MATH 5610/6860
PRACTICE MIDTERM EXAM

Problem 1. Let $\alpha_n \to 0$, $x_n = \mathcal{O}(\alpha_n)$ and $y_n = \mathcal{O}(\alpha_n)$.
Show that $x_n y_n = o(\alpha_n)$.

Problem 2.

(a) Write the Taylor expansion of $\ln(1+x)$ about $x = 0$ with the Lagrange form of the residual term.

(b) Assume the Taylor series for $\ln(1+x)$ is truncated after the term involving x^{10} and is used to approximate the number $\ln 2$. What bound on the error can be given?

Problem 3. Consider the number $x = 2^6 + 2^{-16} + 2^{-19}$.

(a) Write x in scientific base 2 (binary) notation of the form $(1.b_1b_2b_3\ldots b_n)\times 2^S$.

(b) If IEEE single precision is used, the number of bits above is limited to 23. What is x_+ (floating point number immediately above x) and x_- (floating point number immediately below x).

(c) What is $\mathrm{fl}(x)$ (the floating point representation of x, assuming round to nearest)

(d) What is machine precision ϵ in this system?

(e) Verify that the relative error between x and $\mathrm{fl}(x)$ is less than machine precision.

Problem 4. Halley's method for solving $f(x) = 0$ uses the iteration formula

$$x_{n+1} = x_n - \frac{f_n f_n'}{(f_n')^2 - (f_n f_n'')/2},$$

where $f_n = f(x_n)$ and so on. Show that this formula results from applying Newton's method to the function $f/\sqrt{f'}$.

Problem 5. Consider the polynomial $p(z) = z^4 + 2z^3 + 3z^2 + 3z + 2$.

(a) Compute $p(2)$ using Horner's method

(b) Compute $p'(2)$ using Horner's method

(c) Write $p(z)$ in the form $p(z) = (z-2)q(z) + r$, specifying $q(z)$ and r.

Problem 6. Consider a smooth function f with the following values.

x	-1	0	1	2
$f(x)$	-1	0	3	-1

(a) Write the polynomial interpolating $f(x)$ at the *first three nodes* in Lagrange form.

(b) Use divided differences to find the polynomial $p(x)$ interpolating $f(x)$ in Newton form.

(c) Assuming all derivatives of f are available, give an expression of the interpolation error $f(t) - p(t)$ for some $t \in [-1, 2]$.

Problem 7. Let $f(x)$ be a function of x and $x_0, \ldots, x_n$ be $n+1$ distinct nodes. For $k = 0, \ldots, n$, let p_k be the polynomial interpolating f at the nodes $x_0, x_1, \ldots, x_k$. Let q be the polynomial interpolating f at the nodes $x_1, \ldots, x_n$. Show that:

$$p_n(x) = q(x) + \frac{x - x_n}{x_n - x_0}(q(x) - p_{n-1}(x)).$$