# Linear algebra preliminaries

- **Linear space**

  $E$ is a linear space iff $u, v \in E \Rightarrow \alpha u + \beta v \in E$ for all $\alpha, \beta \in \mathbb{R}$

  (note: one can change $\mathbb{R}$ here by $\mathbb{C}$, but we will stick w/ $\mathbb{R}$)

  in english: $E$ is closed under linear operations (add and mult. by scalar)

  Examples: $\mathbb{R}^n$, $\Pi_n$ = poly of degree $\leq n$
  $C[0,1]$, $C^2[0,1]$, etc...

- **Linear span:** Let $v_1, v_2, \ldots, v_k$ be vectors in a linear space $E$.

  $$\text{span}\{v_1\ v_2 \ldots v_k\} = \left\{ \sum_{i=1}^{k} \alpha_i v_i \;\middle|\; \alpha_i \in \mathbb{R}, i = 1..k \right\}$$

  = set of all linear comb. of $v_i$.
  = another example of linear space

- **Linear independence** A family of vectors $v_1, v_2, \ldots, v_k$ is linearly indep. iff:

  $$\sum_{i=1}^{k} \alpha_i v_i = 0 \iff \alpha_i = 0, i = 1..k$$

  in english: one cannot express one vector as a lin. comb of the others

- **Basis**: A basis for a linear space $E$ is a set of linearly indep vectors $\{v_1, v_2, \ldots, v_k\}$ such that

$$E = \text{span}\{v_1, v_2, \ldots, v_k\}$$

$$\dim E = k = \text{dimension}$$
$= \#$ of linearly indep vectors spanning $E$.

Examples:
- $\mathbb{R}^n = \text{span}\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$

where $\underline{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ = ith canonical basis vector

$\leftarrow$ ith pos.

$\Rightarrow$ $\underline{\dim \mathbb{R}^n = n}$

- $\Pi_n = \text{span}\{1, x, x^2, \ldots, x^n\}$ (lin. indep can be shown using FTA)

$\Rightarrow$ $\underline{\dim \Pi_n = n+1}$.

- $C[0,1]$ is infinite dimensional.

A <u>normed linear space</u> is a vector space $E$ endowed with a <u>norm</u> $\|\cdot\| : E \to \mathbb{R}_+$ which satisfies the following properties:

i) $\|x\| \geq 0 \quad \forall x \in E$

ii) $\|x\| = 0 \iff x = 0$

iii) $\|\lambda x\| = |\lambda| \, \|x\|$

iv) $\|x + y\| \leq \|x\| + \|y\|$ (triangle ineq.)

Norms can be used to measure distance.

| <u>Examples in $\mathbb{R}^n$</u> | <u>Examples in $C[0,1]$</u> |
|---|---|
| $\|x\|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$ Euclidian ($l_2$ norm) | $\|f\|_2 = \left( \int_0^1 (f(x))^2 dx \right)^{\frac{1}{2}}$ |
| $\|x\|_1 = \sum_{i=1}^{n} \lvert x_i \rvert$ ($l_1$ norm) | $\|f\|_1 = \int_0^1 \lvert f(x) \rvert dx$ |
| $\|x\|_\infty = \max_{i=1..n} \lvert x_i \rvert$ ($l_\infty$ norm) | $\|f\|_\infty = \max_{x \in [0,1]} \lvert f(x) \rvert$ |
| $\|x\|_p = \left( \sum_{i=1}^{n} \lvert x_i \rvert^p \right)^{\frac{1}{p}}$ | $\|f\|_p = \left( \int_0^1 \lvert f(x) \rvert^p dx \right)^{\frac{1}{p}}$ |
| $1 \leq p \leq \infty$ ($l_p$ norm) | $1 \leq p \leq \infty$ |

which norm we use depends on application.

An $\left|\begin{array}{l}\text{inner}\\ \text{dot}\\ \text{scalar}\end{array}\right|$ product is a symm. positive bilinear form defined over a linear space $E$; i.e.:

$$(\cdot,\cdot) : E \times E \to \mathbb{R}$$

with properties:

i) $(x, y) = (y, x)$ (symmetric)

ii) $(x, x) \geq 0$ and
$(x, x) = 0 \iff x = 0$ (positive)

iii) $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$

(bilinear)

(need only lin. wrt one argument as form is symmetric)

An inner product defines a norm

$$\|x\| = (x, x)^{\frac{1}{2}}$$

Examples of inner products:

$\mathbb{R}^n$: $\quad (x,y) = x^T y = \sum_{i=1}^{n} x_i y_i$

$C[0,1]$ $\quad (f,g) = \int_0^1 f g \, dx$

Inner product convey angle and distance information.

$f$ is orthogonal to $g$ iff $(f,g) = 0$

$f$ is orthogonal to $G \subset E$ iff:

$$\forall g \in G \quad (f,g) = 0$$

If $G = \text{span}\{g_1 \dots g_k\}$ then:

$f$ is orthogonal to $G \Leftrightarrow (f, g_i) = 0, \ i = 1..k$

We often abbreviate orthogonal with $\perp$.