Name KEY **2250-2**

Differential Equations and Linear Algebra 2250-2

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Ch3. (Linear Systems and Matrices)

☐ [50%] Ch3(a): Find the **first entry on the fourth row** of the inverse matrix $B^{-1} = \mathbf{adj}(B)/\det(B)$. Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The only allowed use of Sarrus' rule is the 2×2 case.

$$B = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

☐ [50%] Ch3(b): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has infinitely many solutions [25%] and then for all such k display the solution formula for $\mathbf{x}$ [25%].

$$R = \begin{bmatrix} 2 & 1 & -4k & 0 \\ 2 & 2k & -3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1-2k \\ 0 \end{pmatrix}$$

☐ [50%] Ch3(c): Let A be a 50×51 matrix. Is it possible that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions $\mathbf{x}$.? Explain your answer fully.

(a) $C_{41} = \text{cofactor}(B,1,4)/\det(B) = \boxed{-1/2}$ $\quad \det(B) = 1 \cdot 1 \cdot \begin{vmatrix} 1 & 1 & -2 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 4$, $\text{Cof} = (-1)^5 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -2$

(b) $C = \text{aug}(R,f) = \left(\begin{array}{cccc|c} 2 & 1 & -4k & 0 & 0 \\ 2 & 2k & -3 & 0 & 1-2k \\ 0 & 0 & 4 & 0 & 0 \end{array}\right) \approx \left(\begin{array}{cccc|c} 2 & 1 & -4k & 0 & 0 \\ 0 & 2k-1 & 4k-3 & 0 & 1-2k \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \approx \left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 2k-1 & 0 & 0 & 1-2k \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$

There are 2 leading ones, To obtain ∞-many sols, need consistency plus one free var. Required: $2k-1=0$. Then RREF =

$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$, sol is $\boxed{\begin{cases} x = -t_1/2 \\ y = t_1 \\ z = 0 \\ w = t_2 \end{cases}}$ $\boxed{k = 1/2}$

(c) Always. There are 51 variables and the rank = # lead vars ≤ 50. So at least one variable is a free var.

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Ch4. (Vector Spaces)

☐ [40%] Ch4(a): State an RREF test (not a determinant test) to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}3$ in $\mathcal{R}^4$ [10%]. Apply the test to the vectors below [25%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 2 \\ 6 \\ 0 \end{pmatrix}.$$

☐ [60%] Ch4(b): Define V to be the set of all functions $f(x)$ defined on $0 \le x \le 1$ such that $f(0) = f(1) = 0$. Prove that V is a subspace of the vector space W of all functions $g(x)$ defined on $0 \le x \le 1$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathcal{R}^4$ for the column space of the 4×4 matrix A below. The reported basis must consist of columns of A.

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 11 \\ 2 & 2 & -2 & 1 \end{pmatrix}.$$

☐ [40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition of the fraction given below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{x^2 - 3x + 1}{(x+1)^2(x-2)^2}$$

(a) $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are indep in $\mathbb{R}^4$ $\Leftrightarrow$ $\text{rank}(\text{aug}(\vec{v}_1, \vec{v}_2, \vec{v}_3)) = 3$. Test: $A = \begin{pmatrix} -1 & 3 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ $\cong \begin{pmatrix} -1 & 3 & 4 \\ 0 & 3 & 6 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So the rank is 2. $\boxed{\text{Dependent}}$

(b) Subspace criterion: the zero function satisfies the condition, so 0 in V. If f_1, f_2 are in V, then $(c_1 f_1 + c_2 f_2)(0) = c_1 f_1(0) + c_2 f_2(0) = 0$ and $(c_1 f_1 + c_2 f_2)(1) = c_1 f_1(1) + c_2 f_2(1) = 0$. So $c_1 f_1 + c_2 f_2$ is in V. The proof is complete.

(c) Cols 1, 2, 3 are proportional, hence dependent. $\boxed{\text{Cols } 1, 4 \text{ are indep}}$

(d) $$\frac{5/27}{x-2} + \frac{-1/9}{(x-2)^2} + \frac{-5/27}{x+1} + \frac{5/9}{(x+1)^2}$$

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Ch5. (Linear Equations of Higher Order)

☐ [25%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations given below.

1.[10%] $4y'' + 8y' + y = 0$,
2.[15%] characteristic equation $(r-2)^2(r^2+2r+17)^2(r^2-4)^3 = 0$

☐ [25%] Ch5(b): Given $10x''(t) + 11x'(t) + 3x(t) = 0$, which represents a damped spring-mass system with $m = 10$, $c = 11$, $k = 3$, solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate the model in a figure [5%].

☐ [50%] Ch5(c): Determine the **final form** of a trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv} + 18y'' + 81y = x^2e^{3x} + x\cos 3x + e^{-3x} + \quad \sin 3x$$

☐ [25%] Ch5(d): Find the steady-state periodic solution for the equation

$$x'' + 2x' + 17x = \cos(3t).$$

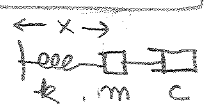
(a) 1. $4r^2+8r+1=0 \quad r = -\frac{8}{8} \pm \frac{1}{8}\sqrt{64-16} = -1 \pm \frac{1}{8}\sqrt{48} = -1 \pm \sqrt{3}/2$

$$\boxed{y = c_1 e^{-x+\sqrt{3}x/2} + c_2 e^{-x-\sqrt{3}x/2}}$$

2. $(r-2)^5(r+2)^3((r+1)^2+16)^2 = 0$

$$\boxed{\begin{array}{l} y = u_1 e^{2x} + u_2 e^{-2x} + u_3 e^{-x}\cos 4x + u_4 e^{-x}\sin 4x \\ u_1 = c_1+c_2x+c_3x^2+c_4x^3+c_5x^4,\ u_2 = c_6+c_7x+c_8x^2,\ u_3 = c_9+c_{10}x,\ u_4 = c_{11}+c_{12}x \end{array}}$$

(b) $(2r+1)(5r+3)=0$, $\boxed{x(t) = c_1 e^{-t/2} + c_2 e^{-3t/5}}$ $\boxed{\text{overdamped}}$



(c) atoms of RHS $= e^{3x}, xe^{3x}, x^2e^{3x}, e^{-3x}, \cos 3x, \sin 3x, x\cos 3x, x\sin 3x$

$y = y_1+y_2+y_3$, $y_1 = (d_1+d_2x+d_3x^2)e^{3x}$, $y_2 = d_4e^{-3x}$, $y_3 = (d_5+d_6x)\cos 3x + (d_7+d_8x)\sin 3x$

homog eq: $(r^2+9)^2$, atoms $= \cos 3x, \sin 3x, x\cos 3x, x\sin 3x$.

$\boxed{\text{corrected trial sol} = y_1+y_2+x^2y_3}$

(d) $\boxed{x_{ss} = d_1\cos 3t + d_2\sin 3t,\quad d_1 = \frac{2}{25},\ d_2 = \frac{3}{50}}$

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Ch6. (Eigenvalues and Eigenvectors)

☐ [30%] Ch6(a): Find the eigenvalues of the matrix A. Don't find eigenvectors!

$$A = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

☐ [35%] Ch6(b): Let A be a 2×2 matrix. Assume Fourier's method for A says that

$$\mathbf{x} = x_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

implies

$$A\mathbf{x} = 3x_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} - 7x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the matrix A.

☐ [35%] Ch6(c): Assume $\det(A - \lambda I) = \det(B - \lambda I)$ for two 5×5 matrices A, B. Let A have eigenvalues 1, 2, 6, a, b and let $5B$ have eigenvalues 30, $10 + 5i$, $10 - 5i$, c, d. Find the eigenvalues of A (determine a, b).

☐ [35%] Ch6(d): Display a 9×9 matrix A which is not diagonal and has eigenvalues 1.1, 1.2, $\cdots$, 1.9 [15%]. Justify your claim by citing applicable determinant evaluation rules [20%].

(a) $\det(A - \lambda I) = (3-\lambda)(2-\lambda)\left((4-\lambda)^2 - 1\right) \Rightarrow$ eigenvalues are $\boxed{2, 3, 3, 5}$

(b) $AP = PD \Rightarrow A = PDP^{-1}$. $P = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$, $P^{-1} = -\frac{1}{6}\begin{pmatrix} -1 & -1 \\ -1 & 5 \end{pmatrix}$

$A = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix} \cdot \frac{1}{6}$

$= \begin{pmatrix} 15 & -7 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix}\frac{1}{6} = \begin{pmatrix} 8/6 & 50/6 \\ 10/6 & -32/6 \end{pmatrix} = \boxed{\begin{pmatrix} 4/3 & 25/3 \\ 5/3 & -16/3 \end{pmatrix}}$

(c) $\det(5B - \lambda I) = \det(5I)\det(B - \frac{\lambda}{5}I) = 5^5 \det(A - \frac{\lambda}{5}I)$ so

$\{\frac{30}{5}, \frac{10+5i}{5}, \frac{10-5i}{5}, \frac{c}{5}, \frac{d}{5}\} = \{1, 2, 6, a, b\}$. To match, $\boxed{a = 2+i,\ b = 2-i}$

(d) Let $A_1 = \mathrm{diag}(1.1, 1.2, \ldots, 1.9)$. Let $A = A_1$ with one zero changed to 1. Then A is not diagonal and by the triangular rule $\det(A - \lambda I) = (1.1-\lambda)(1.2-\lambda)\cdots(1.9-\lambda)$ so A has the required eigenvalues.

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Ch7. (Linear Systems of Differential Equations)

☐ [40%] Ch7(a): Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 5$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\ y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\ x(0) = 0, \quad y(0) = 10. \end{cases}$$

Find the maximum amount of salt ever in tank B.

☐ [60%] Ch7(b): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

☐ [40%] Ch7(c): Assume A is 3×3 and has eigenvalues 3, $2 \pm \sqrt{5}i$. In the system $\mathbf{u}' = A\mathbf{u}$ where $\mathbf{u}(t)$ has components $x(t)$, $y(t)$, $z(t)$, determine the list of atoms used to express the components x, y, z in the general solution.

(a) $x(t) = 0$ by uniqueness. Then $y(t) = 10e^{-rt/100}$. The $\boxed{\text{max is } y(0) = 10}$.

(b) Eigenpairs are $\left(-5, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right), \left(-1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \left(5, \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}\right)$

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} e^{5t}}$$

(c) $\vec{u}(t) = c_1 \vec{v}_1 e^{3t} + c_2 \vec{w}_1 e^{2t} \cos\sqrt{5}t + c_2 \vec{w}_2 e^{2t} \sin\sqrt{5}t$

$$\boxed{\text{atoms} = e^{3t},\ e^{2t}\cos\sqrt{5}t,\ e^{2t}\sin\sqrt{5}t}$$

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Differential Equations and Linear Algebra 2250-2

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic Laplace integral table and know the basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [30%] Ch10(a): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{2s^2+6}{(s-1)(s-3)(s-1)(1-s)}.$$

☐ [30%] Ch10(b): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. Do not solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{v} + x''' = t^3e^{4t} + e^t\cos 2t, \quad x(0) = x'(0) = x''(0) = x'''(0) = 0, \quad x^{iv}(0) = 2.$$

☐ [35%] Ch10(c): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 2x + 3y + t^2, \\ y'' &= 4x + 3y, \\ x(0) &= 3, \quad x'(0) = 1, \\ y(0) &= 2, \quad y'(0) = 0. \end{aligned}$$

☐ [35%] Ch10(d): Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\frac{s}{(s^2-2s+5)} + \frac{s^2}{(s+1)^3} + \frac{2+s}{s^2+5s}.$$

☐ [35%] Ch10(e): Find $\mathcal{L}(f(t))$, given $f(t) = e^t(\cos(3t)-1)/t$.

(a) $\boxed{f(t) = (3+4t+2t^2)e^t - 3e^{3t}}$

(b) $\boxed{(s^5+s^3)\mathcal{L}(x) = 2 + \frac{6}{(s-4)^4} + \frac{s-1}{(s-1)^2+4}}$

(c) $\begin{cases} s^2\mathcal{L}(x) - 3s - 1 = 2\mathcal{L}(x) + 3\mathcal{L}(y) + \frac{2}{s^3} \\ s^2\mathcal{L}(y) - 2s = 4\mathcal{L}(x) + 3\mathcal{L}(y) \end{cases}$

$\mathcal{L}(y) = \frac{\Delta_1}{\Delta}$, $\Delta = \begin{vmatrix} s^2-2 & -3 \\ -4 & s^2-3 \end{vmatrix}$

$\boxed{\Delta = (s^2-2)(s^2-3) - 12}$

$\Delta_1 = \begin{vmatrix} s^2-2 & 2/s^3+3s+1 \\ -4 & 2s \end{vmatrix}$

$\boxed{\Delta_1 = 2s^3 - 4s + 8/s^3 + 12s + 4}$

(d)
$$\boxed{\begin{aligned} x(t) &= x_1 + x_2 + x_3 \\ x_1 &= (-t)\left[e^t\cos 2t + \tfrac{1}{2}e^t\sin 2t\right] \\ x_2 &= (1-2t+t^2/2)e^{-t} \\ x_3 &= (3/5)e^{-5t} + (2/5)\cdot 1 \end{aligned}}$$

(e) $\boxed{\mathcal{L}(f) = -\frac{1}{2}\ln((s-1)^2+9) + \ln(s-1)}$

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