Name KEY

2250 Midterm 3 7:30

Differential Equations and Linear Algebra 2250-1

Midterm Exam 3, Fall 2004

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations.

1. (ch4) (a) [50%] Let V be the set of all vectors $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $x = 4y$. Prove that V is a subspace of $\mathcal{R}^2$.

(b) [50%] Find a 3×3 system of linear equations for the constants a, b, c in the partial fractions decomposition below. Solve for a, b, c by RREF methods.

$$\frac{x^2+x-1}{(x-1)^2(x+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2}$$

(a) proof: Let $A = \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix}$. Then $V = \{ \vec{u} : A\vec{u} = \vec{0} \}$. By superposition, V is a subspace.

(b) $x^2+x-1 = a(x-1)(x+2) + b(x+2) + c(x-1)^2$

$x=1$: $1 = 0 + 3b + 0$

$x=-2$: $1 = 0 + 0 + 9c$

$x=0$: $-1 = -2a + 2b + c$

$$\left(\begin{array}{ccc|c} 0 & 3 & 0 & 1 \\ 0 & 0 & 9 & 1 \\ -2 & 2 & 1 & -1 \end{array}\right) \cong \left(\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 9 & 1 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/9 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1+2/3+\frac{1}{9} \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/9 \end{array}\right)$$

$$\boxed{a = 8/9 \quad b = \frac{1}{3} \quad c = \frac{1}{9}}$$

Chk; mult by $x-1$

$1 = a + c$

$1 = \frac{8}{9} + \frac{1}{9}$ ✓

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2. (ch5)
(a) [50%] Use the *recipe* for higher order constant-coefficient differential equations to write out the general solution of a fifth order linear differential equation with characteristic equation $r^5+2r^4+r^3=0$.
(b) [50%] Given $mx''(t)+cx'(t)+kx(t)=0$ and damped spring-mass system constants $m=2$, $c=4$, $k=1$, solve the differential equation [70%] and classify the answer as over-damped, critically damped or under-damped [30%].

(a) $r^3(r^2+2r+1)=0$

$r^3(r+1)^2=0$

$$\boxed{y=(c_1+c_2x+c_3x^2)e^{0x}+(c_4+c_5x)e^{-x}}$$

(b) $2x''+4x'+x=0$

$2r^2+4r+1=0$

$r=-\frac{4}{4}\pm\frac{1}{4}\sqrt{16-8}$

$=-1\pm\frac{1}{4}\sqrt{8}$

$=-1\pm\frac{1}{2}\sqrt{2}$

$$\boxed{x(t)=c_1e^{-t+t/\sqrt{2}}+c_2e^{-t-t/\sqrt{2}}}$$

overdamped

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3. (ch5)
(a) [50%] Determine for $y^v + y''' = x\sin x + 1 + 3x^2 + \cos x$ the final form of a trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!
(b) [50%] Apply variation of parameters to write an integral formula for a particular solution $x_p(t)$ for the equation $3x'' + 15x' + 12x = f(t)$. Don't integrate!

(a) $r^5 + r^3 = 0$

$r^3(r^2+1) = 0$

$y_h = c_1 + c_2 x + c_3 x^2 + c_4 \cos x + c_5 \sin x$

$y_1 = (d_1 + d_2 x)\cos x + (d_3 + d_4 x)\sin x$

$y_2 = d_5 + d_6 x + d_7 x^2$

Fixup rule:

$y_1 = x(d_1 + d_2 x)\cos x + x(d_3 + d_4 x)\sin x$

$y_2 = x^3(d_5 + d_6 x + d_7 x^2)$

$y = y_1 + y_2$
= Final trial sol

(b) $x'' + 5x' + 4x = \frac{f}{3}$

$r^2 + 5r + 4 = 0$

$(r+4)(r+1) = 0$

$x_1 = e^{-4t}$

$x_2 = e^{-t}$

$W = x_1 x_2' - x_1' x_2 = -e^{-5t} + 4e^{-5t} = 3e^{-5t}$

$$\boxed{x_p = e^{-4t}\int \frac{e^{-t}(-f)}{9e^{-5t}}\,dt + e^{-t}\int \frac{e^{-4t} f}{9e^{-5t}}\,dt}$$

4. (ch6)
Give an example of a 3×3 matrix A with eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

and eigenvalues 4, 3, 7. Justify your answer.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & -5 & -3 \end{pmatrix}$$

$$T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$AP = PT$ defines A.

$$A = PTP^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & -5 & -3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} -1 & +2 & -4 \\ 0 & +3 & +5 \\ 0 & -1 & 2 \end{pmatrix}^T \cdot \frac{1}{-1}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 7 \\ 8 & -15 & -21 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & -3 & 1 \\ 4 & -5 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ -12+28 & -18-35 & 6-14 \\ 8+30-84 & 45+105 & -15+42 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 16 & -53 & -8 \\ -46 & 150 & 27 \end{pmatrix}$$

- Justify: $A = PTP^{-1} \Rightarrow AP = PT \Rightarrow$ eigenpairs are $4, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$; $3, \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}$; $7, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$

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5. (ch7)

Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given $A = \begin{bmatrix} 4 & 1 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$$0 = \begin{vmatrix} 4-\lambda & 1 & 1 & 0 \\ 1 & 4-\lambda & 1 & 0 \\ 0 & 0 & 4-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (-\lambda)(4-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix}$$

$$0 = (-\lambda)(4-\lambda)((4-\lambda)^2-1)$$

$$0 = \lambda(4-\lambda)(3-\lambda)(5-\lambda)$$

$$\boxed{\lambda = 0, 3, 4, 5}$$

$\lambda=0$: $\begin{bmatrix} 4 & 1 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ eigenpair $= 0, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda=3$: $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ eigenpair $= 3, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda=4$: $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ eigenpair $= 4, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda=5$: $\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cong \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ eigenpair $= 5, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{x} = e^{0t}c_1\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + e^{3t}c_2\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + e^{4t}c_3\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + e^{5t}c_4\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$