Name ______________________ **Class time** _________

Applied Differential Equations 2250-1 and 2250-2
Midterm Exam 2, Spring 2004

Take-home problems #1 and #2: March 8 and March 12, 2004
In-class Exam Date: 10 March, 2004

1. (Gaussian algorithm) Due Monday, March 8, 2004 at class time.
Consider a 3×3 linear system $A\mathbf{x} = \mathbf{b}$, which contains the symbols a, b, c.

`Version A-D:`

$$\begin{cases} 2x + (a+b)y + cz = b \\ 3x + 0y + cz = 2b \\ 7x + 2(a+b)y + 3cz = 4b \end{cases}$$

`Version E-K:`

$$\begin{cases} x + (a+c)y + cz = b \\ x + 2cy + cz = b \\ 3x + (a+5c)y + 3cz = 3b \end{cases}$$

`Version L-Q:`

$$\begin{cases} 2x + (a+c)y + cz = 2b \\ x + 2ay + cz = b \\ 4x + (5a+c)y + 3cz = 4b \end{cases}$$

`Version R-Z:`

$$\begin{cases} 3x + (a+c)y + 2cz = b \\ x + 2cy + cz = b \\ 5x + (a+5c)y + 4cz = 3b \end{cases}$$

(a) Determine conditions on a, b, c such that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$. Report the several conditions as *Cases 1, 2, 3, ...* . (25%)

(b) For each condition in (a), display the general solution (25%), rank (5%) and nullity (5%) of the system.

(c) Check each solution in (b). Maple answer checks are acceptable. (20%)

(d) Verify that for all a, b, c not listed in (a), there is no solution of $A\mathbf{x} = \mathbf{b}$. (20%)

Maple Answer Check Example

```
with(linalg):unassign('r','t'):
A:=(a,b,c)->matrix([[2,2*a+c,c],[3,a,2*c],[5,3*a+c,3*c]]);
B:=b->vector([b,2*b,3*b]);
C:=augment(A(a,b,c),B(b)): 'C'=eval(C),RREF=rref(C);
C1:=augment(A(-3*c/4,b,c),B(b)): 'C1'=eval(C1),RREF=rref(C1);
[x,y,z]=linsolve(A(-3*c/4,b,c),B(b),'r','t'),rank=r,nullity=3-r;
# This checks one answer in (b), not all answers.
```

Staple this page to the front of your submitted problem #1.
The version is to match the first letter of your last name.

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Applied Differential Equations 2250-1 and 2250-2
Midterm Exam 2, Spring 2004

Take-home problem #2: March 12, 2004

2. (Jules Verne Problem) Due Friday, March 12, 2004 at class time.
Assume a model

$$\frac{d^2r}{dt^2} = -\frac{Gm_1}{(R_1+r)^2} + \frac{Gm_2}{(R_2-R_1-r)^2}, \quad r(0)=0, \quad r'(0)=v_0,$$

where R_2 is the mean center-to-center distance from the earth to the moon and R_1 is the mean radius of the earth. The mass m_1 of the earth and m_2 of the moon appear, plus the universal gravitation constant G. All units are MKS.

(a) Display the two gravitational forces on the projectile due to the moon and the earth (10%). Identify the corresponding terms in the model (10%)

(b) Calculate the distance r^* at which the projectile has net acceleration zero. Give a symbolic answer (10%) and also a numerical answer $\approx 3.39 \times 10^8$ meters (10%).

(c) Calculate the minimal launch velocity $v_0^* = \sqrt{2F(0) - 2F(r^*)}$, where

$$F(r) = \frac{Gm_1}{R_1+r} + \frac{Gm_2}{R_2 - R1 - r}.$$

Display the answer in meters per second and miles per hour. (10%)

(d) Conduct a numerical experiment to find the flight time to the moon. Let the radius of the moon be $R_3 = 1.74e6$ meters. Assume the launch velocity $r'(0)$ is v_1 meters per second faster than the minimal launch velocity v_0^*, where v_1 is given by: (50%)

Version A-D: v_1=22	Version E-K: v_1=61
Version L-Q: v_1=83	Version R-Z: v_1=97

Staple this page to the front of your submitted problem #2.
The version is to match the first letter of your last name.
Sample `maple` code follows.

Maple code for the Jules Verne problem

```
# Group 1: initialize
G:=6.6726e-11: m1:=5.975e24: m2:=7.36e22:
R1:=6.378e6: R2:=3.84e8: R3:=1.74e6: R4:=R2-R1-R3:
de:=diff(r(t),t,t)=-G*m1/(r(t)+R1)^2+G*m2/(R2-R1-r(t))^2:

# Group 2: Search for flight time T.
v0:=11200: T:=162800: ic:=r(0)=0,D(r)(0)=v0:
p:=dsolve({de,ic},r(t),type=numeric,method=lsode);
Y:=t->rhs(p(t)[2]):
plot({'Y(t)',R4},t=0..T,0..R4); Y(T)-R4;
# Plot done. Change v0, T and re-execute group 2, until
# the plotted curve reaches the upper right corner or
# Y(T)-R4 is approximately zero.

# Group 3: Solve for flight time Tstar.
T1:=T-200: T2:=T:
# If T is close enough to the flight time to the moon,
# then fsolve() prints a precise answer for the flight time.
Tstar:=fsolve('Y(x)'=R4,x,T1..T2);
# Answer check: Y(Tstar)-R4 should be approximately zero.
Y(Tstar)-R4;
```

Do not submit this page with your solution.

Name ______________________ Class time _________

Applied Differential Equations 2250-1 and 2250-2
Sample In-Class Midterm Exam 2, Version A-L

Wednesday, 10 March, 2004

Instructions. The exam is in-class, 50 minutes. No books, notes, calculators, computers or outside materials allowed.

3. **(Inverse matrix)** Consider for symbols a, b the matrix

$$A = \begin{pmatrix} 1 & b & 0 \\ a & 0 & b \\ 0 & 1 & 1 \end{pmatrix}.$$

(a) Display a determinant condition for the existence of the inverse A^{-1}. (20%)

(b) A basic non-determinant theorem exists to test the non-existence of A^{-1} when the condition in (a) fails. State such a theorem and illustrate how it applies to this problem. (20%)

(c) Find A^{-1} under the conditions in (a). (60%)

Staple this page to the front of problem #3.

Name ______________________ Class time _________

Applied Differential Equations 2250-1 and 2250-2
Sample In-Class Midterm Exam 2, Version A-L

Wednesday, 10 March, 2004

4. (rref and inverses)

(a) Determine all values of symbol a such that the system has infinitely many solutions: (35%)

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & a \\ 5x & + & y & + & 2z & = & 3a \\ 6x & + & 3y & + & 3z & = & 1+a \end{array}$$

(b) Determine all values of x for which A^{-1} exists: (35%)

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & x & 1 \end{pmatrix}.$$

(c) Assume given 3×3 matrices A, B. Suppose $B = E_1E_2A$ and E_1, E_2 are elementary matrices representing swap rules. Explain precisely why $\det(B) = \det(A)$. (30%)

Staple this page to the front of problem #4.

Name ________________________ **Class time** __________

Applied Differential Equations 2250-1 and 2250-2
Sample In-Class Midterm Exam 2, Version A-L

Wednesday, 10 March, 2004

5. (Cramer's rule and adjoints)
Define

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(a) Display the formulas for Cramer's rule for the system $A\mathbf{x} = \mathbf{b}$, without evaluating the determinants. (40%)

(b) Find x_2 explicitly in (a). (30%)

(c) Display the adjoint of the matrix A. (30%)

Staple this page to the front of problem #5.