbf{R}^m)$. This is generally a proper Banach subspace.

The spaces $W^{k,2}(\Omega, \mathbf{R}^m)$ and $W_0^{k,2}(\Omega, \mathbf{R}^m)$ are Hilbert spaces with inner product $\langle f, g\rangle$ given by $\langle f, g\rangle = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha f \cdot D^\alpha g dx$.

The completion in $W^{k,p}(\Omega, \mathbf{R}^m)$ of the subspace $C^k(\bar{\Omega}, \mathbf{R}^m)$ is denoted by $H^{k,p}(\Omega, \mathbf{R}^m)$. If $p = 2$, then it is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The space $H_0^{k,p}(\Omega, \mathbf{R}^m)$ is the completion of $C_0^\infty(\Omega, \mathbf{R}^m)$ in $H^{k,p}(\Omega, \mathbf{R}^m)$.

Spaces of linear operators

Let E and X be normed linear spaces with norms $\|\cdot\|_E$ and $\|\cdot\|_X$, respectively. Let $\mathbf{L}(E;X)$ be the set of all linear continuous functions $f : E \to X$. For $f \in \mathbf{L}(E;X)$, let $\|f\|_{\mathbf{L}} = \sup_{\|x\|_E \leq 1} \|f(x)\|_X$. Then $\|\cdot\|_{\mathbf{L}}$ is a norm for $\mathbf{L}(E;X)$. This space is a Banach space, whenever X is.

Gâteaux and Fréchet differentiability

Let E and X be Banach spaces and let U be an open subset of E. Let $f : U \to X$ be a function. Let $x_0 \in U$, then f is said to be **Gâteaux differentiable** (G-differentiable) at x_0 in direction h, if the limit

$$\lim_{t \to 0} \frac{1}{t}\{f(x_0 + th) - f(x_0)\}$$

exists. It said to be **Fréchet differentiable** (F-differentiable) at x_0, if there exists $T \in \mathbf{L}(E; X)$ such that for $\|h\|$ small

$$f(x_0 + h) - f(x_0) = T(h) + o(\|h\|).$$

The Landau symbol $o(\|h\|)$ is defined by the relation $\lim_{\|h\| \to 0} \frac{o(\|h\|)}{\|h\|} = 0$.

Fréchet derivative

The Fréchet–derivative of f at x_0, if it exists, is unique.

The following symbols are used interchangeably for the Fréchet–derivative of f at x_0:

$$Df(x_0), \quad f'(x_0), \quad df(x_0).$$

The symbol df is usual for the case $X = \mathbf{R}$.

A function f is said to be of class C^1 in a neighborhood of x_0 if f is Fréchet differentiable there and the mapping $Df : x \mapsto Df(x)$ is a continuous mapping into the Banach space $\mathbf{L}(E; X)$.

Taylor's formula

Theorem. Let $f : E \to X$ and all of its Fréchet–derivatives of order less than m, $m > 1$, be of class C^1 on an open set U. Let x and $x+h$ be such that the line segment connecting these points lies in U. Then

$$f(x+h) - f(x) = \sum_{k=1}^{m-1} \frac{D^k f(x) h^k}{k!} + \frac{D^m f(z) h^m}{m!},$$

where z is a point on the line segment connecting x to $x+h$. The remainder $\frac{1}{m!} D^m f(z) h^m$ is also given by

$$\frac{1}{(m-1)!} \int_0^1 (1-s)^{m-1} D^m f(x_0 + sh) h^m ds.$$

Euler-Lagrange equations

Let $g : [a,b] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be twice continuously differentiable. Let $E = C_0^2[a,b]$ and let $T : E \to \mathbf{R}$ be given by $T(u) = \int_a^b g(t, u(t), u'(t))dt$.

Lemma. The operator $T : E \to \mathbf{R}$ is of class C^1 with Fréchet derivative given by

$$T'(u_0)h = \int_a^b g_u h\, dt + \int_a^b g_{u'} h'\, dt,$$

and all g-partials are evaluated at $(t, u_0(t), u_0'(t))$.

Lemma. If $T(u_0) \leq T(u)$ for $\|u - u_0\| \leq r$ (u_0 is an extremal of T), then $T'(u_0) = 0$.

Theorem (Euler-Lagrange). If u_0 is an extremal of T, then $\frac{\partial g}{\partial u} - \frac{d}{dt}\frac{\partial g}{\partial u'} = 0$, where the g-partials are evaluated at $(t, u_0(t), u_0'(t))$.

Completely continuous mappings

Let E and X be Banach spaces and let Ω be an open subset of E, let $f : \Omega \to X$ be a mapping. The function f is called **compact**, whenever $f(\Omega')$ has compact closure in X for every bounded subset Ω' of Ω ($f(\Omega')$ is **precompact**). The function f is called **completely continuous** whenever f is compact and continuous. If f is linear and compact, then f is completely continuous.

Lemma. Let Ω be an open set in E and let $f : \Omega \to X$ be completely continuous and F-differentiable at a point $x_0 \in \Omega$. Then the linear mapping $f'(x_0)$ is compact, hence completely continuous.

Proper mappings

Let $M \subset E$, $Y \subset X$ and let $f : M \to Y$ be continuous, then f is called a **proper** mapping if for every compact subset K of Y, $f^{-1}(K)$ is compact in M. The subsets M and Y are treated as metric spaces with metrics induced by the norms of E and X, respectively.

Lemma. Let $h : E \to X$ be completely continuous and let $g : E \to X$ be proper, then $f = g - h$ is a proper mapping, provided

$\lim_{\|x\| \to \infty} \|f(x)\| = \infty$ (f is **coercive**).

Lemma. Let $h : E \to E$ be a completely continuous mapping and let $f = \text{id} - \text{h}$ be coercive (id is the identity map). Then f is proper.

Lemma. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous. Then f is proper if and only if f is coercive.

Contraction mappings

The Banach fixed point theorem

Theorem (Banach). Let M be a closed subset of the Banach space E. Assume $0 \leq k < 1$ and $f : M \to M$ satisfies $\|f(x)-f(y)\| \leq k\|x-y\|$ for all x, y in M (f is a **contraction**). Then the equation $f(x) = x$ has a unique solution $x \in M$. Moreover, x is the unique limit of the sequence of iterates $f^n(x_0)$ for any point $x_0 \in M$.

An L^p approach

A Dirichlet Problem

Let $T > 0$ be given and let $f : [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a mapping satisfying **Carathéodory conditions**:

> $f(t, u, u')$ is continuous in (u, u') for almost all t and measurable in t for fixed (u, u').

Consider the **Dirichlet problem** of finding a function u satisfying the following differential equation subject to boundary conditions

$$\begin{cases} u'' = f(t, u, u'), & 0 < t < T, \\ u(0) = u(T) = 0. \end{cases}$$

An L^p approach

Theorem (Hai-Schmitt 1994). Let f satisfy $f(x, 0, 0) \in L^2[0, T]$ and

$$|f(x, u, v) - f(x, \tilde{u}, \tilde{v})| \leq a|u - \tilde{u}| + b|v - \tilde{v}|$$

for all $u, \tilde{u}, v, \tilde{v} \in \mathbf{R}$, $0 < t < T$, where a, b are nonegative constants such that $\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1$ and λ_1 is the smallest number λ such that the problem

$$\begin{cases} -u'' = \lambda u, & 0 < t < T, \\ u(0) = u(T) = 0 \end{cases}$$

has a nontrivial solution.

Then there is a unique function $u \in C_0^1([0, T])$ with u' absolutely continuous such that $u'' = f(t, u, u')$ almost everywhere on $0 \leq t \leq T$ and $u(0) = u(T) = 0$.

The implicit function theorem

Assume E, Λ and X are Banach spaces with U open in E and V open in Λ. Let $f : U \times V \to X$ be a continuous mapping such that for each $\lambda \in V$ the map $f(\cdot, \lambda) : U \to X$ is Fréchet-differentiable on U. It is assumed that the mapping $(u, \lambda) \mapsto D_u f(u, \lambda)$ is a continuous mapping from $U \times V$ to $\mathbf{L}(E, X)$.

Theorem (Implicit Function Theorem). Let f satisfy the above assumptions and let there exist $(u_0, \lambda_0) \in U \times V$ such that $D_u f(u_0, \lambda_0)$ is a linear homeomorphism of E onto X (i.e. $D_u f(u_0, \lambda_0) \in \mathbf{L}(E, X)$ and $[D_u f(u_0, \lambda_0)]^{-1} \in \mathbf{L}(X, E)$). Then there exist $\delta > 0$ and $r > 0$ and unique mapping $u : B_\delta(\lambda_0) = \{\lambda : \|\lambda - \lambda_0\| \leq \delta\} \to E$ such that

$$f(u(\lambda), \lambda) = f(u_0, \lambda_0)$$

and $\|u(\lambda) - u_0\| \leq r$, $u(\lambda_0) = u_0$.

A Combustion Model

Consider the nonlinear boundary value problem

$$\begin{aligned} &u'' + \lambda e^u = 0, \qquad 0 < t < \pi, \\ &u(0) = 0 = u(\pi). \end{aligned}$$

This is a mathematical model from the theory of combustion where the scalar variable u represents a dimensionless temperature; see Bebernes–Eberly (1989).

An application of the Implicit Function Theorem shows that for $\lambda \in \mathbf{R}$, in a neighborhood of 0, the problem has a unique solution $u(x)$ with $u(x) > 0$ in $0 < x < \pi$ and $\|u\|$ small in $C^2([0, \pi], \mathbf{R})$. The heat generation parameter λ is known to satisfy $0 < \lambda < 1$.

Inverse Function Theorem

Let E and X be Banach spaces and let U be an open neighborhood of $a \in E$. Let $f : U \to X$ be a C^1 mapping with $Df(a)$ a linear homeomorphism of E onto X. Then there exist open sets U' and V, $a \in U'$, $f(a) \in V$ and a uniquely determined function g such that:

(i) $V = f(U')$,

(ii) f is one to one on U',

(iii) $g : V \to U'$, $g(V) = U'$, $g(f(u)) = u$, for every $u \in U'$,

(iv) $g \in C^1(V; U')$ and $Dg(f(a)) = [Df(a)]^{-1}$.

Forced Nonlinear Oscillator

Consider the forced periodic boundary value problem

$$u'' + \lambda u + u^2 = g, \qquad -\infty < t < \infty,$$
$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

where g is a continuous 2π-periodic function and $\lambda \in \mathbf{R}$, is a parameter.

Let $X = C^0([0, 2\pi], \mathbf{R})$. Let E be the subspace of $C^2([0, 2\pi], \mathbf{R})$ with $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$.

Then for certain values of λ the problem has a unique solution $u \in E$ for each forcing term $g \in X$ of small norm.

Number of Solutions

Let $f : M \to Y$ be continuous, proper and locally invertible (e.g., the inverse function theorem is applicable at each point). For $y \in Y$ let $N(y)$ be the number of points in the set $f^{-1}(y) = \{u : f(u) = y\}$.

Then the mapping $y \mapsto N(y)$ is finite and locally constant.

Locally Finite Refinement

Partition of Unity

Let M be metric space. A collection of open sets $\{\mathbf{O}_\lambda\}$, $\lambda \in \Lambda$, is called an **open cover** of M provided M is the union of the $\mathbf{O}_\lambda$.

The open covering $\{\mathbf{O}_\lambda\}$ is called **locally finite** if every point $u \in M$ has a neighborhood U which intersects at most finitely many elements $\mathbf{O}_\lambda$. A **refinement** of an open cover $\{\mathbf{O}_\lambda\}$ is a second open cover $\{\mathbf{U}_\gamma\}$, $\gamma \in \Gamma$, such that each $\mathbf{U}_\gamma$ is a subset of some $\mathbf{O}_\lambda$.

Lemma. Let M be a metric space. Then every open cover of M has a locally finite refinement.

Proof: This result appears in Dugundj's topology text, where it is attributed to A. H. Stone. The statement is *Every metric space is paracompact*.

Dugundji's Extension Theorem

A set K in a Banach space is called **convex** if $\lambda x + (1-\lambda)y \in K$ for x, y in K and $0 \le \lambda \le 1$.

Theorem (Dugundji). Let E and X be Banach spaces. Assume K is convex in X and C is closed in E. Let $f : C \to K$ be continuous. Then there exists a continuous extension of f of the form

$$F(u) = \begin{cases} f(u) & u \in C, \\ \sum_U \kappa_U(u) f(a_U) & u \in E \setminus C, \end{cases}$$

where $a_U \in C \cap U$, $0 \le \kappa_U(u) \le 1$ and $\sum_U \kappa_U(u) = 1$. The symbol U is an open set from a certain open cover of E.

For $x \in E$, $F(x) \in$ convex hull $(f(C)) \subseteq K$. If defined, $\max_{u \in C} \|f(u)\|_X = \max_{x \in E} \|F(x)\|_X$. The extension of a sum is the sum of the extensions.