Fast Fourier Transform

Recall we found that the exponential polynomial interpolating a function f at $x_j = \frac{2\pi j}{N}$ is given by:

$$P = \sum_{k=0}^{N-1} c_k E_k , \quad c_k = (f, E_k)_N$$

$$c_k = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) (\lambda^k)^j , \quad \lambda = e^{-2i\pi/N}$$

Thus computing P requires $\mathcal{O}(N^2)$ computations (N operations to compute each of the N c_k).

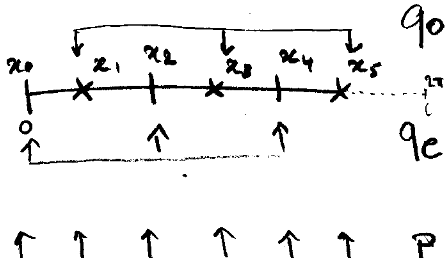
The FFT is based on a tremendous simplification that allows to compute P in $\mathcal{O}(N \log N)$ operations.

For example: $N = 2^{20} \approx 1M$

$N^2 = 2^{40} \approx 1G$, $N \log_2 N \approx 20M$

Basically factor of a million improvement! (in this case)

Here is the basic fact that is used by FFT:



$q_o \equiv$ trig interpolant at odd nodes

$q_o(x_{2j}) = f(x_{2j+1}) \quad 0 \le j \le n-1$

$q_e =$ trig interpolant at even nodes

$q_e(x_{2j}) = f(x_{2j}) \quad 0 \le j \le n-1$

$P \equiv$ trig interpolant at all nodes

$P(x_j) = f(x_j) \quad 0 \le j \le 2n-1$

$$P(x) = \frac{1}{2}(1 + e^{inx}) q_e(x) + \frac{1}{2}(1 - e^{+inx}) q_o\left(x - \frac{\pi}{n}\right)$$

Proof :

$$\left.\begin{array}{l} q_o,\ q_e \text{ have degree } \le n-1 \\ (e^{ix})^n \text{ has degree } n \end{array}\right\} \Rightarrow p \text{ has degree} \le 2n-1$$

We need to show p interpolates f at the $2n$ nodes

$$x_0, x_1, \dots, x_{2n-1}; \quad x_j = \frac{\pi j}{n}, \quad 0 \le j \le 2n-1$$

$$P(x_j) = \frac{1}{2}\left[1 + E_n(x_j)\right] q_e(x_j) + \frac{1}{2}\left[1 - E_n(x_j)\right] q_o\left(x_j - \frac{\pi}{n}\right)$$

$0 \le j \le 2n-1$

$$E_n(x_j) = \exp\left[i n \frac{\pi j}{n}\right] = e^{ij\pi} = \begin{cases} +1 & j \text{ even} \\ -1 & j \text{ odd} \end{cases}$$

thus for j even:

$$p(x_j) = q_e(x_j) = f(x_j)$$

— for j odd:

$$p(x_j) = q_o\left(x_j - \frac{\pi}{n}\right) = q_o(x_{j-1}) = f(x_j) \qquad \underline{\text{QED}}$$

Now this relation is also useful in practice since we can get coeff of P from those of q_e and q_o easily.

Theorem

let

$$\left.\begin{array}{l} q_e = \sum_{j=0}^{n-1} \alpha_j E_j \\ q_o = \sum_{j=0}^{n-1} \beta_j E_j \\ p = \sum_{j=0}^{2n-1} \gamma_j E_j \end{array}\right\} \quad \text{Then for } 0 \le j \le n-1: \quad \begin{array}{l} \gamma_j = \frac{1}{2}\alpha_j + \frac{1}{2} e^{-ij\pi/n} \beta_j \\ \gamma_{j+n} = \frac{1}{2}\alpha_j - \frac{1}{2} e^{-ij\pi/n} \beta_j \end{array}$$

proof:

$$q_o\left(x-\frac{\pi}{n}\right) = \sum_{j=0}^{n-1} \beta_j E_j\left(x-\frac{\pi}{n}\right)$$

$$= \sum_{j=0}^{n-1} \beta_j e^{ij(x-\pi/n)} = \sum_{j=0}^{n-1} \beta_j e^{-i\pi j/n} E_j(x)$$

$$P(x) = \frac{1}{2}\left(1+E_n(x)\right)p(x) + \frac{1}{2}\left(1-E_n(x)\right) q\left(x-\frac{\pi}{n}\right)$$

$$P = \frac{1}{2}\sum_{j=0}^{n-1}\left[(1+E_n)\alpha_j E_j + (1-E_n)\beta_j e^{-i\pi j/n} E_j\right]$$

$$= \frac{1}{2}\sum_{j=0}^{n-1} \underbrace{\left[\alpha_j+\beta_j e^{-i\pi j/n}\right]} E_j + \underbrace{\left[\alpha_j - \beta_j e^{-i\pi j/n}\right]} E_{j+n}$$

(since $E_j E_n = E_{j+n}$) QED.

The relation between odd and even interpolants can be made more general by using the following operators: (linear)

$L_n f$ = trig poly interpol nodes $x_j = \frac{2\pi j}{n}$
$0 \le j \le n-1$

$(T_h f)(x) = f(x+h)$ (translation)

Of course we have:

$$L_n f = \sum_{k=0}^{n-1} (f, E_k)_n E_k$$

And in our formula:

$$P = L_{2n} f$$

$$q^e = L_n f$$

$$q^o = L_n T_{\pi/n} f$$

Thus:

$$\boxed{L_{2n} f = \tfrac{1}{2}(1+E_n) L_n f + \tfrac{1}{2}(1-E_n) T_{-\pi/n} L_n T_{\pi/n} f} \quad (*)$$

We now want to design an algorithm to compute $L_N f$ with $N = 2^m$.

<u>Notation</u>: interp on 2^n nodes

$$P_k^{(n)} = L_{2^n} T_{\frac{2k\pi}{N}} f \qquad 0 \leq n \leq m, \quad 0 \leq k \leq 2^{m-n} - 1$$

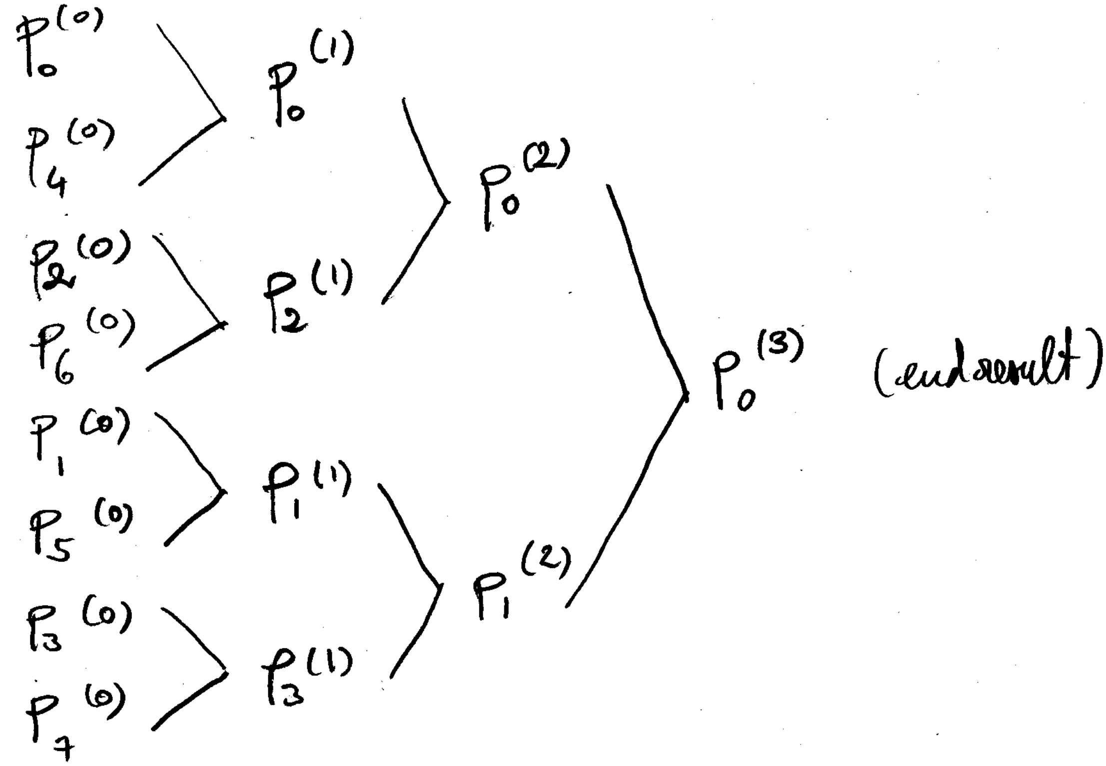
starting at node k

$$= \text{trig interp of degree } 2^n - 1 \text{ s.t.}$$

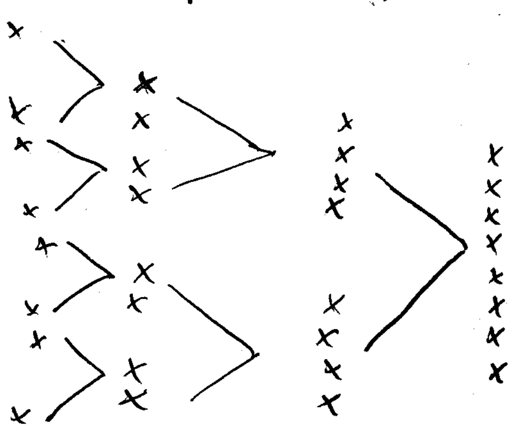
$$P_k^{(n)}\left(\frac{2\pi j}{2^n}\right) = f\left(\frac{2\pi k}{N} + \frac{2\pi j}{2^n}\right), \quad 0 \leq j \leq 2^n - 1$$

Applying $(*)$,

$$P_k^{(n+1)}(x) = \tfrac{1}{2}\left(1+e^{i2^n x}\right) P_k^{(n)}(x) + \tfrac{1}{2}\left(1-e^{i2^n x}\right) P_{k+2^{m-n-1}}^{(n)}\left(x - \frac{\pi}{2^n}\right)$$



2^0 pts	2^1 pts	2^2 pts	2^3 pts.
2^2 sep	2^1 sep.	2^0 sep	0 sep



How do we get an $N \log N$ operation count?

Let $R(n)$ be the # of multiplication needed to compute coeff in interp polynomial for points $\frac{2\pi j}{n}$, $0 \le j \le n-1$.

Then clearly:

$$\underbrace{R(2n)}_{\substack{\text{interp poly}\\ \text{coef for } 2n \text{ pts.}}} \le \underbrace{2R(n)}_{\substack{\text{compute interp}\\ \text{poly on even \& odd}}} + \underbrace{2n}_{\substack{\text{mult to convert}\\ \text{even \& add into true ans.}}}$$

We shall show that $R(2^m) \leq m 2^m$ by induction.

$m=0$: no multiplications are involved because constant interpolant

$\Rightarrow R(2^0) = 0$

Assume $R(2^m) \leq m 2^m$ holds:

$$R(2^{m+1}) = R(2\,2^m) \leq 2R(2^m) + 2\,2^m$$
$$\leq 2m2^m + 2^{m+1} = (m+1)2^{m+1}$$

So total number of operations is $\mathcal{O}(N \log_2 N) = \mathcal{O}(m 2^m)$

In Matlab: fft and ifft

fftw3 (fastest Fourier transform in the west)

not restricted to powers of 2, but N should have many prime factors

⚠ careful with normalizations, e.g. matlab uses:

$$X_k = \sum_{j=0}^{N-1} x_j \, e^{-2i\pi k j/N} = N(x, E_k)_N$$

$$x_j = \frac{1}{N} \sum_{k=0}^{N-1} X_k \, e^{2i\pi k j/N} \quad \text{(inverse fast Fourier transform)}$$

$$= (X, \overline{E_j})_N$$

Convolution using FFT

Convolution has many uses in e.g. signal processing:

$$h = g * f = f * g$$

$$h(x) = \int_{-\infty}^{\infty} g(y)\, f(x-y)\, dy = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

h = output signal
g = input signal
f = filter "impulse response"

$$g \longrightarrow \boxed{f} \longrightarrow h = f * g$$

In the discrete case we can think of h, g as sets of N samples:

$$h_n = \sum_{m=0}^{N-1} g_m f_{n-m} \quad \equiv \text{ discrete convolution}$$

Or in matrix form:

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{N-1} \end{bmatrix} = \begin{bmatrix} f_0 & f_{-1} & f_{-2} & \cdots & f_{1-N} \\ f_1 & f_0 & f_{-1} & \cdots & f_{2-N} \\ f_2 & f_1 & f_0 & \cdots & f_{3-N} \\ \vdots & & & & \\ f_{N-1} & & \cdots & & f_0 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix} \qquad (\star)$$

$$\underline{h} = T_N \, \underline{g} \quad \text{where } T_N = \text{Toeplitz matrix}$$

naive implementation would cost $O(N^2)$

Convolution can be done efficiently by using FFT.

Here is the key result we need:

$$\left(\text{ifft}(F.*G)\right)_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k G_k e^{\frac{2i\pi kn}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} f_j e^{-2i\pi kj/N}\right)\left(\sum_{j'=0}^{N-1} g_{j'} e^{-2i\pi kj'/N}\right) e^{2i\pi kn/N}$$

$$= \sum_{j=0}^{N-1} f_j \sum_{j'=0}^{N-1} g_{j'} \underbrace{\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2i\pi k}{N}(n-j-j')}}_{(E_{n-j}, E_{j'})_N}$$

Now note we have:

$$(E_{n-j}, E_{j'})_N = \begin{cases} 1 & \text{if } n-j-j' \text{ is divisible by } N \\ 0 & \text{otherwise} \end{cases}$$

i.e. $n-j-j' = 0 \mod N$

$j' = n-j \mod N$

Thus:

$$\boxed{\left(\text{ifft}(F.*G)\right)_n = \sum_{j=0}^{N-1} f_j \, g_{(n-j) \bmod N}}$$

So we can evaluate a slightly different convolution efficiently using fft:

1. $F = \text{fft}(f)$; $G = \text{fft}(g)$; $\quad O(N \log N)$
2. $H = F.*G$ $\quad O(N)$
3. $h = \text{ifft}(H)$ $\quad O(N \log N)$

total $O(N \log N)$ operations

In matrix vector product form this becomes:

$$\begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} f_0 & f_{-1} & f_{-2} & \cdots & f_{1-N} \\ f_1 & f_0 & f_{-1} & \cdots & f_{2-N} \\ \vdots & & & & \\ f_{1-N} & \cdots & & & f_0 \end{bmatrix}}_{C_N \,=\, \text{circulant matrix.}} \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix}$$

So the FFT gives us a tool to compute matrix vector products of circulant matrices in $O(N \log N)$ operations!

How do we use it to compute the mat vec product ($\star$)?

The trick is to formulate ($\star$) as a M-V prod with larger matrix that is circulant.

Let $B_N = \begin{bmatrix} 0 & f_{N-1} & f_{N-2} & \cdots & f_2 & f_1 \\ f_{1-N} & 0 & f_{N-1} & \cdots & & f_2 \\ f_{2-N} & f_{1-N} & 0 & & & f_3 \\ \vdots & & & \ddots & & \vdots \\ f_{-1} & f_{-2} & \cdots & & f_{1-n} & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}$

It is easy to check that $C = \begin{bmatrix} T_N & B_N \\ B_N & T_N \end{bmatrix}$ is a circulant matrix

So we can efficiently evaluate:

$$\begin{bmatrix} T_N g \\ B_N g \end{bmatrix} = \underbrace{\begin{bmatrix} T_N & B_N \\ B_N & T_N \end{bmatrix}}_{C} \begin{bmatrix} g \\ 0 \end{bmatrix}$$

throw away

zero padding

in $O(2N \log(2N)) = O(N \log N)$ operations.