Name ________________________ **2270-1**

Introduction to Linear Algebra 2270-1
Midterm Exam 3 Fall 2003

Problems 1, 2 due Tue, 11 Nov
(Problems 3, 4 due Fri, 14 Nov)
In-class Exam Date: Tuesday, 18 Nov

Instructions. The take-home problems below are to be submitted at class time at the date marked above. Answer checks are expected. If `maple` assist is used, then please attach the `maple` output.
The in-class portion of the exam is the last 15 minutes of class, one problem, of a type similar to one or more parts of the four problems. Calculators are not allowed. Books and notes are not allowed.

1. (Kernel, Independence, Similarity)
(a) Use the identity $\mathbf{rref}(A) = E_1 E_2 \cdots E_k A$ to prove: $\mathbf{ker}(A) = \{\mathbf{0}\}$ if and only if $\det(A) \neq 0$.

(b) Assume $n \times n$ matrix A satisfies $A^k \neq 0$ and $A^k A = 0$ for some integer $k \geq 0$. Choose $\mathbf{v}$ with $A^k\mathbf{v} \neq \mathbf{0}$. Prove (1) and (2):
(1) Vectors $\mathbf{v}$, $A\mathbf{v}$, $A^2\mathbf{v}$, $\ldots$, $A^k\mathbf{v}$ are linearly independent.
(2) Always, $k < n$. Hence $A^n = 0$.

(c) Suppose for matrices A, B the product AB is defined. Prove that $\mathbf{ker}(A) = \mathbf{ker}(B) = \{\mathbf{0}\}$ implies $\mathbf{ker}(AB) = \{\mathbf{0}\}$.

(d) Do there exist matrices A and B such that A is not similar to B but $A - 2I$ is similar to $B - 2I$? Justify.

2. (Abstract vector spaces, Linear transformations) Let W be the set of all infinite sequences of real numbers $\mathbf{x} = \{x_n\}_{n=0}^{\infty}$ (page 150).
(a) Define addition and scalar multiplication for W and prove that W is a vector space.

(b) Let V be the subset of W defined by $\sum_{n=0}^{\infty} |x_n|^2 < \infty$. Prove that V is a subspace of W.

(c) Define $T(\mathbf{x}) = \{x_{n+1}\}_{n=0}^{\infty}$ on V. Show that T is a linear transformation from V to V and determine $\mathbf{ker}(T)$.

(d) Define $S(f) = 2f - f'$ from $X = C^{\infty}[0,1]$ into X. Find the kernel and nullity of S.

Please attach this exam or a copy to the front of your submitted exam on the due date. Kindly write your name on all pages.

Name ________________________ **2270-1**

Introduction to Linear Algebra 2270-1
Midterm Exam 3 Fall 2003

Problems 3, 4 due Fri, 14 Nov

In-class Exam Date: Tuesday, 18 Nov

Instructions. The take-home problems below are to be submitted at class time at the date marked above. Answer checks are expected. If `maple` assist is used, then please attach the `maple` output.
The in-class portion of the exam is the last 15 minutes of class, one problem, of a type similar to one or more parts of the four problems. Calculators are not allowed. Books and notes are not allowed.

3. (Inner product spaces, Orthogonality)

(a) Give an algebraic proof, depending only on inner product space properties, of the triangle inequality $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ in $\mathcal{R}^n$.

(b) Find the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto $V = \textbf{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\}$.

(c) Find the QR-factorization of $A = \begin{pmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{pmatrix}$.

(d) Prove that an invertible matrix A has exactly one QR-factorization.

4. (Least squares, Determinants)

(a) Solve one of p225-40 (least squares) or p221-8 (pseudo-inverse) or p239-24 (inner product spaces).

(b) Suppose an $n \times n$ invertible matrix A is reduced to upper triangular matrix $T = [T_{ij}]$ by elementary row operations, involving any number of combo operations plus s swaps and r row multiplications by nonzero numbers $m_1, \ldots, m_r$. Prove that

$$\det(A) = \frac{(-1)^s T_{11} T_{22} \cdots T_{nn}}{m_1 m_2 \cdots m_r}.$$

(c) Given a matrix $A = [a_{ij}]$ with $a_{ij} = 0$ or 1, what is the least number of zeros possible so that A is invertible?

(d) Find A^{-1} by two methods: the classical adjoint method and the **rref** method applied to $\textbf{aug}(A, I)$:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Please attach this exam or a copy to the front of your submitted exam on the due date. Kindly write your name on all pages.