Name. KEY

Applied Differential Equations 2250-2
Midterm Exam 1
Wednesday, 16 February 2005

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for $y(x)$ in the equation $y' = 2xe^{-2x} - \sec^2 x + \dfrac{x^3}{1+x^2}$.

$$\int y'\,dx = \int \left(2xe^{-2x} - \sec^2 x + \frac{x^3}{1+x^2}\right)dx$$

$$= \int 2xe^{-2x}\,dx - \int \sec^2 x\,dx + \int \frac{x^3}{1+x^2}\,dx$$

$$= -xe^{-2x} + \int e^{-2x}\,dx - \tan x + \int \left(\frac{x^3+x}{1+x^2} + \frac{-x}{1+x^2}\right)dx$$

$$= -xe^{-2x} - \frac{e^{-2x}}{2} - \tan x + \int x\,dx - \int \frac{x}{1+x^2}\,dx$$

$$= \left(-x - \tfrac{1}{2}\right)e^{-2x} - \tan x + \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + c$$

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2. (Separable Equation Test)
The problem $y' = 2x - x^{5/3} - 2xy^2 + x^{5/3}y^3$ may or may not be separable. If it is, then write formulae for F, G and decompose the problem as $y' = F(x)G(y)$. Otherwise, explain in detail why it fails to be separable. Do not solve for y!

$$f(x,y) = 2x - x^{5/3} - 2xy^2 + x^{5/3}y^3$$

$$f(1,0) = 2 - 1 = 1 \neq 0$$

$$F(x) = \frac{f(x,0)}{f(1,0)} = 2x - x^{5/3}$$

$$G(y) = f(1,y) = 1 - 2y^2 + y^3$$

$$FG = (2x - x^{5/3})(1 - 2y^2 + y^3) = 2x - x^{5/3} - 4xy^2 + 2x^{5/3}y^2 + 2xy^3 - x^{5/3}y^3 \neq 2x - x^{5/3} - 2xy^2 + x^{5/3}y^3$$

$FG \neq f \Rightarrow$ Not separable.

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3. (Solve a Separable Equation)

Given $yy' = \dfrac{x^2+2x}{3+2x}(1-5y^2)$, find the non-equilibrium solution in implicit form. Do not solve for y explicitly and do not find equilibrium solutions.

$$\int \frac{yy'\,dx}{1-5y^2} = \int \frac{x^2+2x}{3+2x}\,dx$$

$$\begin{array}{r|l} & \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{2} \\ \hline 3+2x & x^2+2x \\ & x^2+\frac{3}{2}x \\ \hline & \frac{1}{2}x \\ & \frac{1}{2}x+3/4 \\ \hline & -3/4 \end{array}$$

$$-\frac{1}{10}\ln|1-5y^2| = \int\left(\frac{1}{2}x+\frac{1}{4}+\frac{-3/4}{3+2x}\right)dx$$

$$-\frac{1}{10}\ln|1-5y^2| = \frac{x^2}{4}+\frac{x}{4}-\frac{3}{4}\cdot\frac{1}{2}\ln|3+2x|+c$$

$$-\frac{1}{10}\ln|1-5y^2| = \frac{x^2}{4}+\frac{x}{4}-\frac{3}{8}\ln|3+2x|+c$$

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4. (Linear Equations)

Solve (a) $3v'(t) = 15 - \dfrac{5}{t+11}v(t)$, $v(0) = 2$. Show all integrating factor steps.
(b) $y'(t) = v(t)$, $y(0) = 10$. Show all quadrature steps.

(a) [90%]

$v' = 5 - \dfrac{\alpha}{t+11}v \qquad v(0)=2 \qquad [\alpha \equiv 5/3]$

$v' + \dfrac{\alpha}{t+11}v = 5$

$\dfrac{(Qv)'}{Q} = 5 \qquad Q = e^{\int \frac{\alpha\,dt}{t+11}} = (t+11)^{\alpha}$

$Qv = 5\int Q\,dt$

$= \dfrac{5}{1+\alpha}(t+11)^{1+\alpha} + c$

$v = \dfrac{5}{1+\alpha}(t+11) + c\,(t+11)^{-\alpha}$

$2 = \dfrac{55}{1+\alpha} + \dfrac{c}{11^{\alpha}}$

$c = \left(2 - \dfrac{55}{1+\alpha}\right)11^{\alpha} = -\dfrac{149}{8}\left(11^{5/3}\right)$

$v = \dfrac{15}{8}(t+11) + \left(2 - \dfrac{55}{1+\alpha}\right)11^{\alpha}(t+11)^{-\alpha}$

$\boxed{v = \dfrac{15t}{8} + \dfrac{165}{8} + \left(2 - \dfrac{165}{8}\right)\left(\dfrac{11}{t+11}\right)^{5/3}}$

(b) [10%]

$\int_0^t y'\,dt = \int_0^t v(t)\,dt$

$y - 10 = \int_0^t \left(\dfrac{15t}{8} + \dfrac{165}{8} + \left(2 - \dfrac{165}{8}\right)\left(\dfrac{11}{t+11}\right)^{5/3}\right)dt$

$y = 10 + \dfrac{15t^2}{16} + \dfrac{165t}{8} + \left(2 - \dfrac{165}{8}\right)11^{5/3}\left[\dfrac{(t+11)^{-\frac{2}{3}} - 11^{-\frac{2}{3}}}{-2/3}\right]$

Full credit for method. Answer not checked.

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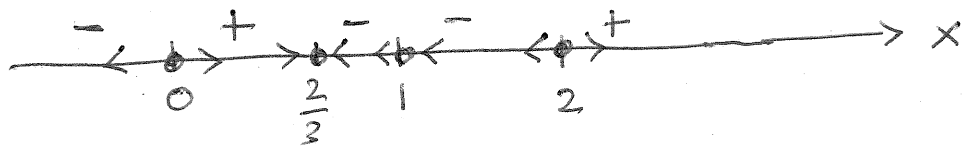
5. (Stability)

(a) Draw a phase line diagram for the chemical reaction equation $dx/dt = (2 - 3x)^5(1-x)^2(2-x)x^3$. Expected in the diagram are equilibrium points, signs of x' and flow direction markers (< and >).

(b) Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, stable, unstable.

(a) equilibria $= \frac{2}{3}, 1, 2, 0$



$f(x) = (2-3x)^5(1-x)^2(2-x)x^3$

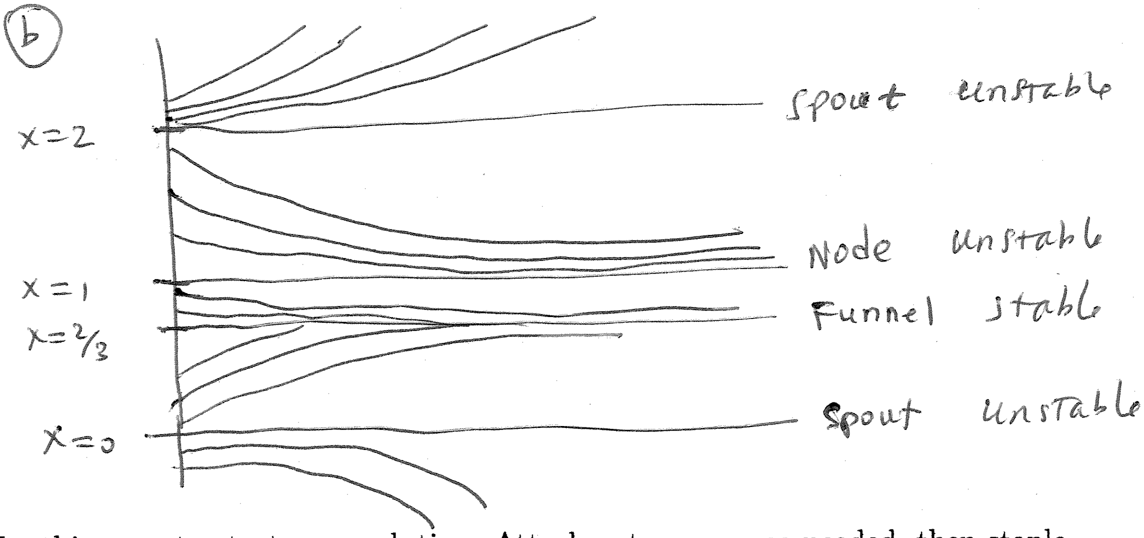
$f(-1) = (+)(+)(+)(-1)$
$= (-)$

$f(.25) = (+)(+)(+)(+)$
$= (+)$

$f(1.5) = (-)(+)(+)(+)$
$= (-)$

$f(3) = (-)(+)(-)(+)$
$= (+)$

(b)



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