Name ________________________ **2250-1**

Differential Equations and Linear Algebra 2250-1

Final Exam 8:00am 4 May 2004

Instructions. The time allowed is 120 minutes. The examination consists of six problems, one for each of chapters 3, 4, 5, 6, 7, 10, each problem with multiple parts. A chapter represents 20 minutes on the final exam.

Each problem represents several textbook problems numbered (a), (b), (c), $\cdots$. Choose the problems to be graded by check-mark $\boxed{\text{X}}$; the credits should add to 100.

Calculators, books, notes and computers are not allowed.

Answer checks are not expected or required. First drafts are expected, not complete presentations.

Please submit **exactly six** separately stapled packages of problems, one package per chapter.

Ch3. (Linear Systems and Matrices)

☐ [40%] Ch3(a): Find the inverse matrix B^{-1} by the RREF method.

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -2 \\ 0 & 2 & 4 \end{bmatrix}$$

☐ [60%] Ch3(b): Find the value of x_4 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and $\mathbf{b}$ below. Evaluate determinants by the method of elimination or cofactor expansion (or both). The use of 3×3 Sarrus' rule is disallowed (2×2 use is allowed).

$$C = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

☐ [40%] Ch3(d): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has infinitely many solutions and then display the solution formula for $\mathbf{x}$.

$$R = \begin{bmatrix} 1 & 1 & -k \\ 1 & k & -2 \\ 0 & 2 & 4 \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ -k \\ 0 \end{pmatrix}$$

Please staple this page to the front of your submitted exam problem Ch3.

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Ch4. (Vector Spaces)

☐ [40%] Ch4(a): Cite or state a determinant test to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}3$ in $\mathcal{R}^3$ [10%]. Apply the test to the vectors below [25%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

☐ [60%] Ch4(b): Let A be a 71×71 matrix. Assume V is the set of all vectors $\mathbf{x}$ such that $A^2\mathbf{x} = 3\mathbf{x}$. Prove that V is a subspace of $\mathcal{R}^{71}$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathcal{R}^4$ for the solution space of $A\mathbf{x} = \mathbf{0}$:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 2 & 4 & 0 \\ 1 & 3 & 2 & 1 \end{bmatrix}$$

☐ [40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{x^2 + 2x - 1}{(x+1)^2(x^2+6x+10)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c(x+3)+d}{x^2+6x+10}$$

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Ch5. (Linear Equations of Higher Order)

☐ [30%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the two equations below.

$$\begin{array}{ll} \textbf{1}.[15\%] & y'' + y' + y = 0 \\ \textbf{2}.[15\%] & y^{iv} + 4y'' = 0 \end{array}$$

☐ [30%] Ch5(b): Given $4x''(t) + 4x'(t) + x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 4$, $k = 1$, solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a diagram the meaning of m, c, k in the physical model [5%].

☐ [40%] Ch5(c): Determine (from the table on page 331 of the textbook) the final form of a trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

$$y^{iv} - 9y'' = xe^{3x} + x^3 + e^{-3x}$$

☐ [30%] Ch5(d): Find the steady-state periodic solution for the equation

$$x'' + 2x' + 6x = 5\cos(3t).$$

Please staple this page to the front of your submitted exam problem Ch5.

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Ch6. (Eigenvalues and Eigenvectors)

☐ [30%] Ch6(a): Find the eigenvalues of the matrix A:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

☐ [35%] Ch6(b): Given a 3×3 matrix A has eigenpairs

$$3, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}; \quad 1, \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}; \quad 0, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix},$$

find an invertible matrix P and a diagonal matrix D such that $AP = PD$.

☐ [35%] Ch6(c): Let A, B be square matrices such that $AP = PD$ for some invertible matrix P. Prove that $\det(A - \lambda I) = \det(B - \lambda I)$, hence A and B have the same eigenvalues.

☐ [30%] Ch6(d): Give an example of a 3×3 matrix C which has exactly one eigenpair

$$2, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Please staple this page to the front of your submitted exam problem Ch6.

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Ch7. (Linear Systems of Differential Equations)

☐ [40%] Ch7(a): Solve for $x(t)$, $y(t)$ in the system below. The answers depend upon two arbitrary constants, because $x(0)$ and $y(0)$ are not supplied.

$$\begin{aligned} x' &= x - y, \\ y' &= 10x + y. \end{aligned}$$

☐ [40%] Ch7(b): Let the real 2×2 matrix A have a complex eigenpair $7i, \begin{pmatrix} 1+i \\ -1 \end{pmatrix}$. Find all real solutions $\mathbf{x}(t)$ of the system $\mathbf{x}' = A\mathbf{x}$.

☐ [60%] Ch7(c): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

☐ [40%] Ch7(d): Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 10$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model

$$\begin{cases} x'(t) = -\dfrac{r}{25}x(t), \\ y'(t) = \dfrac{r}{25}x(t) - \dfrac{r}{50}y(t), \\ x(0) = 0, \quad y(0) = 15. \end{cases}$$

Find the maximum amount of salt ever in tank B.

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic Laplace integral table and know the basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [30%] Ch10(a): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{10s - 3}{(s-1)s^2}.$$

☐ [30%] Ch10(b): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. Do not solve for $x(t)$! Document steps by reference to tables and rules.

$$x'' + 3x' + 16x = \pi e^{-t} + \sin 2t, \quad x(0) = 0, \quad x'(0) = 1.$$

☐ [35%] Ch10(c): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Do not solve for $y(t)$! Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it for $\mathcal{L}(y)$ [15%].

$$\begin{aligned} &x' = 2x + 3y, \\ &y' = 6x + 3y, \\ &x(0) = 1, \quad y(0) = -2. \end{aligned}$$

☐ [35%] Ch10(d): Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\frac{s+2}{(s^2+4)} + \frac{1}{s^3} + \frac{1}{s^2+2s+5}.$$

☐ [35%] Ch10(e): Find $\mathcal{L}(f(t))$, given $f(t) = \dfrac{e^{3t} - 1}{t}$.

Please staple this page to the front of your submitted exam problem Ch10.