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2250 Midterm 2 Ver 2 [7:30]

1. **(rref)**
Determine a, b such that the system has infinitely many solutions:

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1-a \\ 5x & + & y & + & 2z & = & 3-3a \\ 6x & + & 3y & + & 3bz & = & 2-a \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1-a \\ 5 & 1 & 2 & 3-3a \\ 6 & 3 & 3b & 2-a \end{array}\right)$$

3 combo + 1 mult

$$\cong \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1-a \\ 0 & 1 & 1/3 & (-2+2a)/(-9) \\ 0 & 0 & x & y \end{array}\right)$$

where $x = 3b-3$, $y = -2+3a$

∞-many solutions $\Leftrightarrow$ last row is all zeros
$\Leftrightarrow$ $x=0$, $y=0$
$\Leftrightarrow$ $\boxed{b=1,\ a=2/3}$

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2. (vector spaces)

(a) [25%] Give an example of a vector space of functions of dimension four.

(b) [25%] Let S be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let V be the subset of S given by the equation $x_1 = 4(x_2 - x_3)$. Prove that V is a subspace of S. Edwards and Penney Theorem 2 may be referenced in the proof, in order to shorten details. If you cite Theorem 2, then please state the Theorem.

(c) [50%] Find a basis for the subspace of $\mathcal{R}^3$ given by the system of equations

$$\begin{aligned} x + 2y - 2z &= 0, \\ x + y - 3z &= 0, \\ y + z &= 0, \end{aligned}$$

(a) $V =$ all linear combinations of atoms $1, x, x^2, x^3$

(b) Use Thm 2. Define $A = \begin{pmatrix} 1 & -4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A\vec{x} = \vec{0}$ defines S.
By Thm 2, S is a subspace.

(c) $\left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 1 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$ Frame one

$\cong \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\cong \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Last Frame, rref

Gen. Sol.
$\begin{cases} x = 4t_1 \\ y = -t_1 \\ z = t_1 \end{cases}$

Basis $= \{ \partial_{t_1}(\text{Gen sol}) \}$
$= \boxed{\left\{ \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \right\}}$

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3. (independence) Do **only two** of the following.

(a) [50%] Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$. State and apply a test that decides independence or dependence of the list of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

(b) [50%] State the pivot theorem [10%], then extract from the list below a largest set of independent vectors [40%].

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 5 \\ -3 \\ 0 \\ -1 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix},$$

(c) [50%] Assume that matrix D is invertible. Prove:

If $D\mathbf{x}_1$, $D\mathbf{x}_2$, $\ldots$, $D\mathbf{x}_n$ are dependent, then $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$ are dependent.

(a) Test: $\vec{u}, \vec{v}, \vec{w}$ are independent $\Leftrightarrow$ rank$(\text{aug}(\vec{u}, \vec{v}, \vec{w})) = 3$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\text{rank} = 2 \quad \text{Dependent}}$

(b) Pivot Theorem. The pivot columns of A are independent and any other column of A is dependent upon them.

$$\begin{pmatrix} 1 & 1 & 2 & 5 & 2 \\ 1 & -1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & -2 & -1 & 2 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 2 & 5 & 2 \\ 0 & -2 & -4 & -8 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -8 & -16 & -4 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 2 & 5 & 2 \\ 0 & -2 & -4 & -8 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot cols are 1, 2. $\boxed{\text{Independent cols of } A \text{ are } 1, 2}$

(c) Let $c_1, \ldots, c_n$ satisfy

$$\sum c_k D\vec{x}_k = \vec{0}$$

not all $c_1, \ldots, c_n$ are zero

Then

$$D\left(\sum c_k \vec{x}_k\right) = \vec{0}$$

$$D^{-1} D \left(\sum c_k \vec{x}_k\right) = \vec{0}$$

$$\sum c_k \vec{x}_k = \vec{0}$$

Therefore, $\vec{x}_1, \ldots, \vec{x}_n$ are dependent.

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4. (determinants and elementary matrices)
(a) [50%] Assume given invertible 3×3 matrices A, B. Suppose $B = E_3E_2E_1A$ and E_1, E_2, E_3 are elementary matrices representing repectively a swap, a combination and a multiply by 2. Compute the value of $\det(-(AB^{-1})^2)$.
(b) [50%] Let A, B and C be three 5×5 matrices such that ABC contains two rows all of whose entries are ones. Explain precisely why at least one of the three matrices has zero determinant.

(a)

$$
\begin{aligned}
\det(-(AB^{-1})^2) &= \det((-I)(AB^{-1})(AB^{-1})) \\
&= \det(-I)(\det(AB^{-1}))^2 \qquad \text{prod. Thm. for determinant} \\
&= (-1)^5[\det(AB^{-1})]^2 \\
&= (-1)^5\left[\det(E_1^{-1}E_2^{-1}E_3^{-1})\right]^2 \qquad E_1^{-1}E_2^{-1}E_3^{-1} = AB^{-1} \text{ was given.} \\
&= (-1)^5\left[\det E_1^{-1} \det E_2^{-1} \det E_3^{-1}\right]^2 \\
&= (-1)^5\left[(-1)(1)(\tfrac{1}{2})\right]^2 \\
&= \boxed{\frac{-1}{4}}
\end{aligned}
$$

Because $\det E^{-1} = \dfrac{1}{\det E}$ and $\det E_1 = -1$, $\det E_2 = 1$, $\det E_3 = 2$ the next-to-last display is justified.

(b) A determinant with equal rows is zero [a Theorem], which is proved by applying combo to get a row of zeros. Then

$$\det(ABC) = 0$$

$$\det(A)\det(B)\det(C) = 0 \qquad \text{prod. Thm.}$$

implies one of the three matrices has determinant zero.

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5. (inverses and Cramer's rule)

(a) [50%] Determine all values of x and y for which A^{-1} fails to exist: $A = \begin{pmatrix} 1 & 2x-1 & 0 \\ 2 & 0 & -3 \\ 0 & y & 1 \end{pmatrix}$.

(b) [50%] Solve for y in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(a) A^{-1} exists $\iff \det(A) \neq 0$.

$\iff \begin{vmatrix} 1 & 2x-1 & 0 \\ 2 & 0 & -3 \\ 0 & y & 1 \end{vmatrix} \neq 0$

$\iff (1)(3y) - 2(2x-1) \neq 0$

$\iff 3y - 4x + 2 \neq 0$

$\boxed{A^{-1} \text{ Fails to exist} \iff 3y-4x+2=0}$

(b) $y = \dfrac{\Delta_2}{\Delta}$

$\boxed{y = \dfrac{3}{-26}}$

$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{vmatrix}$

$= (1)(-24) - 2(21-20)$

$= -24 - 2$

$= -26$

$\Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 4 \\ 5 & -1 & 7 \end{vmatrix}$

$= (1)(4) - (1)(21-20)$

$= 4 - 1$

$= 3$

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