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# Differential Equations and Linear Algebra 2250 [10:45]

**Midterm Exam 1**

**Version 3: Tuesday, 14 February 2006**

**Instructions**: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

**1. (Quadrature Equation)**

Solve for the general solution $y(x)$ in the equation $y' = 2\cot x + \dfrac{1250x^3}{1+25x^2} + x\ln(1+x^2)$.

$$\int y'\,dx = \int F(x)\,dx$$
$$y = 2\int \cot x\,dx + \int \frac{1250x^3}{1+25x^2}\,dx + \int x\ln(1+x^2)\,dx$$
$$y = I_1 + I_2 + I_3 + C$$ (work split below)

$$I_1 = 2\int \cot x\,dx = 2\int \frac{\cos x}{\sin x}\,du = 2\ln|\sin x|$$

division algorithm, college algebra

$$1+25x^2 \;\overline{)\,1250x^3}$$ quotient: $50x$
$$1250x^3 + 50x$$
$$-50x$$ ← remainder

$$I_2 = \int \frac{1250x^3\,dx}{1+25x^2} = \int\left(50x + \frac{-50x}{1+25x^2}\right)dx = 25x^2 - \ln(1+25x^2)$$

$$I_3 = \int x\ln(1+x^2)\,dx = \int \ln(u)\,\frac{du}{2} = \frac{1}{2}(u\ln u - u) = \frac{1}{2}\left((1+x^2)\ln(1+x^2) - (1+x^2)\right)$$

Details: $u = 1+x^2$, $du = 2x\,dx$

$$y = I_1 + I_2 + I_3 + C$$
$$\boxed{y = (2\ln|\sin x|) + (25x^2 - \ln(1+25x^2)) + \left(\tfrac{1}{2}(1+x^2)\ln(1+x^2) - \tfrac{1}{2}(1+x^2)\right) + C}$$

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**2. (Separable Equation Test)**
The problem $y' = f(x,y)$ is said to be separable provided $f(x,y) = F(x)G(y)$ for some functions $F$ and $G$.

(a) [75%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

| | | | |
|---|---|---|---|
| ☒ | $y' = -y(2xy+1) + (2x+3)y^2$ | ☒ | $yy' = xy^2 + 5x$ |
| ☐ | $y' = e^x + e^y$ | ☒ | $3y' + 5y = 10$ |

(b) [25%] State a test which can verify that an equation is not separable. Use the test to verify that $y' = x + \sqrt{|y|}$ is not separable.

ⓐ

| | |
|---|---|
| $y' = -y(2xy+1) + (2x+3)y^2$ <br> $= -2xy^2 - y + 2xy^2 + 3y^2$ <br> $= -y + 3y^2$ <br> autonomous $\Rightarrow$ sep. | $yy' = xy^2 + 5x$ <br> $= x(y^2+5)$ <br> sep. |
| $y' = e^x + e^y$ <br> not sep. | $3y' + 5y = 10$ <br> $y' = \frac{10-5y}{3}$ <br> autonomous $\Rightarrow$ sep |

ⓑ Let $F(x) = \frac{f(x,y_0)}{f(x_0,y_0)}$, $G(y) = f(x_0,y)$, $f(x_0,y_0) \neq 0$. Then $FG \neq f$ implies $y' = f(x,y)$ is not separable.

application: Choose $x_0 = 1$, $y_0 = 0$. Then $f(x,y) = x + \sqrt{|y|}$ implies
$F(x) = \frac{f(x,0)}{f(1,0)} = x$, $G(y) = 1 + \sqrt{|y|}$, Then

$$FG = x(1+\sqrt{|y|}) = x + x\sqrt{|y|} \neq x + \sqrt{|y|} = f$$

By the test, $y' = x + \sqrt{|y|}$ is not separable.

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**3. (Solve a Separable Equation)**

Given $y^2y' = \dfrac{2x^2+3x}{1+x^2}\left(\dfrac{125}{64} - y^3\right)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for $y$ explicitly.

(a) $F(x) = \dfrac{2x^2+3x}{1+x^2} = 2 + \dfrac{3x-2}{1+x^2}$

$$1+x^2 \overline{)\,2x^2+3x\,} \quad \text{quotient } 2$$
$$2x^2+2$$
$$3x-2 = \text{remainder}$$

$G(y) = \left(\frac{125}{64} - y^3\right)\frac{1}{y^2}$

$G(y) = 0 \iff y = \sqrt[3]{\frac{125}{64}}$ or $\boxed{y = \frac{5}{4}}$

(b) $\dfrac{y'}{G(y)} = F(x)$

$\displaystyle\int \frac{y'\,dx}{G(y)} = \int F(x)\,dx$

$\displaystyle\int \frac{y^2y'\,dx}{\frac{125}{64} - y^3} = \int \left(2 + \frac{3}{2}\left(\frac{2x}{1+x^2}\right) + \frac{-2}{1+x^2}\right)dx$

$$\boxed{-\frac{1}{3}\ln\left|\frac{125}{64} - y^3\right| = 2x + \frac{3}{2}\ln(1+x^2) - 2\arctan(x) + C}$$

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**4. (Linear Equations)**

(a) [60%] Solve $2v'(t) = -32 + \dfrac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x+2}\,\dfrac{dy}{dx} = y$. The answer contains symbol $c$.

(c) [10%] The problem $2\sqrt{x+2}\,y' = y - 5$ can be solved using the answer $y_h$ from (b) plus superposition $y = y_h + y_p$. Find $y_p$. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.

(a) $v' - \left(\frac{1}{3t+1}\right)v = -16$, $\quad v(0) = -8$

$Q = e^{-\int dt/(3t+1)}$

$= e^{-\frac{1}{3}\ln|3t+1|}$

$= (3t+1)^{-1/3}$ constant factor dropped / valid near $t=0$

$(Qv)'/Q = -16$

$Qv = -16\int Q + c$ quadrature

$= -16\int (3t+1)^{-1/3}dt + c$

$= -16\,\dfrac{(3t+1)^{2/3}}{2/3}\cdot\dfrac{1}{3} + c$

$= -8(3t+1)^{2/3} + c$

$v = -8(3t+1) + c(3t+1)^{1/3}$

$-8 = -8 + c$

$c = 0$

$\boxed{v = -8(3t+1)}$ $\quad v = -24t - 8$

(b) $y' - \dfrac{1}{2\sqrt{x+2}}y = 0$

$Q = e^{-\int \frac{1}{2}(x+2)^{-1/2}dx}$

$= e^{-\sqrt{x+2}}$

$(Qy)'/Q = 0$

$y = \dfrac{c}{Q}$ or $\boxed{y = ce^{\sqrt{x+2}}}$

(c) $\boxed{y_p = 5}$ is an equilibrium solution.

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**5. (Stability)**

(a) [50%] Draw a phase line diagram for the differential equation

$$dx/dt = 1000\,(2 - \sqrt[5]{x})^3\,(2 + 3x)(9x^2 - 4)^8.$$

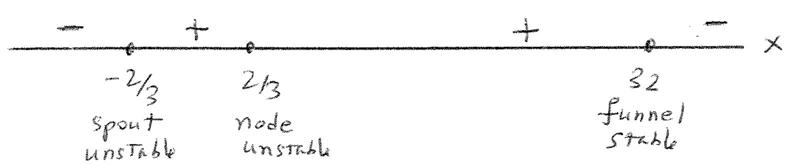
Expected in the diagram are equilibrium points and signs of $x'$ (or flow direction markers $<$ and $>$).

(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

(a) equilibrium solutions are found from

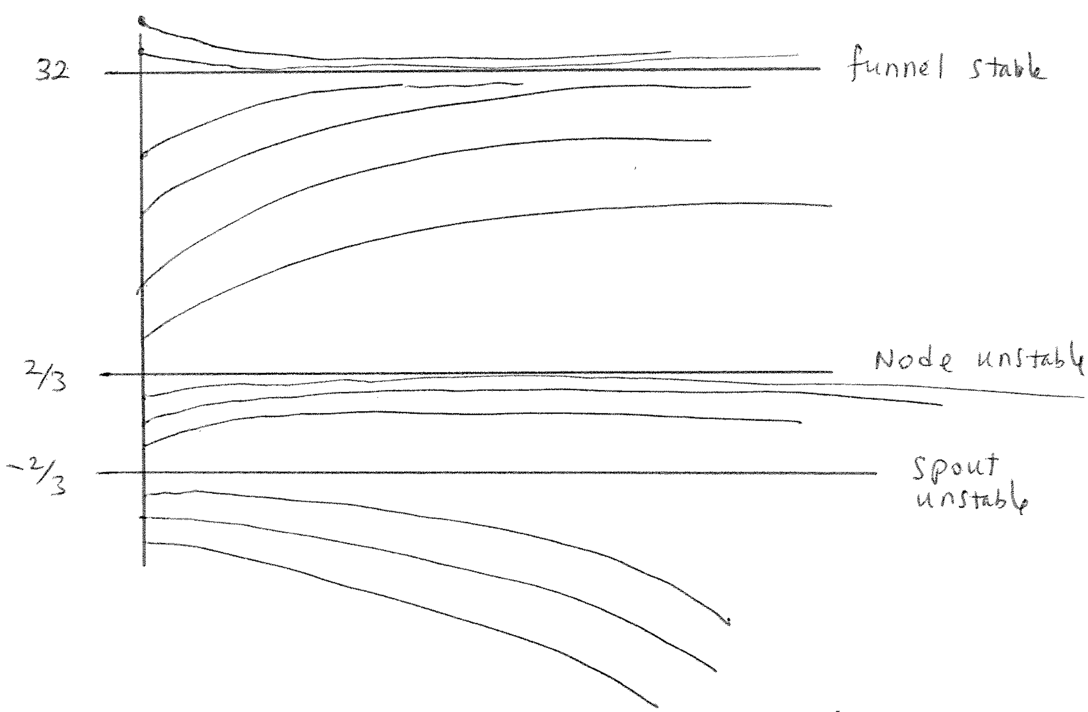
$$1000(2 - x^{1/5})^3 (2+3x)^9 (3x-2)^8 = 0$$

$$x = 32, \quad x = -2/3, \quad x = 2/3$$




phase line diagram

(b)

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