

Differential Equations and Linear Algebra 2250-1

Final Exam 8:00am 14 Dec 2004

Instructions. The time allowed is 120 minutes. The examination consists of six problems, one for each of chapters 3, 4, 5, 6, 7, 10, each problem with multiple parts. A chapter represents 20 minutes on the final exam.

Each problem represents several textbook problems numbered (a), (b), (c), \dots . Choose the problems to be graded by check-mark ; the credits should add to 100.

Calculators, books, notes and computers are not allowed.

Answer checks are not expected or required. First drafts are expected, not complete presentations.

Please submit **exactly six** separately stapled packages of problems, one package per chapter.

Ch3. (Linear Systems and Matrices)

[50%] Ch3(a): Find the value of x_2 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and \mathbf{b} below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of 3×3 Sarrus' rule is disallowed (2×2 use is allowed).

$$C = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

[50%] Ch3(b): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has infinitely many solutions [25%] and then for all such k display the solution formula for \mathbf{x} [25%].

$$R = \begin{bmatrix} 2 & 1 & -k \\ 2 & 2k & -2 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1 - 2k \\ 0 \end{pmatrix}$$

[50%] Ch3(c): Prove that if an $n \times n$ triangular matrix is invertible, then all of its diagonal entries are nonzero.

Ch4. (Vector Spaces)

[40%] Ch4(a): State an RREF test to detect the independence or dependence of fixed vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in \mathcal{R}^3 [20%]. Apply the test to the vectors below [15%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

[60%] Ch4(b): Let A be a 5×5 matrix. Assume V is the set of all vectors \mathbf{x} such that $(A + I)\mathbf{Ax} = (I - A)\mathbf{x}$. Prove that V is a subspace of \mathcal{R}^5 .

[60%] Ch4(c): Find a basis of fixed vectors in \mathcal{R}^4 for the solution space of $\mathbf{Ax} = \mathbf{0}$:

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -3 & 1 \\ 0 & 0 & 3 & 1 \\ 2 & -2 & 4 & 1 \end{bmatrix}$$

[40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d , showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{x^2 + 3x + 1}{(x + 1)^2(x - 2)^2} = \frac{a}{x + 1} + \frac{b}{(x + 1)^2} + \frac{c}{x - 2} + \frac{d}{(x - 2)^2}$$

Ch5. (Linear Equations of Higher Order)

[30%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

$$\begin{aligned} 1. [15\%] \quad & (r^2 - 3r)^2(r - 3) = 0, \\ 2. [15\%] \quad & (r - 2)^2(r^2 + 2r + 5)^2(r^2 - 4) = 0 \end{aligned}$$

[30%] Ch5(b): Given $6x''(t) + 7x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 6, c = 7, k = 2$, solve the differential equation [25%] and classify the answer as over-damped, critically damped or under-damped [5%].

[40%] Ch5(c): Determine the **final form** of a trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv} - 32y'' + 256y = x^2e^{4x} + \cos 4x + e^{-4x}$$

[30%] Ch5(d): Find the steady-state periodic solution for the equation

$$x'' + 2x' + 10x = 5 \cos(3t).$$

Ch6. (Eigenvalues and Eigenvectors)

[30%] Ch6(a): Find the eigenvalues of the matrix A :

$$A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

[35%] Ch6(b): Let A be a 3×3 matrix with eigenpairs

$$(1, \mathbf{v}_1), \quad (2, \mathbf{v}_2), \quad (3, \mathbf{v}_3).$$

Let $P = \mathbf{aug}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3)$. Report the answer for $P^{-1}AP$ [15%]. Justify your claims [20%].

[35%] Ch6(c): Assume A is a given 5×5 matrix with eigenvalues $3, 2 \pm 3i, 2 \pm \sqrt{7}i$. Find the eigenvalues of $A - 2I$, where I is the identity matrix.

[35%] Ch6(d): Give an example of a 3×3 matrix C which has eigenvalues $3, 4, 4$ but only two independent eigenvectors [15%]. Justify your claim [20%].

Ch7. (Linear Systems of Differential Equations)

[40%] Ch7(a): Solve for the general solution $x(t), y(t)$ in the system below.

$$\begin{aligned} x' &= x + y, \\ y' &= -3x + y. \end{aligned}$$

[60%] Ch7(b): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

[40%] Ch7(c): Assume A is 2×2 and has eigenvalues $2 \pm \sqrt{7}i$. In the system $\mathbf{u}' = A\mathbf{u}$ where $\mathbf{u}(t)$ has components $x(t)$, $y(t)$, explain why

$$x(t) = c_1 e^{2t} \cos \sqrt{7}t + c_2 e^{2t} \sin \sqrt{7}t.$$

Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic Laplace integral table and know the basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[30%] Ch10(a): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{s+4}{(s^2-1)(s+1)^2}.$$

[30%] Ch10(b): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. Do not solve for $x(t)$! Document steps by reference to tables and rules.

$$x''' + 3x' = te^{-2t} + \sin \sqrt{3}t, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = -1.$$

[35%] Ch10(c): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 2x + 3y, \\ y'' &= 3x + 4y, \\ x(0) &= 0, \quad x'(0) = -1, \\ y(0) &= 1, \quad y'(0) = 0. \end{aligned}$$

[35%] Ch10(d): Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds} \frac{d}{ds} \left(\frac{s+1}{s^2+2s+2} \right) + \frac{s-1}{s^4} + \frac{2+s}{s^2+3s}.$$

[35%] Ch10(e): Find $\mathcal{L}(f(t))$, given $f(t) = \frac{e^{5t} - e^{3t}}{t}$.