PRACTICE FINAL MATH 5610

Prob 1

x	1	2	4	5
$f(x)$	0	3	-3	0

(a) We use divided differences

1	0	3	-2	1
2	3	-3	2	
4	-3	3		
5	0			

$$\boxed{p(x) = 0 + 3(x-1) - 2(x-1)(x-2) + (x-1)(x-2)(x-4)}$$

(b) The interpolation error is:

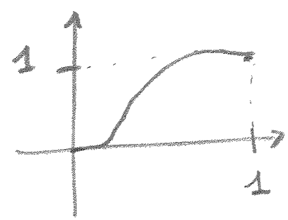
$$f(t) - p(t) = \frac{1}{4!} (x-1)(x-2)(x-4)(x-5) f^{(4)}(\xi)$$

for some $\xi \in [1,5]$.

Prob 2

$S(0) = 0 \qquad S(1) = 1$

$S'(0) = 0 \qquad S'(1) = 0$



Use ansatz $\quad S(x) = ax^3 + bx^2 + cx + d$

$\qquad\qquad\quad S'(x) = 3ax^2 + 2bx + c$

$S(0) = d = 0$

$S'(0) = c = 0$

$$\left.\begin{aligned} S(1) &= a+b = 1 \\ S'(1) &= 3a + 2b \end{aligned}\right\} \Rightarrow a = -2,\ b = 3$$

$$\Rightarrow \boxed{S(x) = -2x^3 + 3x^2}$$

note: this is a clamped cubic spline.

The natural or free cubic spline passing through two points is a line

Prob3 We use Taylor's theorem:

① $$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f^{(3)}(\xi_+)$$

② $$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3!} f^{(3)}(\xi_-)$$

where $\xi_+ \in [x, x+h]$

and $\xi_- \in [x-h, x]$

① - ② gives:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \frac{h^2}{12}\left[f^{(3)}(\xi_+) + f^{(3)}(\xi_-)\right]$$

Using intermediate value theorem, there is some $\xi \in [x-h, x+h]$

s.t.

$$f^{(3)}(\xi) = (1/2)\left(f^{(3)}(\xi_+) + f^{(3)}(\xi_-)\right)$$

hence:

$$\boxed{f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \frac{h^2}{6} f^{(3)}(\xi)}$$

for some $\xi \in [x-h, x+h]$

Problem 4

$$(*) \quad f'(x) = \frac{1}{2h}\left(f(x+h) - f(x-h)\right) - \frac{h^2}{6} f^{(3)}(x) - \frac{h^4}{120} f^{(5)}(x) \ldots$$

What we need to do is apply centered differences w/ h and $h/2$ and find a way of cancelling leading error term.

We have

$$(-1) \quad f'(x) = \underbrace{\frac{1}{2h}\left[f(x+h) - f(x-h)\right]}_{R(h)} - \frac{h^2}{6} f^{(3)}(x) - \frac{h^4}{120} f^{(5)}(x) \ldots$$

$$(4) \quad f'(x) = R(h/2) - \frac{h^2}{24} f^{(3)}(x) - \frac{h^4}{16 \times 120} f^{(5)}(x) \ldots$$

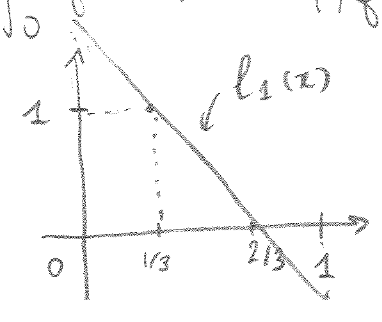
$$3 f'(x) = 4R(h/2) - R(h) + \frac{3}{4 \times 120} h^4 f^{(5)}(x) \ldots$$

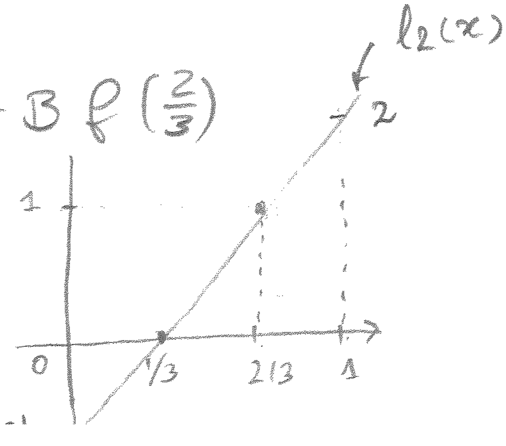
$$\Rightarrow \boxed{f'(x) = \frac{4R(h/2) - R(h)}{3} + \frac{1}{480} h^4 f^{(5)}(x)}$$

Problem 5

Note: this is one of the few case where finding quadrature formulas w/ Lagrange interp is as easy as using undetermined coeff method.

$$\int_0^1 f(x)\,dx \approx A f\left(\tfrac{1}{3}\right) + B f\left(\tfrac{2}{3}\right)$$





we could either write expressions of $l_i(x)$ and compute integrals

$$\int_0^1 l_i(x)\,dx \quad \text{directly.}$$

In this case we can simply look at area under curve.

$$\Rightarrow \quad \int_0^1 l_1(x)\,dx \underset{\uparrow}{=} \int_0^1 l_2(x)\,dx = \frac{1}{2}$$

by symm

$$\Rightarrow \quad \boxed{\int_0^1 f(x)\,dx \approx \frac{1}{2} f\left(\frac{1}{3}\right) + \frac{1}{2} f\left(\frac{2}{3}\right)}$$

To use formula on any interval $[a,b]$ we simply do a change of variables

$$\boxed{\int_a^b f(x)\,dx} = \int_0^1 f\left((b-a)\tilde{x} + a\right)(b-a)\,d\tilde{x}$$

$$\tilde{x} = \frac{x-a}{b-a}$$

$$x = \tilde{x}(b-a) + a$$

$$\boxed{\approx \frac{b-a}{2}\left[f\left(a + \frac{b-a}{3}\right) + f\left(a + \frac{2}{3}(b-a)\right)\right]}$$

Problem 6:

$$I \equiv \int_u^v f(x)\,dx = T(u,v) - \frac{1}{2}(v-u)^3 f''(\xi)$$

(a) Let $w = \dfrac{u+v}{2}$

$$I = \int_u^v f(x)\,dx = \int_u^w f(x)\,dx + \int_w^v f(x)\,dx$$

$$= T(u,w) + T(w,v)$$

$$- \frac{1}{2}(w-u)^3 f''(\xi_1)$$

$$- \frac{1}{2}(v-w)^3 f''(\xi_2)$$

To simplify let $h = v - u$. $\Rightarrow w - u = v - w = \frac{h}{2}$

Thus:

$$I = T(u,w) + T(w,v) - \frac{1}{2\times 8} h^3 \left[f''(\xi_1) + f''(\xi_2) \right]$$

Since $f''(x)$ is continuous, by IVT:

$$f''(\tilde{\xi}) = \frac{1}{2}\left[f''(\xi_1) + f''(\xi_2) \right] \text{ for some } \xi \in (u,v)$$

$$\Rightarrow \boxed{I = T(u,w) + T(w,v) - \frac{h^3}{8} f''(\tilde{\xi})}$$

$$\boxed{C = \frac{-1}{8}}$$

6

We now combine the two eq. as to cancel out the term involving $f''(\xi) \approx f''(\tilde{\xi})$.

$$(-1/3) \qquad I = T(u,v) - \frac{1}{2} h^3 f''(\xi)$$

$$(4/3) \qquad I = T(u,w) + T(w,v) - \frac{1}{8} h^3 f''(\tilde{\xi})$$

$$I \approx \frac{1}{3}\left[4(T(u,w)+T(w,v)) - T(u,v)\right]$$

$$= T(u,w)+T(w,v) + \underbrace{\frac{1}{3}\left[T(u,w)+T(w,v) - T(u,v)\right]}_{\approx -\frac{1}{8}h^3 f''(\tilde{\xi})}$$

Problem 7

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[E_3 = E_3 - E_1]{E_2 = E_2 - E_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 0 & 1 \end{bmatrix}$$

Problem 8 Note this problem is long for an exam

and typo: $p_2(x) = x^2 - \frac{1}{3}$

$$p_4(x) = x p_3(x) - \frac{(xp_3, p_3)}{(p_3, p_3)} p_3(x) - \frac{(xp_3, p_2)}{(p_2, p_2)} p_2(x)$$

$$(xp_3, p_3) = \int_{-1}^{1} \underbrace{x p_3^2(x)}_{\text{odd}} dx = 0$$

$$(xp_3, p_2) = \int_{-1}^{1} x\left(x^3 - \frac{3}{5}x\right)\left(x^2 - \frac{1}{3}\right) dx$$

$$= \int_{-1}^{1} x^6 - \frac{1}{3}x^4 - \frac{3}{5}x^4 + \frac{1}{5}x^2 \, dx$$

$$= \frac{2}{7} - \frac{2}{15} - \frac{6}{25} + \frac{2}{15} = \frac{50-42}{175} = \frac{8}{175}$$

$$(p_2, p_2) = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9} \, dx$$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{2}{5} - \frac{2}{9} = \frac{18-10}{45} = \frac{8}{45}$$

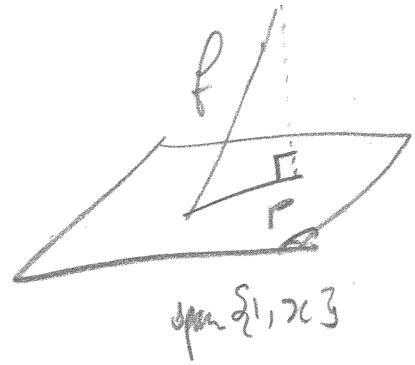
$$\Rightarrow \quad p_4(x) = \underbrace{x^4 - \frac{3}{5}x^2}_{x p_3(x)} - \underbrace{\frac{8}{175} \times \frac{45}{8}}_{9/35} \left(x^2 - \frac{1}{3}\right)$$

$$= x^4 - \underbrace{\left(\frac{3}{5} + \frac{9}{35}\right)}_{= \frac{30}{35} = \frac{6}{7}} x^2 + \frac{3}{35}$$

Problem 9: The condition for optimality is:

$$(p-f, 1) = 0$$
$$(p-f, x) = 0$$

i.e.



where $p(x) = a + bx$

This gives system:

$$\begin{bmatrix} (1,1) & (1,x) \\ (x,1) & (x,x) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (1,x^3) \\ (x,x^3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/5 \end{bmatrix} \Rightarrow \begin{array}{l} a = -\frac{1}{5} \\ b = 9/10 \end{array}$$

Thus $\boxed{p(x) = -\frac{1}{5} + \frac{9}{10}x}$

Problem 7:

$$\sum_{j=0}^{N-1} |p(x_j)|^2 = \sum_{j=0}^{N-1} \left[\sum_{k=0}^{N-1} c_k E_k(x_j)\right] \left[\overline{\sum_{k'=0}^{N-1} c_{k'} E_{k'}(x_j)}\right]$$

$$= \sum_{k=0}^{N-1} \sum_{k'=0}^{N-1} c_k \overline{c_{k'}} \underbrace{\sum_{j=0}^{N-1} E_k(x_j) \overline{E_{k'}(x_j)}}_{N(E_k, E_{k'})_N = N\delta_{kk'}}$$

$$= N \sum_{k=0}^{N-1} |c_k|^2$$