Name <u>KEY</u> **10:45 Midterm Exam 3**

1. **(ch4)** Complete enough of the following to add to 100%.

(a) [100%] Let S be the vector space of all continuous functions defined on $(-\infty, \infty)$. Define V to be the set of all functions $f(x)$ in S such that $\int_0^1 f(x)xe^{-x}dx = 0$. Prove that V is a subspace of S.

(b) [30%] Find a 4×5 augmented matrix representing four equations for the constants a, b, c, d in the partial fractions decomposition for the fraction given below. To save time, **do not solve for** a, b, c, d!

$$\frac{2x-1}{(x-2)^2(x^2-4x+8)}$$

(b) [70%] Solve for the unknowns a, b, c, d in the system of equations below by augmented matrix RREF methods, showing all details.

$$\begin{array}{ccccccccc} a & + & b & - & 2c & + & 2d & = & 2 \\ & + & b & + & 2c & + & & = & 0 \\ a & + & 2b & + & & + & 2d & = & 2 \\ a & + & 3b & + & 2c & + & 2d & = & 2 \end{array}$$

Solution 1(a). Use the subspace criterion: (a) Given f and g in V, write details to show $f+g$ is in V; (b) Given f in V and k constant, write details to show kf is in V. Let $h(x) = xe^{-x}$, which is a function in S. Details for (a): Given $\int_0^1 f(x)h9x)dx = 0$ and $\int_0^1 g(x)h(x)dx = 0$, add the equations to obtain the equation $\int_0^1 (f(x)+g(x))h(x)dx = 0$. This finishes (a). Details for (b): Given $\int_0^1 f(x)h(x)dx = 0$ and k constant, multiply the equation by k and re-arrange factors to obtain the new equation $\int_0^1 (kf(x))h(x)dx = 0$. This proves (b).

Solution 1(b). The decomposition might be

$$\frac{2x-1}{(x-2)^2(x^2-4x+8)} = \frac{a}{x-2} + \frac{b}{(x-2)^2} + \frac{c(x-2)+2d}{x^2-4x+8}$$

although there are other possibilities. Clear the fractions. Set $x = 2$ to get one equation for the constants. Choose 3 other values for x to obtain three other equations. Displav the system of equations.

Solution 1(c). The answer is $\begin{pmatrix} 2+4t\_2t_2 \\ -2t_1 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

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2. **(ch5)** Complete all parts.

(a) [30%] Given $4x''(t) + 2x'(t) + x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 2$, $k = 1$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [70%] Find by variation of parameters or undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + x = 5\cos(2t)$.

(a) $4r^2 + 2r + 1 = 0$ char eq

has roots $-\frac{1}{4} \pm \frac{\sqrt{3}}{4} i$

[10%] under-damped

[20%] Sol $y = c_1 e^{-t/4} \cos(\sqrt{3}\, t/4) + c_2 e^{-t/4} \sin(\sqrt{3}\, t/4)$

(b) $x = d_1 \cos 2t + d_2 \sin 2t$

= corrected trial sol

$-4d_1 \cos 2t - 4d_2 \sin 2t + 2(-2d_1 \sin 2t + 2d_2 \cos 2t) + d_1 \cos 2t + d_2 \sin 2t$

$= 5\cos(2t)$

$$\begin{cases} -4d_1 + 4d_2 + d_1 = 5 \\ -4d_2 - 4d_1 + d_2 = 0 \end{cases} \iff \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Then $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} \frac{1}{\Delta}$

$= \begin{pmatrix} -15/25 \\ 20/25 \end{pmatrix}$

$\Delta = \begin{vmatrix} -3 & 4 \\ -4 & -3 \end{vmatrix} = 9 + 16 = 25$

$x_p = d_1 \cos 2t + d_2 \sin 2t$

$$\boxed{x_p = -\frac{3}{5} \cos 2t + \frac{4}{5} \sin 2t}$$

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3. **(ch5)** Complete all parts below.

(a) [75%] Determine for $y^{v} - 4y''' = xe^{2x} + x^2(1+x) + 2\cos x$ the **corrected** trial solution for y_p according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps $\boxed{1}$ and $\boxed{2}$, but skip steps $\boxed{3}$ and $\boxed{4}$)! Undocumented details will be considered guessing, which earns no credit.

(b) [10%] Using the *recipe* for second order constant-coefficient differential equations, write out a **basis** for the solution space of the equation $y'' + 4y' - y = 0$.

(c) [15%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is $(r-1)^2(r^2-1)^2(r^2+4)^2 = 0$.

(a) $y = y_1 + y_2 + y_3$

$y_1 = (d_1 + d_2 x)e^{2x}$

$y_2 = d_3 + d_4 x + d_5 x^2 + d_6 x^3$

$y_3 = d_7 \cos x + d_8 \sin x$

$r^5 - 4r^3 = 0$

$r^3(r-2)(r+2) = 0$

Fixup required for roots $r=0$ (mult $=3$) and $r=2$ (mult $=1$)

Multiply y_1 by x and y_2 by x^3.

$$\boxed{y = x(d_1 + d_2 x)e^{2x} + x^3(d_3 + d_4 x + d_5 x^2 + d_6 x^3) + d_7 \cos x + d_8 \sin x}$$

(b) $r^2 + 4r - 1 = 0$

$r = -\frac{4}{2} \pm \frac{1}{2}\sqrt{16+4}$

$= -2 \pm \sqrt{5}$

$$\boxed{\text{Basis} = \left\{ e^{-2t+\sqrt{5}t},\ e^{-2t-\sqrt{5}t} \right\}}$$

(c) $(r-1)^4(r+1)^2(r^2+4)^2 = 0$

$y = u_1 e^{x} + u_2 e^{-x} + u_3 \cos 2x + u_4 \sin 2x$

$u_1 = c_1 + c_2 x + c_3 x^2 + c_4 x^3$

$u_2 = c_5 + c_6 x$

$u_3 = c_7 + c_8 x$

$u_4 = c_9 + c_{10} x$

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4. **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$. To save time, **do not** find eigenvectors!

(b) [70%] Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$, then there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Find one possible column of P.

(a) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & -1 & 0 \\ 0 & 1-\lambda & -2 & 1 \\ 0 & 0 & 1-\lambda & -3 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 \left((1-\lambda)^2 + 9 \right)$

$\boxed{\text{Eigenvalues} = 1, 1, 1 \pm 3i}$

(b) $\lambda = 1$ is an eigenvalue. Find an eigenpair $(1, \vec{v})$.

$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right) \cong \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 0 \end{array} \right)$

$\cong \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$

$\boxed{\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

also $\left(4, \begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix}\right)$

and $\left(6, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$

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5. **(ch6)** Complete all parts below.

Consider a given 3×3 matrix A having three eigenpairs

$$6, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}; \quad 1, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

(a) [50%] Display the general solution $\mathbf{x}(t)$ of the system $\mathbf{x}' = A\mathbf{x}$ in vector form.

(b) [20%] Write a matrix algebra formula for the matrix A of (a) above. To save time, do not evaluate anything.

(c) [30%] Describe precisely Fourier's simplification method for the equation $\mathbf{y} = A\mathbf{x}$, using the matrix A of (a) above.

(a) $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{t}$

(b) $AP = PD \qquad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{pmatrix},$

$\boxed{A = PDP^{-1}}$

(c) $\vec{x} = x_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{y} = A\vec{x}$

$= 6x_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + 4x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

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