# Differential Equations and Linear Algebra 2250

Sample Midterm Exam 3, Fall 2005

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations. The midterm exam has 5 problems, some with multiple parts, suitable for 50 minutes.

1. **(ch4)** Let $A$ be a $51 \times 51$ matrix. Assume $V$ is the set of all vectors $\mathbf{x}$ such that $A^2\mathbf{x} = 3\mathbf{x}$. Prove that $V$ is a subspace of $\mathcal{R}^{51}$.

2. **(ch4)** Find a $4 \times 4$ system of linear equations for the constants $a$, $b$, $c$, $d$ in the partial fractions decomposition below [25%]. Solve for $a$, $b$, $c$, $d$, showing all **RREF** steps [60%]. Report the answers [15%].

$$\frac{x^2+2x-1}{(x+1)^2(x^2+6x+10)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c(x+3)+d}{x^2+6x+10}$$

3. **(ch5)** Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions:

   **1.**[25%] $y'' + y' + y = 0$ ,

   **2.**[25%] $y^{iv} + 4y'' = 0$ ,

   **3.**[25%] Char. eq. $(r+2)^3(r^2-4)(r^2+4) = 0$ ,

   **4.**[25%] Char. eq. $(r^2-3)^2(r^2+16)^3 = 0$ .

4. **(ch5)** Given $4x''(t) + 4x'(t) + x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 4$, $k = 1$, solve the differential equation [70%] and classify the answer as over-damped, critically damped or under-damped [15%]. Illustrate in a physical model drawing the meaning of constants $m$, $c$, $k$ [15%].

5. **(ch5)** Determine for $y^{iv} - 9y'' = xe^{3x} + x^3 + e^{-3x} + \sin x$ the corrected trial solution for $y_p$ according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

6. **(ch5)** Find by variation of parameters or undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 6x = 5\cos(3t)$.

7. **(ch6)** Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

8. **(ch6)** Given a $3 \times 3$ matrix $A$ has eigenpairs

$$\left(3, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right); \quad \left(1, \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}\right); \quad \left(0, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}\right),$$

   (a) find an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$ and (b) display Fourier's model for the equation $\mathbf{y} = A\mathbf{x}$.

9. **(ch6)** Give an example of a $3 \times 3$ matrix $C$ which has exactly one eigenpair

$$\left(2, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

**10.** **(ch6)** The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of $A$. Given

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix},$$

then

(1) [75%] Display eigenanalysis details for $A$.

(2) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.