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Differential Equations and Linear Algebra 2250-2

Midterm Exam 3, Fall 2004

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations.

1. **(ch4) (a) [50%]** Let V be the set of all vectors $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $y = 3x$. Prove that V is a subspace of $\mathcal{R}^2$.

(b) [50%] Find a 3×3 system of linear equations for the constants a, b, c in the partial fractions decomposition below. Solve for a, b, c by RREF methods.

$$\frac{x^2+x-3}{(x+1)^2(x+2)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x+2}$$

(a) $V = \{ \vec{u} : \vec{a} \cdot \vec{u} = 0 \}$ where $\vec{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$.
Dot product properties give $\vec{a} \cdot (c_1\vec{u}_1 + c_2\vec{u}_2) = c_1(\vec{a}\cdot\vec{u}_1) + c_2(\vec{a}\cdot\vec{u}_2)$, so V is closed under addition and scalar mult. Hence V is a subspace of $\mathbb{R}^2$.

(b) Clear fractions

$$x^2+x-3 = a(x+1)(x+2) + b(x+2) + c(x+1)^2$$

$x=-1$: $-3 = b$
$x=-2$: $-1 = c$
$x=0$: $-3 = 2a + 2b + c$

The system

RREF method

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left(\begin{array}{ccc|c} 2 & 2 & 1 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right) \end{array} \cong \left(\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\boxed{a = 2,\ b = -3,\ c = -1}$$

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2. (ch5)
(a) [50%] Use the *recipe* for higher order constant-coefficient differential equations to write out the general solution of a fifth order linear differential equation with characteristic equation $r^5 + 4r^4 + 4r^3 = 0$.
(b) [50%] Given $mx''(t) + cx'(t) + kx(t) = 0$ and damped spring-mass system constants $m = 4$, $c = 3$, $k = 1$, solve the differential equation [70%] and classify the answer as over-damped, critically damped or under-damped [30%].

(a) $r^3(r^2+4r+4)=0$

$r^3(r+2)^2=0$

$$\boxed{y_h = (c_1+c_2x+c_3x^2)e^{0x} + (c_4+c_5x)e^{-2x}}$$

(b) $4x''+3x'+x=0$

$4r^2+3r+1=0$

$r = -\frac{3}{8} \pm \frac{1}{8}\sqrt{9-16}$

$= -\frac{3}{8} \pm \frac{1}{8}(\sqrt{7})i$

$$\boxed{x = [c_1\cos(\sqrt{7}t/8) + c_2\sin(\sqrt{7}t/8)]e^{-3t/8}}$$

$\boxed{\text{under damped}}$

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3. (ch5)
(a) [50%] Determine for $y^v + 4y''' = x^2\cos 2x + x + 4x^2 + \sin 2x$ the final form of a trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!
(b) [50%] Apply variation of parameters to write an integral formula for a particular solution $x_p(t)$ for the equation $4x'' + 12x' + 8x = f(t)$. Don't integrate!

(a)

$$\boxed{y_p = (d_1 + d_2 x + d_3 x^2)x^3 + x(d_4 + d_5 x + d_6 x^2)\cos 2x + x(d_7 + d_8 x + d_9 x^2)\sin 2x}$$

$r^3(r^2+4) = 0$

$y_h = c_1 + c_2 x + c_3 x^2 + c_4 \cos 2x + c_5 \sin 2x$

Fixup on polyn : x^3

Fixup on trig : x

(b)

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r = -2,\ r = -1$

$x_h = c_1 e^{-2t} + c_2 e^{-t}$

$W = x_1 x_2' - x_1' x_2$

$= e^{-2t}(-e^{-t}) - (-2)e^{-2t}e^{-t}$

$= e^{-3t}$

$x_p = x_1 \int x_2\left(\frac{-f}{W}\right)\frac{dt}{4} + x_2 \int x_1\left(\frac{f}{W}\right)\frac{dt}{4}$

$$\boxed{x_p = e^{-2t}\int e^{-t}\left(\frac{-f}{e^{-3t}}\right)\frac{dt}{4} + e^{-t}\int e^{-2t}\frac{f}{e^{-3t}}\frac{dt}{4}}$$

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4. (ch6)
Give an example of a 3×3 matrix A with eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}$$

and eigenvalues -4, 3, -1 [60%]. Justify your answer [40%].

100% for:

$AP = PD$

$A = PDP^{-1}$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ -3 & 2 & -5 \end{pmatrix}$$

$$D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

or any permutation of these.
Don't expect matrix mult and inverse to come out right.
Didn't check for accuracy of ans A, only ~~formula~~
$\underline{A = PDP^{-1}}$.

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5. (ch7)

Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given $A = \begin{bmatrix} 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(a)

$$\det(A-\lambda I) = (1-\lambda)\begin{vmatrix} 1-\lambda & 4 & 1 \\ 1 & 1-\lambda & 4 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)(-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-\lambda)\left[(1-\lambda)^2-4\right]$$

$$= (1-\lambda)(-\lambda)(-1-\lambda)(3-\lambda)$$

eigenvalues $= 1, 0, -1, 3$

(b)

$\lambda=1$: $\begin{pmatrix} 0 & 4 & 1 & 1 \\ 1 & 0 & 4 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 & 17 \\ 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad \vec{v}_1 = \begin{pmatrix} -17 \\ -5/4 \\ 4 \\ 1 \end{pmatrix}$

$\lambda=0$: $\begin{pmatrix} 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda=-1$: $\begin{pmatrix} 2 & 4 & 1 & 1 \\ 1 & 2 & 4 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cong \begin{pmatrix} 2 & 4 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad \vec{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda=3$: $\begin{pmatrix} -2 & 4 & 1 & 1 \\ 1 & -2 & 4 & 1 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & -2 \end{pmatrix} \cong \begin{pmatrix} -2 & 4 & 1 & 0 \\ 1 & -2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad \vec{v}_4 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

ans: $\vec{x} = c_1 e^{t}\begin{pmatrix} -17 \\ -5/4 \\ 4 \\ 1 \end{pmatrix} + c_2 e^{0t}\begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_4 e^{3t}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$