MATH 5610/6860
PRACTICE FINAL EXAM

Note: This exam is longer and more difficult than the actual final.

Problem 1. Consider a smooth function f with the following values.

x	1	2	4	5
$f(x)$	0	3	-3	0

(a) Use divided differences to find the polynomial $p(x)$ interpolating $f(x)$ (at all four nodes) in **Newton** form.
(b) Assuming all derivatives of f are available, give an expression of the interpolation error $f(t) - p(t)$ for some $t \in [1, 5]$.

Problem 2. Find the cubic spline $S(x)$ for $x \in [0, 1]$ satisfying the conditions $S(0) = 0$, $S(1) = 1$, $S'(0) = S'(1) = 0$.

Problem 3. Derive the centered difference formula for approximating $f'(x)$ with an error term involving a higher order derivative of f.

Problem 4. (K&C 7.1.15) Derive a numerical differentiation formula of order $\mathcal{O}(h^4)$ by applying Richardson's extrapolation to

$$f'(x) = \frac{1}{2h}[f(x+h) - f(x-h)] - \frac{h^2}{6}f'''(x) - \frac{h^4}{120}f^{(5)}(x) - \cdots$$

What is the error in terms of h^4?

Problem 5. (K&C 7.2.10) Use the Lagrange interpolation polynomial to derive a quadrature formula of the form

$$\int_0^1 f(x)dx \approx Af(1/3) + Bf(2/3).$$

Transform this formula to one for integration over $[a, b]$.

Problem 6. (K&C 7.5.2) The trapezoid rule can be written in the form

$$I \equiv \int_u^v f(x)dx = T(u, v) - \frac{1}{2}(v-u)^3 f''(\xi).$$

(a) Let $w = (u + v)/2$ and assume f is twice continuously differentiable. Find the constant C in the error term below

$$I = T(u, w) + T(w, v) + C(v - u)^3 f''(\tilde{\xi}).$$

(b) Assuming $f''(\tilde{\xi}) \approx f''(\xi)$ find an approximation to the *error term* in the equation from (a) in terms of $T(u, w)$, $T(w, v)$ and $T(u, v)$.

Problem 7. Find the LU factorization (with L being a unit lower triangular matrix) of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Problem 8. The first five Legendre polynomials (monic and orthogonal on [-1,1]) are:

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x \\ p_2(x) &= x^2 - \frac{2}{3} \\ p_3(x) &= x^3 - \frac{3}{5}x \\ p_4(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35} \end{aligned}$$

Find p_4 using $p_0, \cdots, p_3$.

Problem 9. Find the polynomial of the form $p(x) = a + bx$ that best approximates $f(x) = x^3$ in $[0,1]$, where the norm is induced by the product

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Problem 10. Let $x_j = 2\pi j/N$ and $E_j(x) = \exp[2i\pi jx/N]$, $j = 0, \ldots, N-1$. Recall the pseudo-inner product

$$(f, g)_N = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j)\overline{g}(x_j),$$

and its fundamental property

$$(E_n, E_m) = \begin{cases} 1 & \text{if } n - m \text{ is divisible by } N \\ 0 & \text{otherwise.} \end{cases}$$

Consider the trigonometric polynomial

$$p = \sum_{k=0}^{N-1} c_k E_k.$$

Show the discrete Parseval's identity

$$\sum_{j=0}^{N-1} |p(x_j)|^2 = N \sum_{j=0}^{N-1} |c_k|^2.$$

Hint: For $j = 0, \ldots, N-1$ we have,

$$|p(x_j)|^2 = \left(\sum_{k=0}^{N-1} c_k E_k(x_j) \right) \left(\sum_{k'=0}^{N-1} c_{k'} \overline{E}_{k'}(x_j) \right).$$