MATH 5610/6860
HOMEWORK #3, DUE TUE SEP 27

Note: Please use the sample code given in the class website.

Updated version: Prob1: a,b are only cases needed. Prob 5: Corrected references to ninth ed.

1. B&F 2.4.2 and 2.4.4 a,b (comparison of Newton's method with and without modification for multiple roots)
2. Consider the fixed point iteration $x_{n+1} = F(x_n)$ with
$$F(x) = (24x + 25/x^2)/25.$$
   (a) Show that if this iteration converges, it converges to $\sqrt[3]{25}$.
   (b) Use the fixed point theorem to show that the iteration converges to $\sqrt[3]{25}$ for any initial guess in $[2, 3]$ (**Hint:** use derivative of F to show F is a contraction).
   (c) Approximate $\sqrt[3]{25}$ to within 10^{-4} with fixed point iteration and Steffensen's method. Which method converges faster?
3. K&C 3.5.1–3.5.3. Let $p(z) = 3z^5 - 7z^4 - 5z^3 + z^2 - 8z + 2$. Please do the following *by hand*.
   (a) Use Horner's algorithm to find $p(4)$.
   (b) Find the Taylor expansion of $p(z)$ about the point $z_0 = 4$ (see class notes, this can be done by applying Horner's algorithm successively)
   (c) Start Newton's method at the point $z_0 = 4$. What is the next iterate z_1?
4. B&F 2.6.2 a,h. In this problem you are asked to find approximations to all roots (real or complex) of a polynomial using the specialized version of Newton's method we saw in class (which uses Horner's method to evaluate $p(z_0)$ and $p'(z_0)$ efficiently). You can proceed as follows:
   (a) Use different initial guesses to find all the real roots. Please clearly indicate your initial guesses and the root it leads to. Example: In 2.6.1 d starting at $z_0 = 1$ we get approximate root 1.12412. Starting at $z_0 = 0$ we get root -0.87605.
   (b) Then deflate the polynomial to remove all of it's real roots using Horner's algorithm. In all the examples you have there will be at most one pair of complex conjugate roots that you can find by the usual quadratic formula. For example for the polynomial $p(z)$ in 2.6.1 d deflating (i.e. dividing by the factors $z - 1.12412$

and $z + 0.87605$ and ignoring the small residual terms) we get that $p(z) \approx (z - 1.12412)(z + 0.87605)q(z)$ where $q(z) = z^2 + 0.24807z + 3.04632$ which has roots $\approx -0.12403 \pm i1.74096$.

(c) If you want to double check your roots you can use the Matlab command `roots`. For example the roots of the polynomial $p(z) = z^4 + 2x^2 - x - 3$ in B&F 2.6.1 d are given by:

```
>> roots([1 0 2 -1 -3])

ans =

  -0.1240 + 1.7410i
  -0.1240 - 1.7410i
   1.1241
  -0.8761
```

5. [**extra credit**] K&C 3.5.11, 3.5.12 Recall that any polynomial of degree n can be written as

$$p(z) = c(z - z_1)(z - z_2)\dots(z - z_n)$$

and that the multiplicity of a root is the number of times it is repeated in the factored form of $p(z)$. Please show the following statements without using Thm 2.11 or Thm 2.12 in the textbook.

(a) If z_* is a root of multiplicity m of $p(z)$ then

$$p(z_*) = p'(z_*) = \dots = p^{(m-1)}(z_*) = 0$$

but $p^{(m)}(z_*) \neq 0$. Hint: You may use Leibniz's generalized product rule:

$$(fg)^{(n)}(z) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(k)}(z) g^{(n-k)}(z).$$

(b) The converse of previous statement. Hint: Use Taylor expansion of $p(z)$about z_*.